General common fixed point theorems in fuzzy metric spaces

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Abstract. In this paper, we obtain two general common fixed point theorems for two maps in fuzzy metric spaces.

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1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by L. Zadeh [9] in 1965. George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7]. Grabiec [10] proved the contraction principle in the setting of fuzzy metric spaces introduced in [1]. For fixed point theorems in fuzzy metric spaces some of the interesting references are [1, 3, 4, 5, 10, 12-17, 19, 20]. In the sequel, we need the following

Definition 1. [2] A binary operation \(* : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is a continuous t-norm if it satisfies the following conditions

1. \(*\) is associative and commutative,
2. \(*\) is continuous,
3. \(a \ast 1 = a\) for all \(a \in [0, 1]\),
4. \(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d\), for each \(a, b, c, d \in [0, 1]\).

Two typical examples of continuous t-norm are \(a \ast b = ab\) and \(a \ast b = \text{min}\{a, b\}\).

Definition 2. [1] A 3-tuple \((X, M, \ast)\) is called a fuzzy metric space if \(X\) is an arbitrary (non-empty) set, \(*\) is a continuous t-norm and \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\), satisfying the following conditions for each \(x, y, z \in X\) and each \(t\) and \(s > 0\),

1. \(M(x, y, t) > 0\),
2. \(M(x, y, t) = 1\) if and only if \(x = y\),
3. \(M(x, y, t) = M(y, x, t)\),
4. \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)\).
(5): $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Let $(X, M, \ast)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by
\[ B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}. \]

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let $\tau$ denote the family of all open subsets of $X$. Then $\tau$ is called the topology on $X$ induced by the fuzzy metric $M$. This topology is Hausdorff and first countable. A subset $A$ of $X$ is said to be $F$-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

**Lemma 3.** [10] Let $(X, M, \ast)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to $t$, for all $x, y \in X$.

**Definition 4.** Let $(X, M, \ast)$ be a fuzzy metric space. $M$ is said to be continuous on $X^2 \times (0, \infty)$ if
\[ \lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t) \]
whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, i.e., whenever
\[ \lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \to \infty} M(x, y_n, t_n) = M(x, y, t). \]

**Lemma 5.** [8] Let $(X, M, \ast)$ be a fuzzy metric space. Then $M$ is continuous function on $X^2 \times (0, \infty)$.

**Definition 6.** [1]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \ast)$ is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} M(x_n, x, t) = 1$. The sequence $\{x_n\}$ is said to be Cauchy if $\lim_{m,n \to \infty} M(x_m, x_n, t) = 1$. The space $(X, M, \ast)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

**Definition 7.** [18]. A fuzzy metric space $(X, M, \ast)$ is called precompact if for each $0 < r < 1$ and each $t > 0$, there is a finite subset $A \subset X$ such that $X = \bigcup_{a \in A} B(a, r, t)$. A fuzzy metric space $(X, M, \ast)$ is called compact if $(X, \tau)$ is a compact topological space. It is clear that every compact set is closed and $F$-bounded.

**Definition 8.** [6]. Let $f$ and $g$ be self mappings on a fuzzy metric space $(X, d)$. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, $fx = gx$ implies that $f \circ g = g \circ f$.

Generally, several authors obtained fixed point theorems in fuzzy metric spaces for a single map using one of the following contraction conditions.

There exists $k \in (0, 1)$ such that for all $x, y \in X$ and for all $t > 0$,
\[ (1) M(Tx, Ty, kt) \geq M(x, y, t), \]
\[ (2) M(Tx, Ty, kt) \geq \min \{ M(x, y, t), M(x, Tx, t), M(y, Ty, t), \]
\[ M(x, Ty, 2t), M(y, Tx, t) \}, \]
\[ (3) M(Tx, Ty, kt) \geq \min \{ M(x, y, t), M(x, Tx, t), M(y, Ty, t), \]
\[ M(x, Ty, 2t), M(y, Tx, 2t) \}, \]
\[ (4) M(Tx, Ty, kt) \geq \min \{ M(x, y, t), M(x, Tx, t), M(y, Ty, t), \]
\[ M(x, Ty, \alpha t), M(y, Tx, (2 - \alpha) t) \}, \forall \alpha \in (0, 2). \]

In all these types of theorems, the authors assumed that $\lim_{t \to \infty} M(x, y, t) = 1$,
\[ \forall x, y \in X. \]
In this paper, without using this condition, we prove the following general common fixed point theorem in $F$-bounded fuzzy metric spaces.

