Action of the möbius group $M = \langle x, y : x^2 = y^6 = 1 \rangle$ on certain real quadratic fields

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Abstract. Let $C' = C \cup \{\infty\}$ be the extended complex plane and $M = \langle x, y : x^2 = y^6 = 1 \rangle$, where $x(z) = \frac{1}{z^3}$ and $y(z) = \frac{1}{3(z+1)}$ are the linear fractional transformations from $C' \to C'$. Let $m$ be a square-free positive integer. Then $Q^*(\sqrt{m}) = \{a + \sqrt{m} : a \neq 0, b = \frac{a^2-n}{c} \in \mathbb{Z} \}$ for all $k \in \mathbb{N}$. For non-square $n = 3^h \prod_{i=1}^{t} p_i^{k_i}$, it was proved in an earlier paper by the same authors that the set $Q^{***}(\sqrt{m}) = \{a + \sqrt{m} : a \in Q^*(\sqrt{m}) \}$ is a proper subset of $Q(\sqrt{m})$ for all $k \in \mathbb{N}$. In this paper we prove that if $h \geq 2$, then $Q^{***}(\sqrt{m}) = (Q(\sqrt{m}) \setminus Q^{***}(\sqrt{9m})) \cup Q(\sqrt{m}) \cup Q^{***}(\sqrt{9m})$ and also determine its proper $M$-subsets. In particular $Q(\sqrt{m}) \setminus Q = \cup Q^{***}(\sqrt{k^2m})$ for all $k \in \mathbb{N}$.

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1. INTRODUCTION

Throughout the paper we take $m$ as a square free positive integer. Since every element of $Q(\sqrt{m}) \setminus Q$ can be expressed uniquely as $\frac{a + \sqrt{m}}{c}$, where $n = k^2m$, $k$ is any positive...
integer and \(a, b = \frac{a^2 - n}{c}\) and \(c\) are relatively prime integers and we denote it by \(\alpha_n(a, b, c)\) or \(\alpha(a, b, c)\). Then

\[
Q^*(\sqrt{n}) = \left\{ \frac{a + \sqrt{n}}{c} : a, c, b = \frac{a^2 - n}{c} \in \mathbb{Z} \text{ and } (a, b, c) = 1 \right\},
\]

\[
Q''(\sqrt{n}) = \left\{ \frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 3 \right\},
\]

\[
Q'''(\sqrt{n}) = \left\{ \frac{a + \sqrt{n}}{c} : \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 3 \mid c \right\}
\]

are subsets of the real quadratic field \(Q(\sqrt{n})\) for all \(n\) and \(Q(\sqrt{m}) \setminus Q\) is a disjoint union of \(Q^*(\sqrt{n})\) for all \(n\). If \(\alpha(a, b, c) \in Q^*(\sqrt{n})\) and its conjugate \(\overline{\alpha}\) have opposite signs then \(\alpha\) is called an ambiguous number [7]. A non-empty set \(\Omega\) with an action of a group \(G\) on it, is said to be a \(G\)-set. We say that \(\Omega\) is a transitive \(G\)-set if, for any \(p, q\) in \(\Omega\) there exists \(a \in G\) such that \(pa = q\).

We are interested in linear-fractional transformations \(x, y\) satisfying the relations \(x^2 = y^r = 1\), with a view to studying an action of the group \(\langle x, y \rangle\) on real quadratic fields. If \(y : z \to \frac{az + b}{cz + d}\) is to act on all real quadratic fields then \(a, b, c, d\) must be rational numbers, and can be taken to be integers. Thus \(\frac{a + b}{ad - bc}\) is rational. But if \(z \to \frac{az + b}{cz + d}\) is of order \(r\), one must have \(\frac{(a + b)^2}{ad - bc} = \omega + \omega^{-1} + 2\), where \(\omega\) is a primitive \(r\)-th root of unity. Now \(\omega + \omega^{-1}\) is rational, for a primitive \(r\)-th root, only if \(r = 1, 2, 3, 4\) or \(6\), so that these are the only possible orders of \(y\). The group \(\langle x, y : x^2 = y^r = 1 \rangle\) is cyclic of order \(2\) or \(D_n\) (an infinite dihedral group) according as \(r = 1\) or \(2\). For \(r = 3\), the group \(\langle x, y \rangle\) is the modular group \(PSL(2, \mathbb{Z})\). The fractional linear transformations \(x, y\) with \(x(z) = \frac{1}{2z}\) and \(y(z) = \frac{z - 1}{3z + 1}\) generate a subgroup \(M\) of the modular group which is isomorphic to the abstract group \(\langle x, y : x^2 = y^6 = 1 \rangle\). It is a standard example from the theory of the modular group. It has been shown in [10] that the action of \(M\) on the rational projective line \(Q \cup \{\infty\}\) is transitive.

