Numerical Solution of the Riemann–Hilbert Problem

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Abstract. This paper presents an implementation of the integral equations with the generalized Neumann kernel to solve numerically the uniquely and the non-uniquely solvable Riemann-Hilbert problems in Jordan regions with smooth boundaries. The non-uniquely solvable problems are made uniquely solvable by requiring their solutions to satisfy additional constraints. Two type of constraints are presented. Various test numerical examples are presented. The computational efficiency appears significantly excellent.

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1. Introduction

The boundary integral equation method is an inexpensive, flexible technique to solve the elliptic boundary value problems on a simply connected region in the plane $\Omega$. The reformulation of the boundary value problem as an equivalent integral equation over the boundary of $\Omega$ reduces the dimensionality of the problem which makes the method an efficient tool for complicated engineering problems.

Riemann–Hilbert problems on $\Omega$ are the prototypical examples of elliptic systems of differential equations in the plane (see e.g., Wendland [15]). The Dirichlet problem for Laplace’s equation, which is one of the classical elliptic boundary value problems, is a special case of the Riemann-Hilbert problem.

The boundary integral equation method is a classical method for solving the Dirichlet problem (see e.g., Atkinson [1, Ch. 7] and Henrici [4, §15.9]). When the Dirichlet problem solved by the double layer potential representation, a boundary integral equation with a continuous kernel results. The kernel is known as the Neumann kernel [4, pp. 282–286].

The Riemann-Hilbert problem can also be solved using boundary integral equation [11, 8, 6, 7, 14]. Sherman [11] used a generalization of the double layer potential representation to derive a boundary integral equation with a continuous kernel for the interior Riemann-Hilbert problem (see e.g., Gakhov [3, p. 400]). Using
a similar approach, a boundary integral equation can be derived for the exterior
Riemann-Hilbert problem. The kernel of the both boundary integral equations is,
as expected, a generalization of the Neumann kernel.

Recently, Murid and Nasser [8, 6, 7] used a different approach to derive two
new boundary integral equations for the interior and the exterior Riemann-Hilbert
problems. The kernel of these integral equations is the same kernel of the integral
equations derived in [11]. This kernel was called in [7] the generalized Neumann
kernel. The properties of the generalized Neumann kernel has been studied in [7]
and more extensively in [14].

The boundary integral equation method is a popular method for solving the
Dirichlet problem [1]. However, this is not the case for the Riemann-Hilbert prob-
lem. Riemann-Hilbert problems in Jordan regions are often solved by conformal
mapping of the region to the unit disk where the problem can be solved in a closed
form using the harmonic conjugation [2, 3, 4]. Gakhov introduced the concept of
regularizing factor to reduce the Riemann-Hilbert problem to three Dirichlet prob-
lem in the same region of consideration (see e.g., Gakhov [3, p. 222] and Begehr [2,
pp. 45–69]).

The current paper extends the results of the previous papers [8, 6, 7, 14, 10]. The
papers [8, 6, 7, 14] were concentrated on the deriving and studying the solvability
of the integral equations. Although, the paper [10] consider the numerical solution
of the Riemann-Hilbert problems, only the integral equations derived in [7] were
used. The non-uniquely solvable Riemann-Hilbert problem was solved by requiring
their solution to satisfy additional constraints. Only one type of constraints was
present in [14, 10].

The main propose of this paper is to present the numerical treatment of the
integral equations with generalized Neumann kernel and the applications of the
integral equations to solve the Riemann-Hilbert problems. We shall consider the
integral equations derived in [11] as well as the integral equations derived in [8,
6, 7]. For the non-uniquely solvable Riemann-Hilbert problems, we present two
type of constraints to reduce the problems to uniquely solvable problems. Some
results of the original work [7, 14, 10] are included to allow this paper to be read
independently.

This paper is organized as follows. In §2 we review some auxiliary results. The
Riemann–Hilbert problems and the integral equations for the Riemann–Hilbert
problems will be reviewed in §3. The solutions of the interior and the exterior
Riemann–Hilbert problems in terms of the solution of the integral equations will
be given in §4 and §5, respectively. The numerical implementations will be given
in §6. Some numerical examples will be given in §7 and a short conclusion will be
given in §8.

2. Auxiliary Material

Let Ω be a bounded simply connected Jordan region with $0 \in \Omega$. The boundary
$\Gamma := \partial \Omega$ is assumed to have a positively oriented parametrization $\eta(s)$ where $\eta(s)$
is a $2\pi$-periodic twice continuously differentiable function with $\dot{\eta}(s) = \frac{d\eta}{ds} \neq 0$. The
exterior of $\Gamma$ is denoted by $\Omega^-$. For a fixed $\alpha$ with $0 < \alpha < 1$, the Hölder space $H^\alpha$
consists of all $2\pi$-periodic real functions which are uniformly Hölder continuous with exponent $\alpha$. A Hölder
continuous function \( \hat{h} \) on \( \Gamma \) can be interpreted via \( h(s) := \hat{h}(\eta(s)) \) as a Hölder continuous function \( h \) of the parameter \( s \) and vice versa.

Let \( A(s) \) be a complex continuously differentiable \( 2\pi \)-periodic function with \( A \neq 0 \). With \( \gamma, \mu \in H^\alpha \), let the function \( \Phi(z) \) be defined by

\[
\Phi(z) := \frac{1}{2\pi i} \int_\Gamma \frac{\gamma + i\mu}{A} \frac{d\eta}{\eta - z}, \quad z \notin \Gamma. \tag{2.1}
\]

Then \( \Phi(z) \) is analytic in \( \Omega \) as well as in \( \Omega^- \) and the boundary values \( \Phi^+ \) from inside and \( \Phi^- \) from outside belong to \( H^\alpha \) and can be calculated by Plemelj’s formula

\[
\Phi^-(\zeta) = \pm \frac{1}{2} \frac{\gamma(\zeta) + i\mu(\zeta)}{A(\zeta)} + \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(\eta) + i\mu(\eta)}{A(\eta)} \frac{d\eta}{\eta - \zeta}, \quad \zeta \in \Gamma. \tag{2.2}
\]

The integral in (2.2) is a Cauchy principal value integral. The boundary values satisfy the jump relation

\[
A\Phi^+ - A\Phi^- = \gamma + i\mu. \tag{2.3}
\]

We define two real functions \( N \) and \( M \) by

\[
N(s, t) := \frac{1}{\pi} \text{Im} \left( \frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right), \tag{2.4}
\]

\[
M(s, t) := \frac{1}{2\pi} \cot \frac{s - t}{2} - \frac{1}{\pi} \text{Re} \left( \frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right). \tag{2.5}
\]

The function \( N(s, t) \) is called the generalized Neumann kernel formed with \( A \) and \( \eta \) \([7, 14]\).

**Lemma 1** ([14]). (a) The kernel \( N(s, t) \) is continuous with

\[
N(t, t) = \frac{1}{\pi} \text{Im} \left( \frac{1}{2} \frac{\dot{\eta}(t)}{\eta(t)} - \frac{\dot{A}(t)}{A(t)} \right). \tag{2.6}
\]

(b) The kernel \( M(s, t) \) is continuous with

\[
M(t, t) = \frac{1}{\pi} \text{Re} \left( \frac{1}{2} \frac{\dot{\eta}(t)}{\eta(t)} - \frac{\dot{A}(t)}{A(t)} \right). \tag{2.7}
\]

The integral operators with the kernels \( N \) and \( M \) will be denoted by \( \mathcal{N} \) and \( \mathcal{M} \), i.e.,

\[
(\mathcal{N} \mu)(s) := \int_0^{2\pi} N(s, t) \mu(t) dt, \tag{2.8}
\]

\[
(\mathcal{M} \mu)(s) := \int_0^{2\pi} M(s, t) \mu(t) dt. \tag{2.9}
\]

The eigenvalues of the operator \( \mathcal{N} \) have been studied in \([7, 14]\). It turns out that the dimensions of the spaces \( \text{Null}(\mathcal{I} \pm \mathcal{N}) \) depend upon the index of the function \( A \) which is defined as the winding number of \( A \) with respect to 0

\[
\kappa := \text{ind}(A) := \frac{1}{2\pi} \arg(A)|_{t=0}^{2\pi} \tag{2.10}
\]

i.e., the increment of the argument of \( A \) in traversing the curve \( \Gamma \) in the positive sense divided by \( 2\pi \).
Theorem 2 ([14]).

\[
\dim(\text{Null}(I - N)) = \max(0, -2\kappa + 1),
\]

\[
\dim(\text{Null}(I + N)) = \max(0, 2\kappa - 1).
\]

Let the complex-valued functions \( \tilde{A}(t) \) be defined by

\[
\tilde{A}(t) = \hat{\eta}(t)/A(t).
\]

and let \( \tilde{N}(s, t) \) be the generalized Neumann kernel formed with \( \tilde{A} \) and \( \eta \). Then the adjoint kernel \( N^*(s, t) \) of the generalized Neumann kernel \( N(s, t) \) can be written as

\[
N^*(s, t) = N(t, s) = \frac{1}{\pi} \text{Im} \left( \frac{A(t)}{A(s)} \frac{\hat{\eta}(s)}{\eta(s) - \eta(t)} \right) = \frac{1}{\pi} \text{Im} \left( \frac{\hat{A}(s)}{\hat{A}(t)} \frac{\hat{\eta}(t)}{\eta(t) - \eta(s)} \right) = -\tilde{N}(s, t).
\]

Let also the complex-valued functions \( A_0(s) \) and \( A_1(s) \) be defined by

\[
A_0(s) := \eta^{-\kappa}(s) A(s),
\]

\[
A_1(s) := \eta^{1-\kappa}(s) A(s).
\]

Then the generalized Neumann kernel formed with \( A_i \) and \( \eta \) will be denoted by \( N_i(s, t), i = 0, 1 \). Similarly, the continuous kernel \( M_i(s, t) \) is defined as in (2.5) with \( A \) replaced by \( A_i \), \( i = 0, 1 \). The integral operators with the kernels \( \tilde{N}, N_0, M_0, N_1 \) and \( M_1 \) are denoted by \( \tilde{N}, N_0, M_0, N_1 \) and \( M_1 \), respectively.

