The Good Property of Two-Generated Ideals in Integral Domains

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Abstract. In this paper, we introduce and study a class of integral domains \( D \) characterized by the property that whenever \( r, s \in D - \{0\} \) and the ideal \( (r^k, s^k) \) is principal for some \( k \in \mathbb{N} \), then the ideal \( (r, s) \) is principal. We call them Good domains. We show that a Good domain \( D \) is a root closed domain and the converse is true in different cases as follows: (1) \( D \) is quasi-local, (2) \( \text{Pic}(D) = 0 \), (3) \( u^{1/k} \in D \) for all \( u \in D \) and \( k \in \mathbb{N} \), (4) \( D \) is \( t \)-local. We also show that a quasi-local domain \( D \) with the property that \( (r, s)^k = (r^k, s^k) \) for all \( r, s \in D - \{0\} \) and \( k \in \mathbb{N} \), is a Good domain, that a Prüfer Good domain with torsion Picard group is a Bézout domain, and that the integral closure of a domain in an algebraically closed field is a Good domain.

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1. Introduction

In [6], Judith D. Sally showed that for \( J \) any ideal of a quasi-local ring \( R \), if for some integer \( q \geq 1 \), \( v(J^q) = 1 \), then either \( v(J^k) = 1 \) for all positive integers \( k \) or \( J \) consists of zero divisors. If for some integer \( q > 1 \), \( v(J^q) = 2 \), then either \( v(J^k) = 2 \) for all positive integers \( k \) or \( J \) consists of zero divisors, where \( v(J) \) denotes the minimal number, which may be infinite, of generators of \( J \).

In [2], Gerhard Angermüller introduced \( n \)-root closed domains. He called a domain \( D \) with quotient field \( K \) \( n \)-root closed if whenever \( x \in K \) with \( x^n \in D \) for an integer \( n \geq 1 \), then \( x \in D \); \( D \) is called root closed if \( D \) is \( n \)-root closed, for all \( n > 1 \). Obviously, any
intelligently closed domain is a root closed domain. The converse is not true. He showed that if $q$ is a square-free integer, then $\mathbb{Z}[\sqrt{q}]$ is non-integrally closed and root closed iff $q \equiv 1 \pmod{8}$.

In [1], D.D. Anderson and M. Zafrullah introduced almost Bézout domains and almost Prüfer domains. They called $D$ an almost Bézout domain (AB-domain) if for every $r, s \in D - \{0\}$, there exists a positive integer $k = k(r, s)$ such that $(r^k, s^k)$ is principal, and $D$ is an almost Prüfer domain (AP-domain) if for every $r, s \in D - \{0\}$, there exists a positive integer $k = k(r, s)$ such that $(r^k, s^k)$ is invertible. They showed that $D$ is an almost Bézout domain iff $D$ is an almost Prüfer domain with torsion class group. They also showed that an integrally closed AB-domain (respectively, almost AP-domain) is a Prüfer domain with torsion class group (respectively, Prüfer domain).

In this paper, we study the following good property of two-generated ideals in integral domains. We call an integral domain $D$ a Good domain if whenever $r, s \in D - \{0\}$ and the ideal $(r^k, s^k)$ is principal for some $k \in \mathbb{N}$, then the ideal $(r, s)$ is principal.

We show that a Good domain is a root closed domain (Proposition 3). The converse holds in some cases: (1) $D$ is quasi-local (Proposition 5), (2) $\text{Pic}(D) = 0$ (Corollary 6), (3) $u^k \in D$ for all $u \in D$ and $k \in \mathbb{N}$ (Corollary 9), (4) $D$ is $t$-local (Proposition 10). If $D$ is a root closed domain with $(r^k, s^k) = (u^k)$ for some $r, s, u \in D - \{0\}$, then $(r, s) = (u)$ (Proposition 8). A quasi-local domain with the property that $(r, s)^k = (r^k, s^k)$ for all $r, s \in D$ and for all $k \in \mathbb{N}$, is a Good domain (Proposition 11). An almost Bézout domain $D$ is a Bézout domain (Proposition 13). A Prüfer Good domain with torsion Picard group is a Bézout domain (Proposition 14). The integral closure of a domain in an algebraically closed field is a Good domain (Proposition 15).

For the reader’s convenience, we give a working introduction here for the notions involved. Let $D$ be an integral domain with quotient field $K$, and let $F(D)$ denote the set of nonzero fractional ideals of $D$.