In our case the set \(Q(\sqrt{m}) \setminus Q\) is an \(M\)-set. It is noted that \(M\) is the free product of \(C_2 = \langle x : x^2 = 1 \rangle\) and \(C_6 = \langle x : y^6 = 1 \rangle\). The action of the modular group \(PSL(2, \mathbb{Z})\) on the real quadratic fields has been discussed in detail in [1, 6, 8, 9, 11, 12]. The actual number of ambiguous numbers in \(Q^*(\sqrt{n})\) has been discussed in [8] as a function of \(n\).

In a recent paper [11], the authors have investigated that the cardinality of the set \(E_p\), \(p\) a prime factor of \(n\), consisting of all classes \([a, b, c](mod\ p)\) of the elements of \(Q^*(\sqrt{n})\) is \(p^3 - 1\) and obtained two proper \(G\)-subsets of \(Q^*(\sqrt{n})\) corresponding to each odd prime divisor of \(n\). The same authors in [12] have determined the cardinality of the set \(E_{p^r}\), \(r \geq 1\), consisting of all classes \([a, b, c](mod\ p^r)\) of the elements of \(Q^*(\sqrt{n})\) and have determined, for each non-square \(n\), the \(G\)-subsets of an invariant subset \(Q^*(\sqrt{n})\) of \(Q(\sqrt{m}) \setminus Q\) under the modular group action by using classes \([a, b, c](mod\ n)\). Real quadratic irrational numbers under the action of the group \(M\) have been studied in [3, 4, 5, 7, 10]. Closed paths in the coset diagrams under the action of a proper subgroup of \(M\) on \(Q(\sqrt{m})\) have been discussed in [4]. M. Aslam Malik et al. in [2] have studied the action of \(H = \langle x, y : x^2 = y^4 = 1 \rangle\), where \(x(z) = \frac{1}{2z}\) and \(y(z) = \frac{z - 1}{2(z + 1)}\), on \(Q(\sqrt{m}) \setminus Q\). The same authors, in [3], have discussed the properties of real quadratic irrational numbers under the action of the group \(M\). The authors proved, in [3], that if \(n = 1, 3, 4, 6\) or \(7(mod\ 9)\)
then \( \mathbb{Q}^{***}(\sqrt{m}) \) is an \( M \)-subset of \( \mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q} \) and \( \mathbb{Q}'''(\sqrt{m}) = \mathbb{Q}^*(\sqrt{m}) \cup \mathbb{Q}^{***}(\sqrt{9m}) \).

In this paper we extend these results for all non-square integers \( n \) and give some modifications of Lemma 1.1 of [3] for the case \( n \equiv 0 \pmod{9} \) and prove that \( \mathbb{Q}'''(\sqrt{m}) = \left( \mathbb{Q}^*(\sqrt{m}) \setminus \mathbb{Q}^{***}(\sqrt{m}) \right) \cup \mathbb{Q}^{***}(\sqrt{m}) \cup \mathbb{Q}^{***}(\sqrt{9m}) \) which shows that \( \mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q} \) is the union of \( \mathbb{Q}'''(\sqrt{k^2m}) \forall k \in \mathbb{N} \). However if \( n \) and \( n' \) are two distinct non-square positive integers then \( \mathbb{Q}^*(\sqrt{m}) \cap \mathbb{Q}^*(\sqrt{m'}) = \emptyset \) whereas \( \mathbb{Q}'''(\sqrt{m}) \cap \mathbb{Q}'''(\sqrt{m'}) = \mathbb{Q}^*(\sqrt{m}) \cap \mathbb{Q}^*(\sqrt{m'}) \) may not be empty. In particular \( \mathbb{Q}'''(\sqrt{m}) \cap \mathbb{Q}'''(\sqrt{9m}) \) is not empty. In fact we prove that a superset namely

\[
\mathbb{Q}^{***}(\sqrt{9m}) \cup \left\{ \frac{a + \sqrt{9m}}{c} : \alpha = \frac{a + \sqrt{9m}}{c} \in \mathbb{Q}^*(\sqrt{9m}) \setminus \mathbb{Q}^{***}(\sqrt{9m}) \right\}
\]

of \( \mathbb{Q}^{***}(\sqrt{9m}) \) is an \( M \)-subset of \( \mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q} \).