The conjugation operator \( K \) is defined by

\[
(K \mu)(s) := \frac{1}{2\pi} \int_0^{2\pi} \mu(t) \cot \frac{s - t}{2} dt.
\]

The operator \( K \) is also known as the Hilbert transform (see e.g., Henrici [4, p. 107] and [5]).

Let \( z_0 = \alpha_0 - i\beta_0 \) be a given point. Other assumptions on \( z_0 \) will be given latter. We define the continuous real-valued functions \( a(t), b(t), P(t) \) and \( Q(t) \) by

\[
a(t) + ib(t) := \frac{1}{2\pi} \hat{\eta}(t) A_0(t),
\]

\[
P(t) := \alpha_0 a(t) + \beta_0 b(t),
\]

\[
Q(t) := \beta_0 a(t) - \alpha_0 b(t).
\]

Then we define the integral operators \( P \) and \( Q \) by

\[
(P \mu)(s) := \int_0^{2\pi} P(t) \mu(t) dt,
\]

\[
(Q \mu)(s) := \int_0^{2\pi} Q(t) \mu(t) dt.
\]

The functions \( P \mu \) and \( Q \mu \) are constants on \([0, 2\pi]\).
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For a given set of points \( r_i \in [0, 2\pi], \ i = 1, 2, \ldots, m, \ m > 0, \) we define the integral operator \( R_m \) by

\[
(R_m \mu)(s) = \left( \begin{array}{c}
\int_0^{2\pi} N(r_1, t) \mu(t) dt \\
\int_0^{2\pi} N(r_2, t) \mu(t) dt \\
\vdots \\
\int_0^{2\pi} N(r_m, t) \mu(t) dt
\end{array} \right),
\]

(2.23)

3. Integral Equations For The Riemann-Hilbert Problems

3.1. The Riemann-Hilbert Problems. For \( \gamma \in H^\alpha, \) the Riemann-Hilbert (RH) problems are defined as follows:

- **Interior RH problem:** Given functions \( A \) and \( \gamma, \) it is required to find a function \( f \) analytic in \( \Omega \) and continuous on the closure \( \Omega \) such that the boundary values \( f^+ \) satisfy

\[
\text{Re}[A(s)f^+(\eta(s))] = \gamma(s) \quad \text{for all } s.
\]

(3.24)

- **Exterior RH problem:** Given functions \( A \) and \( \gamma, \) it is required to find a function \( g \) analytic in \( \Omega^- \) and continuous on the closure \( \Omega^- \) with \( \text{g}(\infty) = 0 \) such that the boundary values \( g^- \) satisfy

\[
\text{Re}[A(s)g^-(\eta(s))] = \gamma(s) \quad \text{for all } s.
\]

(3.25)

3.2. The Solvability Of The Rh Problems. The solvability of the RH problems depend upon the index \( \kappa = \text{ind}(A) \) [3]. The interior RH problem (3.24) is not necessary solvable for \( \kappa > 0. \) It is solvable only if

\[
\gamma \in \text{Null}(I - \mathcal{N})^\perp.
\]

(3.26)

Similarly, the exterior RH problem (3.25) is solvable for \( \kappa \leq 0 \) only if

\[
\gamma \in \text{Null}(I + \mathcal{N})^\perp.
\]

(3.27)

If \( \gamma \) satisfies these conditions, then the RH problems are uniquely solvable (see e.g., [14]).

The interior RH problem for \( \kappa \leq 0 \) and the exterior RH problem for \( \kappa > 0 \) are non-uniquely solvable. The general solution of the interior RH problem contains \(-2\kappa + 1\) arbitrary real constants and the general solution of the exterior RH problem contains \(2\kappa - 1\) arbitrary real constants.

To reduce the non-uniquely solvable RH problems to uniquely solvable problems, we need to define the following two analytic functions \( Y \) and \( Z. \) Suppose that \( \gamma \in H^\alpha. \) The Schwarz operator \( S_\iota \) for the region \( \Omega \) is an operator which determines the the unique analytic function \( F(z) := (S_\iota \gamma)(z) \) in \( \Omega \) such that \( F(0) \) is real and \( \text{Re}F^+ = \gamma. \) Similarly, the Schwarz operator \( S_\varepsilon \) for the region \( \Omega^- \) is an operator which determines the the unique analytic function \( G(z) := (S_\varepsilon \gamma)(z) \) in \( \Omega^- \cup \{\infty\} \) such that \( G(\infty) \) is real and \( \text{Re}G^- = \gamma \) (see e.g., [3]). Using the Schwarz operators \( S_\iota \) and \( S_\varepsilon, \) we define the two analytical functions \( Y \) in \( \Omega \) and \( Z \) in \( \Omega^- \cup \{\infty\} \) as follows

\[
Y(z) := (S_\iota \text{arg} A_0)(z), \quad z \in \Omega,
\]

(3.28)

\[
Z(z) := (S_\varepsilon \text{arg} A_0)(z), \quad z \in \Omega^-.
\]

(3.29)

The interior RH problem (3.24) can be made uniquely solvable for \( \kappa \leq 0 \) by two methods as in the following two lemmas.
Lemma 3 ([14]). Suppose that \( \kappa \leq 0 \), the point \( z_0 \) satisfies \( \text{Re}[z_0e^{-iz(0)}] \neq 0 \), \( e_j \) (\( j = 0, 1, \ldots, |\kappa| - 1 \)) are given complex numbers and \( e_{|\kappa|} \) is a given real number. Then the interior RH problem (3.24) with the constraints
\[
\text{Im}[z_0f^{(|\kappa|)}(0)] = e_{|\kappa|} \quad f^{(|\kappa|)}(0) = e_j \quad (j = 0, 1, \ldots, |\kappa| - 1)
\]
is uniquely solvable.

Lemma 4 ([2]). Suppose that \( \kappa \leq 0 \), \( r_j \) are given distinct real numbers in \([0, 2\pi]\) and \( d_j \) are prescribed real numbers, \( j = 1, 2, \ldots, 2|\kappa| + 1 \). Then the interior RH problem (3.24) with the side conditions
\[
\text{Im}[A(r_j)f^+(\eta(r_j))] = d_j \quad (j = 1, 2, \ldots, 2|\kappa| + 1)
\]
is uniquely solvable.

Similarly, the exterior RH problem (3.25) can be made uniquely solvable for \( \kappa > 0 \) as in the following two lemmas.

Lemma 5 ([14]). Suppose that \( \kappa > 0 \), the point \( z_0 \) satisfies \( \text{Re}[z_0e^{-iz(\infty)}] \neq 0 \), \( e_j \) (\( j = 1, 2, \ldots, |\kappa| - 1 \)) are given complex numbers and \( e_{|\kappa|} \) is a given real number. Then the exterior RH problem (3.25) with the constraints
\[
\text{Im}\left[ \frac{z_0}{2\pi i} \int \eta^{i-1} g(\eta)d\eta \right] = e_{|\kappa|}, \quad \frac{1}{2\pi i} \int \eta^{i-1} g(\eta)d\eta = e_j \quad (j = 1, 2, \ldots, |\kappa| - 1)
\]
is uniquely solvable.

Lemma 6. Suppose that \( \kappa > 0 \), \( r_j \) are given distinct real numbers in \([0, 2\pi]\) and \( d_j \) are prescribed real numbers, \( j = 1, 2, \ldots, 2\kappa - 1 \). Then the exterior RH problem (3.25) with the side conditions
\[
\text{Im}[A(r_j)g^-(\eta(r_j))] = d_j \quad (j = 1, 2, \ldots, 2\kappa - 1)
\]
is uniquely solvable.

Proof.

