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Integrability properties of the differential-difference Kadomtsev–Petviashvili hierarchy and continuum limits

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Abstract

The paper reveals clear links between the differential-difference Kadomtsev–Petviashvili hierarchy and the (continuous) Kadomtsev–Petviashvili hierarchy. Isospectral differential-difference Kadomtsev–Petviashvili flows $\{\bar{K}_s(\bar{u})\}$ and non-isospectral differential-difference Kadomtsev–Petviashvili flows $\{\bar{\sigma}_s(\bar{u})\}$ are derived through Lax triad approach. The Lax triads also provide simple zero-curvature representations for the obtained flows. The non-isospectral flow $\bar{\sigma}_2(\bar{u})$ acts as a master symmetry to provide recursive relations for the obtained flows. These flows generate a Lie algebra, which is a starting point for investigating more integrability properties. We derive symmetries, Hamiltonians and conserved quantities for the isospectral differential-difference Kadomtsev–Petviashvili hierarchy. The Lie algebras generated respectively by the flows, symmetries, Hamiltonians and conserved quantities have same structures. Finally, we provide a uniform continuum limit which is different from Miwa's transformation. By means of defining *degrees* of some elements w.r.t. the continuum limit, we prove that in the uniform continuum limit the differential-difference Kadomtsev–Petviashvili hierarchies together with their Lax triads, zero-curvature representations and integrability characteristics go to their continuous counterparts. Structure deformation of Lie algebras in the continuum limit is also explained.

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1. Introduction

The Kadomtsev–Petviashvili (KP) equation

$$u_t = \tfrac{1}{4}u_{xxx} + 3uu_x + \tfrac{3}{4}\partial_x^{-1}u_{yy} \tag{1.1}$$

plays a typical role in (2+1)-dimensional integrable systems. It is also the elementary model in the celebrated Sato theory [1, 2]. The KP equation itself and its integrability characteristics, such as infinitely many symmetries and conserved quantities, can be derived from a pseudo-differential operator [3–5],

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots. \tag{1.2}$$

The operator also generates a KP hierarchy [1, 5], of which the recursive structure can be expressed either through a recursion operator [6] or through a master symmetry together with a certain commutator [7]. By means of the recursive structure, symmetries, Hamiltonian structures and conserved quantities of the whole isospectral KP hierarchy were constructed [7–11].

The differential-difference Kadomtsev–Petviashvili (DΔKP) equation reads

$$\bar{u}_{\bar{t}} = (1 + 2\Delta^{-1})\bar{u}_{\bar{x}\bar{x}} - 2h^{-1}\bar{u}_{\bar{x}} + 2\bar{u}\bar{u}_{\bar{x}}, \tag{1.3}$$

where $\bar{u} = \bar{u}(n, \bar{x}, \bar{t})$ with one discrete independent variable n and two continuous ones $\bar{x}$ and $\bar{t}$, the operator Δ is defined as $\Delta f(n) = f(n+1) - f(n)$ and h is a spacing parameter of n. This equation was first derived through a discretization of Sato's theory [12]. The discretization is also known as Miwa's transformation [13]. Based on the transformation it is quite natural to obtain bilinear identities with discrete exponential functions, from which one can derive bilinear equations with discrete variables [12, 14–17]. However, since Miwa's transformation breaks the original continuous dispersion relation, it is hard at the first glance to find the correspondence of an integrable discrete equation and its continuous counterpart. It was shown in [18] that the DΔKP equation is related to the following pseudo-difference operator

$$\bar{L} = \Delta + \bar{u}_0 + \bar{u}_1\Delta^{-1} + \bar{u}_2\Delta^{-2} + \cdots, \tag{1.4}$$

with $\bar{u}_0 = \bar{u}$ and $\bar{t}_1 = \bar{x}$. By using the above pseudo-difference operator some integrability properties of the DΔKP equation, such as symmetries and conservation laws, have been investigated [18–21].

In this paper we will investigate integrability characteristics for the whole DΔKP hierarchy. To do that we need to introduce a master symmetry (see [22]) so as to construct the recursive structure of the DΔKP hierarchy. Usually master symmetries are related to time-dependent spectral parameters and can be derived as non-isospectral flows. Since isospectral and non-isospectral DΔKP flows are considered simultaneously and they are all related to the same spectral problem, we can not take $\bar{t}_1 = \bar{x}$ any longer and we have to consider $\bar{x}$ as a new independent variable. To solve this problem, we use a Lax triad instead of a Lax pair to derive the DΔKP hierarchies and it turns out that this works well. The main results obtained in the paper are the following.

- Isospectral and non-isospectral DΔKP hierarchies

$$\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u}), \qquad \bar{u}_{\bar{t}_s} = \bar{\sigma}_s(\bar{u}), \quad s = 1, 2, \cdots.$$

- A Lie algebra for the flows $\{\bar{K}_l(\bar{u})\}$ and $\{\bar{\sigma}_r(\bar{u})\}$.
- A Lie algebra of symmetries of the equation $\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u})$.

- Hamiltonian structures

$$\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u}) = \partial_{\bar{x}} \frac{\delta \bar{H}_s}{\delta \bar{u}}, \qquad \bar{u}_{\bar{t}_s} = \bar{\sigma}_s(\bar{u}) = \partial_{\bar{x}} \frac{\delta \bar{J}_s}{\delta \bar{u}}$$

  and a Lie algebra generated by $\{\bar{H}_l\}$ and $\{\bar{J}_r\}$.
- A Lie algebra of the conserved quantities of the equation $\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u})$.
- A uniform continuum limit under which the above results go to their continuous counterparts.

Our plan is the following. After introducing some notions and notations in section 2, we revisit the KP hierarchy in section 3. We derive isospectral and non-isospectral KP flows via Lax triad approach. The approach provides simple zero-curvature representations for these flows, by which we can easily obtain a Lie algebra of the flows. The basic structure of the algebra indicates a recursive relation for both the isospectral and non-isospectral KP flows. Integrability properties of the isospectral KP hierarchy, such as symmetries, Hamiltonian structures and conserved quantities and Hamiltonian structures of the non-isospectral KP hierarchy, are also listed out in this section as known results in literature. Next, in section 4, we focus on the DΔKP hierarchy. Through the Lax triad approach we derive isospectral and non-isospectral DΔKP flows and their basic algebraic structure. The structure is used to generate infinitely many symmetries for the isospectral DΔKP hierarchy and provides a recursive relation of flows. Then we investigate Hamiltonian structures and conserved quantities for the isospectral DΔKP hierarchy and Hamiltonian structures for the non-isospectral DΔKP hierarchy. Finally in section 5, by means of continuum limit we discuss possible connections between the DΔKP hierarchies and KP hierarchies together with their Lax triads and integrability characteristics.

2. Basic notions

Here we mainly follow the notions and notations used in [23] (see also [24, 25]).

Consider a (2+1)-dimensional continuous evolution equation

$$u_t = K(u), \qquad u \in M, \tag{2.1}$$

where M is an infinite dimensional linear manifold of C^∞ functions $u(x, y)$ defined on $\mathbb{R}^2$ and vanishing rapidly at infinity. The solution $u(x, y, t)$ is usually asked to depend in a C^∞-way on the time parameter t. Since M is linear, all fibers of the tangent bundle TM are copies of the same vector space S; S can be canonically identified with M, but it is convenient to regard them as different objects for a better geometrical understanding (i.e. M is the manifold under examination, S is the tangent space at any point $u \in M$). Let S^* be the dual space of S w.r.t. the bilinear form $\langle \cdot, \cdot \rangle : S^* \times S \to \mathbb{R}$ defined as

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) g(x, y)\, \mathrm{d}x\, \mathrm{d}y, \qquad f \in S^*, \quad g \in S. \tag{2.2}$$

The Gâteaux derivative of a function (or an operator or a functional) $F(u)$ on M in the direction $g \in S$ is defined as

$$F'(u)[g] = \frac{\partial}{\partial \varepsilon} F(u + \varepsilon g)\Big|_{\varepsilon=0}, \qquad u \in M, \quad g \in S, \tag{2.3}$$

which is usually written as $F'[g]$ for short without confusion. Here we note that generically F can also depend on a time parameter t. In that case both u and t are treated as independent variables. $F(u, t)$ is understood as (see [25]) a family depending in a C^∞-way on the parameter t and the Gâteaux derivative F' is taken by ignoring the parameter t, namely

$$F'[g] = F'(u, t)[g] = \frac{\partial}{\partial \varepsilon} F(u + \varepsilon g, t)\Big|_{\varepsilon=0}, \qquad u \in M, \quad g \in S. \tag{2.4}$$

For the sake of a more generic sense, in the following definitions are given for the time-dependent cases. They are valid as well when we remove the independent time variable t.

Consider a covector field $\gamma : M \times \mathbb{R} \to S^*$, $(u, t) \mapsto \gamma(u, t)$. If there exists a scalar field $H : M \times \mathbb{R} \to \mathbb{R}$, $(u, t) \mapsto H(u, t)$ such that

$$H'[g] = \langle \gamma, g \rangle, \qquad u \in M, \quad \forall g \in S, \tag{2.5}$$

then H is called a potential of γ and γ is called the gradient field of H, denoted by grad H or $\delta H/\delta u$. The following proposition provides a way to verify gradient fields and recover their potentials [23, 26].

Proposition 2.1. *$\gamma = \gamma(u, t)$ is a gradient field if and only if γ' is a self-adjoint operator w.r.t. the dual relation $\langle \cdot, \cdot \rangle$, namely $\gamma'^* = \gamma'$. The corresponding potential $H(u, t)$ is given by*

$$H(u, t) = \int_0^1 \langle \gamma(\lambda u, t), u \rangle \, \mathrm{d}\lambda. \tag{2.6}$$

Consider two vector fields $F, G : M \times \mathbb{R} \to S$, $(u, t) \mapsto F(u, t)$, $G(u, t)$; their standard commutator is the vector field

$$[\![F, G]\!] = F'[G] - G'[F]. \tag{2.7}$$

A vector field $G : M \times \mathbb{R} \to S$ is a symmetry of equation (2.1) if

$$G_t(u, t) + [\![G(u, t), K(u)]\!] = 0 \tag{2.8}$$

holds everywhere in $M \times \mathbb{R}$. Here sub-t (or ∂_t without making any confusion) denotes partial derivative w.r.t. t. A scalar field $I : M \times \mathbb{R} \to \mathbb{R}$, $(u, t) \mapsto I(u, t)$ is a conserved quantity of equation (2.1) if $\frac{\mathrm{d}I(u,t)}{\mathrm{d}t}\big|_{u_t = K(u)} = 0$, or equivalently,

$$I_t + \left\langle \frac{\delta I}{\delta u}, K \right\rangle = 0 \tag{2.9}$$

holds everywhere in $M \times \mathbb{R}$. A covector field $\gamma : M \times \mathbb{R} \to S^*$, $(u, t) \mapsto \gamma(u, t)$ is a conserved covariant of equation (2.1) if

$$\gamma_t + \gamma'[K] + K'^*[\gamma] = 0 \tag{2.10}$$

holds everywhere in $M \times \mathbb{R}$.

Conserved quantities and conserved covariants are related closely to each other through the following proposition (see [23]).

Proposition 2.2. *If $G = G(u, t)$ is a symmetry and $\gamma = \gamma(u, t)$ is a conserved covariant of equation (2.1), then*

$$I(u, t) = \langle \gamma, G \rangle \tag{2.11}$$

is a conserved quantity of (2.1).

In fact, when u satisfied equation (2.1), we have

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= \left\langle \frac{\mathrm{d}G}{\mathrm{d}t}, \gamma \right\rangle + \left\langle G, \frac{\mathrm{d}\gamma}{\mathrm{d}t} \right\rangle \\ &= \langle K'[G], \gamma \rangle + \langle G, -K'^*[\gamma] \rangle \\ &= \langle K'[G], \gamma \rangle + \langle -K'[G], \gamma \rangle \\ &= 0. \end{aligned}$$

Another relation is

Proposition 2.3. *Suppose that* $\gamma = \gamma(u, t) \in S^*$ *is a gradient field and scaler field* $I = I(u, t)$ *is its potential and* $\frac{\mathrm{d}I}{\mathrm{d}t}\big|_{u=0} = 0$. *Then,* I *is a conserved quantity of equation (2.1) if and only if* γ *is a conserved covariant of (2.1).*

Proof. First, since $\gamma = \mathrm{grad}\, I$, i.e. $I'[g] = \langle \gamma, g\rangle$, by using the Leibniz rule we have

$$\partial_t \langle \gamma, g\rangle = \partial_t (I'[g]) = (\partial_t I)'[g] + I'[\partial_t g] = (\partial_t I)'[g] + \langle \gamma, \partial_t g\rangle, \qquad \forall g = g(u, t) \in S,$$

which gives rise to

$$(\partial_t I)'[g] = \langle \partial_t \gamma,\ g\rangle. \tag{2.12}$$

Next, when u satisfies equation (2.1), noting that

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \partial_t I + I'[u_t] = \partial_t I + I'[K] = \partial_t I + \langle \gamma, K\rangle,$$

we have

$$\begin{aligned} \left(\frac{\mathrm{d}I}{\mathrm{d}t}\right)'[g] &= (\partial_t I)'[g] + \langle \gamma, K\rangle'[g] \\ &= \langle \partial_t \gamma, g\rangle + \langle \gamma'[g], K\rangle + \langle \gamma, K'[g]\rangle \\ &= \langle \partial_t \gamma, g\rangle + \langle \gamma'^{*} K, g\rangle + \langle K'^{*}\gamma, g\rangle \\ &= \langle \partial_t \gamma + \gamma' K + K'^{*}\gamma, g\rangle, \qquad \forall g = g(u, t) \in S, \end{aligned}$$

where we have made use of $\gamma' = \gamma'^{*}$. Thus it is clear that if I is a conserved quantity of equation (2.1) then γ is a conserved covariant of (2.1), and vise versa. □

A linear operator $\theta(u) : S^* \to S$ is called a Noether operator of equation (2.1) if

$$\theta'[K] - \theta K'^{*} - K'\theta = 0 \tag{2.13}$$

holds everywhere on M. Equation (2.13) means that the Lie derivative of θ along K is zero[4]. The Noether operator maps conserved covariants of (2.1) to its symmetries; the inverse of the Noether operator (if exists) maps symmetries of (2.1) to its conserved covariants.

An implectic operator (also known as Hamiltonian operator or Poisson tensor (see [23, 25])) is a linear operator $\theta(u) : S^* \to S$ satisfying skew-symmetric property

$$\langle f, \theta g\rangle = -\langle \theta f, g\rangle, \qquad \forall f, g \in S^*, \tag{2.14}$$

and the Jacobi identity

$$\langle f, \theta'[\theta g]h\rangle + \langle h, \theta'[\theta f]g\rangle + \langle g, \theta'[\theta h]f\rangle = 0, \qquad \forall f, g, h \in S^*. \tag{2.15}$$

The evolution equation (2.1) has a Hamiltonian structure if it can be written in the form

$$u_t = K(u) = \theta(u)\frac{\delta H(u)}{\delta u}, \tag{2.16}$$

where $\theta(u)$ is an implectic operator.

Hamiltonian structure, gradient field and Noether operator are related as the following.

