UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – I Annual Examination – 2022

Roll No. ....

Subject: Mathematics (Old & New Course) **Paper: I (Real Analysis)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION – I

Q # 1.

a. Show that the set of rational number $\mathbb{Q}$, is dense in $\mathbb{R}$. (7)

b. Show that there does not exist a rational number r whose square is 2. (6)

c. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are complex numbers, then prove that (7)

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} \left|a_j\right|^2 \sum_{j=1}^{n} \left|b_j\right|^2$$

Q # 2.

a. Prove that every Cauchy sequence of real numbers is bounded. Does the converse of this result hold? Justify your claim. (10)

b. Discuss the convergence of the p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}, p \in [0, \infty)$. (10)

Q # 3.

a. Let $f : A \to R$ and c be a cluster point of A. Show that the following statements are equivalent. (10)

i. $\lim_{x \to c} f(x) = L$

ii. for every sequence (x_n) in A, that converges to c such that $x_n \neq c$ for all n, $f(x_n)$ converges to L.

b. Show that $f(x) = \sin\frac{1}{x}, x \neq 0$ is not uniformly continuous on the interval $(0, 1]$. (10)

Q # 4.

a. Let $I = [a, b]$ and let $f: I \to \mathbb{R}$ be continuous I. If $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then show that there exists a number $c \in (a, b)$ such that $f(c) = 0$. (10)

b. Suppose that f is differentiable at a point c in the domain of f and that g is differentiable at $t = f(c)$, then the composite function $\psi = g \circ f$ is differentiable at c and $\psi'(c) = g'(f(c))f'(c)$. Apply this result to find the derivative of $\sqrt{x^2 + 1}$ at $x = 1$. (10)

Q # 5.

a. State Schwarz's theorem. Show that the function given below does not satisfy the conditions of Schwarz's theorem.

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq 0 \\ 0 & , (x, y) = 0 \end{cases}$$

b. Calculate $f_x, f_y, f_{xy}(0,0), f_{yx}(0, 0)$ for the function

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq 0 \\ 0 & , (x, y) = 0 \end{cases}$$

(10)

SECTION – II

Q # 6.

a. If f is continuous on $[a, b]$, then show that $f \in \Re(\alpha)$ on $[a, b]$. Moreover, to every $\epsilon > 0$, there corresponds a $\delta > 0$ such that (10)

$$\left|S(P, f, \alpha) - \int_a^b f \, d\alpha\right| < \epsilon$$

b. Show that if $f \in \Re(\alpha)$ on $[a, b]$, then $f^2 \in \Re(\alpha)$. (10)

Q # 7.

a. Show that the given integral is convergent if and only if $m > 0$

$$\int_0^1 x^{m-1} e^{-x} \, dx$$

(10)

b. Test the convergence of the integral $\int_0^1 f(x) \, dx$, where $f(x)$ is defined 0n $]0, 1]$ by,

$$f(x) = (-1)^{n+1} n\,(n+1), \quad \frac{1}{n+1} \leq x < \frac{1}{n}, n \in N.$$

(10)

Q # 8.

a. Prove that the function f is function of bounded variation on $[a, b]$ if and only if it is of bounded variation on $[a, c]$ and $[c, b]$, where c is a point of $[a, b]$. (10)

b. Compute the positive negative and total variation functions of (10)

$$f(x) = 3x^2 - 2x^3, \qquad -2 \leq x \leq 2$$

Q # 9.

a. Let $f_n(x)$ be a sequence of functions defined on $[a, b]$, and

$$|f_n(x)| \leq M_n(x), \qquad n = 1, 2, 3, \ldots$$

where $M_n(x)$ are positive real numbers for each $x \in [a, b]$. Show that if the series $\sum_{n=1}^{\infty} M_n(x)$ is convergent, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$. (10)

b. Give an example of the sequence of functions whose every term is continuous, but the limit function is not continuous provided that the sequence is pointwise convergent.

(10)

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M.A./M.Sc. Part – I Annual Examination – 2022

Roll No.