This lemma can be proved with the same arguments as Lemma 4 in [2, p. 55]. \( \square \)

### 3.3. The Integral Equations

For given functions \( \gamma, \mu \in H^\alpha \), let the function \( \Phi(z) \) be defined by (2.1). Based on the application of the Plemelj’s formula, two boundary integral equations with the generalized Neumann kernel have been derived for the interior and the exterior RH problems by Murid and Nasser [8, 6, 7] and Wegmann et al [14] as in the following two lemmas.

Lemma 7 ([14]). Suppose that \( \gamma \in \text{Null}(I - \hat{N})^+ \) for \( \kappa > 0 \) and \( \gamma \) is arbitrary for \( \kappa \leq 0 \). Then the function \( f(z) := \Phi(z) \) is a solution of the interior RH problem (3.24) with the boundary values
\[
A(t)f^+(\eta(t)) = \gamma(t) + i\mu(t)
\]
if and only if \( \mu \) is a solution of the integral equation
\[
(I - \hat{N})\mu = -(M - K)\gamma.
\]

Lemma 8 ([14]). Suppose that \( \gamma \in \text{Null}(I + \hat{N})^+ \) for \( \kappa \leq 0 \) and \( \gamma \) is arbitrary for \( \kappa > 0 \). Then the function \( g(z) := -\Phi(z) \) is a solution of the exterior RH problem (3.25) with the boundary values
\[
A(t)g^-(\eta(t)) = \gamma(t) + i\mu(t)
\]
if and only if $\mu$ is a solution of the integral equation
\[
(I + \mathcal{N})\mu = (M - K)\gamma.
\] (3.37)

Another two boundary integral equations with the generalized Neumann kernel can be derived for the interior and the exterior RH problems (Sherman [11] and Gakhov [3, p. 400]). The derivation of these integral equations based on using a generalization of the double layer potential representation and on using the Plemelj’s formula.

Let $f(z)$ be a solution of the interior RH problem (3.24), then there exists a real function $h$, and for $\kappa \leq 0$, there exist complex constants $c_k \ (k = 0, 1, \ldots, -\kappa)$ such that $f(z)$ can be written as [3, 9]
\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A \eta - z} \, d\eta + \sum_{k=0}^{-\kappa} c_k z^k.
\] (3.38)
The constant $c_{-\kappa}$ has the form (see [3, p. 299])
\[
c_{-\kappa} = (\bar{\alpha} + i\beta) \exp(-iZ(\infty))
\]
where $\beta$ is a real constant depending on $A$ and $f$, and $\bar{\alpha}$ is arbitrary real constant.

In [3, p. 299], the arbitrary constant $\bar{\beta}$ is chosen such that the constant $c_{-\kappa}$ can be written in the form
\[
c_{-\kappa} = ic_{-\kappa} z_0
\] (3.39)
where $c$ is a real constant and $z_0 = 1$ or $z_0 = i$. In this paper, we shall choose $\bar{\alpha}$ such that $c_{-\kappa}$ can be written in the form (3.39) where $z_0$ satisfies the conditions
\[
\text{Re}[z_0 e^{-iY(0)}] \neq 0, \quad \text{Re}[z_0 e^{-iZ(\infty)}] \neq 0, \quad |z_0| = 1.
\] (3.40)
If $\kappa > 0$, the term containing the summation in (3.38) is replaced by zero. Then, by using the Plemelj’s formula, we can prove the following lemma.

**Lemma 9.** The function $f$ given by (3.38) is a solution of the interior RH problem (3.24) if and only if the function $h$ is a solution of the integral equation
\[
(I + \mathcal{N})h + 2\sum_{k=0}^{-\kappa} \text{Re}[c_k A \eta^k] = 2\gamma.
\] (3.41)

Similarly, if $g(z)$ is a solution of the exterior RH problem (3.25), then $g(z)$ can be written as [3, 9]
\[
g(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A \eta - z} \, d\eta + \sum_{k=1}^{\kappa} c_k z^k
\] (3.42)
where $h$ is a real function, and for $\kappa > 0$, $c_k \ (k = 1, 2, \ldots, \kappa)$ are complex constants.

The constant $c_{\kappa}$ has the form
\[
c_{\kappa} = (\bar{\alpha} + i\beta) \exp(-iY(0))
\]
where $\beta$ is a real constant depending on $A$ and $g$, and $\bar{\alpha}$ is arbitrary real constant.

The constant $\bar{\alpha}$ will be chosen such that the constant $c_{\kappa}$ can be written in the form
\[
c_{\kappa} = ic_{\kappa} z_0
\]
where $c$ is a real constant and $z_0$ satisfies the conditions (3.40). If $\kappa \leq 0$, the term containing the summation in (3.42) is replaced by zero. Then using the Plemelj’s formula, we can prove the following lemma.
Lemma 10. The function $g$ defined by (3.42) is a solution of the exterior RH problem (3.25) if and only if the function $h$ is a solution of the integral equation

$$(\mathcal{I} - \mathcal{N})h + 2 \sum_{k=1}^{\kappa} \text{Re}[c_k A\eta^{-k}] = 2\gamma.$$  

(3.43)

4. Solving the Interior RH Problem

In this section, we shall use the integral equations (3.35) and (3.41) to give the solutions for the interior RH problem (3.24). We shall assume the RH problems are uniquely solvable, i.e., we shall assume the right-hand side $\gamma$ satisfies the condition (3.26) for $\kappa > 0$ and the solutions of the RH problems satisfy the constraints (3.30) or (3.31) for $\kappa \leq 0$.

4.1. The Interior RH Problem (3.24) with the Condition (3.26). For the interior RH problem (3.24) with the condition (3.26), we have $\kappa > 0$ and $\gamma \in (\mathcal{I} - \mathcal{N})^{\bot}$. Hence, Theorem 2 implies that $\text{dim Null}(\mathcal{I} - \mathcal{N}) = 0$ and $\text{dim Null}(\mathcal{I} + \mathcal{N}) = 2\kappa - 1 > 0$. Thus, by the Fredholm alternative theorem, the integral equation (3.35) is uniquely solvable. Moreover, the integral equation (3.41) becomes

$$(\mathcal{I} + \mathcal{N})h = 2\gamma.$$  

(4.44)

Since $\mathcal{N}^* = -\mathcal{N}$ and $\gamma \in \text{Null}(\mathcal{I} - \mathcal{N})^{\bot}$, then the Fredholm alternative theorem implies that the integral equation (4.44) is solvable. However, it is non-uniquely solvable.

Since, it is not easy to solve numerically the non-uniquely solvable integral equations. Hence, in this paper, we shall use only the uniquely solvable integral equation (3.35) to solve the interior RH problem (3.24) with the condition (3.26).

Let $\mu$ be the unique solution of the integral equation (3.35) and let $\Phi(z)$ be defined by (2.1), then by Lemma 7, the unique solution of the interior RH problem (3.24) with the condition (3.26) is given by $f(z) := \Phi(z)$. The boundary values of the function $f(z)$ are given by (3.34).

4.2. The Interior RH Problem (3.24) with the Constraints (3.30). For this case, we have $\kappa \leq 0$. Let $f(z)$ be the unique solution of the interior RH problem (3.24) with the constraints (3.30), then $f(z)$ can be written as

$$f(z) = \sum_{j=0}^{|\kappa|-1} \frac{\epsilon_j}{j!} z^j + z^{|\kappa|} f_0(z), \quad z \in \Omega,$$  

(4.45)

where $f_0(z)$ is the unique solution of the interior RH problem

$$\text{Re}[A_0(s)f_0^+(\eta(s))] = \gamma_0(s),$$  

(4.46)

subject to the constraint

$$\text{Im}[z_0 f_0(0)] = l_0,$$  

(4.47)

with $l_0 := e_{|\kappa|}/|\kappa|!$ and $\gamma_0(s) := \gamma(s) - \text{Re} \left[ A(s) \sum_{j=0}^{|\kappa|-1} \epsilon_j \eta(s)^j / j! \right].$

We shall present two methods for solving the the interior RH problem (4.46) with zero index. The first method is based on an integral equation obtained by modifying the integral equation (3.35). In the second method, we develop a new method based on an integral equation obtained by modifying the integral equation (3.41).
4.2.1. The Method 1. Since \( \kappa_0 = \text{ind}(A_0) = 0 \), then Lemma 7 implies that the function
\[
f_0(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\gamma_0 + i\mu}{A_0} \frac{d\eta}{\eta - z}
\]  
(4.48)
is a solution of the interior RH problem (4.46) with boundary values
\[
A_0(s)f^+_0(\eta(s)) = \gamma_0(s) + i\mu(s)
\]  
(4.49)
if and only if \( \mu \) is a solution of the integral equation
\[
\mu - \mathcal{N}_0\mu = -(\mathcal{M}_0 - \mathcal{K})\gamma_0.
\]  
(4.50)

By the definitions of \( a \) and \( b \), we have
\[
f_0(0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\gamma_0 + i\mu}{A_0} \frac{d\eta}{\eta} = \int_0^{2\pi} (\gamma_0(s) + i\mu(s))(a(s) + ib(s))ds.
\]  
(4.51)
Since the function \( f_0(z) \) satisfies the constraint (4.47), then (4.51) and (4.47) imply that \( \mu(s) \) satisfies
\[
\mathcal{P}\mu = \mathcal{Q}\gamma_0 + l_0
\]  
(4.52)
where the integral operators \( \mathcal{P} \) and \( \mathcal{Q} \) are as in (2.21) and (2.22). By adding (4.52) to (4.50), we obtain the following integral equation for the determination of \( \mu \),
\[
(\mathcal{I} - \mathcal{N}_0 + \mathcal{P})\mu = -(\mathcal{M}_0 - \mathcal{Q} - \mathcal{K})\gamma_0 + l_0.
\]  
(4.53)

**Lemma 11.** Suppose that \( \kappa \leq 0 \) and \( z_0 \) satisfies \( \text{Re}[z_0e^{-i\gamma(z_0)}] \neq 0 \), then the integral equation (4.53) is uniquely solvable.