Proposition 2.4. *Suppose* $\theta(u) : S^* \to S$ *is an implectic operator and equation (2.1) has the form*

$$u_t = K(u) = \theta(u)\gamma(u), \qquad \gamma(u) \in S^*. \tag{2.17}$$

Then θ *is a Noether operator of equation (2.1) if and only if* γ *is a gradient field, namely (2.17) is a Hamiltonian structure.*

[4] This was pointed out by the referee.

Proof. It is not difficult to find

$$\langle f, (\theta'[K] - K'\theta - \theta K'^{*})g\rangle$$
$$= \langle \gamma, \theta'[\theta f]g\rangle + \langle g, \theta'[\theta\gamma]f\rangle + \langle f, \theta'[\theta g]\gamma\rangle + \langle \theta f, (\gamma' - \gamma'^{*})[\theta g]\rangle, \quad \forall f, g \in S^{*},$$

which proves the proposition. □

Now we turn to the differential-difference case. By $\bar{M}$ we denote the infinite dimensional linear manifold of functions $\bar{u}(n, \bar{x})$ defined on $\mathbb{Z} \times \mathbb{R}$ and vanishing rapidly at infinity. Let $\bar{S}$ be the fiber of the tangent bundle $T\bar{M}$ at any point $\bar{u} \in \bar{M}$. In principle there is an identification between the linear spaces $\bar{M}$ and $\bar{S}$, but in the sequel we generally regard them as different objects, for the same geometrical reasons pointed out when speaking of M and S. Let $\bar{S}^{*}$ be the dual space of $\bar{S}$ w.r.t. the bilinear form $\langle\cdot,\cdot\rangle : \bar{S}^{*} \times \bar{S} \to \mathbb{R}$ defined as

$$\langle \bar{f}, \bar{g}\rangle = \frac{h^2}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty} \bar{f}(n,\bar{x})\bar{g}(n,\bar{x})\,\mathrm{d}\bar{x}, \qquad \bar{f} \in \bar{S}^{*}, \bar{g} \in \bar{S}, \tag{2.18}$$

where h stands for some spacing parameter for later use. Consider the (2+1)-dimensional evolution equation with one discretized argument (or differential-difference equation)

$$\bar{u}_{\bar{t}} = \bar{K}(\bar{u}), \qquad \bar{u} \in \bar{M}, \tag{2.19}$$

where $\bar{K}$ is some suitable C^{∞} vector field on $\bar{M}$. The solution $u(n, \bar{x}, \bar{t})$ is usually asked to depend in a C^{∞}-way on the time parameter $\bar{t}$.

Then the notions of symmetries, conserved quantities, conserved covariants, Noether operator, implectic operator and Hamiltonian structure can be defined correspondingly, and they are formally the same as defined in the continuous case.

3. The KP system and integrability

The KP hierarchy can be derived from the pseudo-differential operator (1.2) (for example, one can refer to [5].) The whole hierarchy admits Hamiltonian structures, infinitely many symmetries and conserved quantities [7–11]. In the following let us re-derive the KP hierarchy from Lax triad approach and briefly review their integrability properties.

3.1. Lax triad approach and the KP hierarchies

3.1.1. The isospectral KP hierarchy. Let us start from the pseudo-differential operator (1.2), i.e.

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots + u_{j+1}\partial^{-j} + \cdots, \tag{3.1}$$

where $\partial \doteq \partial_x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$ and $u_j = u_j(x, y, \boldsymbol{t})$ with time parameters $\boldsymbol{t} = (t_1, t_2, \cdots)$ are C^{∞} functions on $\mathbb{R}^2$ and vanish rapidly at infinity. The operator ∂^s obeys the Leibniz rule

$$\partial^s f = \sum_{i=0}^{\infty} \mathrm{C}_s^i(\partial^i f)\partial^{s-i}, \qquad s \in \mathbb{Z}, \tag{3.2}$$

where

$$\mathrm{C}_s^i = \frac{s(s-1)(s-2)\cdots(s-i+1)}{i!}. \tag{3.3}$$

Using this formula one can calculate the differential part of L^m, denoted by $A_m = (L^m)_+$, of which the first few are

$$A_1 = \partial, \tag{3.4a}$$
$$A_2 = \partial^2 + 2u_2, \tag{3.4b}$$
$$A_3 = \partial^3 + 3u_2\partial + 3u_3 + 3u_{2,x}, \tag{3.4c}$$
$$A_4 = \partial^4 + 4u_2\partial^2 + (4u_3 + 6u_{2,x})\partial + 4u_4 + 6u_{3,x} + 4u_{2,xx} + 6u_2^2. \tag{3.4d}$$

Indeed, in the conventional formulation of the KP isospectral hierarchy, each u_j is a function of x and $\boldsymbol{t}$ only and equation (1.1) is derived from the compatibility of $L\phi = \eta\phi$ and $\phi_{t_m} = A_m\phi$ setting $t_2 = y, t_3 = t$. Actually, the function u in the KP equation (1.1) depends on two independent spatial variables (x, y) and a time parameter t. A Lax triad is indeed needed to matching these three independent variables (also see [27]). This is crucially important when the non-isospectral case makes sense. In fact, when we derive a master symmetry as a non-isospectral flow we cannot take t_2 to be y any longer and we have to consider y and t_2 as two different and independent variables. This also requires a triad instead of a pair.

For the whole isospectral KP hierarchy we need

$$L\phi = \eta\phi, \qquad \eta_{t_m} = 0, \tag{3.5a}$$
$$\phi_y = A_2\phi, \qquad A_2 = \partial^2 + 2u_2, \tag{3.5b}$$
$$\phi_{t_m} = A_m\phi, \qquad m = 1, 2, \ldots, \tag{3.5c}$$

where we suppose

$$A_m = \partial^m + \sum_{j=1}^{m} a_j\partial^{m-j}, \qquad A_m|_{\boldsymbol{u}=\boldsymbol{0}} = \partial^m, \tag{3.6}$$

with temporarily unknown coefficients $\{a_j\}$. The compatibility of (3.5) reads

$$L_y = [A_2, L], \tag{3.7a}$$
$$L_{t_m} = [A_m, L], \tag{3.7b}$$
$$A_{2,t_m} - A_{m,y} + [A_2, A_m] = 0, \qquad m = 1, 2, \cdots, \tag{3.7c}$$

where $[\cdot,\cdot]$ denotes the commutator $[F, G] = FG - GF$. Among the above compatibility conditions, (3.7*a*) gives rise to

$$u_{2,y} = 2u_{3,x} + u_{2,xx},$$
$$u_{3,y} = 2u_{4,x} + u_{3,xx} + 2u_2u_{2,x},$$
$$\cdots\cdots,$$

which will be used to express $\{u_j\}_{j>2}$ in terms of u_2, in the following way:

$$u_3 = \tfrac{1}{2}(\partial^{-1}u_{2,y} - u_{2,x}), \tag{3.8a}$$
$$u_4 = \tfrac{1}{4}(\partial^{-2}u_{2,yy} - 2u_{2,y} + u_{2,xx} - 2u_2^2), \tag{3.8b}$$
$$\cdots\cdots.$$

The equation (3.7*b*) plays the role to calculate those unknowns $\{a_j\}$ in A_m. In fact, these $\{a_j\}$ are uniquely determined by (3.7*b*) and it turns out that $A_m = (L^m)_+$ (see [28]). The third equation (3.7*c*) provides the isospectral KP hierarchy

$$u_{t_m} = K_m(u) = \tfrac{1}{2}(A_{m,y} - [A_2, A_m]), \qquad m = 1, 2, \cdots, \tag{3.9}$$

where we have taken $u_2 = u$. Here are the first four equations in the KP hierarchy:

$$u_{t_1} = K_1(u) = u_x, \tag{3.10a}$$
$$u_{t_2} = K_2(u) = u_y, \tag{3.10b}$$
$$u_{t_3} = K_3(u) = \tfrac{1}{4}u_{xxx} + 3uu_x + \tfrac{3}{4}\partial^{-1}u_{yy}, \tag{3.10c}$$
$$u_{t_4} = K_4(u) = \tfrac{1}{2}u_{xxy} + 4uu_y + 2u_x\partial^{-1}u_y + \tfrac{1}{2}\partial^{-2}u_{yyy}. \tag{3.10d}$$

3.1.2. The non-isospectral KP hierarchy. To derive a master symmetry we turn to the non-isospectral case, in which we take

$$\eta_{t_m} = \eta^{m-1}, \qquad m = 1, 2, \ldots. \tag{3.11}$$

In this turn the Lax triad is

$$L\phi = \eta\phi, \tag{3.12a}$$
$$\phi_y = A_2\phi, \tag{3.12b}$$
$$\phi_{t_m} = B_m\phi, \qquad m = 1, 2, \cdots, \tag{3.12c}$$

and the compatibility reads

$$L_y = [A_2, L], \tag{3.13a}$$
$$L_{t_m} = [B_m, L] + L^{m-1}, \tag{3.13b}$$
$$A_{2,t_m} - B_{m,y} + [A_2, B_m] = 0, \qquad m = 1, 2, \cdots, \tag{3.13c}$$

where we suppose B_m is an undetermined operator of the form

$$B_m = \sum_{j=0}^{m} b_j\partial^{m-j}. \tag{3.14}$$

Checking the asymptotic results $(3.13b)_{\mathbf{u}=\mathbf{0}}$ and $(3.13c)_{\mathbf{u}=\mathbf{0}}$, respectively, one finds that they together give rise to the necessary asymptotic condition of B_m:

$$B_m|_{\mathbf{u}=\mathbf{0}} = 2y\partial^m + x\partial^{m-1}, \qquad m = 1, 2, \cdots. \tag{3.15}$$

We note that one can also add isospectral asymptotic terms, for example,

$$B_m|_{\mathbf{u}=\mathbf{0}} = 2y\partial^m + x\partial^{m-1} + \partial^{m-2}$$

when $m \geqslant 3$. This will lead to a mixed non-isospectral flow which is actually a combination of a non-isospectral flow σ_m and isospectral flow K_{m-2}. Such a combination does not change the basic algebraic structure of flows (see section 3.2.1).

With the asymptotic condition (3.15) the operator B_m can be uniquely determined from (3.13*b*) and the first few of them are

$$B_1 = 2yA_1 + x, \tag{3.16a}$$
$$B_2 = 2yA_2 + xA_1, \tag{3.16b}$$
$$B_3 = 2yA_3 + xA_2 + (\partial^{-1}u_2), \tag{3.16c}$$
$$B_4 = 2yA_4 + xA_3 + (\partial^{-1}u_2)\partial + 2(\partial^{-1}u_3), \tag{3.16d}$$

where $A_j = (L^j)_+$ are defined as in section 3.1.1. Then, from (3.13*c*) we obtain the following non-isospectral KP hierarchy

$$u_{t_m} = \sigma_m(u) = \tfrac{1}{2}(B_{m,y} - [A_2, B_m]), \qquad m = 1, 2, \cdots. \tag{3.17}$$

The first four equations are

$$u_{t_1} = \sigma_1(u) = 2yK_1, \tag{3.18a}$$

$$u_{t_2} = \sigma_2(u) = 2yK_2 + xK_1 + 2u, \tag{3.18b}$$

$$u_{t_3} = \sigma_3(u) = 2yK_3 + xK_2 + 2\partial^{-1}u_y - u_x, \tag{3.18c}$$

$$u_{t_4} = \sigma_4(u) = 2yK_4 + xK_3 + u_{xx} + 4u^2 + u_x\partial^{-1}u + \tfrac{3}{2}\partial^{-2}u_{yy} - \tfrac{3}{2}u_y, \tag{3.18d}$$

where $\{K_j\}$ are the isospectral flows given in (3.9) and we have taken $u_2 = u$.

$\{K_m(u)\}$ and $\{\sigma_m(u)\}$ are called the isospectral KP flows and non-isospectral KP flows, respectively. They are used to generate symmetries, Hamiltonians and conserved quantities for the isospectral KP hierarchy (3.9). For these flows we have the following.

Proposition 3.1. *For the isospectral and non-isospectral KP flows* $\{K_s(u)\}$ *and* $\{\sigma_s(u)\}$ *we have*

$$K_s = \tfrac{1}{2}(A_{s,y} - [A_2, A_s]), \tag{3.19a}$$

$$\sigma_s = \tfrac{1}{2}(B_{s,y} - [A_2, B_s]), \tag{3.19b}$$

which are called zero-curvature representations of the isospectral flow K_s *and the non-isospectral flow* σ_s*, respectively. Here we particularly note that the asymptotic data is*

$$K_s|_{u=0} = 0, \qquad A_s|_{u=0} = \partial^s, \tag{3.20a}$$

$$\sigma_s|_{u=0} = 0, \qquad B_s|_{u=0} = 2y\partial^s + x\partial^{s-1}, \tag{3.20b}$$

for $s = 1, 2, \ldots$.

Besides, the isospectral flows $\{K_s(u)\}$ can also be expressed in terms of the pseudo-differential operator L.

Proposition 3.2. *The isospectral flows* $\{K_s(u)\}$ *defined by (3.19a) can be expressed as*

$$K_s = \partial\,(\operatorname*{Res}_{\partial} L^s), \tag{3.21}$$

where

$$\operatorname*{Res}_{k}\left(\sum_{j=-m}^{+\infty} c_j k^j\right) = c_{-1}, \qquad m \geqslant 1.$$

Proof. From (3.19a) we have

$$\begin{aligned} 2K_s &= A_{s,y} - [A_2, A_s] \\ &= [(L^s - (L^s)_-)_y - [A_2, L^s - (L^s)_-]]_0 \\ &= [(L^s)_y - [A_2, L^s] - ((L^s)_-)_y + [A_2, (L^s)_-]]_0. \end{aligned}$$

Here $(L^s)_- = L^s - (L^s)_+$ and $(\cdot)_0$ means the constant part of the operator $(\cdot)$. Noting that (3.7a) indicates $(L^s)_y - [A_2, L^s] = 0$, we then have

$$2K_s = [A_2, (L^s)_-]_0 = 2\partial(\operatorname*{Res}_{\partial} L^s). \tag{3.22}$$

We finish the proof. □

3.2. Integrability properties of the KP hierarchies

3.2.1. Algebra of flows, recursive structures and symmetries. Now we return to the notions and notations in section 2. We suppose $u = u(x, y, \{t_j\})$ with time parameters $\{t_j\}$ is a generic point on the manifold M. Then the KP flows $\{K_l(u)\}$ and $\{\sigma_r(u)\}$ are vector fields in S. These flows generate a Lie algebra w.r.t. the commutator $[\![\cdot, \cdot]\!]$ defined by (2.7). This fact can be proved by using the zero-curvature representations of these flows.

Theorem 3.1. *The KP flows $\{K_l(u)\}$ and $\{\sigma_r(u)\}$ span (or generate) a Lie algebra*[5] $\boldsymbol{X}$ *with basic structure*

$$[\![K_l, K_r]\!] = 0, \tag{3.23a}$$
$$[\![K_l, \sigma_r]\!] = lK_{l+r-2}, \tag{3.23b}$$
$$[\![\sigma_l, \sigma_r]\!] = (l - r)\sigma_{l+r-2}, \tag{3.23c}$$

where $l, r \geqslant 1$ and we set $K_0(u) = \sigma_0(u) = 0$.

We prove the theorem through the following two lemmas.