We have also found \( M \)-subsets of \( \mathbb{Q}'''(\sqrt{m}) \) such that these may or may not be transitive. However they help in determining the transitive \( M \)-subsets (\( M \)-orbits). The notation is standard and we follow [3], [9], [11] and [12]. In particular \( (\cdot, \cdot) \) denotes the Legendre symbol and \( x(Y) = \{ \frac{a}{p} : \alpha \in Y \} \) for each subset \( Y \) of \( \mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q} \). Throughout this paper, \( n \) denotes a non-square positive integer and \( \alpha \) denotes \( \frac{\alpha + \sqrt{m}}{c} \) with \( b = \frac{a^2 - n}{c} \) such that \( (a, b, c) = 1 \).

2. Preliminaries

The following results of [3], [11] and [12] will be used in the sequel.

**Lemma 1.** (3). Let \( \alpha = \frac{a + \sqrt{m}}{c} \in \mathbb{Q}^*(\sqrt{m}) \) with \( b = \frac{a^2 - n}{c} \). Then:

1. If \( n \not\equiv 0 \pmod{9} \) then \( \frac{a}{c} \in \mathbb{Q}^{***}(\sqrt{m}) \) if and only if \( 3 \mid b \).
2. \( \frac{a}{c} \in \mathbb{Q}^{***}(\sqrt{9m}) \) if and only if \( 3 \not\mid b \).

**Theorem 2.** (3) The set \( \mathbb{Q}'''(\sqrt{m}) = \{ \frac{a}{c} : \alpha \in \mathbb{Q}^*(\sqrt{m}), t = 1, 3 \} \) is invariant under the action of \( M \).

**Theorem 3.** (see [3]) For each \( n \equiv 1, 3, 4, 6 \) or \( 7 \pmod{9} \),

\[
\mathbb{Q}^{***}(\sqrt{m}) = \left\{ \alpha + \sqrt{m} : \alpha = \frac{a + \sqrt{m}}{c} \in \mathbb{Q}^*(\sqrt{m}) \text{ and } 3 \mid c \right\}
\]

is an \( M \)-subset of \( \mathbb{Q}'''(\sqrt{m}) \).

**Corollary 4.** (3) \( \mathbb{Q}'''(\sqrt{m}) = \emptyset \) if and only if \( n \equiv 2 \pmod{3} \).

It is well known that \( \mathbb{G} = \langle x, y : x^2 = y^3 = 1 \rangle \) represents the modular group, where \( x(z) = \frac{1}{z}, y(z) = \frac{-2z}{z} \) are linear fractional transformations.

**Theorem 5.** ([11]) Let \( p \) be an odd prime factor of \( n \). Then both of \( S_1^p = \{ \alpha \in \mathbb{Q}^*(\sqrt{p}) : (b/p) = 1 \} \) and \( S_2^p = \{ \alpha \in \mathbb{Q}^*(\sqrt{p}) : (c/p) = 1 \} \) are \( G \)-subsets of \( \mathbb{Q}^*(\sqrt{m}) \). In particular these are the only \( G \)-subsets of \( \mathbb{Q}^*(\sqrt{m}) \) depending upon classes \([a, b, c] \) modulo \( p \).

**Theorem 6.** ([12]) Let \( n = \prod p_i^{k_i} \), where \( p_1, p_2, ..., p_r \) are distinct odd primes such that \( n \) is not equal to a single prime congruent to 1 modulo 8. Then the number of \( G \)-subsets of \( \mathbb{Q}^*(\sqrt{m}) \) is \( 2^r \) namely \( S_{1,2,3,...,r} \) if \( k = 0 \) or 1. Moreover if \( k \geq 2 \), then each \( G \)-subset \( X \) of these \( G \)-subsets further splits into two proper \( G \)-subsets \( \{ \alpha \in X : b \text{ or } c \equiv 1 \pmod{4} \} \) and \( \{ \alpha \in X : b \text{ or } c \equiv -1 \pmod{4} \} \). Thus the number of \( G \)-subsets of \( \mathbb{Q}^*(\sqrt{m}) \) is \( 2^{r+1} \) if \( k \geq 2 \). More precisely these are the only \( G \)-subsets of \( \mathbb{Q}^*(\sqrt{m}) \) depending upon classes \([a, b, c] \) modulo \( n \).
3. ACTION OF $M = \langle x, y : x^2 = y^3 = 1 \rangle$ ON $Q'''(\sqrt{n})$