**Proof.**
Suppose that \( \mu \in \text{Null}(\mathcal{I} - \mathcal{N}_0 + \mathcal{P}) \), i.e.,
\[
\mu(s) = \int_0^{2\pi} N_0(s,t)\mu(t)dt + \int_0^{2\pi} P(t)\mu(t)dt = 0.
\]  
(4.54)
Since \( \kappa_0 = \text{ind}(A_0) = 0 \), then by the Fredholm alternative theorem and by Theorem 2, \( \dim \text{Null}(\mathcal{I} - \mathcal{N}_0^2) = \dim \text{Null}(\mathcal{I} - \mathcal{N}_0) = 1 \). Let \( \phi \in \text{Null}(\mathcal{I} - \mathcal{N}_0^2) \). By Theorem 7 in [14], we can assume \( \phi \) to be a strictly positive function. Multiplying both sides of (4.54) by \( \phi(s) \) then integrating with respect to \( s \), we obtain
\[
\int_0^{2\pi} P(t)\mu(t)dt \int_0^{2\pi} \phi(s)ds = 0.
\]
Since \( \phi(s) \) is a positive real function, we obtain \( \mathcal{P}\mu = 0 \); Hence (4.54) implies that \( \mu \in \text{Null}(\mathcal{I} - \mathcal{N}_0) \). Let \( F(z) := \Phi(z) \) where \( \Phi(z) \) be formed with \( \gamma = 0 \) and \( \mu \) by (2.1), then Lemma 7 implies that the function \( F(z) \) is a solution of the homogeneous interior RH problem
\[
\text{Re}[A_0(s)F^+(\eta(s))] = 0,
\]
with \( A_0(s)F^+(\eta(s)) = i\mu(s) \). By the definition of the functions \( a \) and \( b \), we have
\[
F(0) = \int_0^{2\pi} i\mu(s)(a(s) + ib(s))ds \quad \text{which implies in view of } \mathcal{P}\mu = 0 \quad \text{that } \text{Im}[z_0F(0)] = 0.
\]
Since \( \text{Re}[z_0e^{-i\gamma(z_0)}] \neq 0 \), it follows from Lemma 3 that \( F(z) = 0 \) for all \( z \in \Omega \). Hence, \( \mu = 0 \), which implies in view of the Fredholm alternative theorem that the integral equation (4.53) is uniquely solvable. \( \square \)

Consequently, by the solving the uniquely solvable integral equation (4.53) for \( \mu \), the unique solution of the interior RH problem (3.24) with the constraints (3.30) is given by (4.45) where \( f_0 \) is given by (4.48). The boundary values of the function \( f(z) \) can be calculated from (4.49) and (4.45).
4.2.2. The Method 2. Since \( \kappa_0 = \text{ind}(A_0) = 0 \), then Lemma 9 implies that the solution \( f_0(z) \) of the interior RH problem (4.46) with the constraint (4.47) is given by

\[
 f_0(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A_0} \frac{d\eta}{\eta - z} + i\kappa_0 \tag{4.55}
\]

where \( c \) is a real constant, \( z_0 \) satisfies the conditions (3.40), and \( h \) is a solution of the integral equation

\[
 (I + N_0)h - 2i\text{Im}\left[\overline{\kappa_0}A_0\right] = 2\gamma_0. \tag{4.56}
\]

By the Cauchy integral formula and by the definition of the functions \( a \) and \( b \), we have

\[
 f_0(0) = \int_{0}^{2\pi} h(t)(a(t) + ib(t))dt + icz_0. \tag{4.57}
\]

Since \( f_0(z) \) satisfies the constraint (4.47) and \( |z_0| = 1 \), the constant \( c \) is given by

\[
 c = l_0 + Qh. \tag{4.58}
\]

By substituting (4.57) into (4.56), we obtain

\[
 (I + N_2)h = 2l_0\text{Im}\left[\overline{\kappa_0}A_0\right] + 2\gamma_0, \tag{4.59}
\]

where \( N_2 \) is the integral operator with the kernel

\[
 N_2(s, t) := N_0(s, t) - 2Q(t)\text{Im}\left[\overline{\kappa_0}A_0(s)\right]. \tag{4.60}
\]

**Lemma 12.** Suppose that \( \kappa \leq 0 \) and \( z_0 \) satisfies the conditions (3.40), then the integral equation (4.59) is uniquely solvable.

**Proof.**

Let \( h \in \text{Null}(I - N_2) \), i.e.,

\[
 h(s) + \int_{0}^{2\pi} \left( N_0(s, t) - 2Q(t)\text{Im}[\overline{\kappa_0}A_0(s)] \right) h(t)dt = 0, \tag{4.61}
\]

and let the function \( F(z) \), \( z \notin \Gamma \), be defined by

\[
 F(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A_0} \frac{d\eta}{\eta - z} + i\kappa_0 \int_{0}^{2\pi} Q(t)h(t)dt. \tag{4.62}
\]

Then by mean of the Plemelj’s formula, (4.60) implies that \( F(z) \) is a solution of the homogeneous interior RH problem

\[
 \text{Re}\left[ A_0(s)F^+(\eta(s)) \right] = 0. \tag{4.63}
\]

By the definition of the functions \( P \) and \( Q \), we have

\[
 z_0F(0) = \int_{0}^{2\pi} P(s)h(s)ds \tag{4.64}
\]

which implies that \( F(z) \) satisfies the constraint \( \text{Im}[z_0F(0)] = 0 \).

Since \( \kappa_0 = 0 \), Lemma 3 implies that \( F(z) = 0 \) for all \( z \in \Omega \). Consequently, we have

\[
 \frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A_0} \frac{d\eta}{\eta - z} = -i\kappa_0 \int_{0}^{2\pi} Q(t)h(t)dt, \quad z \in \Omega, \tag{4.65}
\]

which implies that \( F(z) \) is analytic in \( \Omega^- \) with \( A_0(s)F^-(\eta(s)) = -h(s) \) and
Let $G(z) := -iF(z)/z$, then $G$ is analytic in $\Omega^-$ with $A_1(s)G^-(\eta(s)) = ih(s)$ and satisfies the homogeneous exterior RH problem

\[ \mathrm{Re}[A_1(s)G^-(\eta(s))] = 0. \tag{4.66} \]

By the definition of the function $G$ and by (4.65), we have

\[ \frac{1}{2\pi i} \int_{\Gamma} G^-(\eta) d\eta = -\frac{1}{2\pi i} \int_{\Gamma} \frac{F^-(\eta)}{\eta} d\eta = -iF(\infty) = \frac{1}{2\pi i} \int_{0}^{2\pi} Q(t)h(t)dt, \tag{4.67} \]

which implies that the function $G$ satisfies

\[ \mathrm{Im} \left[ z_0 \frac{1}{2\pi i} \int_{\Gamma} G^-(\eta) d\eta \right] = 0. \]

Since $\mathrm{Re}[z_0 e^{-iz(\infty)}] \neq 0$, then by Lemma 5, $G(z) = 0$ for all $z \in \Omega^-$. Hence $h = 0$. Then the Fredholm alternative theorem implies that the integral equation (4.58) is uniquely solvable.

Let $\mu$ be the unique solution of the uniquely solvable integral equation (4.58) and let $f_0$ be given by (4.55), then the unique solution of the interior RH problem (3.24) with the constraints (3.30) is given by (4.45).

In the above two methods, the values of the unique solution $f$ of the interior RH problem (3.24) with the constraints (3.30) at any point $z \in \Omega$ can be calculated using the Cauchy integral formula. However, on the one hand, the method 1 provides us with the boundary values $f^+$ of the unique solution $f$ without any extra calculations as we need for the method 2. This is an advantage of the method 1 over the method 2. On the other hand, the right-hand side of the integral equation (4.58) is given explicitly and the right-hand side of the integral equation (4.53) requires extra calculations which is an advantage of the method 2 over the method 1. This remark is true of the methods 1 and 2 which will be given in the next section of the exterior RH problem (3.25) with the constraints (3.32).