Lemma 3.1. *For the vector field $X(u) \in S$ and differential operator*

$$N = a_0\partial^m + a_1\partial^{m-1} + \cdots + a_{m-1}\partial + a_m, \qquad N|_{u=0} = 0,$$

the equation

$$2X - N_y + [A_2, N] = 0 \tag{3.24}$$

has only zero solution $X = 0$, $N = 0$. Here $A_2 = \partial^2 + 2u$ in which we have taken $u_2 = u$.

Proof. Comparing the coefficient of the highest power of ∂ in (3.24), we find $a_0 = 0$. Then, step by step, one can successfully obtain $a_1 = a_2 = \cdots = a_m = 0$, which, in turn, gives rise to $N = 0$ and consequently $X = 0$. □

Lemma 3.2. *The KP flows $\{K_l(u)\}$ and $\{\sigma_r(u)\}$ and operators $\{A_l\}$ and $\{B_r\}$ satisfy*

$$2[\![K_l, K_r]\!] = (|A_l, A_r|)_y - [A_2, (|A_l, A_r|)], \tag{3.25a}$$
$$2[\![K_l, \sigma_r]\!] = (|A_l, B_r|)_y - [A_2, (|A_l, B_r|)], \tag{3.25b}$$
$$2[\![\sigma_l, \sigma_r]\!] = (|B_l, B_r|)_y - [A_2, (|B_l, B_r|)], \tag{3.25c}$$

where

$$(|A_l, A_r|) = A_l'[K_r] - A_r'[K_l] + [A_l, A_r], \tag{3.26a}$$
$$(|A_l, A_r|) = A_l'[\sigma_r] - B_r'[K_l] + [A_l, B_r], \tag{3.26b}$$
$$(|B_l, B_r|) = B_l'[\sigma_r] - B_r'[\sigma_l] + [B_l, B_r]. \tag{3.26c}$$

These operators satisfy

$$(|A_l, A_r|)|_{u=0} = 0, \tag{3.27a}$$
$$(|A_l, B_r|)|_{u=0} = l\partial^{l+r-2}, \tag{3.27b}$$
$$(|B_l, B_r|)|_{u=0} = (l - r)(2y\partial^{l+r-2} + x\partial^{l+r-3}). \tag{3.27c}$$

[5] By this we mean that $\{K_l\}$ and $\{\sigma_l\}$ generate a linear space $\boldsymbol{X} = \left\{\sum_j \alpha_j K_j + \sum_j \beta_j \sigma_j \middle| \alpha_j, \beta_j \in \mathbb{R}\right\}$ which is closed w.r.t. the commutator $[\![\cdot, \cdot]\!]$.

Proof. We only prove (3.25*b*). The proofs of the other equations are similar. By direct calculation from (3.19) we find

$$\begin{aligned} 2K_l'[\sigma_r] &= (A_l'[\sigma_r])_y - [2\sigma_r, A_l] - [A_2, A_l'[\sigma_r]] \\ &= (A_l'[\sigma_r])_y - [B_{r,y}, A_l] + [[A_2, B_r], A_l] - [A_2, A_l'[\sigma_r]] \end{aligned}$$

and

$$\begin{aligned} 2\sigma_r'[K_l] &= (B_r'[K_l])_y - [2K_l, B_r] - [A_2, B_r'[K_l]] \\ &= (B_r'[K_l])_y - [A_{l,y}, B_r] + [[A_2, A_l], B_r] - [A_2, B_r'[K_l]]. \end{aligned}$$

Then, by substraction of the above two equalities and making using of the Jacobi identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0,$$

we obtain (3.25*b*). Besides, substituting the asymptotic data (3.20*a*) and (3.20*b*) into (3.26*b*) yields (3.27*b*). We note that the method used to prove this lemma has been used for many systems (for example, see [29–35]). □

These two lemmas together with the zero-curvature representations (3.19) immediately lead to theorem 3.1. Theorem 3.1 directly yields the following two corollaries.

Corollary 3.1. *In the isospectral KP hierarchy (3.9), any equation* $u_{t_s} = K_s(u)$ *has two sets of symmetries*

$$\{K_l = K_l(u)\}, \qquad \{\tau_r^s = \tau_r^s(u, t_s) = st_sK_{s+r-2}(u) + \sigma_r(u)\}, \tag{3.28}$$

which generate a Lie algebra with basic structure

$$[\![K_l, K_r]\!] = 0, \tag{3.29a}$$
$$[\![K_l, \tau_r^s]\!] = lK_{l+r-2}, \tag{3.29b}$$
$$[\![\tau_l^s, \tau_r^s]\!] = (l-r)\tau_{l+r-2}^s, \tag{3.29c}$$

where $l, r, s \geqslant 1$ *and we set* $K_0(u) = \tau_0^s(u, t_s) = 0$.

Corollary 3.2. *The vector field* $\sigma_3(u)$ *is a master symmetry, i.e., it acts as a flow generator via the following relations:*

$$K_{s+1} = \frac{1}{s}[\![K_s, \sigma_3]\!], \tag{3.30a}$$
$$\sigma_{s+1} = \frac{1}{s-3}[\![\sigma_s, \sigma_3]\!], \qquad s \neq 3, \tag{3.30b}$$

with the initial flows $K_1 = u_x$ *given in (3.10) and* σ_1 *and* σ_4 *given in (3.18).*

We note that σ_3 and the recursive relation (3.30*a*) are the same as those given in [7], which means that the KP hierarchy derived from the Lax triad approach and the KP hierarchy generated from the recursive structure in [7] are the same.

3.2.2. Hamiltonian structures and conserved quantities. In the literature [7, 9–11] it has been proved that both isospectral and non-isospectral KP hierarchies have Hamiltonian structures and each equation in the isospectral KP hierarchy has two sets of conserved quantities. We list out these results through the following two theorems.

Theorem 3.2.

(1) Equation $u_{t_s} = K_s(u)$ in the isospectral KP hierarchy has a Hamiltonian structure, namely

$$u_{t_s} = K_s(u) = \partial\,\gamma_s = \partial\,\frac{\delta H_s}{\delta u}, \tag{3.31}$$

where the gradient field $\gamma_s = \gamma_s(u)$ is defined by

$$\gamma_s = \partial^{-1}K_s = \begin{cases} u, & s = 1, \\ \frac{1}{s-1}\,\mathrm{grad}\,\langle\gamma_{s-1}, \sigma_3\rangle, & s > 1. \end{cases} \tag{3.32}$$

The corresponding Hamiltonian $H_s = H_s(u)$ is

$$H_s = \begin{cases} \dfrac{1}{2}\langle u, u\rangle, & s = 1, \\ \dfrac{1}{s-1}\langle\gamma_{s-1}, \sigma_3\rangle, & s > 1. \end{cases} \tag{3.33}$$

(2) Equation $u_{t_s} = \sigma_s(u)$ in the non-isospectral KP hierarchy (3.17) has a Hamiltonian structure

$$u_{t_s} = \sigma_s(u) = \partial\,\omega_s = \partial\frac{\delta J_s}{\delta u}, \tag{3.34}$$

where the gradient field $\omega_s = \omega_s(u)$ is defined by

$$\omega_s = \begin{cases} 2yu, & s = 1, \\ \dfrac{1}{s-4}\,\mathrm{grad}\langle\omega_{s-1}, \sigma_3\rangle, & s > 1, s \neq 4, \\ 2y\gamma_4 + x\gamma_3 + \dfrac{3}{4}u_x + \dfrac{3}{2}\partial^{-1}u^2 + \dfrac{3}{4}\partial^{-3}u_{yy} + u\partial^{-1}u - \dfrac{3}{2}\partial^{-1}u_y, & s = 4. \end{cases} \tag{3.35}$$

The corresponding Hamiltonian $J_s = J_s(u)$ is

$$J_s = \begin{cases} \langle yu, u\rangle, & s = 1, \\ \dfrac{1}{s-4}\langle\omega_{s-1}, \sigma_3\rangle, & s > 1, s \neq 4, \\ \displaystyle\int_0^1 \langle\omega_4(\lambda u), u\rangle \mathrm{d}\lambda, & s = 4. \end{cases} \tag{3.36}$$

The key identity that leads to the above theorem is [7]

$$\sigma_3'\partial + \partial\sigma_3'^{*} = 0. \tag{3.37}$$

Theorem 3.3. *Hamiltonians $\{H_l(u)\}$ and $\{J_r(u)\}$ generate a Lie algebra w.r.t. the Poisson bracket $\{\cdot, \cdot\}$*

$$\{H, J\} = \left\langle\frac{\delta H}{\delta u}, \partial\frac{\delta J}{\delta u}\right\rangle.$$

The basic structure of the Lie algebra is

$$\{H_l, H_r\} = 0, \tag{3.38a}$$

$$\{H_l, J_r\} = l H_{l+r-2}, \tag{3.38b}$$

$$\{J_l, J_r\} = (l - r) J_{l+r-2}, \tag{3.38c}$$

where $l, r, s \geqslant 1$ and we set $H_0(u) = J_0(u) = 0$.

Corollary 3.3. *Equation $u_{t_s} = K_s(u)$ in the isospectral KP hierarchy (3.9) has two sets of conserved quantities*

$$\{H_l = H_l(u)\}, \qquad \{I_r^s = I_r^s(u, t_s) = st_s H_{s+r-2}(u) + J_r(u)\} \tag{3.39}$$

and they generate a Lie algebra with basic structure

$$\{H_l, H_r\} = 0, \tag{3.40a}$$

$$\{H_l, I_r^s\} = l H_{l+r-2}, \tag{3.40b}$$

$$\{I_l^s, I_r^s\} = (l-r) I_{l+r-2}^s, \tag{3.40c}$$

where $l, r, s \geqslant 1$ and we set $H_0(u) = I_0^s(u, t_s) = 0$.

4. The DΔKP system and integrability

In this section, we will construct the DΔKP hierarchy and discuss their recursive structure, symmetries, Hamiltonian structures and conserved quantities.

4.1. Lax triad approach and the DΔKP hierarchies

4.1.1. The isospectral DΔKP hierarchy. First, we introduce a discrete independent variable n. The basic operation w.r.t. n is a shift. By E we denote a shift operator defined through $E^j g(n) = g(n+j)$ for $j \in \mathbb{Z}$. Besides, a difference operator $\Delta = E - 1$ is a discrete analogue of derivative and $\Delta^{-1} = (E-1)^{-1}$ satisfies $\Delta\Delta^{-1} = \Delta^{-1}\Delta = 1$. Δ^s obeys the following discrete Leibniz rule,

$$\Delta^s g(n) = \sum_{i=0}^{\infty} \mathrm{C}_s^i \, (\Delta^i g(n+s-i)) \Delta^{s-i}, \qquad s \in \mathbb{Z}, \tag{4.1}$$

where C_s^i is defined in (3.3). In equation (4.1), the notation $\Delta^s g(n)$ indicates the composition between Δ^s and the operator of multiplication by the function $n \mapsto g(n)$; the notation $(\Delta^i g(n+s-i))$ indicates the operator of multiplication by the function $n \mapsto \Delta^i g(n+s-i)$. For example, we have

$$\Delta g(n) = g(n+1)\Delta + (\Delta g(n)), \tag{4.2a}$$

$$\Delta^2 g(n) = g(n+2)\Delta^2 + 2(\Delta g(n+1))\Delta + (\Delta^2 g(n)), \tag{4.2b}$$

$$\Delta^{-1} g(n) = g(n-1)\Delta^{-1} - (\Delta g(n-2))\Delta^{-2} + \cdots + (-1)^{j-1}(\Delta^{j-1} g(n-j))\Delta^{-j} + \cdots. \tag{4.2c}$$

Formula (4.1) can be proved by using mathematical inductive method, and we note that it is also valid for negative integer s.

Now we consider the pseudo-difference operator

$$\bar{L} = h^{-1}\Delta + \bar{u}_0 + h\bar{u}_1\Delta^{-1} + \cdots + h^j \bar{u}_j \Delta^{-j} + \cdots, \tag{4.3}$$

where h is the lattice spacing parameter of n-direction, $\bar{u}_j = \bar{u}_j(n, \bar{x}, \bar{\boldsymbol{t}})$ with time parameters $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \cdots)$ are C^∞ functions w.r.t. $\bar{x} \in \mathbb{R}$, defined on $\mathbb{Z} \times \mathbb{R}$ and vanishing rapidly at infinity. Let us look at the following linear triad

$$\bar{L}\phi = \eta\phi, \qquad \eta_{\bar{t}_m} = 0, \tag{4.4a}$$

$$\phi_{\bar{x}} = \bar{A}_1\phi, \qquad \bar{A}_1 = h^{-1}\Delta + \bar{u}_0, \tag{4.4b}$$

$$\phi_{\bar{t}_m} = \bar{A}_m\phi, \qquad m = 1, 2, \cdots. \tag{4.4c}$$

The compatibility condition is

$$\bar{L}_{\bar{x}} = [\bar{A}_1, \bar{L}], \tag{4.5a}$$
$$\bar{L}_{\bar{t}_m} = [\bar{A}_m, \bar{L}], \tag{4.5b}$$
$$\bar{A}_{1,\bar{t}_m} - \bar{A}_{m,\bar{x}} + [\bar{A}_1, \bar{A}_m] = 0, \tag{4.5c}$$

for $m = 1, 2, \ldots$, where $\bar{A}_m = (\bar{L}^m)_+$ with the form

$$\bar{A}_m = h^{-m}\Delta^m + \sum_{j=1}^{m} h^{-(m-j)}\bar{a}_j\Delta^{m-j}, \qquad \bar{A}_s|_{\bar{u}=\mathbf{0}} = h^{-m}\Delta^m. \tag{4.6}$$

Here $\bar{\boldsymbol{u}} = (\bar{u}_0, \bar{u}_1, \cdots)$ and for a generic difference operator $T = \sum_{j=-k}^{m} a_j(\bar{\mathbf{u}})\Delta^j,\ m, k \geqslant 0$, $(T)_+ = \sum_{j=0}^{m} a_j(\bar{\mathbf{u}})\Delta^j$. The first three of the operators $\bar{A}_m$ are

$$\bar{A}_1 = h^{-1}\Delta + \bar{u}_0, \tag{4.7a}$$
$$\bar{A}_2 = h^{-2}\Delta^2 + h^{-1}(\Delta\bar{u}_0 + 2\bar{u}_0)\Delta + (h^{-1}\Delta\bar{u}_0 + \bar{u}_0^2 + \Delta\bar{u}_1 + 2\bar{u}_1), \tag{4.7b}$$
$$\bar{A}_3 = h^{-3}\Delta^3 + h^{-2}\bar{a}_1\Delta^2 + h^{-1}\bar{a}_2\Delta + \bar{a}_3, \tag{4.7c}$$

where

$$\begin{aligned}
\bar{a}_1 &= \Delta^2\bar{u}_0 + 3\Delta\bar{u}_0 + 3\bar{u}_0,\\
\bar{a}_2 &= 2h^{-1}\Delta^2\bar{u}_0 + 3h^{-1}\Delta\bar{u}_0 + 3\bar{u}_0^2 + \bar{u}_0\Delta\bar{u}_0 + \Delta\bar{u}_0^2 + 3\bar{u}_1 + 3\Delta\bar{u}_1 + \Delta^2\bar{u}_1,\\
\bar{a}_3 &= h^{-2}\Delta^2\bar{u}_0 + \bar{u}_0^3 + h^{-1}\bar{u}_0\Delta\bar{u}_0 + h^{-1}\Delta\bar{u}_0^2 + 5\bar{u}_0\bar{u}_1 + (\Delta\bar{u}_0)\Delta\bar{u}_1 + 3\bar{u}_0\Delta\bar{u}_1\\
&\quad + \bar{u}_1\Delta\bar{u}_0 + \bar{u}_1E^{-1}\bar{u}_0 + 2h^{-1}\Delta^2\bar{u}_1 + 3h^{-1}\Delta\bar{u}_1 + 3\bar{u}_2 + 3\Delta\bar{u}_2 + \Delta^2\bar{u}_2.
\end{aligned}$$