In this section we establish that if $n$ contains $r$ distinct prime factors then $Q'''(\sqrt{n}) \setminus Q''''(\sqrt{n})$ is the disjoint union of $2^r$ subsets which are invariant under the action of $M$. However these $M$ invariant subsets may further split into transitive $M$-subsets ($M$-orbits) of $Q''''(\sqrt{n})$, for example $Q''''(\sqrt{37})$ splits into twelve orbits namely $(\sqrt{37})^M, (-\sqrt{37})^M, (\frac{1+\sqrt{37}}{2})^M, (\frac{1-\sqrt{37}}{2})^M, (-\frac{1+\sqrt{37}}{2})^M, (\frac{1-\sqrt{37}}{2})^M, (\frac{1+\sqrt{37}}{6})^M, (-\frac{1+\sqrt{37}}{6})^M, (\frac{1-\sqrt{37}}{6})^M$ and $(\frac{1+\sqrt{37}}{6})^M$. The first six orbits are contained in $A_1^{37} \cup x(A_1^{37})$ and last four orbits are contained in $A_2^{37} \cup x(A_2^{37})$ where $A_1^{37} = S_1^{37} \setminus Q'''''(\sqrt{37})$ and $A_2^{37} = S_2^{37} \setminus Q'''''(\sqrt{37})$.

**Lemma 7.** Let $n = 1, 3, 4, 6$ or $7(\text{mod } 9)$. Let $Y = S \setminus Q'''''(\sqrt{n})$ where $S$ is any $G$-subset of $Q''(\sqrt{n})$. Then $Y \cup x(Y)$ is an $M$-subset of $Q''''(\sqrt{n}) \setminus Q'''''(\sqrt{n})$.

**Proof.** By Theorem 3, we know that $Q'''(\sqrt{n}) \setminus Q'''''(\sqrt{n})$ is an $M$-set. For any $\alpha \in Q''''(\sqrt{n}) \setminus Q'''''(\sqrt{n})$, Lemma 7 follows from the equations $x(\alpha) = \frac{-1}{\alpha n + 1}, y(\alpha) = 1, y(\frac{1}{\alpha n + 1}) = y(t) = y(t^2)$, where $\alpha \in 1, 3 \setminus Q'''''(\sqrt{n})$, $t \geq 1$ and $\alpha = \frac{-1}{\alpha n + 1} + 1$. Since every element of the group $M = \langle x, y : x^2 = y^3 = 1 \rangle$ is a word in the generators $x, y$ of the group $M$ and the transformations $\alpha \mapsto \alpha + 1, \alpha \mapsto \alpha - 1$ belong to both of the groups $G$ and $M$.

The following corollary is an immediate consequence of Lemma 7 since we know by Corollary 4 that $Q'''''(\sqrt{n}) = \emptyset$ if and only if $n \equiv 2(\text{mod } 3)$.

**Corollary 9.** Let $n = 2(\text{mod } 3)$. Let $S$ be any $G$-subset of $Q''(\sqrt{n})$. Then $S \cup x(S)$ is an $M$-subset of $Q'')(\sqrt{n})$.

**Theorem 10.** Let $n = 2^{k_1}p_1^{k_2}p_2^{k_3} \cdots p_r^{k_r}$, where $p_1, p_2, \ldots, p_r$ are distinct odd primes and $k_i = 0$ or $1$. Let $A_{1,2} = S_1^{37} \setminus Q''''(\sqrt{n})$ and $A_{1} = S_1^{37} \setminus Q''''(\sqrt{n})$. Then both $A_{1,2} \cup A_{1}^{37}$ and $A_{1} \cup x(A_{1,2})$ are $M$-subsets of $Q'''(\sqrt{n}) \setminus Q'''''(\sqrt{n})$. Consequently the action of $M$ on $Q'''(\sqrt{n}) \setminus Q'''''(\sqrt{n})$ is transitive.

