UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – I Supply – 2020 & Annual – 2021

Roll No. ....

Subject: Mathematics (Old & New Course)   Paper: I (Real Analysis)   Time: 3 Hrs.   Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION – I

Q # 1.

a.

i. Let $x, y \in \mathbb{R}$ and $x > 0$. Prove that there exists $m \in \mathbb{Z}^+$ such that $mx > y$.

ii. Let A and B be two non empty bounded subsets of real numbers. Define

$$A - B = \{a - b, a \in A, b \in B\}.$$

Prove that $\text{Sup}(A - B) = \text{Sup}\,A - \text{Inf}\,B$. (5+7)

b. If r is a non-zero rational and $x \neq 0$ is an irrational number. Prove that $r - x$ and r/x both are irrational. (8)

Q # 2.

a. Let $X := \{x_n\}$ and $Y := \{y_n\}$ be strictly positive sequences and suppose that the limit $\lambda = \lim\limits_{n\to\infty} \frac{x_n}{y_n} \neq 0$ exists. Show that the series $\sum x_n$ and $\sum y_n$ converge or diverge together. (10)

b. Prove that every Cauchy sequence of real numbers is convergent. Apply this to check the convergence of the sequence $\{x_n\}$ defined by $x_1 := 1, x_2 := 2$ and $x_n := \frac{1}{2}(x_{n-2} + x_{n+2})$ for $n > 2$. (10)

Q # 3.

a. Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. Let $c \in \mathbb{R}$ be a cluster point of A. Show that

i. If $\lim\limits_{x\to c} f(x) = l$, then $\lim\limits_{x\to c} |f(x)| = l$.

ii. If $f(x) < g(x)$, then $\lim\limits_{x\to c} f(x) \leq \lim\limits_{x\to c} g(x)$. (10)

b. Show that every continuous function on a closed and bounded interval I attains an absolute maximum and an absolute minimum on I. (10)

Q # 4.

a. Show that a function f is uniformly continuous on an interval (a, b) if and only if it can be defined at the end points a and b such that the extended function is continuous on $[a, b]$. (10)

b. Evaluate the following:

i. $\lim\limits_{x\to c} \frac{x^c - c^x}{x^x - c^c}, c > 0$

ii. $\lim\limits_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ (10)

Q # 5.

a. If (a, b) is a point in the domain of a function $f(x, y)$ such that

i. f_x, f_y, f_{xy} all exist in certain neighbourhood of (a, b).

ii. f_{xy} is continuous at (a, b).

Show that $f_{yx}(a, b)$ exists and is equal to $f_{xy}(a, b)$. (10)

b. Find the nearest and farthest points on the surface $x^2 + y^2 + z^2 = 1$ from the point $(1, 2, 3)$. (10)

SECTION – II

Q # 6.

a. If $f \in \Re(\alpha)$ on $[a, b]$, then show that $|f| \in \Re(\alpha)$ and (10)

$$\left|\int_a^b f\, d(\alpha)\right| \leq \int_a^b |f| d(\alpha).$$

b. Show that every continuous function is integrable. Give an example to show that the continuity is only a sufficient condition for integrability not necessary. (10)

Q # 7.

a. Show that $\int_0^\infty \frac{\sin x}{x}\, dx$ is not absolutely convergent. (10)

b. For what value of n the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges? (10)

Q # 8.

a. Prove that the sum as well as the product of two functions of bounded variation is also a function of bounded variation. (10)

b. Determine whether the function:

$$f(x) = \begin{cases} x\cos\left(\frac{\pi x}{2}\right), & 0 < x \leq 1 \\ 0 & , x = 0. \end{cases}$$

is a function of bounded variation on $[0, 1]$. (10)

Q # 9.

a. If a series $\sum f_n$ converges uniformly to f on $[a, b]$, and each f_n continuous on $[a, b]$, then prove that f is integrable on $[a, b]$, and the series $\sum\left(\int_a^x f_n dt\right)$converges uniformly to $\int_a^x f dt$ for all on $x \in [a, b]$, i.e.

$$\int_a^x f dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n dt\right), \forall x \in [a, b].$$

(10)

b. Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{x}{1 + nx^2}, x \text{ being a real number,}$$

converges uniformly on any interval I. (10)

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Roll No. ....

Subject: Mathematics (Old & New Course) Paper: II (Algebra) Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION – I

Q. 1 .... *20*

(a) Show that conjugate elements of a group have the same order. [10]

(b) Let n be an integer other than 1 and let ω be the complex number $e^{\frac{2\pi i}{n}}$ Show that the set of distinct complex numbers $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ forms a multiplicative group. [10]

Q. 2 .... *20*

(a) Let H be a subgroup of a group G and N be a normal subgroup of G. Show that $H \cap N$ is a normal subgroup of H. [10]

(b) Show that the Symmetric group S_3 and the Dihedral group [10]

$$\langle a, b : \; a^3 = 1 = b^2, bab^{-1} = a^{-1} \rangle$$

are isomorphic.

