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Solutions to the modified Korteweg–de Vries equation

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This is a continuation of [Notes on solutions in Wronskian form to soliton equations: Korteweg–de Vries-type, arXiv:nlin.SI/0603008]. In the present paper, we review solutions to the modified Korteweg–de Vries equation in terms of Wronskians. The Wronskian entry vector needs to satisfy a matrix differential equation set which contains complex operation. This fact makes the analysis of the modified Korteweg–de Vries to be different from the case of the Korteweg–de Vries equation. To derive complete solution expressions for the matrix differential equation set, we introduce an auxiliary matrix to deal with the complex operation. As a result, the obtained solutions to the modified Korteweg–de Vries equation are categorized into two types: solitons and breathers, together with their limit cases. Besides, we give rational solutions to the modified Korteweg–de Vries equation in Wronskian form. This is derived with the help of a Galilean transformed version of the modified Korteweg–de Vries equation. Finally, typical dynamics of the obtained solutions are analyzed and illustrated. We also list out the obtained solutions and their corresponding basic Wronskian vectors in the conclusion part.

Keywords: The modified Korteweg–de Vries equation; Wronskian; breathers; rational solutions; dynamics.

Mathematics Subject Classification 2010: 37K40, 35Q35, 35Q35

1. Introduction

It is well known that the modified Kordeweg-de Vries (mKdV) equation,

$$v_t + 6\varepsilon v^2 v_x + v_{xxx} = 0, \quad \varepsilon = \pm 1, \tag{1.1}$$

played an important role in the history of Soliton Theory. It was used to construct infinitely many conservation laws of the Korteweg–de Vries (KdV) equation [2], which triggered off the discovery of the Lax pair of the KdV equation and the breakthrough of the Inverse Scattering Transform (IST) [3]. The mKdV equation is also famous for its special soliton behavior, i.e. breathers. This equation, sometimes appearing as its Galilean transformed version,

$$V_t + 12\varepsilon v_0 V V_X + 6\varepsilon V^2 V_X + V_{XXX} = 0, \quad \varepsilon = \pm 1, \tag{1.2}$$

(referred to as the mixed KdV-mKdV equation), arose in many physics contexts, such as propagation of ultrashort few-optical cycle solitons in nonlinear media [4, 5], anharmonic lattices [6], Alfv́en waves [7], ion acoustic solitons [8–10], traffic jam [11, 12], Schottky barrier transmission lines [13], thin ocean jets [14, 15], internal waves [16, 17], heat pulses in solids [18], and so on. Besides, mathematically, the mKdV hierarchy (both positive order and negative order) exhibit interesting integrable structures (some pioneer investigation of this aspect can be found in [19–21]).

With regard to the exact solutions of the mKdV equation, many classical solving methods, such as Hirota's bilinear approach [22], the IST [23–25], commutation methods [26] and Wronskian technique [27–29] has been used to solve it. For more references on its solutions, one can refer to [26] and the references therein.

In general, for a soliton equation with bilinear form, its solutions can be expressed through a Wronskian by imposing certain conditions on the entry vector of the Wronskian [30–32, 1]. For convenience, in the following we refer to such conditions as condition equation set (CES). Usually for an (1+1)-dimensional soliton equation the crucial role in its CES is a coefficient matrix and this matrix and its any similar form leads to same solutions for the corresponding soliton equation. Thus it is possible to give a complete classification (or structure) for the solutions of the soliton equation according to the canonical form of the coefficient matrix [32, 1]. It has been understood that the solutions generated from a coefficient matrix in Jordan form are related to the solutions generated from a diagonal coefficient matrix via some limiting procedure [1]. Therefore the former solutions can be referred to as limit solutions. From the viewpoint of the IST, N solitons are identified by N distinct eigenvalues of the corresponding spectral problem, or in other words, N distinct simple poles $\{k_j\}$ of the transparent coefficient $\frac{1}{a(k)}$. When $\{k_j\}$ are multiple-poles, the related multiple-pole solutions can be obtained through a limiting procedure like $k_2 \to k_1$ from simple-pole solutions. This limiting procedure is more easily realized for the solutions in Wronskian form [1]. Such a procedure is also helpful to understand the dynamics of the limit solutions [33–35].