A function $A \mapsto A^* : F(D) \to F(D)$ is called a star operation on $D$ if $*$ satisfies the following three conditions for all $0 \neq x \in K$ and for all $A, B \in F(D)$: (1) $D^* = D$ and $(xA)^* = xA^*$, (2) $A \subseteq B^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$, (3) $(A^*)^* = A^*$. An ideal $A \in F(D)$ is called a *-ideal if $A^* = A$. For all $A, B \in F(D)$, we have $(AB)^* = (A^*B^*) = (A^*B^*)^*$. These equations define the so-called *-multiplication. If $\{A_n\}$ is a subset of $F(D)$ such that $\cap A_n \neq 0$, then $\cap A_n^*$ is a *-ideal. Also, if $\{A_n\}$ is a subset of $F(D)$ such that $\sum A_n$ is a fractional ideal, then $\sum A_n^* = (\sum A_n)^*$. The function $* : F(D) \to F(D)$ given by $A^* = \cap B^*$, where $B$ ranges over all nonzero finitely generated sub-ideals of $A$, is also a star operation; $*$ is said to be a star operation of finite character if $*$ is a finite operation of any $*$-ideal $A$ is of finite type if $A$ is a $*$-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal. A *-ideal $A$ is of finite type if $A$ is a *-ideal in $*$, then every proper *-ideal is contained in some maximal *-ideal and every maximal *-ideal is a prime ideal.
of $D$ modulo the subgroup of principal ideals. The $t$-class group of $D$, $Cl_t(D)$, is the group of all $t$-invertible fractional $t$-ideals of $D$ under $t$-multiplication (i.e., the operation sending a pair of $t$-ideals $A, B$ of $D$ to $(AB)_t$) modulo the subgroup of principal ideals [3]. $Pic(D)$ is a subgroup of $Cl_t(D)$.

Throughout this paper, we denote the integral closure of a domain $D$ by $D'$ and the quotient field of a domain $D$ by $K$. Our standard reference for any undefined notation or terminology is [4].

2. Good domain

**Remark 1.** Let $D$ be a Dedekind domain with torsion class group, which is not a PID. Then there exists a two-generated ideal $(r, s)$ of $D$ that is not principal, but the ideal $(r, s)^k$ of $D$ is principal for some $k \in \mathbb{N}$. Now in a Prufer domain, $(r, s)^k = (r^k, s^k)$ for all $r, s \in D$ [4, Theorem 24.3]. For example, let $D = \mathbb{Z}[\sqrt{-5}]$. The ring $\mathbb{Z}[\sqrt{-5}]$ is known to be a non-PID Dedekind domain such as $Cl(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2\mathbb{Z}$. Since $Cl(\mathbb{Z}[\sqrt{-5}]) \neq 0$, there is a prime ideal $P$ of $\mathbb{Z}[\sqrt{-5}]$ which is not principal. Now every ideal of a Dedekind domain which is not principal is always generated by two elements [4, Theorem 38.5]. So we can take $P = (r, s)$. Since $Cl(\mathbb{Z}[\sqrt{-5}]) = 2$, we must have $P^2 = (r^2, s^2)$ principal. For instance, take $r = 2$ and $s = 1 + \sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$. Then $(2, 1 + \sqrt{-5})$ is non-principal and $(2^2, (1 + \sqrt{-5})^2) = (2)$ in $\mathbb{Z}[\sqrt{-5}]$.

**Example 2.** Let $F$ be a field with characteristic $m \neq 0$, $L$ be a purely inseparable extension of $F$ such that $L^m \subset F$ and $X$ be an indeterminate over $L$. Define $D = F + XL[X] = \{a_0 + \sum_{i=1}^m a_iX^i : a_0 \in F$ and $a_i \in L\}$. By [7, Example 2.13], it is clear that $D$ is a non-integrally closed AB-domain. Now let $l_1, l_2 \in L/F$ such that $l_1l_2 \not\in F$. Then $(l_1X, l_2X)D$ is non-principal, but $(l_1^2X^m, l_2^2X^m) = (X^m)$ in $F[X]$ and so in $D$.