Q. 3 .... *20*

(a) Show that every infinite cyclic group is isomorphic to the cyclic group Z. [10]

(b) Write down the elements of the quotient group G/H, where G is the Quaternion group $\langle a, b : \; a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$ and $H = \{1, a^2\}$. Moreover, identify G/H. [10]

Q. 4 .... *20*

(a) Write down the conjugacy classes of the Symmetric group S_4. [10]

(b) Show that a group of order 56 can not be simple. [10]

Q. 5 .... *20*

(a) Let G be a group of order $p^n.m$, where p is prime and $p \nmid m$. Show that G has a subgroup of order p^n. [10]

(b) Find the normalizer of the subgroup $H = \{1, (1,2,3), (1,3,2)\}$ of the Symmetric group S_3. [10]

SECTION – II

Q. 6 .... *20*

(a) Given an ideal I of the ring R, define the set $C(I)$ by [10]

$$C(I) = \{r \in R : \; ra - ar \in I, \; \forall a \in R.\}$$

Verify that $C(I)$ forms a subring of R.

(b) If I_i, $(i = 1, 2, \dots)$ is a collection of ideals of the ring R such that [10]

$$I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots,$$

prove that $\bigcup_i I_i$ is also an ideal of R.

Q. 7 .... *20*

(a) Assuming that R is a division ring, show that center of R forms a field. [10]

(b) Determine if the elements $(1,2,3), (0,4,5), (\frac{1}{2}, 3, \frac{21}{4})$ in V, the vector space of 3-tuples over $\mathbb{R}$, are linearly independent over $\mathbb{R}$. [10]

Q. 8 .... *20*

(a) Find all ideals of the ring $Z_2 \times Z_2$. [10]

(b) Let R be a commutative ring with unit element and I be a proper ideal of R. Show that I is a prime ideal of R if and only if R/I is an Integral Domain. [10]

Q. 9 .... *20*

(a) Find a matrix P, if possible, for which $P^{-1}AP$ is diagonal, where [10]

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) Let V be a vector space over field F with $dim(V) = n$. Prove that any n linearly independent elements of V form a basis of V over F. [10]

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Roll No. ....

Subject: Mathematics (Old & New Course) | Time: 3 Hrs. Marks: 100

Paper: III (Complex Analysis and Differential Geometry)

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section. All Questions carry equal marks.

SECTION – I

Q1. a) If z is root of a polynomial equation $a_n z^n + a_{n-1} z^n - 1 + \cdots\ a_1 z^1 + a_0 z^0 = 0$, then show that conjugate of z is also root.

b) Use the Cauchy-Riemann equations to prove that if f is analytic in D and its first derivative is constant, then $f(z)$ must be a complex linear function through D.

c) Compute: (07+07+06)

$$\lim_{z \to 1+\sqrt{3}} \left(\frac{z^2 - 2z + 4}{z - 1 - \sqrt{3}\, i} \right)$$

Q2. a) Evaluate (08+08+04)

$$\oint_C \frac{5z+7}{z^2+2z-3}\, dz, \text{ where } C \text{ is circle } |z-2| = 2$$

b) Find all roots of Sinh$z = 0.1$

c) Compute the principal value of complex logarithm for $z = 1 + i$

Q3. a) Determine whether the sequence $\left\{\frac{n(1+i^n)}{n+1}\right\}$ converges or diverges. (07+07+06)

b) Define residue. Find the residue of

$$f(z) = \frac{\cos z e^z}{(z^2 + a^2)^5}$$

c) State only Moreira's Theorem, Cauchy's Residue Theorem, Cauchy's Inequality Theorem.

Q4. Evaluate (07+07+06)

$$\int_0^{\infty} \frac{x \sin x}{x^2+9} dx, \quad \int_{-\infty}^{+\infty} \frac{\sin^2 x}{(x^2+1)^4} dx, \quad \int_0^{2\pi} \frac{d\theta}{2+\text{Cos}\theta}$$

Q5. a) Derive the Cauchy-Riemann equations in polar form. (08+08+04)

b) Construct a linear fractional transformation that maps the points $1, i$ and -1 on the unit circle onto the points $-1, 0, 1$ on the real axis.

c) State why a composition of two entire functions is entire.

SECTION – II

Q6. (12 + 08)

a) On the right helicoid given by $x = u\cos\emptyset, y = u\sin\emptyset, z = c\emptyset$ show that the parametric curves are circular helices and straight lines.

b) For the curve $x = a\cos u$, $y = 3a\sin u$, $z = a\cos 2u$. Find the curvature and torsion of the curve.