**Proof.** follows from Theorem 5 and Lemma 7.

We now extend Theorem 9 for each non-square $n$.

**Theorem 11.** Let $n = 2^{k_1}p_1^{k_2}p_2^{k_3} \cdots p_r^{k_r}$, where $p_1, p_2, \ldots, p_r$ are distinct odd primes and $k_i \geq 2$. If $S$ is any of the $G$-subsets given in Theorem 6. Let $A = S \setminus Q'''''(\sqrt{n})$. Then $A \cup x(A)$ is an $M$-subset of $Q'''(\sqrt{n}) \setminus Q'''''(\sqrt{n})$. More precisely these are the only $M$-subsets of $Q'''(\sqrt{n}) \setminus Q'''''(\sqrt{n})$ depending upon classes $[a, b, c]$ modulo $n$.

**Proof.** follows from Theorem 6 and Lemma 7.
and \(-S\) are not \(G\)-subsets of \(Q^* (\sqrt{n})\). However the following lemma shows that \(S \cup x(S)\) and \(-S \cup x(-S)\) are distinct \(M\)-subsets of \(Q^* (\sqrt{n})\).

**Lemma 12.** If \(n \not\equiv 0 (\text{mod } 9)\) and \(Y\) be any of the \(G\)-subsets of \(Q^* (\sqrt{n})\). Let \(X = Y \setminus Q^* (\sqrt{n})\). Let \(S = \{a : c \in \{1, 3, 3, 6, 7\}\} \) and \(-S = \{a : c \in \{1, 3, 3, 3, 3\}\} \). Then \(S \cup x(S)\) and \(-S \cup x(-S)\) are both disjoint \(M\)-subsets of \(X \cup x(X)\). Consequently the action of \(M\) on each of \(X \cup x(X)\) is intransitive.

If \(n \equiv 2, 5\) or \(8 (\text{mod } 9)\) then, by Corollary 4, \(Q^* (\sqrt{n})\) is empty. But if \(n \equiv 1, 3, 6, 7\) \(\text{(mod } 9)\), then, by Theorem 3, \(Q^* (\sqrt{n})\) is an \(M\)-subset of \(Q^* (\sqrt{n})\). If \(n \equiv 0 (\text{mod } 9)\), then \(Q^* (\sqrt{n})\) is not an \(M\)-subset of \(Q^* (\sqrt{n})\). Instead we later prove that \(Q^* (\sqrt{n})\) is an \(M\)-subset of \(Q^* (\sqrt{n})\). For this we need to establish the following results.

**Lemma 13.** Let \(n \equiv 1, 3, 6, 7\) \(\text{(mod } 9)\). Then
1. \(Q^* (\sqrt{n}) \equiv Q^* (\sqrt{n}) \setminus Q^* (\sqrt{n})\) and
2. \(Q^* (\sqrt{n}) \equiv Q^* (\sqrt{n}) \setminus Q^* (\sqrt{n})\).

**Proof.** 1. Let \(\alpha \in Q^* (\sqrt{n})\) and \(\beta \in Q^* (\sqrt{n})\). Then if \(\alpha \) and \(\beta \) are both integers and \(\alpha, \beta \equiv 0 (\text{mod } 3)\), then \(\alpha \) and \(\beta \) are both divisible by 3.

2. Conversely suppose that \(\alpha \) and \(\beta \) are not divisible by 3.

The following corollary is an immediate consequence of Corollary 4 and Lemma 13.

**Theorem 14.** Let \(n \equiv 0 (\text{mod } 9)\). Then
1. \(Q^* (\sqrt{n}) \equiv Q^* (\sqrt{n}) \setminus Q^* (\sqrt{n})\) and \(Q^* (\sqrt{n}) \equiv Q^* (\sqrt{n}) \setminus Q^* (\sqrt{n})\) and
2. \(Q^* (\sqrt{n}) \equiv Q^* (\sqrt{n}) \setminus Q^* (\sqrt{n})\).

The following corollary is an immediate consequence of Corollary 4 and Lemma 13.
Corollary 15. Let \( n \equiv 2, 5 \) or \( 8 \) (mod 9). Then:
1. \( Q^{**}((\sqrt{3}n)) = Q^{*}((\sqrt{3}) \setminus Q^{*}((\sqrt{n})) \) and
2. \( \{ \alpha / 3 : \alpha = 3a + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \} = Q^{*}((\sqrt{n})) \).