**Proposition 3.** A Good domain is a root closed domain.

**Proof.** Let $D$ be a Good domain and $x \in K - \{0\}$ with $x^k \in D$ for some $k > 1$. Say $x = r/s$, where $r, s \in D - \{0\}$; so $x^k = r^k/s^k$. Now $r^k/s^k \in D$ implies that $s^k|p^k$, which gives $(r^k, s^k)$ is principal and so gives $(r, s)$ is principal. We claim $(r, s) = (s)$. Suppose not and let $(r, s) = (d)$. Then $r = ad$ and $s = bd$, where $(a, b) = D$. Since $s^k|p^k$, we have $b^k|a^k$. But this can happen only if $b^k$ and hence $b$ is a unit. Thus $(d) = (s)$ and $(r, s) = (s)$ implies that $s|r$, so $x = r/s \in D$. Hence $D$ is root closed. □

**Remark 4.** The converse of Proposition 3 is false. Indeed, $\mathbb{Z}[\sqrt{-5}]$ is an integrally closed domain and so is also root closed [2]. Note that the ideal $(2, 1 + \sqrt{-5})$ is not principal but $(2^2, (1 + \sqrt{-5})^2) = (4, 2\sqrt{-5}) = (4, 4 + 2\sqrt{-5} + 4) = (4, 2\sqrt{-5}) = 2(2, \sqrt{-5}) = 2D = (2)$. Hence $D$ is not a Good domain.

**Proposition 5.** A root closed quasi-local domain is a Good domain.

**Proof.** Let $D$ be a root closed quasi-local domain, and let $(r^k, s^k) = (u)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Since $D$ is quasi-local, so $(u) = (r^k)$ or $(u) = (s^k)$ implies that $r^k | s^k$ or $s^k | r^k$. As $D$ is also root closed, so $r | s$ or $s | r$. Finally, $(r, s) = (r)$ or $(r, s) = (s)$. Hence $D$ is a Good domain. □

**Corollary 6.** Let $D$ be a root closed domain and $r, s \in D - \{0\}$. If the ideal $(r^k, s^k)$ is principal for some $k \in \mathbb{N}$, then the ideal $(r, s)$ is invertible. In particular, a root closed domain $D$ with $Pic(D) = 0$ is a Good domain.
Proof. Since $D$ is root closed, $D_M$ is also root closed for all $M \in \text{Max}(D)$ [2, Lemma 2]. Now $(r^k, s^k)$ is principal implies that $(r^k, s^k)D_M$ is principal. By Proposition 5, $(r, s)D_M$ is principal. Thus $(r, s)$ is locally principal, and hence invertible [5, Theorem 62].

Remark 7. Corollary 6 can be used to give an example of a Good domain which is not quasi-local. Indeed, if $S$ is a subfield of another field $L$, then the $t$-class group of a domain $S + XL[X]$ is zero [3, Example 1.10]. As the Picard group is a subgroup of the $t$-class group, we have $\text{Pic}(S + XL[X]) = 0$. Now it is well known that $S + XL[X]$ is not a quasi-local domain. So, if $S + XL[X]$ is root closed then by Corollary 6, $S + XL[X]$ is a Good domain. For instance, let $L$ be an algebraic closure of $\mathbb{Q}$ and let $S$ be the subfield of $L$ consisting of all elements $\theta$ of $L$ such that the minimal polynomial for $\theta$ over $\mathbb{Q}$ is solvable by radicals over $\mathbb{Q}$. Define $D = S + XL[X]$. Then by [4, Exercise 6, Page 184], $D$ is a root closed domain with $\text{Pic}(D) = 0$, which is not a quasi-local domain. Hence by Corollary 6, $D$ is a Good domain.

Proposition 8. If $D$ is a root closed domain with $(r^k, s^k) = (u^k)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$, then $(r, s) = (u)$.

Proof. Let $(r^k, s^k) = (u^k)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$ implies that $u^k | r^k, u^k | s^k$. Since $D$ is root closed, so $u | r, u | s$. Then $r = au$, $s = bu$ for some $a, b \in D$, where $(a, b) = D$. Hence $(r, s) = (u)$.

Corollary 9. If $D$ is a root closed domain in which $u^{1/k} \in D$ for all $u \in D$ and $k \in \mathbb{N}$, then $D$ is a Good domain.

Proof. Let $(r^k, s^k) = (u) = (u^{1/k})^k$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Then by Proposition 8, $(r, s) = (u^{1/k})$. Hence $D$ is a Good domain.

Recall from [1] that a domain $D$ is called a $t$-local domain if $D$ has a unique maximal t-ideal, equivalently, if $D$ has a unique maximal ideal $M$ which is also a t-ideal.

Recall from [7] that $r, s \in D - \{0\}$ are called $v$-coprime if $(r, s)_v = D$.