4.3. The Interior RH Problem (3.24) with the Side Conditions (3.31). Let $\kappa \leq 0$ and let $f(z)$ be the unique solution of the interior RH problem (3.24) with the side conditions (3.31) and let $\mu$ be a solution of the integral equation (3.35), then Lemma 7 implies that $f(z) = \Phi(z)$ where $\Phi(z)$ is defined by (2.1). Furthermore, the boundary values of $f(z)$ are given by (3.34).

From (3.34), the side conditions (3.31) on the solution $f(z)$ of the interior RH problem (3.24) require the solution of the integral equation (3.35) to satisfy the constraints

\[ \mu(r_i) = d_i, \quad i = 1, 2, \ldots, 2|\kappa| + 1. \tag{4.68} \]

Substituting (4.68) into (3.35) implies that

\[ (\mathcal{R}_{2|\kappa|+1}\mu)(t) = c_{2|\kappa|+1}, \tag{4.69} \]

where $c_{2|\kappa|+1}$ is the $2|\kappa| + 1 \times 1$ vector with the elements

\[ (c_{2|\kappa|+1})_i = d_i + (\mathcal{M}\gamma)(r_i) - (\mathcal{K}\gamma)(r_i), \quad i = 1, 2, \ldots, 2|\kappa| + 1. \]

Consequently, the function $\mu$ satisfies the system of integral equations

\[ \left( \begin{array}{c} \mathcal{R}_{2|\kappa|+1} \\ \mathcal{I} - \mathcal{N} \end{array} \right) \mu = \left( \begin{array}{c} c_{2|\kappa|+1} \\ -(\mathcal{M} - \mathcal{K}\gamma) \end{array} \right). \tag{4.70} \]
Lemma 13. *The system of integral equations (4.70) is uniquely solvable.*

*Proof.*

Since the interior RH problem (3.24) with the side conditions (3.31) is uniquely solvable, then the system of integral equations (4.70) is solvable. Thus to show that (4.70) is uniquely solvable, it is sufficient to show that the homogenous system

\[
\begin{pmatrix}
\mathcal{R}_{2|\kappa|+1} \\
\mathcal{I} - \mathcal{N}
\end{pmatrix} \mu_0 = 0
\]

(4.71)

has only the trivial solution \( \mu_0 = 0 \). Let \( \mu_0 \) be any solution of the homogenous system (4.71), then \( \mu_0 = 0 \) satisfies

\[
\mu_0 - \mathcal{N} \mu_0 = 0 \quad \text{and} \quad \mathcal{R}_{2|\kappa|+1} \mu_0 = 0.
\]

Hence \( \mu_0(r_i) = (\mathcal{N} \mu_0)(r_i) = 0 \) for \( i = 1, 2, \ldots, 2|\kappa| + 1 \). Let \( f_0(z) := \Phi(z) \) where \( \Phi \) formed with \( \gamma = 0 \) and \( \mu_0 \) as in (2.1), then Lemma 7 implies that \( f_0 \) is a solution of the homogenous interior RH problem

\[
\text{Re}[A(s) f_0(\eta(s))] = 0
\]

(4.72)

with \( A(t) f_0(\eta(t)) = i \mu_0(t) \). Since \( \mu_0(r_i) = 0, \) hence

\[
\text{Im}[A(r_i) f_0(\eta(r_i))] = \mu_0(r_i) = 0, \quad i = 1, 2, \ldots, 2|\kappa| + 1.
\]

(4.73)

Then by Lemma 5, the homogenous interior RH problem (4.72) with the side conditions (4.73) has the unique solution \( f_0 = 0 \) which implies that \( \mu_0 = 0 \). \( \square \)

Let \( \mu \) be the unique solution of the system of integral equations (4.70) and let \( \Phi(z) \) be defined by (2.1), then the unique solution of the interior RH problem (3.24) with the side conditions (3.31) is given by \( f(z) := \Phi(z) \). The boundary values of the function \( f(z) \) are given by (3.34).

5. Solving the Exterior RH Problems

In this section, we shall give formulas for the solutions of the exterior RH problem (3.25) in terms of the solution of the integral equations (3.37) and (3.43). We shall assume the right-hand side \( \gamma \) satisfies the condition (3.27) for \( \kappa \leq 0 \) and the solutions of the exterior RH problem satisfy the constraints (3.32) or (3.33) for \( \kappa > 0 \) so the problem is always uniquely solvable.

5.1. The Exterior RH Problem (3.24) with the Condition (3.26).

Since \( \kappa \leq 0 \) and \( \gamma \in \text{Null}(\mathcal{I} + \mathcal{N})^\perp \), then Theorem 2 implies that \( \dim \text{Null}(\mathcal{I} + \mathcal{N}) = 0 \) and \( \dim \text{Null}(\mathcal{I} - \mathcal{N}) = 2|\kappa| + 1 > 0 \). Hence, the integral equation (3.43) becomes

\[
(\mathcal{I} - \mathcal{N}) h = 2 \gamma.
\]

(5.74)

Since \( \mathcal{N}^* = -\mathcal{N} \) and \( \gamma \in \text{Null}(\mathcal{I} + \mathcal{N})^\perp \), then the Fredholm alternative theorem implies that the integral equation (5.74) is non-uniquely solvable.

However, by the Fredholm alternative theorem, the integral equation (3.37) is uniquely solvable. Hence, we shall use only the uniquely solvable integral equation (3.37) to solve the exterior RH problem (3.25) with the condition (3.27).

By Lemma 8, the unique solution of the exterior RH problem (3.25) with the condition (3.27) is given by \( g(z) := -\Phi(z) \) where \( \Phi(z) \) is defined by (2.1) with \( \mu \) being the unique solution of the integral equation (3.37). The boundary values of the function \( g(z) \) are given by (3.36).
5.2. The Exterior RH Problem (3.25) with the Constraints (3.32). Let $\kappa > 0$ and let $g(z)$ be the unique solution of the exterior RH problem (3.25) with the constraints (3.32). Then $g(z)$ can be written as

$$
g(z) = \sum_{j=1}^{\kappa-1} \frac{e_j}{z^{\kappa-1}}, \quad z \in \Omega, \tag{5.75}
$$

where $g_1(z)$ is the unique solution of the exterior RH problem

$$
\text{Re}[A_1(s)g_1^-(\eta(s))] = \gamma_1(s) \tag{5.76}
$$

subject to the constraint

$$
\text{Im} \left[ \frac{z_0}{2\pi i} \int_{\Gamma} g_1(\eta)d\eta \right] = l_1 \tag{5.77}
$$

with $l_1 := e_\kappa$ and $\gamma_1(s) := \gamma(s) - \text{Re} \left[ A(s) \sum_{j=1}^{\kappa-1} e_j/\eta(\eta(s)) \right]$.

As for the interior RH problem, we shall present two methods for solving the exterior RH problem (5.76) with the constraint (5.77). The first method is based on a uniquely solvable integral equation obtained by modifying the integral equation (3.37). In the second method, we modify the integral equation (3.43) to obtain a new uniquely solvable integral equation.

5.2.1. The Method 1. Since $\kappa_1 = \text{ind}(A_1) = 1$, then Lemma 8 implies that the function

$$
g_1(z) := -\frac{1}{2\pi i} \int_{\Gamma} \frac{\gamma_1 + i\mu}{A_1(\eta)} d\eta \tag{5.78}
$$

is a solution of the exterior RH problem (5.76) with boundary values

$$
A_1(s)g_1^-(\eta(s)) = \gamma_1(s) + i\mu(s) \tag{5.79}
$$

if and only if $\mu$ is a solution of the integral equation

$$
(\mathcal{I} + \mathcal{N}_1)\mu = (\mathcal{M}_1 - \mathcal{K})\gamma_1. \tag{5.80}
$$

By the definition of $a$ and $b$, we have

$$
\frac{1}{2\pi i} \int_{\Gamma} g_1^-(\eta)d\eta = \frac{1}{2\pi i} \int_{\Gamma} \frac{\gamma_1 + i\mu}{A_1} d\eta = \int_0^{2\pi} (\gamma_1(s) + i\mu(s))(a(s) + ib(s))ds.
$$

Since the function $g_1(z)$ satisfies the constrain (5.77), then (5.77) implies that $\mu(s)$ satisfies

$$
\mathcal{P}\mu = \mathcal{Q}\gamma_1 + l_1. \tag{5.81}
$$

By adding (5.81) to (5.80), we obtain the following integral equation for the determination of $\mu$,

$$
(\mathcal{I} + \mathcal{N}_1 + \mathcal{P})\mu = (\mathcal{M}_1 + \mathcal{Q} - \mathcal{K})\gamma_1 + l_1. \tag{5.82}
$$

The following lemma can be proved along the same lines as Lemma 11.

Lemma 14. Suppose that $\kappa > 0$ and $z_0$ satisfies $\text{Re}[z_0e^{-iZ(\infty)}] \neq 0$, then the integral equation (5.82) is uniquely solvable.