Equation (4.5*a*) is used to express $\bar{u}_j (j > 0)$ in terms of $\bar{u}_0$, i.e.

$$\bar{u}_1 = \Delta^{-1}\frac{\partial\bar{u}_0}{\partial\bar{x}}, \tag{4.8a}$$
$$\bar{u}_2 = \Delta^{-2}\frac{\partial^2\bar{u}_0}{\partial\bar{x}^2} - h^{-1}\Delta^{-1}\frac{\partial\bar{u}_0}{\partial\bar{x}} - \Delta^{-1}\Big(\bar{u}_0\Delta^{-1}\frac{\partial\bar{u}_0}{\partial\bar{x}}\Big) + \Delta^{-1}\Big(\Big(\Delta^{-1}\frac{\partial\bar{u}_0}{\partial\bar{x}}\Big)E^{-1}\bar{u}_0\Big), \tag{4.8b}$$

$\cdots\cdots.$

Equation (4.5*b*) actually plays a role to determine the operator $\bar{A}_m$. In fact, if starting from the assumption (4.6) with unknown $\{\bar{a}_j\}$, then (4.5*b*) uniquely determines $\bar{A}_m = (\bar{L}^m)_+$. With $\{\bar{A}_m\}$ ready, equation (4.5*c*) provides the isospectral DΔKP hierarchy (with $\bar{u}_0 = \bar{u}$)

$$\bar{u}_{\bar{t}_m} = \bar{K}_m(u) = \bar{A}_{m,\bar{x}} - [\bar{A}_1, \bar{A}_m], \qquad m = 1, 2, \cdots. \tag{4.9}$$

The first three equations are

$$\bar{u}_{\bar{t}_1} = \bar{K}_1(\bar{u}) = \bar{u}_{\bar{x}}, \tag{4.10a}$$
$$\bar{u}_{\bar{t}_2} = \bar{K}_2(\bar{u}) = (1 + 2\Delta^{-1})\bar{u}_{\bar{x}\bar{x}} - 2h^{-1}\bar{u}_{\bar{x}} + 2\bar{u}\bar{u}_{\bar{x}}, \tag{4.10b}$$
$$\begin{aligned}
\bar{u}_{\bar{t}_2} = \bar{K}_3(\bar{u}) &= (3\Delta^{-2} + 3\Delta^{-1} + 1)\bar{u}_{\bar{x}\bar{x}\bar{x}} + 3\Delta^{-1}\bar{u}_{\bar{x}}^2 + 3\bar{u}\Delta^{-1}\bar{u}_{\bar{x}\bar{x}}\\
&\quad - 6h^{-1}\Delta^{-1}\bar{u}_{\bar{x}\bar{x}} + 3h^{-2}\bar{u}_{\bar{x}} + 3\bar{u}_{\bar{x}}\Delta^{-1}\bar{u}_{\bar{x}} + 3\Delta^{-1}(\bar{u}\bar{u}_{\bar{x}\bar{x}})\\
&\quad + 3\bar{u}\bar{u}_{\bar{x}\bar{x}} - 3h^{-1}\bar{u}_{\bar{x}\bar{x}} + 3\bar{u}_{\bar{x}}^2 + 3\bar{u}^2\bar{u}_{\bar{x}} - 6h^{-1}\bar{u}\bar{u}_{\bar{x}},
\end{aligned} \tag{4.10c}$$

in which (4.10*b*), i.e. (1.3), is first derived in [12] from a discrete Sato's approach and is referred to as the DΔKP equation.

4.1.2. The non-isospectral D∆KP hierarchy. For the non-isospectral case, we take

$$\eta_{\bar{t}_m} = h\eta^m + \eta^{m-1} \tag{4.11}$$

and assume that

$$\bar{B}_m = \sum_{j=0}^{m} h^{-(m-j)}\bar{b}_j\Delta^{m-j} \tag{4.12}$$

with unknown functions $\{\bar{b}_j\}$. In this turn, we consider the Lax triad

$$\bar{L}\phi = \eta\phi, \tag{4.13a}$$

$$\phi_{\bar{x}} = \bar{A}_1\phi, \tag{4.13b}$$

$$\phi_{\bar{t}_m} = \bar{B}_m\phi, \qquad m = 1, 2, \cdots, \tag{4.13c}$$

together with the spectrum evolution (4.11). The compatibility reads

$$\bar{L}_{\bar{x}} = [\bar{A}_1, \bar{L}], \tag{4.14a}$$

$$\bar{L}_{\bar{t}_m} = [\bar{B}_m, \bar{L}] + h\bar{L}^m + \bar{L}^{m-1}, \tag{4.14b}$$

$$\bar{A}_{1,\bar{t}_m} - \bar{B}_{m,\bar{x}} + [\bar{A}_1, \bar{B}_m] = 0. \tag{4.14c}$$

Looking at (4.14b) and (4.14c) asymptotically, namely, $(4.14b)|_{\bar{u}=\mathbf{0}}$ and $(4.14c)|_{\bar{u}=\mathbf{0}}$, from them one can find

$$(\Delta\bar{b}_0)|_{\bar{u}=\mathbf{0}} = 0, \quad (\Delta\bar{b}_1)|_{\bar{u}=\mathbf{0}} = h, \quad (\Delta\bar{b}_j)|_{\bar{u}=\mathbf{0}} = 0, \qquad j = 2, 3, \cdots, m;$$

$$(\partial_{\bar{x}}\bar{b}_0)|_{\bar{u}=\mathbf{0}} = h, \quad (\partial_{\bar{x}}\bar{b}_1)|_{\bar{u}=\mathbf{0}} = 1, \quad (\partial_{\bar{x}}\bar{b}_j)|_{\bar{u}=\mathbf{0}} = 0, \qquad j = 2, 3, \cdots, m.$$

This provides the necessary asymptotic condition for $\bar{B}_m$[6]:

$$\bar{B}_m|_{\bar{u}=\mathbf{0}} = h^{-(m-1)}\bar{x}\Delta^m + h^{-(m-1)}(\bar{x} + hn)\Delta^{m-1}. \tag{4.15}$$

With this condition, $\bar{B}_m$ can be uniquely determined by (4.14b) and here we write out the first three of them:

$$\bar{B}_1 = h\bar{x}\bar{A}_1 + \bar{x} + hn, \tag{4.16a}$$

$$\bar{B}_2 = h\bar{x}\bar{A}_2 + (\bar{x} + hn)\bar{A}_1 + h\Delta^{-1}\bar{u}_0, \tag{4.16b}$$

$$\begin{aligned}\bar{B}_3 = {} & h\bar{x}\bar{A}_3 + (\bar{x} + hn)\bar{A}_2 + \Delta^{-1}\bar{u}_0\Delta + h\bar{u}_0\Delta^{-1}\bar{u}_0 \\ & + 2h\Delta^{-1}\bar{u}_1 - \Delta^{-1}\bar{u}_0 + h\Delta^{-1}\bar{u}_0^2,\end{aligned} \tag{4.16c}$$

where $\bar{A}_j = (\bar{L}^j)_+$.

Now, (4.14a) plays a role similar to (4.5a) and provides transformations that are the same as in (4.8). (4.14c) provides the non-isospectral D∆KP hierarchy (with $\bar{u}_0 = \bar{u}$)

$$\bar{u}_{\bar{t}_m} = \bar{\sigma}_m(\bar{u}) = \bar{B}_{m,\bar{x}} - [\bar{A}_1, \bar{B}_m], \tag{4.17}$$

i.e.

$$\bar{u}_{\bar{t}_1} = \bar{\sigma}_1(\bar{u}) = h\bar{x}\bar{K}_1 + h\bar{u}, \tag{4.18a}$$

$$\bar{u}_{\bar{t}_2} = \bar{\sigma}_2(\bar{u}) = h\bar{x}\bar{K}_2 + (\bar{x} + hn)\bar{K}_1 + h\bar{u}_{\bar{x}} + 3h\Delta^{-1}\bar{u}_{\bar{x}} + h\bar{u}^2 - \bar{u}, \tag{4.18b}$$

$$\begin{aligned}\bar{u}_{\bar{t}_2} = \bar{\sigma}_3(\bar{u}) = {} & h\bar{x}\bar{K}_3 + (\bar{x} + hn)\bar{K}_2 + 5h\Delta^{-2}\bar{u}_{\bar{x}\bar{x}} - 6\Delta^{-1}\bar{u}_{\bar{x}} + 5h\Delta^{-1}(\bar{u}\bar{u}_{\bar{x}}) \\ & + h\bar{u}_{\bar{x}}\Delta^{-1}\bar{u} + 4h\bar{u}\Delta^{-1}\bar{u}_{\bar{x}} - 2\bar{u}^2 + h^{-1}\bar{u} + h\bar{u}^3 + 3h\bar{u}\bar{u}_{\bar{x}} \\ & + 3h\Delta^{-1}\bar{u}_{\bar{x}\bar{x}} + h\bar{u}_{\bar{x}\bar{x}} - 2\bar{u}_{\bar{x}},\end{aligned} \tag{4.18c}$$

$$\cdots\cdots,$$

where $\{\bar{K}_j\}$ are the isospectral D∆KP flows defined in (4.9).

$\{\bar{K}_m(\bar{u})\}$ and $\{\bar{\sigma}_m(\bar{u})\}$ are respectively called the isospectral and non-isospectral D∆KP flows. For them we have the following.

[6] In [20] the asymptotic condition for B_m is $B_m|_{\bar{u}=\mathbf{0}} = h^{-(m-1)}\bar{x}\Delta^m + h^{-(m-2)}n\Delta^{m-1}$. We note that that is not sufficient due to missing (4.13b) in the Lax triad (4.13).

Proposition 4.1. *The isospectral and non-isospectral D∆KP flows* $\{\bar{K}_s(\bar{u})\}$ *and* $\{\bar{\sigma}_s(\bar{u})\}$ *can be expressed through the following zero curvature representations together with asymptotic data,*

$$\bar{K}_s = \bar{A}_{s,\bar{x}} - [\bar{A}_1, \bar{A}_s], \qquad \bar{K}_s|_{\bar{u}=0} = 0, \qquad \bar{A}_s|_{\bar{u}=0} = h^{-s}\Delta^s, \tag{4.19a}$$

$$\bar{\sigma}_s = \bar{B}_{s,\bar{x}} - [\bar{A}_1, \bar{B}_s], \quad \bar{\sigma}_s|_{\bar{u}=0} = 0, \qquad \bar{B}_s|_{\bar{u}=0} = h^{-(s-1)}\bar{x}\Delta^s + h^{-(s-1)}(\bar{x}+hn)\Delta^{s-1}, \tag{4.19b}$$

for $s = 1, 2, \ldots$.

Besides, similar to proposition 3.2, we have the following.

Proposition 4.2. *The isospectral D∆KP flows* $\{\bar{K}_s(u)\}$ *defined by (4.9) can be expressed in terms of the pseudo-difference operator* $\bar{L}$ *as*

$$\bar{K}_s = \Delta(\operatorname*{Res}_{\Delta} \bar{L}^s). \tag{4.20}$$

The proof is similar to that for proposition 3.2 and here we skip it.

4.2. Integrability properties

We have derived the isospectral and non-isospectral D∆KP hierarchies (4.9) and (4.17). To investigate their integrability properties, in the rest of this subsection we indicate with $\bar{u} = \bar{u}(n, \bar{x})$ a generic point of the manifold $\bar{M}$ (that might depend as well on the time parameters $\{\bar{t}_j\}$). Of course, the flows $\{\bar{K}_l(\bar{u})\}$ and $\{\bar{\sigma}_r(\bar{u})\}$ are vector fields on $\bar{M}$.

4.2.1. Algebra of flows, recursive structures and symmetries. Similar to the continuous case, the D∆KP flows $\{\bar{K}_l(\bar{u})\}$ and $\{\bar{\sigma}_r(\bar{u})\}$ can span a Lie algebra with recursive structures, and the algebra leads to symmetries of the isospectral D∆KP hierarchy. The proofs for the results of this part are also similar to the continuous case (see section 3.2.1). We just list these results without giving proofs.

Lemma 4.1. *For the function* $\bar{X}(\bar{u}) \in \bar{S}$ *and difference operator*

$$\bar{N} = \bar{a}_0\Delta^m + \bar{a}_1\Delta^{m-1} + \cdots + \bar{a}_{m-1}\Delta + \bar{a}_m, \qquad \bar{N}|_{\bar{u}=0} = 0,$$

the equation

$$\bar{X} - \bar{N}_{\bar{x}} + [\bar{A}_1, \bar{N}] = 0 \tag{4.21}$$

only admits the zero solution $\bar{X} = 0,\ \bar{N} = 0$.

Lemma 4.2. *Define the operators*

$$(|\bar{A}_l, \bar{A}_r|) = \bar{A}_l'[\bar{K}_r] - \bar{A}_r'[\bar{K}_l] + [\bar{A}_l, \bar{A}_r], \tag{4.22a}$$

$$(|\bar{A}_l, \bar{B}_r|) = \bar{A}_l'[\bar{\sigma}_r] - \bar{B}_r'[\bar{K}_l] + [\bar{A}_l, \bar{B}_r], \tag{4.22b}$$

$$(|\bar{B}_l, \bar{B}_r|) = \bar{B}_l'[\bar{\sigma}_r] - \bar{B}_r'[\bar{\sigma}_l] + [\bar{B}_l, \bar{B}_r]. \tag{4.22c}$$

Then we have

$$[\![\bar{K}_l, \bar{K}_r]\!] = (|\bar{A}_l, \bar{A}_r|)_{\bar{x}} - [\bar{A}_1, (|\bar{A}_l, \bar{A}_r|)], \tag{4.23a}$$

$$[\![\bar{K}_l, \bar{\sigma}_r]\!] = (|\bar{A}_l, \bar{B}_r|)_{\bar{x}} - [\bar{A}_1, (|\bar{A}_l, \bar{B}_r|)], \tag{4.23b}$$

$$[\![\bar{\sigma}_l, \bar{\sigma}_r]\!] = (|\bar{B}_l, \bar{B}_r|)_{\bar{x}} - [\bar{A}_1, (|\bar{B}_l, \bar{B}_r|)], \tag{4.23c}$$

and

$$(|\bar{A}_l, \bar{A}_r|)|_{\bar{u}=0} = 0, \tag{4.24a}$$

$$(|\bar{A}_l, \bar{B}_r|)|_{\bar{u}=0} = h^{-(l+r-2)}l(\Delta^{l+r-1} + \Delta^{l+r-2}), \tag{4.24b}$$

$$(|\bar{B}_l, \bar{B}_r|)|_{\bar{u}=0} = h^{-(l+r-3)}(l-r)\big(\bar{x}\Delta^{l+r-1} + (2\bar{x}+hn)\Delta^{l+r-2} + (\bar{x}+hn)\Delta^{l+r-3}\big). \tag{4.24c}$$

Theorem 4.1. *The flows* $\{\bar{K}_l(\bar{u})\}$ *and* $\{\bar{\sigma}_r(\bar{u})\}$ *span a Lie algebra* $\bar{\boldsymbol{X}}$ *with basic structure*

$$[\![\bar{K}_l, \bar{K}_r]\!] = 0, \tag{4.25a}$$

$$[\![\bar{K}_l, \bar{\sigma}_r]\!] = l(h\bar{K}_{l+r-1} + \bar{K}_{l+r-2}), \tag{4.25b}$$

$$[\![\bar{\sigma}_l, \bar{\sigma}_r]\!] = (l-r)(h\bar{\sigma}_{l+r-1} + \bar{\sigma}_{l+r-2}), \tag{4.25c}$$

where $l, r \geqslant 1$ *and we set* $\bar{K}_0(\bar{u}) = \bar{\sigma}_0(\bar{u}) = 0$.