Q7. (10+10)

a) If the plane of curvature at every point of a curve passing through a fixed point, show that the curve is plane.

b) On the surface generated by the binormals of a twisted curve, the position vector of the current point may be expressed $\vec{r} + u\vec{b}$, where $\vec{r}$ and $\vec{b}$ are functions of s. Taking u and s as parameters, calculate the **Fundamental Magnitudes and Unit Normal** for the surface.

Q8. (10+10)

a) At a point of intersection of the paraboloid $xy = cz$ with hyperboloid $x^2 + y^2 - z^2 + c^2 = 0$, find the principal radii of the paraboloid.

b) Show that the locus of centers of curvatures of a curve is an evolute of the curve if and only if the curve is a plane curve.

Q9. (08+06+06)

a) Derive the Weingarten equations.

b) Prove that the geodesic curvature vector of any curve is orthogonal to the curve.

c) Find the geodesic curvature pf the parametric lines on the surface

$$x = a(u+v), \quad y = b(u+v), \quad z = uv$$

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Roll No. ....

Subject: Mathematics (Old & New Course) Paper: IV (Mechanics) Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section. All questions carry equal marks.

SECTION I

1. (A) If $\mathbf{A} = 5t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$ and $\mathbf{B} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$, find (a) $\frac{d}{dt}(\mathbf{A}\bullet\mathbf{B})$, (b) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B})$, (c) $\frac{d}{dt}(\mathbf{A}\bullet\mathbf{A})$.

   (B) Find (a) $\frac{\partial^2\mathbf{A}}{\partial x^2}$, (b) $\frac{\partial^2\mathbf{A}}{\partial y^2}$, (c) $\frac{\partial^2\mathbf{A}}{\partial x\partial y}$, (d) $\frac{\partial^2\mathbf{A}}{\partial y\partial x}$, for $\mathbf{A} = \cos xy\,\mathbf{i} + (3xy - 2x)\,\mathbf{j} - (3x + 2y)\,\mathbf{k}$. [10+10=20]

2. (A) Find constants a, b, c so that $\mathbf{V} = (x + 2y + az)\,\mathbf{i} + (bx - 3y - z)\,\mathbf{j} + (4x + cy + 2z)\,\mathbf{k}$ is irrotational. Show that $\mathbf{V}$ can be expressed as the gradient of a scalar function ϕ.

   (B) Give a definition of $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ over a surface S in terms of limit of a sum. Suppose that the surface S has projection R on the xy-plane. Show that $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$. [10+10=20]

3. (A) Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\,\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

   (B) State and prove Green's theorem in plane. Use Green's theorem to find (a) $\oint_C f(x)dx + g(y)dy$ and (b) $\oint_C k\,y dx + h\,x dy$, where k, h are constants. [10+10=20]

4. (A) Find the work done in moving a particle in the force field $\mathbf{F} = 3x^2\,\mathbf{i} + (2xz - y)\,\mathbf{j} + z\,\mathbf{k}$ along (a) the straight line from $(0,0,0)$ to $(2,1,3)$, (b) the space curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$, (c) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

   (B) Find the scale factors and the volume element dV in oblate spheroidal coordinates (μ, ν, φ) $x = a\cosh\mu\cos\nu\cos\varphi$, $y = a\cosh\mu\cos\nu\sin\varphi$, $z = a\sinh\mu\sin\nu$, where μ is a non-negative real number and the angle $\nu \in [-\pi/2, \pi/2]$. The azimuthal angle φ can fall anywhere on a full circle, between $\pm\pi$. [10+10=20]

5. (A) Find the covariant and contravariant components of a tensor in spherical coordinates r, θ, φ if its covariant components in rectangular components are $2xz$, x^2y, yz.

   (B) If A_r^{pq} and B_r^{pq} are tensors, prove that $C_r^{pq} = A_r^{pq} + B_r^{pq}$ and $D_{ru}^{pqst} = A_r^{pq} + B_u^{st}$ are also tensors. What are their covariant and contravariant ranks? [10+10=20]

SECTION II

6. (A) A rigid body consists of 3 particles of masses 2,1,4 located at $(1,-1,1)$, $(2,0,2)$, $(-1,1,0)$ respectively. Find the angular momentum of the body if it is rotated about the origin with angular velocity $\omega = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

   (B) Let $\mathbf{r}'_\nu$ and $\mathbf{V}'_\nu$ be respectively the position vector and velocity of the particle ν relative to the center of mass. Prove that the angular momentum Ω and total kinetic energy T satisfy the relations: (a) $\Omega = \sum_\nu m_\nu(\mathbf{r}'_\nu \times \mathbf{V}'_\nu) + M(\bar{\mathbf{r}} \times \bar{\mathbf{V}})$ (b) $T = \frac{1}{2}\sum_\nu m_\nu(\mathbf{V}'_\nu)^2 + \frac{1}{2}M(\bar{\mathbf{V}})^2$. [10+10=20]

7. (A) Find the moment of inertia of a disc of radius a and mass m about an axis passing through its center and perpendicular to its plane.