Proposition 10. Let $D$ be a $t$-local domain. Then the following assertions hold:

1. Any two nonzero nonunit elements of $D$ are not $v$-coprime.
2. If $(r, s)_v = (u)$ for some $r, s, u \in D - \{0\}$, then either $(u) = (r)$ or $(u) = (s)$.
3. If $D$ is a root closed domain, then $D$ is also a Good domain.

Proof. (1) Let $D$ be a $t$-local domain with maximal ideal $M$, and let $r, s \in D - \{0\}$ be nonunits. Then $(r, s)_v \subseteq M$ implies that $(r, s)_v \subseteq M$. Hence $r, s$ are not $v$-coprime.

(2) Let $(r, s)_v = (u)$ for some $r, s, u \in D - \{0\}$. Then $(r/u, s/u)_v = D$ implies by (1) that $r/u$ or $s/u$ is a unit. Therefore, $(r/u) = D$ or $(s/u) = D$ gives $(u) = (r)$ or $(u) = (s)$.

(3) Let $(r^k, s^k) = (u)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Then by (2), $(u) = (r^k)$ or $(u) = (s^k)$ gives $r^k | s^k$ or $s^k | r^k$. Since $D$ is root closed; so $r | s$ or $s | r$ implies that $(r, s) = (r) \text{ or } (r, s) = (s)$. Hence $D$ is a Good domain.

Proposition 11. If $D$ is a quasi-local domain with property $P : (r, s)^k = (r^k, s^k)$ for all $r, s \in D - \{0\}$ and for all $k \in \mathbb{N}$, then $D$ is a Good domain.

Proof. Let $(r^k, s^k) = (u)$ for some $r, s, u \in D - \{0\}$ and $k \in \mathbb{N}$. Then by property $P$, $(r, s)^k = (r^k, s^k) = (u)$ implies that $(r, s)$ is principal [6]. Hence $D$ is a Good domain.
Corollary 12. If $D$ is a domain with property $P : (r, s)^k = (r^k, s^k)$ for all $r, s \in D - \{0\}$ and for all $k \in \mathbb{N}$, then $D$ is root closed.

Proof. Suppose $D$ has property $P$. Then $D$ locally also has property $P$. Therefore, by Proposition 11, Proposition 3, and [2, Lemma 2], $D$ is root closed. □

Recall from [1] that a domain $D$ is called an almost Bézout domain (AB-domain) if for each pair $r, s \in D - \{0\}$, there exists a positive integer $k = k(r, s)$ such that $(r^k, s^k)$ is principal.

Proposition 13. A domain $D$ is a Bézout domain if and only if it is an almost Bézout and a Good domain.

Proof. Clearly a Bézout domain is an almost Bézout and a Good domain. Conversely, let $r, s \in D - \{0\}$. Since $D$ is an almost Bézout domain, there exists a positive integer $k = k(r, s)$ such that $(r^k, s^k)$ is principal. As $D$ is also a Good domain, we get that $(r, s)$ is principal. Hence $D$ is a Bézout domain. □

Proposition 14. If $D$ is a Prüfer Good domain with torsion Picard group, then $D$ is a Bézout domain.

Proof. Let $r, s \in D - \{0\}$. Since $D$ is a Prüfer domain with torsion Picard group, there exists $k \in \mathbb{N}$ such that $(r, s)^k = (r^k, s^k) = (u)$ for some $u \in D$. As $D$ is also a Good domain, we get that $(r, s)$ is principal. Hence $D$ is a Bézout domain. □

Proposition 15. The integral closure of a domain $D$ in an algebraically closed field is a Good domain.

Proof. Let $E = D[t]$, where $L$ is an algebraically closed field containing the quotient field of $D$, and let $(e^k, f^k) = (g)$ for some $e, f, g \in E - \{0\}$ and $k \in \mathbb{N}$. Take $h = g^{1/k}$. Since $E$ being integrally closed is also root closed and $h^k \in E$, then $h \in E$. We have $(e^k, f^k) = (h^k)$ implies that $h^k \mid e^k, h^k \mid f^k$ in $E$. As $E$ is integrally closed, we have $h \mid e, h \mid f$ in $E$. Say $e = xh$ and $f = yh$ for some $x, y \in E$ with $(x, y) = D$, this gives $(xh, yh) = hD = (h)$, which implies that $(e, f) = (h)$. Hence $E$ is a Good domain. □

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