By the solving the uniquely solvable integral equation (5.82) for $\mu$, the unique solution of the exterior RH problem (3.25) with the constraints (3.32) can be calculated from (5.75) where $g_1$ is given by (5.78). The boundary values of the function $g(z)$ can be calculated from (5.79) and (5.75).
5.2.2. The Method 2. By Lemma 10 and since \( \kappa_1 = \text{ind}(A_1) = 1 \), the solution of the exterior RH problem (5.76) with the constraint (5.77) is given by

\[
g_\ell(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A_1} \frac{d\eta}{\eta - z} + \frac{i z_0}{z},
\]

(5.83)

where \( c \) is a real constant, \( z_0 \) satisfies the conditions (3.40) and \( h \) is a solution of the integral equation

\[
(I - N_1)h - 2\text{Im}[\overline{z_0}]A_0 = 2\gamma_1.
\]

(5.84)

The function \( g_\ell \) can be written in the form

\[
g_\ell(z) = \frac{1}{z} \left( -\frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A_0} \frac{d\eta}{\eta - z} + \int_0^{2\pi} (a(s) + ib(s))h(s)ds + i z_0 \right).
\]

(5.85)

Hence

\[
\frac{1}{2\pi i} \int_{\Gamma} g_\ell^{-}(\eta)d\eta = \int_0^{2\pi} (a(s) + ib(s))h(s)ds + i z_0.
\]

(5.86)

Since \( g_\ell \) satisfies the constraint (5.77) and \( |z_0| = 1 \), then (5.86) implies that the constant \( c \) is given by

\[
c = l_1 + Qh.
\]

(5.87)

By substituting (5.87) into (5.84), we obtain

\[
(I - N_3)h = 2l_1\text{Im}[\overline{z_0}]A_0 + 2\gamma_1,
\]

(5.88)

where \( N_3 \) is the integral operator with the kernel

\[
N_3(s, t) = N_1(s, t) + 2Q(t)\text{Im}[\overline{z_0}A_0(s)].
\]

(5.89)

**Lemma 15.** Suppose that \( \kappa > 0 \) and \( z_0 \) satisfies the conditions (3.40), then the integral equation (5.88) is uniquely solvable.

**Proof.**

Let \( h \) be a solution of the homogenous equation

\[
h(s) - \int_0^{2\pi} (N_1(s, t) + 2Q(t)\text{Im}[\overline{z_0}]A_0(s))] h(t)dt = 0,
\]

(5.90)

and let the function \( G(z) \), \( z \notin \Gamma \), be defined by

\[
G(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{h}{A_1} \frac{d\eta}{\eta - z} - \frac{i z_0}{z} \int_0^{2\pi} Q(t)h(t)dt.
\]

(5.91)

Then it follows from the jump relation (2.3) that

\[
G^+(\eta(s)) - G^-(\eta(s)) = \frac{h(s)}{A_1(s)} - \frac{i z_0}{\eta(s)} \int_0^{2\pi} Q(t)h(t)dt.
\]

(5.92)

By mean of the Plemelj’s formula, (5.90) implies that the function \( G \) is a solution of the homogeneous exterior RH problem

\[
\text{Re}[A_1(s)G^-(\eta(s))] = 0.
\]

(5.93)

By the definition of the function \( a \) and \( b \), we have

\[
\frac{z_0}{2\pi i} \int_{\Gamma} G^-(\eta)d\eta = -\int_0^{2\pi} a(s)h(s)ds,
\]

(5.94)
which implies that $G$ satisfies the constraint $\text{Im } [z_0 \frac{1}{2\pi} \int_{\Gamma} G^-(\eta)d\eta] = 0$. Since $\kappa_1 = 1$, Lemma 5 implies that $G(z) = 0$ on $\Omega^-$. Hence, $G^-(\eta) = 0$ on $\Gamma$, which implies in view of (5.92) that the function $G(z)$ is analytic in $\Omega$ with

$$G^+(\eta(s)) = \frac{h(s)}{A_1(s)} - \frac{iz_0}{\eta(s)} \int_0^{2\pi} Q(t)h(t)dt.$$  \hfill (5.95)

Hence the function

$$F(z) := izG(z) - \frac{z_0}{2\pi} \int_0^{2\pi} Q(t)h(t)dt,$$  \hfill (5.96)

is a solution of the homogeneous interior RH problem

$$\text{Re}[A_0(s)F^+(\eta(s))] = 0,$$  \hfill (5.97)

with the condition $\text{Im}[z_0 F(0)] = 0$. Since $\kappa_0 = 0$, it follows from Lemma 3 that $F(z) = 0$ for all $z \in \Omega$. Consequently,

$$G(z) = \frac{z_0}{iz} \int_0^{2\pi} Q(t)h(t)dt.$$  \hfill (5.98)

Since $G(z)$ is analytic in $\Omega$, it follows from (5.98) that $\int_0^{2\pi} Q(t)h(t)dt = 0$ which implies that $G(z) = 0$ on $\Omega$. In view of (5.95), we obtain $\mu = 0$. Hence, by the Fredholm alternative theorem, the integral equation (5.88) is uniquely solvable. \hfill \square

### 5.3. The Exterior RH Problem (3.25) with the Side Conditions (3.33).

Let $\kappa \leq 0$ and let $g(z)$ be the unique solution of the exterior RH problem (3.25) with the side conditions (3.33), then Lemma 8 implies that $g(z) = -\Phi(z)$ where $\Phi(z)$ defined by (2.1) with $\mu$ is a solution of the integral equation (3.37). Furthermore, the boundary values of $g(z)$ are given by (3.36).

By (3.36), the side conditions (3.33) on the solution $g(z)$ of the exterior RH problem (3.25) require the solution $\mu(t)$ of the integral equation (3.37) to satisfy the constraints

$$\mu(r_i) = d_i, \quad i = 1, 2, \ldots, 2\kappa - 1.$$  \hfill (5.99)

Substituting (5.99) into (3.37) implies that

$$(R_{2\kappa-1} - \mu)(t) = d_{2\kappa-1},$$  \hfill (5.100)

where $d_{2\kappa-1}$ is the $2\kappa - 1 \times 1$ vector with the elements

$$(d_{2\kappa-1})_i = -d_i + (M\gamma)(r_i) - (K\gamma)(r_i), \quad i = 1, 2, \ldots, 2\kappa - 1.$$  \hfill \square

Consequently, the function $\mu$ satisfies the system of integral equations

$$\left( R_{2\kappa-1} \right) \mu = \left( \begin{array}{c} d_{2\kappa-1} \\ I + N \end{array} \right).$$  \hfill (5.101)

#### Lemma 16. The the system of integral equations (5.101) is uniquely solvable.

**Proof.**

This lemma can be proved with the same arguments as Lemma 13. \hfill \square

By solving the uniquely solvable system of integral equations (5.101) for $\mu$, the unique solution of the exterior RH problem (3.25) with the side conditions (3.33)
is given by $g(z) := -\Phi(z)$ where $\Phi(z)$ is defined by (2.1). The boundary values of the function $g(z)$ are given by (3.36).

6. The Numerical Implementations

Since the integrals in the integral equations of this paper are over $2\pi$-periodic functions, they can be best discretized on an equidistant grid by the trapezoidal rule, i.e., the integral equations are solved by the Nyström method with the trapezoidal rule as the quadrature rule (Atkinson [1]).

Suppose that $n$ is an even integer and define the the $n$ equidistant collocation points $t_j$ by

$$t_j := (j - 1)\frac{2\pi}{n}, \quad j = 1, 2, \ldots, n.$$  \hspace{1cm} (6.102)

For a $2\pi$-periodic function $h$, then the trapezoidal rule approximate the integral $I(h) := \int_0^{2\pi} h(t)dt$ by

$$I_n(h) := \frac{2\pi}{n} \sum_{j=1}^{n} h(t_j).$$  \hspace{1cm} (6.103)

Then the trapezoidal rule (6.103) with the grid (6.102) will be used to discretize the integrals in the integral equations (3.35) for $\cdot > 0$, (4.53), (4.58), (3.37) for $\cdot \cdot \cdot 0$, (5.82), (5.88) and the system of integral equations (4.70) for $\cdot \cdot \cdot 0$ and (5.101) for $\cdot > 0$.

The discretization operator $N_n$ of the operator $\mathcal{N}$ is given by

$$(\mathcal{N}_n h)(s) := \frac{2\pi}{n} \sum_{j=1}^{n} N(s, t_j) h(t_j).$$  \hspace{1cm} (6.104)

Then we define the matrix $\mathbf{N}$ to be the $n \times n$ matrix with the elements

$$\mathbf{N}_{ij} := \frac{2\pi}{n} N(t_i, t_j), \quad i, j = 1, 2, \ldots, n.$$  \hspace{1cm} (6.105)

The discretization of the operators $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{M}, \mathcal{M}_0$ and $\mathcal{M}_1$ is defined as in (6.104) and will be denoted by $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{M}, \mathcal{M}_0, \mathcal{M}_1$, respectively. Similarly, we define the matrices $\mathbf{N}_0, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{M}, \mathbf{M}_0$ and $\mathbf{M}_1$ as in (6.105) with $N$ replaced by $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{M}, \mathcal{M}_0$ and $\mathcal{M}_1$, respectively.