   (B) For a system of N particles, show that the components L_x, L_y and L_z of angular momentum L in terms of moments and products of inertia are: $L_x = \omega_x I_{xx} + \omega_y I_{xy} + \omega_z I_{xz}$, $L_y = \omega_x I_{xy} + \omega_y I_{yy} + \omega_z I_{yz}$ and $L_z = \omega_x I_{xz} + \omega_y I_{yz} + \omega_z I_{zz}$. [10+10=20]

8. (A) Find the expression for the kinetic energy of rotation of a rigid body with respect to the principal axes in terms of the Euler angles. Hence, find the result in case $I_1 = I_2$.

   (B) The moments and products of inertia of a rigid body about the x, y and z axes are $I_{xx} = 3$, $I_{yy} = 10/3$, $I_{zz} = 8/3$, $I_{xy} = 4/3$, $I_{xz} = -4/3$, $I_{yz} = 0$. Find the principal moments of inertia and the directions of principal axes. [10+10=20]

9. (A) Find a set of three rotation matrices for Euler angles and express the components of angular velocity in terms of these angles.

   (B) Using Euler's equations of motion for a rigid body having zero external torque show that $I_1\omega_1{}^2 + I_2\omega_2{}^2 + I_3\omega_3{}^2$ and $I_1{}^2\omega_1{}^2 + I_2{}^2\omega_2{}^2 + I_3{}^2\omega_3{}^2$ are conserved quantities. What do they represent? [10+10=20]

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M.A./M.Sc. Part – I Supply – 2020 & Annual – 2021

Roll No. ....

Subject: Mathematics (Old & New Course)
Paper: V (Topology and Functional Analysis)

Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION-I

Q.1 (a) Let $X=\mathbb{R}, \Im=$ Usual Topology. Find the interior, closure, exterior, derived set and frontier of the set $A=\left\{\frac{1}{n} \mid n\in \mathbf{Z}^{+}\right\}$. Is A dense in X? (10)

(b) Let $X=\{a,b,c,d,e\}, \Omega=\{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{a,e\}\}$. Find the topology generated by Ω. Is X with this topology normal? (10)

Q.2 (a) Show that every Hausdorff space is T_1 – space but converse is not true. (10)

(b) Show that a regular Lindelöf space is normal. (10)

Q.3 (a) Let X be Hausdorff space, C a compact subset of X and x an element of X which is not in C. Then show that there are disjoint open sets U_x and V_x in X such that $x\in U_x$ and $C\subseteq V_x$. (10)

(b) Show that a closed subset of a compact space is compact. (10)

Q.4 (a) Prove that image of a connected space under a continuous map is connected. (10)

(b) Show that product of connected spaces is connected. (10)

SECTION-II

Q.5 (a) Prove that the space l^{∞} is complete metric space but not separable. (10)

(b) Let X be the set of all continuous real-valued functions on $I=[0,1]$, and let (10)

$$d(x,y)=\int_0^1 |x(t)-y(t)|\,dt.$$

Show that (X,d) is not complete.

Q.6 (a) If (X,d) is complete, show that $(X,\tilde{d})$ is complete, where $\tilde{d}=d/(1+d)$. (10)

(b) State and prove Cantor's intersection theorem. (10)

Q.7 (a) Show that equivalent norms on a vector space X induce the same topology for X. (10)

(b) Show that the dual space of l^1 is l^{∞}. (10)

Q.8 (a) Consider a bounded linear operator $T:D(T)\to Y$, where domain of T i.e. $D(T)$ lies in a normed space X and Y be a Banach space. Show that T has an extension (a bounded linear operator) $\tilde{T}:\overline{D(T)}\to Y$ with norm $\|\tilde{T}\|=\|T\|$. (10)

(b) Let N be a normed space and $f:X\to Y$ be a linear functional. Let x_0 be any fixed point of $N\setminus Ker f$ then show that every element x of N has a unique representation of the form $x=ax_0+y, a\in F, y\in Ker f$. (10)

Q.9 (a) Show that the space l^{p} is an inner product space iff $p=2$. (10)

(b) For any inner product space X there exists a Hilbert space H and an isomorphism A from X onto a dense subspace $W\subset H$. The space H is unique except for isomorphisms. (10)