Corollary 4.1. *In the isospectral DΔKP hierarchy (4.9), each equation* $\bar{u}_{t_s} = \bar{K}_s$ *possesses two sets of symmetries*

$$\{\bar{K}_l = \bar{K}_l(\bar{u})\}, \qquad \{\bar{\tau}_r^s = s\bar{t}_s(h\bar{K}_{s+r-1}(\bar{u}) + \bar{K}_{s+r-2}(\bar{u})) + \bar{\sigma}_r(\bar{u})\}, \tag{4.26}$$

which generate a Lie algebra with basic structure

$$[\![\bar{K}_l, \bar{K}_r]\!] = 0, \tag{4.27a}$$

$$[\![\bar{K}_l, \bar{\tau}_r^s]\!] = l(h\bar{K}_{l+r-1} + \bar{K}_{l+r-2}), \tag{4.27b}$$

$$[\![\bar{\tau}_l^s, \bar{\tau}_r^s]\!] = (l-r)(h\bar{\tau}_{l+r-1}^s + \bar{\tau}_{l+r-2}^s), \tag{4.27c}$$

where $l, r, s \geqslant 1$ *and we set* $\bar{K}_0(\bar{u}) = \bar{\tau}_0^s(\bar{u}, \bar{t}_s) = 0$.

Corollary 4.2. $\bar{\sigma}_2$ *is a master symmetry, i.e., it acts as a flow generator via the following relations:*

$$\bar{K}_{s+1} = \frac{1}{h}\left(\frac{1}{s}[\![\bar{K}_s, \bar{\sigma}_2]\!] - \bar{K}_s\right), \tag{4.28a}$$

$$\bar{\sigma}_{s+1} = \frac{1}{h}\left(\frac{1}{s-2}[\![\bar{\sigma}_s, \bar{\sigma}_2]\!] - \bar{\sigma}_s\right), \qquad s \neq 2, \tag{4.28b}$$

with initial flows $\bar{K}_1 = \bar{u}_{\bar{x}}$ *given in (4.10) and* $\bar{\sigma}_1$ *and* $\bar{\sigma}_3$ *given in (4.18).*

4.2.2. Hamiltonian structures and conserved quantities. (1+1)-dimensional Lax integrable systems usually have their own recursion operators that play important roles in investigating integrability characteristics (see [23, 35–37]). For the DΔKP hierarchy, so far there is no explicit recursion operator but their recursive structure (4.28) will play a role which is similar to a recursion operator. We will show that each member in the isospectral DΔKP hierarchy (4.9) and each member in the non-isospectral DΔKP hierarchy (4.17) has a Hamiltonian structure, respectively. These Hamiltonians lead to two sets of conserved quantities for the isospectral DΔKP hierarchy (4.9). Let us prove this step by step.

Lemma 4.3. *The following formula*

$$\mathrm{grad}\langle\bar{\gamma}, \bar{\sigma}\rangle = \bar{\gamma}'^{*}\bar{\sigma} + \bar{\sigma}'^{*}\bar{\gamma} \tag{4.29}$$

holds for any $\bar{\gamma} \in \bar{S}^*$ *and* $\bar{\sigma} \in \bar{S}$.

In fact, one can verify that

$$\langle\bar{\gamma}, \bar{\sigma}\rangle'[\bar{g}] = \langle\bar{\gamma}'[\bar{g}], \bar{\sigma}\rangle + \langle\bar{\gamma}, \bar{\sigma}'[\bar{g}]\rangle = \langle\bar{\gamma}'^{*}\bar{\sigma} + \bar{\sigma}'^{*}\bar{\gamma}, \bar{g}\rangle, \qquad \forall \bar{g} \in \bar{S}.$$

Lemma 4.4. *The master symmetry* $\bar{\sigma}_2(\bar{u})$ *given in (4.18b) satisfies*

$$\bar{\sigma}_2'\partial_{\bar{x}} + \partial_{\bar{x}}\bar{\sigma}_2'^{*} = 0. \tag{4.30}$$

In other words, $\partial_{\bar{x}}$ *is a Noether operator of the master symmetry equation* $\bar{u}_{\bar{t}_2} = \bar{\sigma}_2$.

This identity is important for us to obtain Hamiltonian structures for the equations in the isospectral and non-isospectral DΔKP hierarchies. The proof of (4.30) needs a lengthy but direct calculation and here we skip it.

Now we arrive at some main theorems of this part.

Theorem 4.2. *In the whole isospectral D∆KP hierarchy (4.9), each equation $\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u})$ has a Hamiltonian structure*

$$\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u}) = \partial_{\bar{x}} \frac{\delta \bar{H}_s}{\delta \bar{u}}. \tag{4.31}$$

The gradient $\bar{\gamma}_s = \bar{\gamma}_s(\bar{u}) = \frac{\delta \bar{H}_s}{\delta \bar{u}}$ is given by

$$\bar{\gamma}_s = \begin{cases} \bar{u}, & s = 1, \\ \dfrac{1}{h}\left(\dfrac{1}{s-1}\mathrm{grad}\langle \bar{\gamma}_{s-1}, \bar{\sigma}_2\rangle - \bar{\gamma}_{s-1}\right), & s > 1. \end{cases} \tag{4.32}$$

The Hamiltonian $\bar{H}_s = \bar{H}_s(\bar{u})$ is given by

$$\bar{H}_s = \begin{cases} \dfrac{1}{2}\langle \bar{u}, \bar{u}\rangle, & s = 1, \\ \dfrac{1}{h}\left(\dfrac{1}{s-1}\langle \bar{\gamma}_{s-1}, \bar{\sigma}_2\rangle - \bar{H}_{s-1}\right), & s > 1. \end{cases} \tag{4.33}$$

Proof. Obviously, $\partial_{\bar{x}}$ is an implectic operator. Next we need to prove $\bar{\gamma}_s$ is a gradient field. Let us do that by means of mathematical inductive method. Obviously, $\gamma_1 = \bar{u}$ is a gradient field. We suppose $\bar{\gamma}_s$ is a gradient field, i.e. $\bar{\gamma}_s' = \bar{\gamma}_s'^*$. Then, from the recursive relation (4.28*a*) we have

$$\begin{aligned} \bar{\gamma}_{s+1} &= \partial_{\bar{x}}^{-1}\bar{K}_{s+1} \\ &= \frac{1}{h}\left(\frac{1}{s}\,\partial_{\bar{x}}^{-1}[\![\bar{K}_s, \bar{\sigma}_2]\!] - \partial_{\bar{x}}^{-1}\bar{K}_s\right) \\ &= \frac{1}{h}\left(\frac{1}{s}\,\partial_{\bar{x}}^{-1}\big((\partial_{\bar{x}}\bar{\gamma}_s)'[\bar{\sigma}_2] - \bar{\sigma}_2'[\partial_{\bar{x}}\bar{\gamma}_s]\big) - \bar{\gamma}_s\right) \\ &= \frac{1}{h}\left(\frac{1}{s}\,\partial_{\bar{x}}^{-1}(\partial_{\bar{x}}\bar{\gamma}_s'\bar{\sigma}_2 - \bar{\sigma}_2'\partial_{\bar{x}}\,\bar{\gamma}_s) - \bar{\gamma}_s\right). \end{aligned}$$

It then follows from (4.30) that

$$\begin{aligned} \bar{\gamma}_{s+1} &= \frac{1}{h}\left(\frac{1}{s}\,\partial_{\bar{x}}^{-1}(\partial_{\bar{x}}\,\bar{\gamma}_s'\bar{\sigma}_2 + \partial_{\bar{x}}\bar{\sigma}_2'^*\bar{\gamma}_s) - \bar{\gamma}_s\right) \\ &= \frac{1}{h}\left(\frac{1}{s}(\bar{\gamma}_s'\bar{\sigma}_2 + \bar{\sigma}_2'^*\,\bar{\gamma}_s) - \bar{\gamma}_s\right) \\ &= \frac{1}{h}\left(\frac{1}{s}\mathrm{grad}\langle \bar{\gamma}_s, \bar{\sigma}_2\rangle - \bar{\gamma}_s\right). \end{aligned}$$

Here we also made use of lemma 4.3. Thus, it is clear that if $\bar{\gamma}_s$ is a gradient field, so is $\bar{\gamma}_{s+1}$. For the Hamiltonians, $\bar{H}_1 = \frac{1}{2}\langle \bar{u}, \bar{u}\rangle$ is derived by using proposition 2.1 where $\bar{\gamma}_1 = \bar{u}$. When $s > 1$ $\bar{H}_s$ is derived from the recursive relation of $\bar{\gamma}_s$ given in (4.32). □

Theorem 4.3. *In the non-isospectral D∆KP hierarchy (4.17), each equation $\bar{u}_{t_s} = \bar{\sigma}_s(\bar{u})$ has a Hamiltonian structure*

$$\bar{u}_{\bar{t}} = \bar{\sigma}_s(\bar{u}) = \partial_{\bar{x}} \frac{\delta \bar{J}_s}{\delta \bar{u}}. \tag{4.34}$$

More precisely, the covector fields

$$\bar{\omega}_s(\bar{u}) = \partial_{\bar{x}}^{-1}\bar{\sigma}_s \tag{4.35}$$

are given by

$$\bar{\omega}_1 = h\bar{x}\bar{u}, \tag{4.36a}$$

$$\bar{\omega}_2 = h\bar{x}\bar{\gamma}_2 + (\bar{x}+hn)\bar{\gamma}_1 + h\Delta^{-1}\bar{u}, \tag{4.36b}$$

$$\bar{\omega}_3 = h\bar{x}\bar{\gamma}_3 + (\bar{x}+hn)\bar{\gamma}_2 + 2h\Delta^{-2}\bar{u}_{\bar{x}} + h\Delta^{-1}\bar{u}^2 + h\bar{u}\Delta^{-1}\bar{u} - 2\Delta^{-1}\bar{u}, \tag{4.36c}$$

$$\bar{\omega}_s = \frac{1}{h}\left(\frac{1}{s-3}\,\mathrm{grad}\langle\bar{\omega}_{s-1},\bar{\sigma}_2\rangle - \bar{\omega}_{s-1}\right), \qquad s = 4,5,\cdots, \tag{4.36d}$$

and $\bar{\omega}_s = \frac{\delta \bar{J}_s}{\delta \bar{u}}$, *where the Hamiltonian* $\bar{J}_s = \bar{J}_s(\bar{u})$ *are given by*

$$\bar{J}_s = \begin{cases} \frac{h}{2}\langle\bar{x}\bar{u},\bar{u}\rangle, & s=1,\\ \frac{1}{h}\left(\frac{1}{s-3}\langle\bar{\omega}_{s-1},\bar{\sigma}_2\rangle - \bar{J}_{s-1}\right), & s>1, s\neq 3,\\ \int_0^1 \langle\bar{\omega}_3(\lambda\bar{u}),\bar{u}\rangle \mathrm{d}\lambda, & s=3. \end{cases} \tag{4.37}$$

Proof. The proof is quite similar to the isospectral case. In this turn we need to start from the recursive relation (4.28*b*). However, $\bar{\sigma}_3$ cannot be derived from (4.28*b*). By direct verification we can find that $\bar{\omega}_s' = \bar{\omega}_s'^{*}$ holds for $s = 1, 2, 3$. Now we suppose that $\bar{\omega}_s$ is a gradient function. Then, if $s > 2$, from the recursive relation (4.28*b*) we have

$$\begin{aligned}\bar{\omega}_{s+1} = \partial_{\bar{x}}^{-1}\bar{\sigma}_{s+1} &= \frac{1}{h}\left(\frac{1}{s-2}\,\partial_{\bar{x}}^{-1}[\![\bar{\sigma}_s,\bar{\sigma}_2]\!] - \partial_{\bar{x}}^{-1}\bar{\sigma}_s\right)\\ &= \frac{1}{h}\left(\frac{1}{s-2}\,\partial_{\bar{x}}^{-1}(\partial_{\bar{x}}\,\bar{\omega}_s'\bar{\sigma}_2 - \bar{\sigma}_2'\partial_{\bar{x}}\,\bar{\omega}_s) - \bar{\omega}_s\right)\\ &= \frac{1}{h}\left(\frac{1}{s-2}\,\partial_{\bar{x}}^{-1}(\partial_{\bar{x}}\,\bar{\omega}_s'\bar{\sigma}_2 + \partial_{\bar{x}}\bar{\sigma}_2'^{*}\,\bar{\omega}_s) - \bar{\omega}_s\right)\\ &= \frac{1}{h}\left(\frac{1}{s-2}\,(\bar{\omega}_s'\bar{\sigma}_2 + \bar{\sigma}_2'^{*}\,\bar{\omega}_s) - \bar{\omega}_s\right)\\ &= \frac{1}{h}\left(\frac{1}{s-2}\,\mathrm{grad}(\bar{\omega}_s,\bar{\sigma}_2) - \bar{\omega}_s\right),\end{aligned}$$

where we have made use of (4.30) and lemma 4.3. Since $\bar{\omega}_s$ is a gradient function, so is $\bar{\omega}_{s+1}$, and therefore (4.34) holds. The Hamiltonian $\bar{J}_s$ is determined using proposition 2.1 and lemma 4.3. Thus the proof is completed. □

As a corollary of the above two theorems and proposition 2.4, we have

Corollary 4.3. $\partial_{\bar{x}}$ *is a Noether operator of the whole isospectral DΔKP hierarchy (4.9) and non-isospectral DΔKP hierarchy (4.17), namely*

$$\bar{K}_s'\partial_{\bar{x}} + \partial_{\bar{x}}\bar{K}_s'^{*} = 0, \tag{4.38a}$$

$$\bar{\sigma}_s'\partial_{\bar{x}} + \partial_{\bar{x}}\bar{\sigma}_s'^{*} = 0 \tag{4.38b}$$

hold for $s = 1, 2, \ldots$.

These two equations play an important role in deriving the following theorem.