In [1] we mentioned four topics related to solutions in Wronskian form: to find the CES, to solve the CES, to describe relations between different kinds of solutions, and to discuss dynamics of the solutions. In the present paper, following this line, we will review the Wronskian solutions of the mKdV ($\varepsilon = 1$) equation

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{1.3}$$

together with its Galilean transformed version,

$$V_t + 12 v_0 V V_X + 6V^2 V_X + V_{XXX} = 0. \tag{1.4}$$

The main results of the paper are the following.

- The CES of the mKdV equation can be given by

$$\varphi_x = \mathbb{B}\bar{\varphi}, \tag{1.5a}$$

$$\varphi_t = -4\varphi_{xxx}, \tag{1.5b}$$

  where φ is the Nth order Wronskian entry vector, bar stands for complex conjugate and $\mathbb{B}$ is a non-trivial $N \times N$ constant complex matrix. We note that there is a complex conjugate involved in the CES and this makes difficulties when solving the CES.
- There are only two kinds of solutions led by the CES (1.5): Solitons (together with their limit case) and breathers (together with their limit case). No rational solution arises from (1.5) due to $|\mathbb{B}| \neq 0$.
- Rational solutions to the mKdV equation (1.3) can be derived with the help of the KdV-mKdV equation (1.4).
- Dynamics of obtained solutions are analyzed and illustrated.

The paper is organized as follows. In Sec. 2, we give a general CES of the mKdV equation and simplify the CES by introducing an auxiliary equation. Then in Sec. 3, we solve the CES and classify the solutions into solitons and breathers. In Sec. 4, we derive rational solutions to the mKdV equation. This is done with the help of the KdV-mKdV equation (1.4). Section 5 consists of dynamic analysis and illustrations. Finally, in the conclusion section we list out the obtained solutions and their corresponding basic Wronskian vectors.

2. Wronskian Solutions of the mKdV Equation

2.1. *Preliminary*

A $N \times N$ Wronskian is defined as

$$W(\phi_1, \phi_2, \ldots, \phi_N) = |\phi, \phi^{(1)}, \ldots, \phi^{(N-1)}| = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \tag{2.1}$$

where $\phi_j^{(l)} = \partial^l \phi_j / \partial x^l$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$ is called the entry vector of the Wronskian. Usually we use the compact form [30]

$$W(\phi) = |\phi, \phi^{(1)}, \ldots, \phi^{(N-1)}| = |0, 1, \ldots, N-1| = |\widehat{N-1}|, \tag{2.2}$$

where $\widehat{N-j}$ indicates the set of consecutive columns $0, 1, \ldots, N-j$. In the paper we also employ the notation $\widetilde{N-j}$ to indicate the set of consecutive columns $1, 2, \ldots, N-j$.

A Wronskian can provide simple forms for its derivatives and this advantage admits direct verification of solutions that are expressed in terms of Wronskians. The following matrix properties are usually necessary in Wronskian verification.

Proposition 2.1 ([36, 1]). *Suppose that Ξ is a $N\times N$ matrix with column vector set $\{\Xi_j\}$; Ω is a $N\times N$ operator matrix with column vector set $\{\Omega_j\}$ and each entry $\Omega_{j,s}$ being an operator. Then we have*

$$\sum_{j=1}^{N}|\Omega_j*\Xi|=\sum_{j=1}^{N}|(\Omega^T)_j*\Xi^T|, \tag{2.3}$$

where for any Nth-order column vectors A_j and B_j we define

$$A_j\circ B_j=(A_{1,j}B_{1,j},A_{2,j}B_{2,j},\ldots,A_{N,j}B_{N,j})^T \tag{2.4}$$

and

$$|A_j*\Xi|=|\Xi_1,\ldots,\Xi_{j-1},A_j\circ\Xi_j,\Xi_{j+1},\ldots,\Xi_N|. \tag{2.5}$$