The discretization operator $\mathcal{P}_n$ of the operator $\mathcal{P}$ is defined by

$$(\mathcal{P}_n h)(s) = \frac{2\pi}{n} \sum_{j=1}^{n} P(t_j) h(t_j).$$  \hspace{1cm} (6.106)

The discretization operator $\mathcal{Q}_n$ of the operator $\mathcal{Q}$ is defined as in (6.106). Then we define the matrices $\mathbf{P}$ and $\mathbf{Q}$ to be the $n \times n$ matrices with the elements

$$\mathbf{P}_{ij} := \frac{2\pi}{n} P(t_j), \quad \mathbf{Q}_{ij} := \frac{2\pi}{n} Q(t_j), \quad i, j = 1, 2, \ldots, n.$$  \hspace{1cm} (6.107)

For $\kappa \leq 0$, we define the $(2|\kappa| + 1) \times n$ matrix $\mathbf{R}_1$ and the $(2|\kappa| + 1) \times 1$ vector $\mathbf{\bar{c}}$ for $i = 1, 2, \ldots 2|\kappa| + 1$ and $j = 1, 2, \ldots, n$ by

$$\mathbf{R}_{1,ij} := \frac{2\pi}{n} N(r_i, t_j), \quad \mathbf{\bar{c}}_i = d_i + (\mathcal{M}_\gamma)(r_i) - (\mathcal{K}_\gamma)(r_i).$$
where \( r_i \) and \( d_i \) are as in Lemma 4. Similarly, for \( \kappa > 0 \), we define the \((2\kappa - 1) \times n\) matrix \( R_2 \) and the \((2\kappa - 1) \times 1\) vector \( d \) by

\[
R_{2,ij} := \frac{2\pi}{n} N(r_i, t_j), \quad d_i = -d_i + (\mathcal{M}\gamma)(r_i) - (\mathcal{K}\gamma)(r_i)
\]

where \( r_i \) and \( d_i \) are as in Lemma 6, \( i = 1, 2, \ldots 2\kappa - 1 \) and \( j = 1, 2, \ldots, n \).

We denote by \( \mu_n \) to the approximate solution of the integral equations (3.35) for \( \kappa > 0 \), (4.53), (4.58), (3.37) for \( \kappa \leq 0 \), (5.82), (5.88) and the system of integral equations (4.70) for \( \kappa \leq 0 \) and (5.101) for \( \kappa > 0 \). Then we define the \( n \times 1 \) vector \( x \) by

\[
x_i := \mu_n(t_i), \quad i = 1, 2, \ldots, n.
\]

We define also the \( n \times 1 \) vectors \( y, y_0 \) and \( y_1 \) by

\[
y_i := \gamma(t_i), \quad y_{0,i} := \gamma_0(t_i), \quad y_{1,i} := \gamma_1(t_i), \quad y_{2,i} := \text{Im}[h_0(t_i)].
\]

Let \( K_n \) be the discretization of the operator \( K \), then we can calculate \((K_n h)(t)\) efficiently for all \( t \in [0, 2\pi] \) using the FFT. For the collocation points \( t_i \) \( (i = 1, 2, \ldots, n) \), if \( y_3 \) is the \( n \times 1 \) vector with the elements

\[
y_{3,i} = (K_n h)(t_i),
\]

then

\[
y_3 = Kx
\]

where the matrix \( K \) is known as the Wittich’s matrix and is given for \( i, j = 1, 2, \ldots, n \) by

\[
K_{ij} = \begin{cases} 
0, & \text{if } j - i \text{ is even} \\
\frac{n}{\pi} \cot \frac{(i-j)\pi}{n}, & \text{if } j - i \text{ is odd}.
\end{cases}
\]

Let \( I \) be the \( n \times n \) identity matrix and let \( I_1 \) be the \( n \times 1 \) vector whose elements are all ones. Hence, the applying of the Nyström method to the uniquely solvable integral equations (3.35) for \( \kappa > 0 \), (4.53), (4.58), (3.37) for \( \kappa \leq 0 \), (5.82) and (5.88) leads, respectively, to the following linear systems

\[
(I - N)x = -(M - K)y,
\]

\[
(I - N_0 + P)x = -(M_0 - Q - K)y_0 + l_0 I_1,
\]

\[
(I + N_2)x = 2l_0 y_2 + 2y_0,
\]

\[
(I + N)x = (M - K)y.
\]

\[
(I + N_1 + P)x = (M_1 + Q - K)y_1 + l_1 I_1,
\]

\[
(I - N_3)x = 2l_1 y_2 + 2y_1.
\]

Since the integral equations are uniquely solvable, then the resulting linear system (6.108)–(6.113) are uniquely solvable for sufficiently large \( n \) [1, p. 107].

By using the trapezoidal rule (6.103) to discretize the integrals in the system of integral equations (4.70) for \( \kappa \leq 0 \) and (5.101) for \( \kappa > 0 \) then collocating at the node points (6.102), we obtain, respectively, the following over-determined linear system

\[
\begin{pmatrix}
R_1 \\
I - N \end{pmatrix} x = \begin{pmatrix}
\tilde{c} \\
-(M - K)y \end{pmatrix},
\]

\[
\begin{pmatrix}
R_2 \\
I + N \end{pmatrix} x = \begin{pmatrix}
\tilde{d} \\
(M - K)y \end{pmatrix}.
\]
The above linear systems (6.108)–(6.115) are either uniquely solvable square linear systems or over-determined linear systems. In our numerical calculations, both type of linear systems are solved using the MATLAB’s \ operator that makes use of the Gauss elimination method for square systems and the QR factorization with column pivoting method for over-determined systems [12].

By solving the linear systems, we obtain the solutions of the integral equations at the collocation points \( t_i, i = 1, 2, \ldots, n \). Then the Nyström interpolating formula provides us with approximate solutions \( \mu_n(t) \) to the integral equations for all \( t \in [0, 2\pi] \). The approximate solutions \( \mu_n(t) \) of these integral equations can be then used to obtain approximate solutions to the RH problems.

7. Numerical Examples

In this section we apply the proposed method to six examples contain three interior RH problems (Examples 1–3) and three exterior RH problems (Examples 4–6) in the interior and the exterior of the smooth Jordan curve \( \Gamma \) with the parameterization \( \eta(s) = (3 + \cos 3s + \sin 5s)e^{is}, \ 0 \leq s \leq 2\pi \). The graphs of \( \Gamma \) is shown in Figure 1.

Tables 1–8 show the values of the approximate solutions of the RH problems at the test points \( z_1 = -1 - i \) and \( z_2 = 1 - i \) for the interior problems and at the test points \( z_3 = -3 - i \) and \( z_4 = 3 - i \) for the exterior problems.

Tables 1, 2, 4–6 and 8 list also the sup-norm error \( \|f^+ - f_n^+\|_{\infty} \) and \( \|g^+ - g_n^+\|_{\infty} \) where \( f^+ \), \( g^+ \) are the boundary values of the exact solutions and \( f_n^+ \), \( g_n^+ \) are the boundary values of the approximate solutions. The sup-norm is computed numerically by comparing \( f^+(\eta(t)), g^+(\eta(t)) \) and \( f_n^+(\eta(t)), g_n^+(\eta(t)) \) at 100 equally spaced points in \([0, 2\pi]\), most of which are not collocation points.

The exact solution of the RH problem in Example 1 is \( f(z) = z \) and the exact solution of the RH problem in Example 4 is \( g(z) = 1/z \). The exact solutions of the RH problems in the remaining examples are not known. For this case, we consider the approximate solution obtained with \( n = 1024 \) as the exact solution.
### Table 1. The numerical results for Example 17.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$|f - f_n|_\infty$</th>
<th>$f(z_1)$</th>
<th>$f(z_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>$2.67(-01)$</td>
<td>-1.030373 - 1.001346i</td>
<td>1.001033 - 0.966912i</td>
</tr>
<tr>
<td>64</td>
<td>$3.95(-02)$</td>
<td>-1.002061 - 0.998145i</td>
<td>0.999868 - 0.996886i</td>
</tr>
<tr>
<td>128</td>
<td>$1.13(-03)$</td>
<td>-0.999988 - 1.000000i</td>
<td>1.000008 - 1.000012i</td>
</tr>
<tr>
<td>256</td>
<td>$4.92(-07)$</td>
<td>-1.000000 - 1.000000i</td>
<td>1.000000 - 1.000000i</td>
</tr>
<tr>
<td>512</td>
<td>$4.14(-11)$</td>
<td>-1.000000 - 1.000000i</td>
<td>1.000000 - 1.000000i</td>
</tr>
</tbody>
</table>