2. MAIN RESULTS

**Theorem 9.** Let $T$ and $f$ be self maps of on a $F$-bounded fuzzy metric space $(X, M, \ast)$ satisfying

1. $T(X) \subseteq f(X)$, $(T, f)$ is a weakly compatible pair and $f(X)$ is complete,
2. 

$$M(Tx, Ty, t) \geq \min \left\{ M(fx, fy, t), M(fx, Tx, t), M(fy, Ty, t), M(fx, Ty, t) \right\}$$

for all $x, y \in X$ and for all $t > 0$, where $\phi : [0, 1] \to [0, 1]$ is continuous and monotonically increasing such that $\phi(s) > s$, for all $s \in [0, 1]$.

Then $f$ and $T$ have a unique common fixed point in $X$.

**Proof.** Let $x_0 \in X$. From (9.1), there exists a sequence $\{x_n\}$ in $X$ such that

$Tx_n = fx_{n+1} = y_n$, say.

Case (i): Suppose $y_{n+1} = y_n$ for some $n$.

Then $Tz = fz$, where $z = x_{n+1}$. Denote $p = Tz = fz$.

Since $(T, f)$ is a weakly compatible pair, we have $Tp = fp$.

From (9.2), we have

$$M(Tp, p, t) = M(Tp, Tz, t)$$

$$\geq \min \left\{ M(fp, fz, t), M(fp, Tp, t), M(fz, Tz, t), M(fp, Tp, t) \right\}$$

$$= \phi(1) = \phi(M(Tp, p, t))$$

$$\geq M(Tp, p, t), \text{ if } M(Tp, p, t) < 1.$$ 

Hence $Tp = p$. Thus $fp = Tp = p$.

If $q$ is another common fixed point of $f$ and $T$, then

$$M(p, q, t) = M(Tp, Tq, t)$$

$$= \phi(1) = \phi(M(p, q, t))$$

$$\geq M(p, q, t), \text{ if } M(p, q, t) < 1.$$ 

Hence $p = q$. Thus $p$ is the unique common fixed point of $f$ and $T$.

Case (ii): Assume that $y_{n+1} \neq y_n$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, let $\alpha_n(t) = \inf \{ M(y_i, y_j, t) : i \geq n, j \geq n \}$ for all $t > 0$.

Then $\{\alpha_n(t)\}$ is a monotonically increasing sequence of real numbers between 0 and 1 for all $t > 0$.

Hence $\lim_{n \to \infty} \alpha_n(t) = \alpha(t)$ for some $0 \leq \alpha(t) \leq 1$.

For any $n \in \mathbb{N}$ and integers $i \geq n, j \geq n$, we have

$$M(y_i, y_j, t) = M(Tx_i, Tx_j, t)$$

$$\geq \min \left\{ M(y_{i-1}, y_{j-1}, t), M(y_{i-1}, y_j, t), M(y_j, y_{j-1}, t) \right\}$$

$$\geq \min \left\{ M(y_{i-1}, y_{j-1}, t), M(y_{i-1}, y_j, t), M(y_{i-1}, y_j, t) \right\}$$

$$\geq \phi(\alpha_{n-1}(t)), \text{ since } \phi \text{ is monotonically increasing.}$$

Hence $\alpha_n(t) \geq \phi(\alpha_{n-1}(t)).$

Letting $n \to \infty$, we get $\alpha(t) \geq \phi(\alpha(t)) > \alpha(t)$, if $\alpha(t) < 1$.

Hence $\alpha(t) = 1$ so that $\lim_{n \to \infty} \alpha_n(t) = 1$.

Thus for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\alpha_n(t) > 1 - \epsilon$ for all $n \geq n_0$.

Thus for $n \geq n_0, m \in \mathbb{N}$, we have $M(y_n, y_{n+m}, t) > 1 - \epsilon$. 