Proposition 2.2 ([30]). *Suppose that D is a $N\times(N-2)$ matrix and a,b,c,d are Nth-order column vectors, then*

$$|D,a,b||D,c,d|-|D,a,c||D,b,d|+|D,a,d||D,b,c|=0. \tag{2.6}$$

Besides, to write solutions of CES in simple forms one may make use of lower triangular Toeplitz matrices. A Nth-order lower triangular Toeplitz matrix is a matrix in the following form

$$\mathcal{A}=\begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 & 0\\ a_1 & a_0 & 0 & \cdots & 0 & 0\\ a_2 & a_1 & a_0 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_1 & a_0\end{pmatrix}_{N\times N}, \quad a_j\in\mathbb{C}. \tag{2.7}$$

All such matrices form a commutative semigroup $\widetilde{G}_N(\mathbb{C})$ with identity with respect to matrix multiplication and inverse, and the set $G_N(\mathbb{C})=\{\mathcal{A}|\mathcal{A}\in\widetilde{G}_N(\mathbb{C}),|\mathcal{A}|\neq0\}$ is an Abelian group. Besides (2.7), we will also need the following block lower triangular Toeplitz matrix,

$$\mathcal{A}^B=\begin{pmatrix} A_0 & 0 & 0 & \cdots & 0 & 0\\ A_1 & A_0 & 0 & \cdots & 0 & 0\\ A_2 & A_1 & A_0 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_1 & A_0\end{pmatrix}_{2N\times 2N}, \tag{2.8}$$

where $A_j=\begin{pmatrix} a_{j1} & 0\\ 0 & a_{j2}\end{pmatrix}$ and $\{a_{js}\}$ are arbitrary complex numbers. All such block matrices also compose a commutative semigroup with identity and we denote it by $\widetilde{G}^B_{2N}(\mathbb{C})$; and the set $G^B_{2N}(\mathbb{C})=\{\mathcal{A}^B|\mathcal{A}^B\in\widetilde{G}^B_{2N}(\mathbb{C}),|\mathcal{A}^B|\neq0\}$ is an Abelian group. If all the elements are real, then we correspondingly denote the above mentioned matrix sets by $\widetilde{G}_N(\mathbb{R})$, $G_N(\mathbb{R})$, $\widetilde{G}^B_{2N}(\mathbb{R})$ and $G^B_{2N}(\mathbb{R})$, respectively. For more properties of such matrices, please refer to [1].

2.2. *CES of the mKdV equation*

By the transformation

$$v = i\left(\ln \frac{\bar f}{f}\right)_x = i\frac{\bar f_x f - \bar f f_x}{\bar f f}, \tag{2.9}$$

the mKdV equation (1.3) can be bilinearized as [22]

$$(D_t + D_x^3)\bar f \cdot f = 0, \tag{2.10a}$$

$$D_x^2 \bar f \cdot f = 0, \tag{2.10b}$$

where i is the imaginary unit, $\bar f$ is the complex conjugate of f, and D is the well-known Hirota's bilinear operator defined by [37, 38]

$$D_t^m D_x^n a(t,x)\cdot b(t,x) = \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n} a(t+s, x+y) b(t-s, x-y)|_{s=0,y=0},$$
$$m, n = 0, 1, \ldots.$$

To obtain the bilinear form (2.10), one needs first to rewrite the mKdV equation (1.3) into its potential form

$$w_t + 2(w_x)^3 + w_{xxx} = 0, \quad (v = w_x), \tag{2.11}$$

and then substitute $w = i \ln \frac{\bar f}{f}$ into it. It turns out that

$$\bar f f_t - f \bar f_t + \bar f f_{xxx} - f \bar f_{xxx} - \frac{3}{f\bar f}[\bar f f_x(\bar f f_{xx} - 2\bar f_x f_x) - f\bar f_x(f\bar f_{xx} - 2 f_x \bar f_x)] = 0.$$

This form can be rewritten in terms of the Hirota bilinear operator D as the following,

$$(D_t + D_x^3) f \cdot \bar f - \frac{3}{f\bar f}(D_x f\cdot \bar f)(D_x^2 f \cdot \bar f) = 0,$$

which implies the bilinearisation (2.10).

The bilinear mKdV equation (2.10) admits a solution f in Wronskian form.