### Table 2. The numerical results for Example 18 using method 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$|f - f_n|_\infty$</th>
<th>$f(z_1)$</th>
<th>$f(z_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>$1.64(-01)$</td>
<td>0.121506 + 0.388527i</td>
<td>-0.082456 - 0.044672i</td>
</tr>
<tr>
<td>64</td>
<td>$2.92(-02)$</td>
<td>0.139595 + 0.405296i</td>
<td>-0.069501 - 0.056732i</td>
</tr>
<tr>
<td>128</td>
<td>$5.88(-04)$</td>
<td>0.139656 + 0.405110i</td>
<td>-0.068900 - 0.058102i</td>
</tr>
<tr>
<td>256</td>
<td>$1.88(-07)$</td>
<td>0.139653 + 0.405115i</td>
<td>-0.068910 - 0.058077i</td>
</tr>
<tr>
<td>512</td>
<td>$8.81(-12)$</td>
<td>0.139653 + 0.405115i</td>
<td>-0.068910 - 0.058077i</td>
</tr>
</tbody>
</table>

### Table 3. The numerical results for Example 18 using method 2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(z_1)$</th>
<th>$f(z_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>0.116989 + 0.289777i</td>
<td>-0.125253 + 0.013495i</td>
</tr>
<tr>
<td>64</td>
<td>0.142597 + 0.405528i</td>
<td>-0.068479 - 0.058130i</td>
</tr>
<tr>
<td>128</td>
<td>0.139671 + 0.405097i</td>
<td>-0.068861 - 0.058195i</td>
</tr>
<tr>
<td>256</td>
<td>0.139653 + 0.405115i</td>
<td>-0.068910 - 0.058077i</td>
</tr>
<tr>
<td>512</td>
<td>0.139653 + 0.405115i</td>
<td>-0.068910 - 0.058077i</td>
</tr>
</tbody>
</table>

### Table 4. The numerical results for Example 19.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$|f - f_n|_\infty$</th>
<th>$f(z_1)$</th>
<th>$f(z_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>$2.44(-01)$</td>
<td>0.049024 - 0.649521i</td>
<td>-0.608792 - 0.403835i</td>
</tr>
<tr>
<td>64</td>
<td>$4.15(-02)$</td>
<td>0.077166 - 0.672586i</td>
<td>-0.564633 - 0.384698i</td>
</tr>
<tr>
<td>128</td>
<td>$6.31(-04)$</td>
<td>0.086361 - 0.679775i</td>
<td>-0.555738 - 0.381776i</td>
</tr>
<tr>
<td>256</td>
<td>$3.15(-07)$</td>
<td>0.086348 - 0.679774i</td>
<td>-0.555758 - 0.381788i</td>
</tr>
<tr>
<td>512</td>
<td>$1.99(-11)$</td>
<td>0.086348 - 0.679774i</td>
<td>-0.555758 - 0.381788i</td>
</tr>
</tbody>
</table>

Example 17. $A(s) = e^{is}/R(s)$ ($\kappa = 1$) and $\gamma(s) = \cos 2s$.

Example 18. $A(s) = e^{-is}$ ($\kappa = -1$), $e_0 = e_1 = 0$, $z_0 = 1$ and $\gamma(s) = \cos 2s$.

Example 19. $A(s) = e^{-is}$ ($\kappa = -1$), $r_1 = 0$, $r_2 = \pi/2$, $r_3 = \pi$, $d_1 = d_2 = d_3 = 0$ and $\gamma(s) = \cos 2s$.

Example 20. $A(s) = R(s)e^{-is}$ ($\kappa = -1$) and $\gamma(s) = \cos 2s$.

Example 21. $A(s) = e^{is}$ ($\kappa = 1$), $e_1 = 0$, $z_0 = 1$ and $\gamma(s) = \cos 2s$.

Example 22. $A(s) = e^{is}$ ($\kappa = 1$), $r_1 = 0$, $d_1 = 0$ and $\gamma(s) = \cos 2s$. 
Table 5. The numerical results for Example 20.

<table>
<thead>
<tr>
<th>n</th>
<th>|g - g_n|_\infty</th>
<th>g(z_3)</th>
<th>g(z_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.55</td>
<td>-0.356640 + 0.123182i</td>
<td>0.347841 + 0.186617i</td>
</tr>
<tr>
<td>64</td>
<td>2.97(-02)</td>
<td>-0.301208 + 0.099487i</td>
<td>0.301949 + 0.103685i</td>
</tr>
<tr>
<td>128</td>
<td>8.74(-04)</td>
<td>-0.299973 + 0.099983i</td>
<td>0.299980 + 0.099960i</td>
</tr>
<tr>
<td>256</td>
<td>1.71(-07)</td>
<td>-0.300000 + 0.100000i</td>
<td>0.300000 + 0.100000i</td>
</tr>
<tr>
<td>512</td>
<td>3.66(-12)</td>
<td>-0.300000 + 0.100000i</td>
<td>0.300000 + 0.100000i</td>
</tr>
</tbody>
</table>

Table 6. The numerical results for Example 21 using method 1.

<table>
<thead>
<tr>
<th>n</th>
<th>|g - g_n|_\infty</th>
<th>g(z_3)</th>
<th>g(z_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.55</td>
<td>-0.113352 + 0.559966i</td>
<td>0.465106 + 1.480249i</td>
</tr>
<tr>
<td>64</td>
<td>2.85(-01)</td>
<td>-0.252018 + 0.533466i</td>
<td>0.259934 + 1.379282i</td>
</tr>
<tr>
<td>128</td>
<td>1.16(-02)</td>
<td>-0.251439 + 0.535743i</td>
<td>0.260401 + 1.372368i</td>
</tr>
<tr>
<td>256</td>
<td>4.21(-06)</td>
<td>-0.251351 + 0.535805i</td>
<td>0.262485 + 1.372376i</td>
</tr>
<tr>
<td>512</td>
<td>3.49(-11)</td>
<td>-0.251351 + 0.535805i</td>
<td>0.264285 + 1.372376i</td>
</tr>
</tbody>
</table>

Table 7. The numerical results for Example 21 using method 2.

<table>
<thead>
<tr>
<th>n</th>
<th>g(z_3)</th>
<th>g(z_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>-0.179432 + 0.544865i</td>
<td>0.633549 + 1.580277i</td>
</tr>
<tr>
<td>64</td>
<td>-0.248657 + 0.538546i</td>
<td>0.286340 + 1.396616i</td>
</tr>
<tr>
<td>128</td>
<td>-0.251359 + 0.535816i</td>
<td>0.264296 + 1.372618i</td>
</tr>
<tr>
<td>256</td>
<td>-0.251351 + 0.535805i</td>
<td>0.264285 + 1.372376i</td>
</tr>
<tr>
<td>512</td>
<td>-0.251351 + 0.535805i</td>
<td>0.264285 + 1.372376i</td>
</tr>
</tbody>
</table>

Table 8. The numerical results for Example 22.

<table>
<thead>
<tr>
<th>n</th>
<th>|g - g_n|_\infty</th>
<th>g(z_3)</th>
<th>g(z_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.55</td>
<td>-0.447336 - 0.075625i</td>
<td>0.103461 + 1.831099i</td>
</tr>
<tr>
<td>64</td>
<td>3.42(-01)</td>
<td>-0.034616 + 1.192131i</td>
<td>0.482947 + 0.717498i</td>
</tr>
<tr>
<td>128</td>
<td>1.20(-02)</td>
<td>-0.030617 + 1.198096i</td>
<td>0.484756 + 0.710105i</td>
</tr>
<tr>
<td>256</td>
<td>4.58(-06)</td>
<td>-0.030675 + 1.197833i</td>
<td>0.484962 + 0.710348i</td>
</tr>
<tr>
<td>512</td>
<td>7.02(-11)</td>
<td>-0.030675 + 1.197833i</td>
<td>0.484961 + 0.710348i</td>
</tr>
</tbody>
</table>

8. Conclusions

We developed a numerical method for solving numerically the interior and the exterior RH problems. The method is based on the boundary integral equations with the generalized Neumann kernel that have been derived in [11, 8, 6, 7, 14]. The uniquely solvable RH problems were solved using only the integral equations derived in [8, 6, 7, 14] because they are uniquely solvable and the integral equations derived in [11] are non-uniquely solvable.

The non-uniquely solvable RH problems with additional constraints (at z = 0 or at z = \infty) were solved using two methods, the method 1 based on the boundary
integral equation of [7, 14] and the method 2 based on the boundary integral equation of [11]. For the two methods, the solutions of the RH problems were calculated using the Cauchy integral formula. The advantage of method 1 is that it provides us with the boundary values of the solutions of the RH problems without any extra calculations as we need for the method 2. However, the right-hand side of the integral equation of [11] is given explicitly and the right-hand side of the integral equation of [7, 14] requires extra calculations which is an advantage of method 2 over method 1.

Several RH problems were solved using the developed method. The numerical examples show clearly that the developed method gives results of high accuracy.

Acknowledgement

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References