Theorem 4.4. *The Hamiltonians* $\{\bar{H}_l(\bar{u})\}$ *and* $\{\bar{J}_r(\bar{u})\}$ *described in (4.33) and (4.37) span a Lie algebra with basic structure*

$$\{\bar{H}_l,\bar{H}_r\} = 0, \tag{4.39a}$$

$$\{\bar{H}_l,\bar{J}_r\} = l(h\bar{H}_{l+r-1} + \bar{H}_{l+r-2}), \tag{4.39b}$$

$$\{\bar{J}_l,\bar{J}_r\} = (l-r)(h\bar{J}_{l+r-1} + \bar{J}_{l+r-2}), \tag{4.39c}$$

where the Poisson bracket $\{\cdot,\cdot\}$ *is defined as*

$$\{\bar{F},\bar{G}\} = \left\langle \frac{\delta \bar{F}}{\delta \bar{u}}, \partial_{\bar{x}} \frac{\delta \bar{G}}{\delta \bar{u}} \right\rangle \tag{4.40}$$

with scalar fields $\bar{F}$ *and* $\bar{G}$ *on* $\bar{M}$.

Proof. The relations (4.39) can be derived from (4.25). We only prove (4.39*b*), let us look at the relation

$$[\![\bar{K}_l, \bar{\sigma}_r]\!] = l(h\bar{K}_{l+r-1} + \bar{K}_{l+r-2}). \tag{4.41}$$

On one hand,

$$\partial_{\bar{x}}^{-1}[\![\bar{K}_l, \bar{\sigma}_r]\!] = \partial_{\bar{x}}^{-1}(\partial_{\bar{x}}\bar{\gamma}_l'\bar{\sigma}_r - \bar{\sigma}_r'\partial_{\bar{x}}\bar{\gamma}_l) = \bar{\gamma}_l'^{*}\bar{\sigma}_r + \bar{\sigma}_r'^{*}\bar{\gamma}_l = \mathrm{grad}\langle\bar{\gamma}_l, \bar{\sigma}_r\rangle,$$

where we have made use of $\bar{\gamma}_l' = \bar{\gamma}_l'^{*}$, (4.38*b*) and lemma 4.3. On the other hand, applying $\partial_{\bar{x}}^{-1}$ on the r.h.s. of (4.41) yields

$$l\partial_{\bar{x}}^{-1}(h\bar{K}_{l+r-1} + \bar{K}_{l+r-2}) = l(h\bar{\gamma}_{l+r-1} + \bar{\gamma}_{l+r-2}).$$

Thus we have

$$\langle\bar{\gamma}_l, \bar{\sigma}_r\rangle = l(h\bar{H}_{l+r-1} + \bar{H}_{l+r-2}).$$

Meanwhile, noting that

$$\{\bar{H}_l, \bar{J}_r\} = \left\langle \frac{\delta \bar{H}_l}{\delta \bar{u}}, \partial_{\bar{x}} \frac{\delta \bar{J}_r}{\delta \bar{u}} \right\rangle = \langle\bar{\gamma}_l, \partial_{\bar{x}}\bar{\omega}_r\rangle = \langle\bar{\gamma}_l, \bar{\sigma}_r\rangle,$$

we immediately obtain (4.39*b*). The relations (4.39*a*) and (4.39*c*) can be proved in a similar way. □

Now we come to the final theorem of this part.

Theorem 4.5. *In the isospectral D∆KP hierarchy (4.9), each equation*

$$\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u}) \tag{4.42}$$

has two sets of conserved quantities

$$\{\bar{H}_l = \bar{H}_l(\bar{u})\}, \qquad \{\bar{I}_r^s = \bar{I}_r^s(\bar{u}, \bar{t}_s) = s\bar{t}_s(h\bar{H}_{s+r-1}(\bar{u}) + \bar{H}_{s+r-2}(\bar{u})) + \bar{J}_r(\bar{u})\}, \tag{4.43}$$

where $\bar{H}_l$ *and* $\bar{J}_r$ *are defined in (4.33) and (4.37), respectively. They generate a Lie algebra w.r.t. the Poisson bracket (4.40) with basic structure*

$$\{\bar{H}_l, \bar{H}_r\} = 0, \tag{4.44a}$$

$$\{\bar{H}_l, \bar{I}_r^s\} = l(h\bar{H}_{l+r-1} + \bar{H}_{l+r-2}), \tag{4.44b}$$

$$\{\bar{I}_l^s, \bar{I}_r^s\} = (l-r)(h\bar{I}_{l+r-1}^s + \bar{I}_{l+r-2}^s), \tag{4.44c}$$

where $l, r, s \geqslant 1$ *and we set* $\bar{H}_0(\bar{u}) = \bar{I}_0^s(\bar{u}, \bar{t}_s) = 0$.

Proof. Noting that equation (4.42) has two sets of symmetries

$$\{\bar{K}_l\}, \qquad \{\bar{\tau}_r^s = s\bar{t}_s(h\bar{K}_{s+r-1} + \bar{K}_{s+r-2}) + \bar{\sigma}_r\} \tag{4.45}$$

and $\partial_{\bar{x}}$ is a Noether operator of equation (4.42) (namely $\partial_{\bar{x}}^{-1}$ maps symmetries to conserved covariants for (4.42)), these symmetries applied by $\partial_{\bar{x}}^{-1}$ give rise to

$$\{\bar{\gamma}_l = \partial_{\bar{x}}^{-1}\bar{K}_l\}, \qquad \{\bar{\vartheta}_r^s = \partial_{\bar{x}}^{-1}\bar{\tau}_r^s = s\bar{t}_s(h\bar{\gamma}_{s+r-1} + \bar{\gamma}_{s+r-2}) + \bar{\omega}_r\}, \tag{4.46}$$

which are conserved covariants of equation (4.42). Since it is already known that $\{\bar{\gamma}_l\}$ and $\{\bar{\vartheta}_r^s\}$ are all gradient fields, according to proposition 2.3, their potentials $\{\bar{H}_l\}$ and $\{\bar{I}_r^s\}$ are conserved quantities of equation (4.42).

Finally, the relation (4.44) can be easily verified by using the algebra (4.39). □

Conserved quantities of the equation $\bar{u}_{\bar{t}_s} = \bar{K}_s(\bar{u})$ can also be derived from proposition 2.2 about conserved covariants and symmetries; let us note that the definition of the Poisson bracket (4.40) readily gives the relations

$$\{\bar{H}_r, \bar{H}_l\} = \langle\bar{\gamma}_r, \bar{K}_l\rangle, \tag{4.47a}$$

$$\{\bar{I}_r^s, \bar{H}_l\} = \langle\bar{\vartheta}_r^s, \bar{K}_l\rangle, \tag{4.47b}$$

$$\{\bar{I}_r^s, \bar{I}_l^s\} = \langle\bar{\vartheta}_r^s, \bar{\tau}_l^s\rangle. \tag{4.47c}$$

5. Continuum limits

5.1. Backgrounds

Let us write the KP equation and the DΔKP equation below,

$$u_{t_3} = \tfrac{1}{4}u_{xxx} + 3uu_x + \tfrac{3}{4}\partial_x^{-1}u_{yy}, \tag{5.1}$$

$$\bar{u}_{\bar{t}_2} = (1 + 2\Delta^{-1})\bar{u}_{\bar{x}\bar{x}} - 2h^{-1}\bar{u}_{\bar{x}} + 2\bar{u}\bar{u}_{\bar{x}}. \tag{5.2}$$

Following Miwa's transformation, or in practice, comparing exponential parts in the solution of these two equations, one can introduce coordinates relation

$$x = \bar{x} + \tau, \qquad y = \bar{t}_2 - \frac{h}{2}\tau, \qquad t_3 = \frac{h^2}{3}\tau, \qquad \tau = nh. \tag{5.3}$$

The continuum limit is then conducted through replacing $\bar{u}$ by hu and taking $n \to \infty$ and $h \to 0$ simultaneously. The result is that the KP equation (5.1) appears as the leading term of the DΔKP equation (5.2). Similar relationship exists in non-commutative case [38].

However, the continuum limit (5.3) does not fit the whole DΔKP hierarchy. It also breaks those basic algebraic structures and the Hamiltonian structure of the DΔKP equation. In fact, to keep the Hamiltonian structure in a continuum limit, one at least needs $\bar{t}_m \propto t_m$. We need a new scheme for continuum limits.

5.2. Plan for continuum limits

Our plan for the continuum limit is the following.

- $n \to \infty$ and $h \to 0$ simultaneously such that nh is finite.
- Introduce auxiliary continuous variable[7]

$$\tau = nh, \tag{5.4}$$

  and thus, function $f(n + j)$ is mapped to $f(\tau + jh)$.
- Define coordinates relation

$$x = \bar{x} + \tau, \qquad y = -\tfrac{1}{2}h\tau, \qquad t_m = \bar{t}_m, \tag{5.5}$$

  under which one has

$$\partial_{\bar{x}} = \partial_x, \qquad \partial_\tau = \partial_x - \tfrac{1}{2}h\partial_y, \qquad \partial_{\bar{t}_m} = \partial_{t_m}. \tag{5.6}$$

- Define functions relation

$$\bar{u}_0(n, \bar{x}, \bar{t}_m) = \bar{u}(n, \bar{x}, \bar{t}_m) = hu(x, y, t_m), \tag{5.7a}$$

$$\bar{u}_j(n, \bar{x}, \bar{t}_m) = u_{j+1}(x, y, t_m), \qquad j = 1, 2, \cdots. \tag{5.7b}$$

[7] In fact, we can take $\tau = \tau_0 + nh$ with constant τ_0. Here we take $\tau_0 = 0$ for convenience and without loss of generality.

5.3. Pseudo-difference operator and the DΔKP equation

Under the continuum limit designed in the above subsection, the pseudo-difference operator $\bar{L}$ and pseudo-differential operator L satisfy

$$\bar{L} = L + O(h). \tag{5.8}$$

In fact, applying Δ on a test function $f(n)$ and making use of Taylor expansion, one finds

$$\begin{aligned}\Delta &= h\partial_\tau + \frac{1}{2!}h^2\partial_\tau^2 + \frac{1}{3!}h^3\partial_\tau^3 + O(h^4)\\ &= h\partial_x + \frac{h^2}{2}(\partial_x^2 - \partial_y) + \frac{h^3}{6}(\partial_x^3 - 3\partial_x\partial_y) + O(h^4).\end{aligned} \tag{5.9}$$

Consequently,

$$\Delta^{-1} = h^{-1}\partial_x^{-1} + \frac{1}{2}(\partial_x^{-2}\partial_y - 1) + h\left(\frac{1}{2}\partial_x^{-3}\partial_y^2 + \frac{1}{12}\partial_x\right) + O(h^2), \tag{5.10a}$$

$$\Delta^{-2} = h^{-2}\partial_x^{-2} + h^{-1}(\partial_x^{-3}\partial_y - \partial_x^{-1}) + \left(\frac{3}{4}\partial_x^{-4}\partial_y^2 - \frac{1}{2}\partial_x^{-2}\partial_y - \frac{5}{12}\right) + O(h), \tag{5.10b}$$

$$\cdots\cdots.$$

Thus, it is clear that

$$h^{-j}\Delta^j = \partial_x^j + O(h), \qquad j \in \mathbb{Z}. \tag{5.11}$$

Making use of (5.11) together with the relation (5.7) one immediately reaches (5.8).

Let us have a look at some lower order flows. In the continuum limit designed in section 5.2, we find

$$\bar{K}_1 = hu_x = hK_1, \tag{5.12a}$$

$$\bar{K}_2 = hu_y + O(h^2) = hK_2 + O(h^2), \tag{5.12b}$$

$$\bar{K}_3 = h\left(\frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}\partial_x^{-1}u_{yy}\right) + O(h^2) = hK_3 + O(h^2). \tag{5.12c}$$

In our continuum limit, it is not the so-called DΔKP equation $\bar{u}_{\bar{t}_2} = \bar{K}_2$ but the next member in the DΔKP hierarchy, namely $\bar{u}_{\bar{t}_3} = \bar{K}_3$ that goes to the continuous KP equation $u_{t_3} = K_3$.

In addition, for the first three non-isospectral flows, we find

$$\bar{\sigma}_1 = h(2yK_1) + O(h) = h\sigma_1 + O(h^2), \tag{5.13a}$$

$$\bar{\sigma}_2 = h(2yK_2 + xK_1 + 2u) + O(h^2) = h\sigma_2 + O(h^2), \tag{5.13b}$$

$$\bar{\sigma}_3 = h(2yK_3 + xK_2 + 2\partial_x^{-1}u_y - u_x) + O(h^2) = h\sigma_3 + O(h^2). \tag{5.13c}$$

Let us, taking (5.13*a*) as an example, explain how the variable y appears. In fact,

$$\bar{\sigma}_1 = h\bar{x}\bar{u}_{\bar{x}} = h^2(x-\tau)u_x = h^2xu_x + 2hyu_x = h\sigma_1 + O(h^2).$$

Thus, we have seen that, in our continuum limit, the first three DΔKP isospectral and non-isospectral flows go to their continuous counterparts and the leading terms are of $O(h)$.

5.4. Degrees

In order to investigate the continuum limit for the whole DΔKP hierarchies together with their integrability properties, let us introduce *degrees* of functions and operators (see [37]).

Definition 5.1. *Assume that, under the plan described in section 5.2, a function $\bar{f}(n, \bar{x}, \bar{t}_m)$ (or an operator $\bar{P}(\bar{u}, \Delta)$) can be expanded in a series in terms of h; in this case, the order of the leading term is called the degree of $\bar{f}(n, \bar{x}, \bar{t}_m)$ (or $\bar{P}(\bar{u}, \Delta)$) and denoted by* deg $\bar{f}$ *(or* deg $\bar{P}(\bar{u}, \Delta)$*).*

By this definition, from the previous discussion we immediately have

$$\deg \bar{L} = 0, \tag{5.14}$$
$$\deg \Delta^j = j, \qquad j \in \mathbb{Z}, \tag{5.15}$$
$$\deg \bar{u} = 1, \qquad \deg \bar{u}_j = 0, \quad j = 1, 2, \ldots \tag{5.16}$$

and

$$\deg \bar{K}_j = 1, \qquad \deg \bar{\sigma}_j = 1, \quad j = 1, 2, 3.$$

Hereafter in this paper, by continuum limit we mean the one we designed in section 5.2 without any confusion. Let us first give some properties about degrees of functions and operations. We particularly note that propositions 5.1–5.5 are still valid when the involved functions, i.e. $\bar{f}(\bar{u})$, $\bar{g}(\bar{u})$, $\bar{\gamma}(\bar{u})$, $\bar{\vartheta}(\bar{u})$, $\bar{H}(\bar{u})$ and $\bar{I}(\bar{u})$, are time dependent.

Proposition 5.1. *For the functions (or operators) $\bar{f}(\bar{u})$ and $\bar{g}(\bar{u})$, it holds that*

$$\deg \bar{f} \cdot \bar{g} = \deg \bar{f} + \deg \bar{g}, \tag{5.17a}$$
$$\deg (\bar{f} + \bar{g}) \geqslant \min\{\deg \bar{f}, \deg \bar{g}\}. \tag{5.17b}$$

Proposition 5.2. *For the vector fields $\bar{f}(\bar{u})$ and $\bar{g}(\bar{u})$ defined on $\bar{M}$ and satisfying $\bar{f}(\bar{u})|_{\bar{u}=0} = 0$ and $\bar{g}(\bar{u})|_{\bar{u}=0} = 0$, suppose that in the continuum limit*

$$\bar{f}(\bar{u}) = f(u)h^i + O(h^{i+1}), \qquad \bar{g}(\bar{u}) = g(u)h^j + O(h^{j+1}), \tag{5.18}$$

namely

$$\deg \bar{f} = i, \qquad \deg \bar{g} = j.$$

It then holds that

$$[\![\bar{f}(\bar{u}), \bar{g}(\bar{u})]\!]_{\bar{u}} = [\![f(u), g(u)]\!]_u\, h^{i+j-1} + O(h^{i+j}), \tag{5.19}$$
$$\deg [\![\bar{f}(\bar{u}), \bar{g}(\bar{u})]\!]_{\bar{u}} \geqslant \deg \bar{f}(\bar{u}) + \deg \bar{g}(\bar{u}) - 1. \tag{5.20}$$

Here the subscripts $\bar{u}$ and u indicate the commutator $[\![\cdot, \cdot]\!]$ are defined in terms of the Gâteaux derivatives w.r.t. $\bar{u}$ and u, respectively.