Theorem 16. Let \( n \not\equiv 0 \) (mod 9). Then
\( Q^{**}((\sqrt{3}n)) \cup \{ \alpha / 3 : \alpha = 3a + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \} \) is an M-subset of \( Q^{**}((\sqrt{3}n)) \).
Proof: Let \( n \equiv 1, 3, 4, 6 \) or \( 7 \) (mod 9). Then by Lemma 13, \( Q^{**}((\sqrt{3}n)) = Q^{*}((\sqrt{3}) \setminus Q^{*}((\sqrt{n})) \) and \( \{ \alpha / 3 : \alpha = 3a + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \} = Q^{*}((\sqrt{n}) \setminus Q^{**}((\sqrt{n})).
However, if \( n \equiv 2, 5 \) or \( 8 \) (mod 9) then, as mentioned earlier, \( Q^{**}((\sqrt{n})) \) is empty. Hence the above result holds for all \( n \not\equiv 0 \) (mod 9). Thus if \( n \not\equiv 0 \) (mod 9) then
\( Q^{**}((\sqrt{3}n)) \cup \{ \alpha / 3 : \alpha = 3a + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \} = Q^{*}((\sqrt{n}) \setminus Q^{**}((\sqrt{n})).
If \( n \equiv 2, 5 \), or \( 8 \) (mod 9) then, by Corollary 4, \( Q^{**}((\sqrt{n})) \) is empty. However, if \( n \equiv 1, 3, 4, 6 \) or \( 7 \) (mod 9) then, by Theorem 3, \( Q^{**}((\sqrt{n})) \) is an M-subset of \( Q^{*}((\sqrt{n})). \) Also since \( Q^{*}((\sqrt{n})) \) is not an M-subset so \( Q^{*}((\sqrt{n}) \setminus Q^{**}((\sqrt{n}) \) and \( Q^{**}((\sqrt{n}) \setminus Q^{**}((\sqrt{n}) \) are not M-subsets of \( Q^{*}((\sqrt{n})). \) By Theorems 2 and 3, we know that \( Q^{*}((\sqrt{n}) \setminus Q^{**}((\sqrt{n}) \) is an M-subset of \( Q^{*}((\sqrt{n}) \) for all \( n \not\equiv 0 \) (mod 9).
Thus \( Q^{**}((\sqrt{3}n)) \cup \{ \alpha / 3 : \alpha = 3a + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \} \) is an M-subset of \( Q^{**}((\sqrt{3}n)) \) for all \( n \not\equiv 0 \) (mod 9).
Following theorem is an extension of Theorem 16 for each non-square \( n \) and its proof follows from Theorem 14.

Theorem 17. Let \( n \equiv 0 \) (mod 9). Then
\( Q^{**}((\sqrt{3}n)) \cup \{ \alpha / 3 : \alpha = 3a + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \} \) is an M-subset of \( Q^{**}((\sqrt{3}n)) \).

Theorem 18. Let \( n \equiv 0 \) (mod 9). Let \( \alpha = \frac{a}{c} + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \) with \( b = \frac{a^2 - n}{c} \). Then:
1. If \( 3 \mid a \) then \( \frac{a}{3} \) belongs to \( Q^{**}((\sqrt{3}n)) \).
2. If \( 3 \mid a \) then \( \frac{a}{3} \) belongs to \( Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \) or \( Q^{**}((\sqrt{3}n)) \) according as \( \alpha \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \) or \( Q^{**}((\sqrt{3}n)) \).

Proof. Let \( n \equiv 0 \) (mod 9). Let \( \alpha = \frac{a}{c} + \sqrt{3}n \in Q^{*}((\sqrt{3}n)) \) with \( b = \frac{a^2 - n}{c} \). Then:
(1) If \( 3 \nmid a \) then \( bc = (a^2 - n) \equiv 1, 4 \) or \( 7 \) (mod 9) so \( 3 \nmid b \). Therefore, by Lemma 1(2), \( \frac{a}{3} \) belongs to \( Q^{**}((\sqrt{3}n)) \).