Theorem 2.1. *A Wronskian solution to the bilinear mKdV equation* (2.10) *is given as*

$$f = W(\phi) = |\widehat{N-1}|, \tag{2.12}$$

provided that its entry vector ϕ satisfies

$$\phi_x = B(t)\bar\phi, \tag{2.13a}$$

$$\phi_t = -4\phi_{xxx} + C(t)\phi, \tag{2.13b}$$

where $B(t) = (B_{ij}(t))_{N\times N}$ and $C(t) = (C_{ij}(t))_{N\times N}$ are two $N\times N$ matrices of t but independent of x, and satisfy

$$|B(t)| \neq 0, \quad |\bar B(t)|_t = 0, \quad \mathrm{tr}(C(t)) \in \mathbb{R}(t), \tag{2.14a}$$

$$B_t(t) + B(t)\bar C(t) = C(t)B(t). \tag{2.14b}$$

The proof is given in Appendix A.

2.3. *Simplification of the CES* (2.13)

To solve the CES (2.13) with arbitrary $B(t)$ and $C(t)$ which satisfy (2.14), we first introduce a non-singular $N \times N$ complex matrix $H(t) \in \mathbb{C}_{N\times N}(t)$ such that ([39], also see [1])

$$H_t(t) = -H(t)C(t). \tag{2.15}$$

By $H(t)$ we then introduce a new Wronskian entry vector

$$\psi = H(t)\phi, \tag{2.16}$$

which transfers the CES (2.13) to the following,

$$\psi_x = \widetilde{B}\bar{\psi}, \tag{2.17a}$$

$$\psi_t = -4\psi_{xxx}, \tag{2.17b}$$

where $\widetilde{B} = H(t)B(t)\bar{H}^{-1}(t)$. $\widetilde{B}$ has to be a constant matrix independent of both x and t due to the compatibility condition (2.14b) (noting that now $C(t) = 0$ in (2.17b)).

We note that the Wronskians composed by ϕ and ψ, which we respectively denote by $f(\phi)$ and $f(\psi)$, yield same solutions to the mKdV equation through the transformation (2.9) due to $f(\psi) = |H(t)|f(\phi)$. That means in the following one only needs to focus on the CES (2.17). However, since there exists a complex operation in (2.17), solutions can not be classified in terms of the canonical form of $\widetilde{B}$, as done in [32, 1]. To overcome the difficulty we introduce an auxiliary equation

$$\psi_{xx} = \widetilde{A}\psi, \tag{2.18}$$

where $\widetilde{A} = \widetilde{B}\bar{\widetilde{B}}$. Our plan now is to first solve the equation set composed by (2.18) and (2.17b), which is nothing but the CES of the KdV equation and have been well studied in [32, 1]. Then in the second step we impose the condition (2.17a) on the obtained solution ψ and finally get solutions for (2.17).

In addition, we can prove that $\widetilde{A}$ and its any similar form generate same solutions to the mKdV equation. In fact, suppose that $\mathbb{A} = P^{-1}\widetilde{A}P$ where P is the similarity transformation matrix. By introducing

$$\varphi = P^{-1}\psi, \quad \mathbb{B} = P^{-1}\widetilde{B}\bar{P},$$

we write (2.18) and (2.17) as

$$\varphi_{xx} = \mathbb{A}\varphi, \tag{2.19a}$$

$$\varphi_x = \mathbb{B}\bar{\varphi}, \tag{2.19b}$$

$$\varphi_t = -4\varphi_{xxx}. \tag{2.19c}$$

As a consequence, first, $\mathbb{A}$ and $\mathbb{B}$ are connected through

$$\mathbb{A} = \mathbb{B}\bar{\mathbb{B}}. \tag{2.20}$$

In addition, it is easy to find the Wronskians generated by ψ and φ satisfy

$$f(\psi) = |P|f(\varphi),$$

which means ψ determined by (2.18) and (2.17) and φ determined by (2.19) lead to same solutions to the mKdV equation (1.3) through the transformation (2.9). Moreover, based on the above analysis, hereafter we only need to focus on the CES (2.19) and we only need to deal with the case where $\mathbb{A}$ is in canonical form.

In the next section we will see that solutions to the mKdV equation can be classified in terms of the canonical form of $\mathbb{A}$ (rather than the canonical form of $\mathbb{B}$).