Subject: Mathematics (Old & New Course) **Paper: II (Algebra)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION – I

Q. 1 .... **20**

(a) Show that every group is isomorphic to a subgroup of the Symmetric group. [10]

(b) Let G be a group and p is the smallest prime dividing $|G|$. Show that if G has a subgroup H of index p then H is a normal subgroup of G. [10]

Q. 2 .... **20**

(a) Let H be a subgroup of a group G and N be a normal subgroup of G. Show that $HN = \{hn : h \in H, n \in N\}$ is a subgroup of G. [10]

(b) Let $G = Z$, $H = \langle 3 \rangle$, and $K = \langle 4 \rangle$. Find $(H + K)/K$ and $H/(H \cap K)$ and show that they are isomorphic. [10]

Q. 3 .... **20**

(a) Let G be a cyclic group of order n generated by element g and $a = g^k$. Show that a generates a cyclic subgroup H of order n/d, where $d = gcd(n, k)$. [10]

(b) Write down the elements of the quotient group G/H, where G is the Quaternion group $\langle a, b : \; a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$ and $H = \{1, a^2\}$. Moreover, identify G/H. [10]

Q. 4 .... **20**

(a) Write down the conjugacy classes of the Symmetric group S_4. [10]

(b) Show that a group of order 48 can not be simple. [10]

Q. 5 .... **20**

(a) Let G be a group of order $p^n.m$, where p is prime and $p \nmid m$. Show that any two Sylow p subgroups of G are conjugate in G, [10]

(b) Find the normalizer of the subgroup $H = \{1, (1, 2, 3), (1, 3, 2)\}$ of the Symmetric group S_3. [10]

SECTION – II

Q. 6 .... ***20***

(a) Given an ideal I of the ring R, define the set $C(I)$ by [10]

$$C(I) = \{r \in R : \ ra - ar \in I, \ \forall a \in R.\}$$

Verify that $C(I)$ forms a subring of R.

(b) If I_i, $(i = 1, 2, \ldots)$ is a collection of ideals of the ring R such that [10]

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots,$$

prove that $\bigcup_i I_i$ is also an ideal of R.

Q. 7 .... ***20***

(a) Assuming that R is a division ring, show that center of R forms a field. [10]

(b) Determine if the elements $(1, 2, 3), (0, 4, 5), (\frac{1}{2}, 3, \frac{21}{4})$ in V, the vector space of 3-tuples over $\mathbb{R}$, are linearly independent over $\mathbb{R}$. [10]

Q. 8 .... ***20***

(a) Find all ideals of the ring $Z_2 \times Z_2$. [10]

(b) Let R be a commutative ring with unit element and I be a proper ideal of R. Show that I is a prime ideal of R if and only if R/I is an Integral Domain. [10]

Q. 9 .... ***20***

(a) Find a matrix P, if possible, for which $P^{-1}AP$ is diagonal, where [10]

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) Let V be a vector space over field F with $dim(V) = n$. Prove that any n linearly independent elements of V form a basis of V over F. [10]

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M.A./M.Sc. Part – I Annual Examination – 2022

Subject: Mathematics (Old & New Course)
Paper: III (Complex Analysis and Differential Geometry)

Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section. All Questions carry equal marks.

SECTION – I

Q1.

a) Determine the set of points S where the function $f(z) = |z - i|^2$ is continuous.
b) Suppose $z_1 \neq z_2$. Interpret $\text{Re}(z_1\overline{z_2}) = 0$ geometrically in terms of vectors z_1 and z_2
c) Draw the pair of points $z = a + ib$ and $\bar{z} = a - ib$ in the complex plane if

$$a > 0, b > 0; \;\; a >, b < 0; \;\; a < 0, b > 0; \;\; a, 0, b < 0$$

(06+06+08)

Q2.

a) Show that the function $f(z) = z^4$ is analytic in the domain $D = C$, and the function $g(z) = |z|^2$ is not analytic in any domain.
b) Evaluate

$$\oint_C \frac{9z^2 - iz + 4}{z(z^2+1)} dz$$

where the contour is a circle of radius 2, described in the positive direction.