Proof. Noting that $\bar{u} = hu$, we have

$$\begin{aligned}
\bar{f}'[\bar{g}] &= \frac{\partial}{\partial\varepsilon}\bar{f}(\bar{u} + \varepsilon\bar{g}(\bar{u}))\Big|_{\varepsilon=0} \\
&= \frac{\partial}{\partial\varepsilon}\bar{f}(hu + \varepsilon(g(u)h^j + O(h^{j+1})))\Big|_{\varepsilon=0} \\
&= \frac{\partial}{\partial\varepsilon}\bar{f}(h(u + \varepsilon(g(u)h^{j-1} + O(h^j))))\Big|_{\varepsilon=0} \\
&= \frac{\partial}{\partial\varepsilon}\Big(f(u + \varepsilon(g(u)h^{j-1} + O(h^j)))h^i + \cdots\Big)\Big|_{\varepsilon=0} \\
&= f'[g]h^{i+j-1} + O(h^{i+j}).
\end{aligned} \tag{5.21a}$$

Similarly,

$$\bar{g}'[\bar{f}] = g'[f]h^{i+j-1} + O(h^{i+j}), \tag{5.21b}$$

which, together with (5.21*a*), yields (5.19). (5.20) is valid in light of (5.17*b*). □

Proposition 5.3. *For $\bar{f}(\bar{u}) \in \bar{S}^*$ and $\bar{g}(\bar{u}) \in \bar{S}$, if in the continuum limit,*

$$\bar{f}(\bar{u}) = f(u)h^i + O(h^{i+1}), \qquad \bar{g}(\bar{u}) = g(u)h^j + O(h^{j+1}),$$

then

$$\langle \bar{f}(\bar{u}), \bar{g}(\bar{u}) \rangle = \langle f(u), g(u) \rangle h^{i+j} + O(h^{i+j+1}), \tag{5.22a}$$

$$\deg \langle \bar{f}(\bar{u}), \bar{g}(\bar{u}) \rangle = \deg \bar{f}(\bar{u}) + \deg \bar{g}(\bar{u}). \tag{5.22b}$$

Here on lhs and rhs of (5.22a) the inner products are defined as in (2.18) for the semi-discrete case and (2.2) for the continuous case, respectively.

Proof. First,

$$\begin{aligned} \langle \bar{f}(\bar{u}), \bar{g}(\bar{u}) \rangle &= \frac{h^2}{2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \bar{f}(\bar{u}) \bar{g}(\bar{u}) \, \mathrm{d}\bar{x} \\ &= \frac{h^2}{2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(u)g(u)h^{i+j} + O(h^{i+j+1})) \, \mathrm{d}\bar{x} \\ &= \frac{h}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(u)g(u)h^{i+j} + O(h^{i+j+1})) \, \mathrm{d}\bar{x} \, \mathrm{d}\tau. \end{aligned}$$

Next, noting that the coordinates transformation (5.5) indicates the Jacobian

$$J = \frac{\partial(\bar{x}, \tau)}{\partial(x, y)} = -\frac{2}{h},$$

we have

$$\begin{aligned} \langle \bar{f}(\bar{u}), \bar{g}(\bar{u}) \rangle &= \frac{h}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(u)g(u)h^{i+j} + O(h^{i+j+1})) \, |J| \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(u)g(u)h^{i+j} + O(h^{i+j+1})) \, \mathrm{d}x \, \mathrm{d}y \\ &= \langle f(u), g(u) \rangle h^{i+j} + O(h^{i+j+1}). \end{aligned}$$

The proof is finished. □

Proposition 5.4. *In the continuum limit if*

$$\bar{\gamma}(\bar{u}) = \frac{\delta \bar{H}(\bar{u})}{\delta \bar{u}} = \gamma(u)h^i + O(h^{i+1}),$$

then we have

$$\deg \bar{H}(\bar{u}) = \deg \bar{\gamma}(\bar{u}) + 1. \tag{5.23}$$

In addition, if $\gamma(u)$ is also a gradient field and we recover its potential by

$$H(u) = \int_0^1 \langle \gamma(\lambda u), u \rangle \, \mathrm{d}\lambda, \tag{5.24}$$

then we have

$$\bar{H}(\bar{u}) = H(u)h^{i+1} + O(h^{i+2}), \qquad \gamma(u) = \frac{\delta H(u)}{\delta u}. \tag{5.25}$$

Proof. In light of proposition 5.3 and noting that

$$\bar{H}(\bar{u}) = \int_0^1 \langle \bar{\gamma}(\lambda\bar{u}), \bar{u}\rangle \mathrm{d}\lambda = h^{i+1}\int_0^1 \langle \gamma(\lambda u), u\rangle \mathrm{d}\lambda + O(h^{i+2}), \tag{5.26}$$

one has

$$\deg \bar{H}(\bar{u}) = \deg \bar{\gamma}(\bar{u}) + \deg \bar{u} = \deg \bar{\gamma}(\bar{u}) + 1.$$

If $\gamma(u)$ is a gradient field, after defining $H(u)$ in (5.24), from (5.26) we reach to (5.25). □

Proposition 5.5. *Suppose that in continuum limit*

$$\bar{\gamma}(\bar{u}) = \frac{\delta \bar{H}(\bar{u})}{\delta \bar{u}} = \gamma(u)h^i + O(h^{i+1}), \qquad \bar{\vartheta}(\bar{u}) = \frac{\delta \bar{I}(\bar{u})}{\delta \bar{u}} = \vartheta(u)h^j + O(h^{j+1}),$$

and both $\gamma(u)$ and $\vartheta(u)$ are still gradient fields. Then, according to proposition 5.4 we have

$$\bar{H}(\bar{u}) = H(u)h^{i+1} + O(h^{i+2}), \qquad \bar{I}(\bar{u}) = I(u)h^{j+1} + O(h^{j+2}) \tag{5.27}$$

with $\gamma(u) = \frac{\delta H(u)}{\delta u}$ and $\vartheta(u) = \frac{\delta I(u)}{\delta u}$ and

$$\{\bar{H}(\bar{u}), \bar{I}(\bar{u})\} = \{H(u), I(u)\}h^{i+j} + O(h^{i+j+1}), \tag{5.28a}$$

$$\deg\{\bar{H}(\bar{u}), \bar{I}(\bar{u})\} = \deg \bar{H}(\bar{u}) + \deg \bar{I}(\bar{u}) - 2. \tag{5.28b}$$

Proof. In light of proposition 5.3, one has

$$\begin{aligned}\{\bar{H}(\bar{u}), \bar{I}(\bar{u})\} = \langle \bar{\gamma}(\bar{u}), \partial_{\bar{x}}\bar{\vartheta}(\bar{u})\rangle &= \langle \gamma(u), \partial_x \vartheta(u)\rangle h^{i+j} + O(h^{i+j+1}) \\ &= \{H(u), I(u)\}h^{i+j} + O(h^{i+j+1}),\end{aligned}$$

which also indicates the degree relation (5.28*b*). □

Besides, the following lemmas will be helpful for investigating the degrees of $\bar{A}_m$, $\bar{B}_m$, $\bar{K}_j$ and $\bar{\sigma}_j$.

Lemma 5.1. *Suppose that $\bar{W}_m$ is a difference operator*

$$\bar{W}_m = \sum_{j=0}^{m} \bar{w}_j(\bar{\boldsymbol{u}})\Delta^{m-j}, \qquad \text{with } \bar{W}_m|_{\bar{\boldsymbol{u}}=\boldsymbol{0}} = 0$$

and $\bar{w}_j(\bar{\boldsymbol{u}})$ are polynomials in $\partial_{\bar{x}}^i(\Delta^k \bar{\boldsymbol{u}})$, $i, k \in \mathbb{Z}$. If $\bar{W}_m$ satisfies $[\bar{W}_m, \bar{L}] = 0$, then $\bar{W}_m = 0$.

Proof. Arrange the terms of $[\bar{W}_m, \bar{L}]$ in terms of Δ. The highest order term reads $(\Delta\bar{w}_0)\Delta^{m+1}$, which indicates $\Delta\bar{w}_0 = 0$. This yields $\bar{w}_0 = 0$ due to $\bar{W}_m|_{\bar{\boldsymbol{u}}=\boldsymbol{0}} = 0$. Thus, in the remains the highest order term is $(\Delta\bar{w}_1)\Delta^m$, which should be zero. Then we obtain $\bar{w}_1 = 0$ due to $\bar{W}_m|_{\bar{\boldsymbol{u}}=\boldsymbol{0}} = 0$. Repeating the procedure we will finally reach $\bar{W}_m = 0$. □

Similarly, we have

Lemma 5.2. *For the differential operator*

$$W_m = \sum_{j=0}^{m} w_j(\boldsymbol{u})\partial^{m-j}, \qquad \text{with } W_m|_{\boldsymbol{u}=\boldsymbol{0}} = 0$$

and $w_j(\boldsymbol{u})$ are polynomials in $\partial_x^i \boldsymbol{u}$, if $[W_m, L] = 0$, then $W_m = 0$.

Now let us present more results on degrees.

Proposition 5.6. *For the difference operator*

$$\bar{W}_m = \sum_{j=0}^{m} \bar{w}_j(\bar{\mathbf{u}})\Delta^{m-j}$$

with $\bar{w}_j(\bar{\mathbf{u}})$ *being polynomials in* $\partial_{\bar{x}}^i(\Delta^k \bar{\mathbf{u}}),\ i, k \in \mathbb{Z}$, *we have*

$$\deg\,[\bar{W}_m, \bar{L}] \geqslant \deg \bar{W}_m, \tag{5.29}$$

and if $\bar{W}_m|_{\bar{u}=\mathbf{0}} = 0$, *then*

$$\deg\,[\bar{W}_m, \bar{L}] = \deg \bar{W}_m, \tag{5.30}$$

Proof. (5.29) holds by virtue of proposition 5.1 and the fact $\deg \bar{L} = 0$. Let us prove (5.30). Suppose that

$$\bar{W}_m = p(\partial_x)h^s + O(h^{s+1}),$$

i.e.

$$\deg \bar{W}_m = s,$$

where $p(\partial_x)$ is some differential operator polynomial and $p(\partial_x) \neq 0$. Then one has

$$[\bar{W}_m, \bar{L}] = [p(\partial_x), L]h^s + O(h^{s+1})$$

with leading term $[p(\partial_x), L]$. If

$$\deg\,[\bar{W}_m, \bar{L}] > \deg \bar{W}_m, \tag{5.31}$$

then the leading term $[p(\partial_x), L]$ of $[\bar{W}_m, \bar{L}]$ has to be zero, i.e.

$$[p(\partial_x), L] = 0. \tag{5.32}$$

Noting that $\bar{W}_m|_{\bar{u}=\mathbf{0}} = 0$ indicates $p(\partial_x)|_{u=\mathbf{0}} = 0$, thus in light of lemma 5.2, from (5.32) one obtain $p(\partial_x) = 0$. This is contradictory to the assumption $\deg \bar{W}_m = s$, which means the assumption (5.31) is not correct, and consequently (5.30) holds. □

Proposition 5.7. *In the continuum limit, we have*

$$\deg \bar{A}_m = 0, \tag{5.33a}$$

$$\bar{A}_m = A_m + O(h). \tag{5.33b}$$

Proof. First, $\bar{A}_m$ can be written out in the following form

$$\bar{A}_m = \frac{\Delta^m}{h^m} + \sum_{j=1}^{m} \bar{a}_j \frac{\Delta^{m-j}}{h^{m-j}} = \partial_x^m + O(h) + \sum_{j=1}^{m} \bar{a}_j(\partial_x^{m-j} + O(h)). \tag{5.34}$$

Note that $\bar{A}_m = (\bar{L}^m)_+$. That is to say, $\bar{a}_j$ only contains shifted $\bar{u}_s$ without any integration terms (for example, $\Delta^{-1}\bar{u}_s$). That means that $\deg \bar{a}_j \geqslant 0$ for $j = 1, 2, \ldots m$, and therefore (5.33*a*) holds.

Next, in light of the above discussion, one can write $\bar{A}_m$ as

$$\bar{A}_m = A_m^{(0)} + O(h),$$

where $A_m^{(0)}$ is a pure differential operator independent of h and $A_m^{(0)}|_{u=\mathbf{0}} = \partial_x^m$. Now, from $\bar{L}_{\bar{t}_m} = [\bar{A}_m, \bar{L}]$ we have

$$L_{t_m} = [A_m^{(0)}, L] + O(h)$$

and taking $h \to 0$ it goes to

$$L_{t_m} = [A_m^{(0)}, L].$$

Finally, noting that $A_m^{(0)}|_{u=\mathbf{0}} = A_m|_{u=\mathbf{0}} = \partial_x^m$ and making use of lemma 5.2, we obtain $A_m^{(0)} = A_m$, namely (5.33*b*) holds. □

Proposition 5.8. *In the continuum limit, we have*

$$\deg \bar{B}_m = 0, \tag{5.35a}$$
$$\bar{B}_m = B_m + O(h). \tag{5.35b}$$

Proof. In light of lemma 5.1, $\bar{B}_m$ can be written out in the following form

$$\bar{B}_m = \bar{D}_m + \bar{C}_{m-2}, \tag{5.36}$$

where $\bar{D}_m = \bar{x}h\bar{A}_m + (\bar{x} + nh)\bar{A}_{m-1}$, $\bar{C}_{m-2}$ is a pure difference operator and $\bar{C}_{m-2}|_{\bar{\mathbf{u}}=\mathbf{0}} = 0$. Suppose that

$$\bar{C}_{m-2} = C^{(0)}_{m-2}h^s + O(h^{s+1}),$$

i.e. $\deg \bar{C}_{m-2} = s$. From (4.14*b*) we have

$$\bar{L}_{\bar{t}_m} = [\bar{D}_m, \bar{L}] + [\bar{C}_{m-2}, \bar{L}] + h\bar{L}^m + \bar{L}^{m-1}, \tag{5.37}$$

where for the term $[\bar{C}_{m-2}, \bar{L}]$ we have $\deg [\bar{C}_{m-2}, \bar{L}] = \deg \bar{C}_{m-2} = s$ due to $\bar{C}_{m-2}|_{\bar{\mathbf{u}}=\mathbf{0}} = 0$ and proposition 5.6, and the remaining terms altogether have degree zero. If $s < 0$, there must have $[C^{(0)}_{m-2}, L] = 0$, which yields $C^{(0)}_{m-2} = 0$ in light of lemma 5.2. This is in contradiction with the assumption $\deg \bar{C}_{m-2} = s$. Consequently, we must have $s \geqslant 0$. Thus, noting that $\deg \bar{D}_m = 0$, (5.35*a*) holds.