3. Solutions of the mKdV Equation

In this section, we list several possible choices of the matrix $\mathbb{A}$ and derive the related Wronskian entry vectors, one of which is for breathers. We will also discuss the limiting relationship of some solutions.

First, for the matrix $\mathbb{A}$ defined by (2.20), we have the following result on its eigenvalues.

Proposition 3.1. *The eigenvalues of* $\mathbb{A}$ *defined by* (2.20) *are either real or*, *if there are some complex ones*, *appear as conjugate pairs.*

We leave the proof in Appendix B.

According to the eigenvalues of $\mathbb{A}$, we can categorize solutions to the mKdV equation as solitons and breathers, which correspond to real eigenvalues and complex eigenvalues of conjugate pairs, respectively.

3.1. *Solitons*

Case I. Solitons: When $\mathbb{A}$ has N distinct real positive eigenvalues $\{\lambda_j^2\}$, its canonical form reads

$$\mathbb{A} = \mathrm{Diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_N^2), \tag{3.1}$$

where, for convenient to discuss, we let $\lambda_j = \varepsilon_j \|k_j\| \neq 0$, in which $\varepsilon_j = \pm 1$ and k_j can either real or complex numbers with distinct absolute values. We consider two subcases.

(1) $k_j \in \mathbb{R}$, i.e.

$$\mathbb{A} = \mathrm{Diag}(k_1^2, k_2^2, \ldots, k_N^2). \tag{3.2}$$

Following the relation (2.20), we can take

$$\mathbb{B} = \mathrm{Diag}(\pm k_1, \pm k_2, \ldots, \pm k_N). \tag{3.3}$$

We neglect the sign $\pm$ because this can be compensated by the arbitrariness of k_j. Therefore, we take

$$\mathbb{B} = \mathrm{Diag}(k_1, k_2, \ldots, k_N), \quad k_j \in \mathbb{R}. \tag{3.4}$$

For the above matrix $\mathbb{B}$, the solution to the CES (2.19) can be

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_N)^T, \tag{3.5a}$$

where

$$\varphi_j = a_j^+ e^{\xi_j} + i a_j^- e^{-\xi_j}, \quad \xi_j = k_j x - 4k_j^3 t + \xi_j^{(0)}, \quad a_j^+, a_j^-, k_j, \xi_j^{(0)} \in \mathbb{R}. \tag{3.5b}$$

When $a_j^+ = (-1)^{j-1}, a_j^- = 1$, (3.5b) reads

$$\varphi_j = (-1)^{j-1} e^{\xi_j} + i e^{-\xi_j}. \tag{3.6}$$

In this case, the corresponding Wronskian solution $f(\varphi)$ can be written as [40]

$$f = \left(\prod_{j=1}^{N} e^{\xi_j}\right)\left(\prod_{1\le j<l\le N}(k_j - k_l)\right) \times \sum_{\mu=0,1} \exp\left\{\sum_{j=1}^{N} \mu_j \left(2\eta_j + \frac{\pi}{2} i\right) + \sum_{1\le j<l\le N} \mu_j \mu_l A_{jl}\right\},$$

where the sum over $\mu = 0, 1$ refers to each of $\mu_j = 0, 1$ for $j = 1, 2, \ldots, N$, and

$$\eta_j = -\xi_j - \frac{1}{4}\sum_{l=1, l\neq j}^{N} A_{jl}, \quad e^{A_{jl}} = \left(\frac{k_l - k_j}{k_l + k_j}\right)^2.$$

This coincides with the N-soliton solution in Hirota's exponential polynomial form given in [41].