(12+08)

Q3.

a) Show that under the transformation $w = sinz$, the line $x = p$, where $2p/\pi$ is not an integer, maps onto a branch of the hyperbola

$$|w + 1| - |w - 1| = 2\, sinp$$

b) Prove that the expansion of an analytic function in a power series is unique.

$(12 + 08)$

Q4.

Evaluate

$$\int_0^\infty \frac{\ln(1 + x^2)}{1 + x^{22}} dx$$

$$\int_0^\infty \frac{x - sinx}{x^3(a^2 + x^2)} dx \qquad a > 0$$

(10+10)

Q5.

a) Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2|$.

b) Evaluate $\int_c \bar{z}\, dz$, where c is parameterized by $z = 3t + it^2, -1 \le t \le 4$

c) Show that the power series $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges at every point on its circle of convergence.

(07+07+06)

 P.T.O

Q4.

SECTION – II

Q6.

a) Define curvature. Find the curvature of a normal section of a helicoid.

b) Show that the locus of the mid points of the chords of a circular helix is a right helicoid.

(10+10)

Q7.

a) Find the principal curvatures and the lines of curvature on the right helicoid.

$$x = u\, cosa, \quad y = u\, sina, \quad z = ca$$

b) Show that the perpendicular from the vertex of a right circular cone, to a tangent to a given geodesic, is of constant length.

(10+10)

Q8.

a) Prove that, in order that the principal normals of a curve be binormals of another, the relation

$$a(k^2 + \tau^2) = k$$

must hold, where a is constant.

b) Show that the radius of spherical curvature of a circular helix is equal to the radius of circular curvature.

(10+10)

Q9.

a) Show that the locus of the centre of curvature is an evolute only when the curve is plane.

b) Prove that at a point of intersection of the paraboloid $xy = cz$ with the hyperboloid $x^2 + y^2 - z^2 + c^2 = 0$ the principal radii of the paraboloid are $z^2(1 \pm \sqrt{2})/c$

(10+10)

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M.A./M.Sc. Part – I Annual Examination – 2022

Subject: Mathematics (Old & New Course) **Paper: IV (Mechanics)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section. All questions carry equal marks.

Section I

1. (a) If $\mathbf{A} = x^2yz\ \mathbf{i} - 2xz^3\ \mathbf{j} + xz^2\ \mathbf{k}$ and $\mathbf{B} = 2z\ \mathbf{i} + y\ \mathbf{j} - x^2\ \mathbf{k}$, then prove that $\frac{\partial^2}{\partial y \partial x}(\mathbf{A} \times \mathbf{B}) = \frac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B})$.

(b) Find the constants a and b, so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$. [10+10=20]

2. (a) Prove that the vector field $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = 0$.

(b) Evaluate $\iint_S \mathbf{A}.\mathbf{n}\ dS$ where $\mathbf{A} = 18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$ and S is the part of plane $2x + 3y + 6z = 12$ which is located in the first octant. The projection R of the surface S is in xy-plane. [10+10=20]

3. (a) Verify the Stoke's theorem for $\mathbf{A} = (2x - y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

(b) Evaluate $\oint(x^2 - 2xy)dx + (x^2y + 3)dy$ around the boundary of the region defined by $y^2 = 8x$ and $x = 2$ by using Green's theorem. [10+10=20]

4. (a) Prove that cylindrical coordinate system is orthogonal.

(b) Find the square of the element of arc length in cylindrical coordinates and determine the corresponding scale factors. [10+10=20]

5. (a) A covariant tensor has components xy, $2y - z^2$, xz in rectengular coordinates. Find the covariant components in spherical coordinate system.

(b) Determine the Christoffel symbols of second kind in spherical coordinate system. [10+10=20]

SECTION II

6. (a) Find the kinetic energy of rotation of a rigid body with respect to the principle axes in terms of Euler's angles. Hence find the result in case $I_1 = I_2$.

(b) Using Euler's equations of motion for a rigid body having zero external torque show that $I_1{\omega_1}^2 + I_2{\omega_2}^2 + I_3{\omega_3}^2$ and ${I_1}^2{\omega_1}^2 + {I_2}^2{\omega_2}^2 + {I_3}^2{\omega_3}^2$ are conserved quantities. What do they represent? [10+10=20]

7. (a) Find moment of inertia of a uniform rod about an axis passing through its end points and perpendicular to the length of rod.