Hence \( \{ y_n \} \) is a Cauchy sequence in \( X \). Since \( f(X) \) is complete, it follows that \( y_n \to z \) for some \( z \in f(X) \). Hence there exists \( u \in X \) such that \( z = fu \). Now,

\[
M(Tu, Tx_n, t) \geq \phi \left( \min \left\{ M(fu, fx_n, t), M(fu, Tu, t), M(fx_n, Tx_n, t), M(fx_n, Tu, t) \right\} \right)
\]

Letting \( n \to \infty \), we get

\[
M(Tu, z, t) \geq \phi \left( \min \{1, M(z, Tu, t), 1, 1, M(z, Tu, t)\} \right)
\]

Thus

\[
M(Tu, z, t) = \phi(M(z, Tu, t)) > M(z, Tu, t)
\]

Hence \( Tu = z \). Thus \( fu = Tu = z \).

The rest of the proof follows as in case(i).

\[ \square \]

**Corollary 10.** Let \( T \) be a self map of on a \( F \)-bounded complete fuzzy metric space \((X, M, \ast)\) satisfying

\[
M(Tx, Ty, t) \geq \phi \left( \min \left\{ M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t) \right\} \right)
\]

for all \( x, y \in X \) and for all \( t > 0 \), where \( \phi : [0, 1] \to [0, 1] \) is continuous and monotonically increasing such that \( \phi(s) > s \), for all \( s \in [0, 1] \).

Then \( T \) has a unique fixed point in \( X \).

Now using the technique of Shih and Yeh [11] in metric spaces, we prove the following theorem in compact fuzzy metric spaces.

**Theorem 11.** Let \((X, M, \ast)\) be a compact fuzzy metric space, \( f, T : X \to X \) be satisfying (11.1)\( T \) is continuous, \( fT = Tf \) and

(11.2) \( M(Tx, Ty, t) > \min \{M(x, y, t), x_1, y_1 \in O(x) \cup O(y)\} \)

for all \( x, y \in X \) with \( x \neq y \), \( \forall t > 0 \), where

\[
O(x) = \{ hx : h \in \tau \}, \tau \text{ being the semi group of self maps on } X \text{ generated by } \{f, T, I\}
\]

\((I \text{ is the identity map on } X)\).

Then \( f \) and \( T \) have a unique common fixed point \( z \in X \).

\[ \text{Proof.} \]

We know that \( T^n X \) is compact and \( T^{n+1} X \subseteq T^n X \) for \( n = 1, 2, 3, \ldots \)

Let \( X_0 = \bigcap_{n=1}^{\infty} T^n X \).

Then \( X_0 \) is a nonempty compact subset of \( X \).

Since \( M \) is continuous on \( X_0 \times (0, \infty) \) and \( X_0 \) is compact, it follows that for each \( t > 0 \), \( M(., ., t) \) has a minimum value. Hence there exist \( z_1, z_2 \in X_0 \) such that

\[
M(z_1, z_2, t) = \inf \{ M(x, y, t) : x, y \in X_0 \} \text{ for each } t > 0.
\]

Since \( TX_0 = X_0 \), there exist \( x_1, x_2 \in X_0 \) such that \( Tx_1 = z_1 \) and \( Tx_2 = z_2 \).

Suppose \( x_1 \neq x_2 \). Then from (11.2), we have

\[
M(z_1, z_2, t) = M(Tx_1, Tx_2, t) > \min \{ M(x, y, t) : x, y \in O(x_1) \cup O(x_2) \}
\]

\[
\geq M(z_1, z_2, t)
\]

It is a contradiction.

Thus \( z_1 = z_2 \) and so \( z_1 = z_2 \). Hence \( X_0 \) is a singleton set, say, \( \{z\} \).

From (11.2), it is clear that \( z \) is the unique common fixed of \( f \) and \( T \).

\[ \square \]

**Corollary 12.** Let \( T \) be a continuous self map on a compact fuzzy metric space \((X, M, \ast)\) satisfying

\[
M(Tx, Ty, t) > \min \left\{ M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t) \right\}
\]
for all \( x, y \in X \) with \( x \neq y \) and for all \( t > 0 \).

Then \( T \) has a unique fixed point in \( X \).

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