(2) $k_j = k_{j1} + i k_{j2} \in \mathbb{C}$, i.e.

$$\mathbb{A} = \mathrm{Diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_N^2), \quad \lambda_j^2 = k_{j1}^2 + k_{j2}^2. \tag{3.7}$$

In this case we have

$$\mathbb{B} = \mathrm{Diag}(k_1, k_2, \ldots, k_N), \quad k_j \in \mathbb{C}, \tag{3.8}$$

and the solution to the CES (2.19) can be given by (3.5a) with

$$\varphi_j = \gamma_j (a_j^+ e^{\xi_j} + i a_j^- e^{-\xi_j}), \quad \xi_j = \lambda_j x - 4\lambda_j^3 t + \xi_j^{(0)}, \quad a_j^+, a_j^-, \xi_j^{(0)} \in \mathbb{R}, \tag{3.9a}$$

where

$$\gamma_j = 1 + \frac{i(\lambda_j - k_{j1})}{k_{j2}}, \quad \lambda_j = \varepsilon_j \|k_j\|. \tag{3.9b}$$

Now, if we compare (3.5b) and (3.9a), we can find that both of them provide same solution to the mKdV equation through the transformation (2.9). In particular, when $k_{j1} = 0$, i.e.

$$\mathbb{B} = \mathrm{Diag}(i k_{12}, i k_{22}, \ldots, i k_{N2}), \tag{3.10}$$

and we take $\lambda_j = k_{j2}$, the entry function (3.9a) reduces to

$$\varphi_j = (1+i)(a_j^+ e^{\xi_j} + i a_j^- e^{-\xi_j}), \quad \xi_j = k_j x - 4k_j^3 t + \xi_j^{(0)}, \quad a_j^+, a_j^-, \xi_j^{(0)} \in \mathbb{R}. \tag{3.11}$$

Such $\{\phi_j\}$ were first given by Nimmo and Freeman [28] as Wronskian entries for soliton solutions.

Let us remark this case as follows.

Remarks.

- When $\mathbb{B}$ in the CES (2.19b) and (2.19c) is a diagonal matrix (3.8) of which the diagonal elements have different absolute values, no matter these diagonal elements are real or complex, the related Wronskian $f(\varphi)$ generates N-soliton solutions to the mKdV equation.
- Besides, we specify the non-degenerate condition[a]

$$\|k_i\| \neq \|k_j\|, \quad (i \neq j), \tag{3.12}$$

i.e. k_i and k_j $(i \neq j)$ do not appear on the same circle with the original point as the center of the circle. Otherwise, the solution degenerates.

Case II. Limit Solutions of Solitons: Corresponding to Case I, we discuss two subcases.
(1) Let

$$\mathbb{A} = \begin{pmatrix} k_1^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2k_1 & k_1^2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2k_1 & k_1^2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 2k_1 & k_1^2 \end{pmatrix}_N, \quad k_1 \in \mathbb{R}. \tag{3.13}$$

In this subcase, general solution to the equation set (2.19a) and (2.19c) is [1]

$$\hat{\phi} = \mathcal{A}\mathcal{Q}_0^+ + \mathcal{B}\mathcal{Q}_0^-, \quad \mathcal{A}, \mathcal{B} \in \widetilde{G}_N(\mathbb{C}), \tag{3.14}$$

where

$$\mathcal{Q}_0^\pm = (\mathcal{Q}_{0,0}^\pm, \mathcal{Q}_{0,1}^\pm, \ldots, \mathcal{Q}_{0,N-1}^\pm)^T, \quad \mathcal{Q}_{0,s}^\pm = \frac{1}{s!}\partial_{k_1}^s e^{\pm\xi_1}, \tag{3.15}$$

ξ_1 is defined in (3.5b), $\widetilde{G}_N(\mathbb{C})$ is the commutative set of all the Nth-order lower triangular Toeplitz matrices, see Sec. 2.1.

For the matrix $\mathbb{B}$ satisfying (2.20) we take

$$\mathbb{B} = \begin{pmatrix} k_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & k_1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & k_1 \end{pmatrix}_N. \tag{3.16}$$

[a]The above non-degenerate relation can also be described as follows. Define the equivalent relation $\sim$ on the complex plane $\mathbb{C}$:

$$k_i \sim k_j \quad \text{iff } \|k_i\| = \|k_j\|.$$

The quotient space $\mathbb{C}/\!\sim$ denotes the positive half real axis. Then to get a non-degenerate soliton solution one needs to take $k_i \nsim k_j$, $(i \neq j)$.