(b) State and prove parallel axis theorem for the moment and product of inertia for a continuous distribution of mass. Illustrate it by means of an example. [10+10=20]

8. (a) A rigid body is rotating about a fixed point 'o'. The points $\mathrm{A}(0, -1, 2)$ and $\mathrm{B}(2, 0, 0)$ are moving with velocities $V_A = [7, -2, -1]$ and $V_B = [0, 6, -4]$ respectively. Find the angular velocity of the body.

(b) Consider a uniform solid rectangular block of mass m and dimensions $2a$, $2b$, $2c$. Find moment of inertia about coordinate axes with 'O' as center of mass. [10+10=20]

9. (a) Four particles of mass m, $2m$, $3m$ and $4m$ are located at the points (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$ and $(-a, -a, a)$ respectively. Find the principle moment of inertia at origin.

(b) Derive the equation for ellipsoid of inertia for a solid ellipsoid of mass M with distinct semi-axes of length a, b and c. [10+10=20]

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M.A./M.Sc. Part – I Annual Examination – 2022

Subject: Mathematics (Old & New Course)
Paper: V (Topology and Functional Analysis)

Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION-I

Q.1 **(a)** Let $X=\{a,b,c,d,e\}, \tau=\{\phi,\{b\},\{b,c\},\{a,b,d\},\{a,b,c,d\},X\}$. Then find the interior, exterior, closure and frontier of the sets $A=\{a,c,e\}$. (10)

(b) Prove that any uncountable set X with co-finite topology is not first countable. Is X second countable? Justify your answer. (10)

Q.2 **(a)** Let X be a T_1 – space and A be a subset of X. If x is a limit point of A then every open set containing x contains an infinite number of distinct points of A. (10)

(b) Prove that every metric space is completely regular, Tychonoff, and Hausdorff space. (10)

Q.3 **(a)** Show that compact subset of a Hausdorff space is closed. (10)

(b) Prove that every sequentially compact space countably compact space. (10)

Q.4 **(a)** A space X is connected if and only if there does not exist a surjective continuous function f from X onto the two point discrete space. (10)

(b) Let F be a family of subsets of X whose union is all of X. If each member of F is connected, and if no two members of F is separated from one another in X, then show that X is connected. (10)

SECTION-II

Q.5 **(a)** Prove that $C(X,\mathbb{R})$ is complete under the metric defined by: $d(f,g)=\sup_{x\in X}|f(x)-g(x)|$. (10)

(b) Prove that the space $B[a,b]$ is not separable. (10)

Q.6 **(a)** Prove that any two equivalent norms on a linear space N define the same topology on N. (10)

(b) State and prove the Baire's Category theorem. (10)

Q.7 **(a)** If $\sum_{n=1}^{\infty} x_n$ be absolutely convergent series in a Banach space N. Then show that $\sum_{n=1}^{\infty} x_n$ is convergent. (10)

(b) Let M be a proper closed subspace of a normed space N and $a\in(0,1)$. Then prove that there exists a $x_a \in N$ such that $\|x_a\|=1$ and $\|x-x_a\|\geq a$ for all $x\in M$. (10)

Q.8 **(a)** Let $T:N\to M$ be a surjective linear operator. Then prove that (10)

(i) T^{-1} exists if and only if Tx = 0 implies x = 0

(ii) If T is bijective and dim N = n, then show that M also has dimension n.

(b) Let S be a closed subspace of a Banach space N. Then show that the quotient space N/S with the norm $\|x+S\|_1=\inf_{s\in S}\|x+s\|$ is also a Banach space. (10)

Q.9 **(a)** Define direct sum and prove that if Y is a closed subspace of a Hilbert space H then H can be written as a direct sum of Y and its orthogonal complement. (10)

(b) Let H be a Hilbert space and f be any linear functional on H, then there exists a unique $y\in H$ such that $f(x)=\langle x,y\rangle \ \forall x\in H$ (10)