(2) If \( 3 \mid a \) then \( (a^2 - n) \equiv 0 \) (mod 9). So \( b, c \) cannot be both divisible by 3, as otherwise \( (a, b, c) \neq 1 \). Thus exactly one of \( b, c \) is divisible by 3. Therefore, again by second part of Lemma 1, if \( b \) is not divisible by 3 then \( \frac{a}{3} \) belongs to \( Q^{**}((\sqrt{3}n)) \). But if \( b \) is divisible by 3 then, from the proof of Lemma 13(2), \( \frac{a}{3} \) belongs to \( Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)). \) That is, \( \frac{a}{3} \) belongs to \( Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \) or \( Q^{**}((\sqrt{3}n)) \) according as \( \alpha \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)) \) or \( Q^{**}((\sqrt{3}n)) \).

Following example illustrates the above theorem.

Example 19. Let \( n = 27 \). Then \( \alpha = \frac{1 + \sqrt{27}}{3} \in Q^{*}((\sqrt{3}n)) \) but \( \frac{a}{3} = \frac{1 + \sqrt{27}}{3} = \frac{3 + \sqrt{27}}{3} \in Q^{**}((\sqrt{3}n)) \). Also \( \beta = \frac{2 + \sqrt{27}}{3} \in Q^{*}((\sqrt{3}n)) \) but \( \frac{a}{3} = \frac{1 + \sqrt{27}}{3} \in Q^{*}((\sqrt{3}n)) \setminus Q^{**}((\sqrt{3}n)). \)
Similarly \( \gamma = \frac{1 + \sqrt{27}}{3} \in Q^{**}((\sqrt{3}n)) \).\( \) whereas \( \frac{a}{3} = \frac{3 + \sqrt{27}}{3} \in Q^{**}((\sqrt{3}n)). \)

Summarizing the above results we have the following
Theorem 20. Let $n \equiv 0 \pmod{9}$. Then $Q'''(\sqrt{n}) = (Q^*(\sqrt{9}) \setminus Q^{**}(\sqrt{9})) \cup Q^*(\sqrt{n}) \cup Q^{**}(\sqrt{9n})$.

Proof. Follows from Theorems 17 and 18. \hfill \square

We conclude this paper with the following observations.

If $n \equiv 2, 3, 5, 7 \pmod{9}$, then $Q'''(\sqrt{n}), Q''(\sqrt{n}) \setminus Q'''(\sqrt{n})$ are both $M$-subsets of $Q''(\sqrt{n})$ and in particular $Q''(\sqrt{n}) \subset Q'''(\sqrt{n})$. If $n \equiv 1, 3, 4, 6, 8 \pmod{9}$, then $Q''(\sqrt{n}) \setminus Q^{**}(\sqrt{n})$, and $Q''(\sqrt{n}) \setminus Q'''(\sqrt{n})$ are all $M$-subsets of $Q'''(\sqrt{n})$.

In particular $Q''(\sqrt{n}) \setminus Q^{**}(\sqrt{n}) \subset Q'''(\sqrt{n})$. That is $Q'''(\sqrt{n}) \cap Q''(\sqrt{n}) = Q''(\sqrt{n}) \setminus Q^{**}(\sqrt{n})$. For the cases $n \not\equiv 0 \pmod{9}$. For $n = 2, 9n = 18$, $Q^{**}(\sqrt{2}) = \emptyset$, $Q'''(\sqrt{2}) = (\sqrt{2})^M \cup (-\sqrt{2})^M$, and $Q'''(\sqrt{18}) \setminus Q''(\sqrt{2}) = (\sqrt{18})^M \cup (-\sqrt{18})^M$.

So $Q'''(\sqrt{18})$ has exactly 4 orbits under the action of $M$. Also if $n = 3, 9n = 27$, $Q''(\sqrt{3}) \setminus Q^{**}(\sqrt{3}) = (\sqrt{3})^M \cup (-\sqrt{3})^M$, $Q'''(\sqrt{27}) \setminus Q''(\sqrt{3}) = (\sqrt{27})^M \cup (-\sqrt{27})^M$.

So $Q'''(\sqrt{27})$ has exactly 4 orbits under the action of $M$. Similarly if $n = 5, 9n = 45$, $Q'''(\sqrt{5}) = (\sqrt{5})^M \cup (-\sqrt{5})^M \cup (\frac{1+\sqrt{5}}{2})^M \cup (\frac{1-\sqrt{5}}{2})^M$, $Q'''(\sqrt{45}) = \emptyset$.

So $Q'''(\sqrt{45})$ splits into exactly 8 orbits under the action of $M$.

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