With (5.35*a*) in hand, we can write

$$\bar{B}_m = B^{(0)}_m + O(h), \tag{5.38}$$

where

$$B^{(0)}_m = 2yA_m + xA_{m-1} + C_{m-2},$$

and C_{m-2} is a pure differential operator independent of h, that satisfies $C_{m-2}|_{\mathbf{u}=\mathbf{0}} = 0$. Substituting (5.38) into (4.14*b*) and taking the leading terms we get

$$L_{t_m} = [B^{(0)}_m, L] + L^{m-1}.$$

Obviously, B_m and $B^{(0)}_m$ satisfy the same equation and have same asymptotic condition

$$B_m|_{\mathbf{u}=0} = B^{(0)}_m|_{\mathbf{u}=0} = 2y\partial_x^m + x\partial_x^{m-1},$$

which gives $B^{(0)}_m = B_m$ in light of lemma 5.2. Therefore the relation (5.35*b*) holds as well. □

Proposition 5.9. *In the continuum limit, we have*

$$\deg \bar{K}_m = 1, \tag{5.39a}$$
$$\bar{K}_m = hK_m + O(h^2). \tag{5.39b}$$

Proof. First, we would like to stress the following relation,

$$\bar{A}_1 = A_1 + \frac{h}{2}(A_2 - \partial_y) + O(h^2). \tag{5.40}$$

This can be derived by substituting (5.9) and $\bar{u}_0 = \bar{u} = hu$ into equation (4.7*a*) for $\bar{A}_1$. Actually, to derive (5.39) we need to make higher order expansions. Let us write

$$\bar{A}_m = A_m + A^{(1)}_m h + O(h^2). \tag{5.41}$$

Inserting (5.40) and (5.41) into the zero-curvature representation (4.19*a*) we obtain

$$\begin{aligned}\bar{K}_m &= \bar{A}_{m,\bar{x}} - [\bar{A}_1, \bar{A}_m] \\ &= \frac{h}{2}(A_{m,y} - [A_2, A_m]) + O(h^2) \\ &= hK_m + O(h^2),\end{aligned}$$

which proves (5.39). □

In a quite similar way, by using (5.40), (4.19*b*) and the expression

$$\bar{B}_m = B_m + B_m^{(1)}h + O(h^2), \tag{5.42}$$

we have the following.

Proposition 5.10. *In the continuum limit, we have*

$$\deg \bar{\sigma}_m = 1, \tag{5.43a}$$

$$\bar{\sigma}_m = h\sigma_m + O(h^2). \tag{5.43b}$$

5.5. Lax triads

From the previous discussions we have known that

$$\bar{L} = L + O(h), \tag{5.44a}$$

$$\bar{A}_1 = A_1 + \frac{h}{2}(A_2 - \partial_y) + O(h^2), \tag{5.44b}$$

$$\bar{A}_m = A_m + A_m^{(1)}h + O(h^2), \tag{5.44c}$$

$$\bar{B}_m = B_m + B_m^{(1)}h + O(h^2). \tag{5.44d}$$

Substituting them into the Lax triads and their compatibility equations given in section 4 we immediately reach the following results.

Proposition 5.11. *For the isospectral DΔKP hierarchy we have*

$$\bar{L}\phi - \eta\phi = L\phi - \eta\phi + O(h), \tag{5.45a}$$

$$\phi_{\bar{x}} - \bar{A}_1\phi = \frac{h}{2}(\phi_y - A_2\phi) + O(h^2), \tag{5.45b}$$

$$\phi_{\bar{t}_m} - \bar{A}_m\phi = \phi_{t_m} - A_m\phi + O(h), \tag{5.45c}$$

and

$$\bar{L}_{\bar{x}} - [\bar{A}_1, \bar{L}] = \frac{h}{2}(L_y - [A_2, L]) + O(h^2), \tag{5.46a}$$

$$\bar{L}_{\bar{t}_m} - [\bar{A}_m, \bar{L}] = L_{t_m} - [A_m, L] + O(h), \tag{5.46b}$$

$$\bar{A}_{1,\bar{t}_m} - \bar{A}_{m,\bar{x}} + [\bar{A}_1, \bar{A}_m] = \frac{h}{2}(A_{2,t_m} - A_{m,y} + [A_2, A_m]) + O(h^2). \tag{5.46c}$$

Proposition 5.12. *For the non-isospectral DΔKP hierarchy we have*

$$\bar{L}\phi - \eta\phi = L\phi - \eta\phi + O(h), \tag{5.47a}$$

$$\phi_{\bar{x}} - \bar{A}_1\phi = \frac{h}{2}(\phi_y - A_2\phi) + O(h^2), \tag{5.47b}$$

$$\phi_{\bar{t}_m} - \bar{B}_m\phi = \phi_{t_m} - B_m\phi + O(h), \tag{5.47c}$$

and

$$\bar{L}_{\bar{x}} - [\bar{A}_1, \bar{L}] = \frac{h}{2}(L_y - [A_2, L]) + O(h^2), \tag{5.48a}$$

$$\bar{L}_{\bar{t}_m} - [\bar{B}_m, \bar{L}] - h\bar{L}^m - \bar{L}^{m-1} = L_{t_m} - [B_m, L] - L^{m-1} + O(h), \tag{5.48b}$$

$$\bar{A}_{1,\bar{t}_m} - \bar{B}_{m,\bar{x}} + [\bar{A}_1, \bar{B}_m] = \frac{h}{2}(A_{2,t_m} - B_{m,y} + [A_2, B_m]) + O(h^2). \tag{5.48c}$$

5.6. Symmetries and algebra deformation

We have shown that both isospectral DΔKP flows $\{\bar{K}_m\}$ and non-isospectral DΔKP flows $\{\bar{\sigma}_m\}$ go to their continuous counterparts in the continuum limit designed in section 5.2. However, comparing the basic algebra structures (3.23) and (4.25), one can see that the structures are different. In fact, this deformation in the basic structures can be well understood with the help of the degrees of the flows. Let us take (4.25*b*) and (3.23*b*) as an example. (4.25*b*) reads

$$[\![\bar{K}_l, \bar{\sigma}_r]\!] = l\,(h\bar{K}_{l+r-1} + \bar{K}_{l+r-2}). \tag{5.49}$$

Among the three terms of (5.49)

$$\deg [\![\bar{K}_l, \bar{\sigma}_r]\!] = 1, \qquad \deg\,(h\bar{K}_{l+r-1}) = 2, \qquad \deg \bar{K}_{l+r-2} = 1. \tag{5.50}$$

Noting that in the continuum limit only the terms with the lowest degrees (i.e. leading terms) remain, and comparing the degrees of each term of (5.49) we have

$$[\![K_l, \sigma_r]\!] = l\,K_{l+r-2}, \tag{5.51}$$

i.e. (3.23*b*). Such degree analysis works as well as in understanding the relationship of symmetries together with their algebras in semi-discrete and continuous cases. Let us conclude these relations in the following.

Theorem 5.1. *In continuum limit, the basic algebra structure (4.25) of flows goes to (3.23), symmetries given in (4.26) are related by*

$$\{\bar{K}_l\} \to \{K_l\}, \qquad \{\bar{\tau}_r^s\} \to \{\tau_r^s\},$$

and their basic structure (4.27) goes to (3.29).

5.7. Hamiltonian structures and conserved quantities

Now let us investigate the continuum limits of Hamiltonian structures and conserved quantities.

Since

$$\bar{K}_m = hK_m + O(h^2), \qquad \partial_{\bar{x}} = \partial_x,$$

it is easy to obtain

$$\bar{K}_m = \partial_{\bar{x}}\bar{\gamma}_m = h\,K_m + O(h^2) = h\,\partial_x\gamma_m + O(h^2),$$

namely

$$\bar{\gamma}_m = h\,\gamma_m + O(h^2), \tag{5.52}$$

and γ_m is still a gradient field. Then, following proposition 5.5 we obtain

$$\bar{H}_m = h^2 H_m + O(h^3).$$

We can conduct the similar discussion in the non-isospectral case and obtain similar results. To sum up, for the Hamiltonian structures we have the following.

Proposition 5.13. *The continuum limit designed in section 5.2 keeps the Hamiltonian structures of equations (4.31) and (4.34), in which*

$$\bar{\gamma}_m = h\gamma_m + O(h^2), \qquad \bar{H}_m = h^2 H_m + O(h^3), \tag{5.53a}$$

$$\bar{\omega}_m = h\omega_m + O(h^2), \qquad \bar{J}_m = h^2 J_m + O(h^3). \tag{5.53b}$$

Next we look at the basic algebraic structure (4.44) composed by the conserved quantities $\{\bar{H}_l\}$ and $\{\bar{I}_r^s\}$. We have seen that in the continuum limit $\bar{\gamma}_m(\bar{u})$ and $\bar{\omega}_m(\bar{u})$ go to $\gamma_m(u)$ and $\omega_m(u)$ that are still gradient fields. Noting that $\gamma_m(u) = \frac{\delta H_m(u)}{\delta u}$ and $\omega_m(u) = \frac{\delta J_m(u)}{\delta u}$, it then follows from proposition 5.5 that in the continuum limit

$$\{\bar{H}_l, \bar{H}_r\} = \{H_l, H_r\}h^2 + O(h^3),$$
$$\{\bar{H}_l, \bar{J}_r\} = \{H_l, J_r\}h^2 + O(h^3),$$
$$\{\bar{J}_l, \bar{J}_r\} = \{J_l, J_r\}h^2 + O(h^3).$$

We use the same trick as in the previous subsection for symmetries. By comparing the degrees of both sides of the basic algebraic relation (4.39), the leading terms yield

$$\{H_l, H_r\} = 0,$$
$$\{H_l, J_r\} = l H_{l+r-2},$$
$$\{J_l, J_r\} = (l-r) J_{l+r-2},$$

i.e. (3.38). This also leads to the basic algebraic relation (3.40). Let us conclude this in the following.

Theorem 5.2. *In the continuum limit designed in section 5.2, we have*

$$\bar{H}_l \to H_l, \qquad \bar{J}_r \to J_r, \qquad \bar{I}_r^s \to I_r^s, \tag{5.54}$$

the basic algebra structure (4.39) goes to (3.38) and the basic structure (4.44) goes to (3.40).

5.8. Deformation of Lie algebras

Now let us see something special of the obtained algebras. The Lie algebra $\bar{\boldsymbol{X}}$ spanned by the DΔKP flows $\{\bar{K}_m\}$ and $\{\bar{\sigma}_m\}$ with the basic structures (4.25) has generators $\{\bar{K}_1, \bar{\sigma}_1, \bar{\sigma}_3\}$ w.r.t. the commutator $[\![\cdot,\cdot]\!]$, while the Lie algebra $\boldsymbol{X}$ spanned by the KP flows $\{K_m\}$ and $\{\sigma_m\}$ with the basic structures (3.23) has generators $\{K_1, \sigma_1, \sigma_4\}$. Obviously, the two algebras have different basic structures: (3.23) is a neat centerless Kac–Moody–Virasoro structure but (4.25) is not. Now let us look at subalgebras. $\boldsymbol{X}$ has infinitely many subalgebras spanned by $\{K_1, K_2, \cdots, K_j, \sigma_1, \sigma_2\}$ for any $j \in \mathbb{Z}^+$; for $\bar{\boldsymbol{X}}$ it also has infinitely many subalgebras spanned by $\{\bar{K}_1, \bar{K}_2, \cdots, \bar{K}_j, \bar{\sigma}_1\}$ for any $j \in \mathbb{Z}^+$. Moreover, by calculating the degrees of the flows, the deformation of the basic algebraic structures in the continuum limit can be understood. However, the continuum limit does not keep generators and subalgebras. In fact, such discontinuity of Lie algebras of the flows (or symmetries), also known as the contraction of algebras, is not rarely seen in some semi-discrete cases when they go to their continuous correspondences in continuous limit [37, 39, 40]. Here, the spacing parameter h acts as a contraction parameter that brings changes of basic algebraic structures.

Since the basic algebraic structures (4.39) for Hamiltonians, (4.44) for conserved quantities and (4.25) for flows are the same, and the basic structures (3.38) for Hamiltonians, (3.40) for conserved quantities and (3.23) for flows are also the same, we conclude that these Lie algebras undergo the same deformation.

Finally, we end up this section by the following comment. We can see that all the analysis on degrees in section 5.4 are done with respect to a generic u on M and the continuum limit keeps equations and their integrability characteristics. Thus, we conclude that the continuum limit is performed at arbitrary points of the phase space but it keeps evolution equations (i.e. maps solutions of the semi-discrete case to those of the continuous case) and their integrability characteristics.

6. Conclusions

We have discussed integrability properties of the DΔKP hierarchy, including symmetries, Hamiltonian structures and conserved quantities. The obtained results have been listed out in section 1. To achieve them, we introduced Lax triads as our starting point. In this approach we consider the spatial variable $\bar{x}$ (y for the KP system) as a new independent variable that is completely independent of the temporal variable $\bar{t}_1$ (t_2 for the KP system). Such a separation of spatial and temporal variables not only enables us to derive master symmetries as non-isospectral flows but also provides simple zero-curvature representations for both isospectral and non-isospectral flows. Compared with the traditional treatments, we believe that the Lax triad approach would be more reasonable in the study of (2+1)-dimensional systems related to pseudo-difference operators and pseudo-differential operators. Besides, explicit recursion operators might exist and be used to investigate integrable (2+1)-dimensional systems [6, 41], but this is absent in discrete case.

Continuum limit acts as a bridge to connect discrete and continuous integrable systems. However, such connections usually are hidden behind integrable discretization [42–44]. It is not easy to find out a uniform continuum limit to connect both equations and their integrability properties [37, 45–47]. Sometimes combinatorics is used. In this paper we designed a continuum limit that connects the DΔKP and KP hierarchies. The continuum limit has been shown to keep their Lax triads, zero-curvature representations, Hamiltonian structures (for both isospectral and non-isospectral cases), symmetries and conserved quantities. We defined and made use of *degrees* of some elements to analyze continuum limits. By calculating and comparing degrees, the deformation of the basic algebraic structures in the continuum limit can be understood. We also want to emphasize that in our continuum limit the traditional DΔKP equation $\bar{u}_{\bar{t}_2} = \bar{K}_2$ goes to the linear equation $u_{t_2} = u_y$ instead of the KP equation. It is the next member $\bar{u}_{\bar{t}_3} = \bar{K}_3$ that corresponds to the continuous KP equation.

The pseudo-difference operator $\bar{L}$ is not a unique means for investigating the DΔKP hierarchy. In a series of papers [48–51] two semi-discrete KP hierarchies together with conserved quantities and Hamiltonian structures were investigated. In their approach a fully discrete KP is a starting point and the semi-discrete KP hierarchies were derived by introducing infinitely many continuous time variables in continuum limits. Infinitely many conserved quantities were derived from a time-independent scattering problem and the obtained conserved quantities can be treated as Hamiltonians to match the semi-discrete KP hierarchies under suitable poisson brackets. Here we have given more conserved quantities and more algebraic structures for the DΔKP hierarchy. The integrable master symmetry played an important role in our paper and in the continuum limit we have fixed time variables so that the continuum limit keeps Hamiltonian structures for the whole hierarchies.

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