Then, substituting into (2.19b) yields

$$\mathcal{A}\mathcal{Q}_{0,x}^{+} + \mathcal{B}\mathcal{Q}_{0,x}^{-} = \mathbb{B}(\bar{\mathcal{A}}\mathcal{Q}_0^{+} + \bar{\mathcal{B}}\mathcal{Q}_0^{-}). \tag{3.17}$$

Meanwhile, it can be verified that

$$\mathcal{Q}_{0,x}^{+} = \mathbb{B}\mathcal{Q}_0^{+}, \quad \mathcal{Q}_{0,x}^{-} = -\mathbb{B}\mathcal{Q}_0^{-}. \tag{3.18}$$

Noting that $\mathbb{B} \in \widetilde{G}_N(\mathbb{R})$, substituting (3.18) into (3.17), and making use of the commutative property of $\widetilde{G}_N(\mathbb{C})$, one finds

$$\mathbb{B}(\mathcal{A}\mathcal{Q}_0^{+} - \mathcal{B}\mathcal{Q}_0^{-}) = \mathbb{B}(\bar{\mathcal{A}}\mathcal{Q}_0^{+} + \bar{\mathcal{B}}\mathcal{Q}_0^{-}). \tag{3.19}$$

Then, compared with (3.17) we immediately get

$$\mathcal{A} = \bar{\mathcal{A}}, \quad \mathcal{B} = -\bar{\mathcal{B}}, \tag{3.20}$$

which means $\mathcal{A}$ is real and $\mathbb{B}$ pure imaginary. Finally, the solution to the CES (2.19) can be described as

$$\varphi = \mathcal{A}^{+}\mathcal{Q}_0^{+} + i\mathcal{A}^{-}\mathcal{Q}_0^{-}, \quad \mathcal{A}^{\pm} \in \widetilde{G}_N(\mathbb{R}). \tag{3.21}$$

We note that solutions generated from (3.21) can also be derived from the solution given in Case I.(1) by a limiting procedure (cf. [1]). Let us explain this procedure by starting from the following Wronskian

$$f(\varphi) = \frac{W(\varphi_1, \varphi_2, \ldots, \varphi_N)}{\prod_{j=2}^{N}(k_j - k_1)^{j-1}} \tag{3.22}$$

with $\varphi_1 = \varphi_1(k_1, x, t) = a_1^{+}e^{\xi_1} + ia_1^{-}e^{-\xi_1}$ as defined in (3.5b) and $\varphi_j = \varphi_1(k_j, x, t)$. (3.22) gives a Wronskian solution to the bilinear mKdV equation (2.10). Taking the limit $k_j \to k_1$ successively for $j = 2, 3, \ldots, N$ and using the L'Hospital rule, the Wronskian (3.22) goes to a Wronskian $W(\varphi)$ with entry vector (3.21), where the arbitrary coefficient matrices $\mathcal{A}, \mathcal{B}$ can come from $a_1^{\pm}$ by considering $a_1^{\pm}$ to be some polynomials of k_1 (see [1] for more details for the case of the KdV equation).

(2) Corresponding to Case I.(2), let us consider

$$\mathbb{A} = \begin{pmatrix} k_{11}^2 + k_{12}^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2k_{12} & k_{11}^2 + k_{12}^2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2k_{12} & k_{11}^2 + k_{12}^2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 2k_{12} & k_{11}^2 + k_{12}^2 \end{pmatrix}_N, \tag{3.23}$$

where $k_{11}, k_{12} \in \mathbb{R}$ and $k_{12} \neq 0$. It then follows from (2.20) that one can take

$$\mathbb{B} = \begin{pmatrix} -ik_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -ik_1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -ik_1 \end{pmatrix}_N, \quad k_1 = k_{11} + ik_{12}. \tag{3.24}$$

For the matrix $\mathbb{A}$ defined by (3.23), the general solution to the equation set (2.19a) and (2.19c) can be written as

$$\varphi = \mathcal{A}\mathcal{P}_0^+ + \mathcal{B}\mathcal{P}_0^-, \quad \mathcal{A}, \mathcal{B} \in \widetilde{G}_N(\mathbb{C}), \tag{3.25}$$

where[b]

$$\mathcal{P}_0^\pm = (\mathcal{P}_{0,0}^\pm, \mathcal{P}_{0,1}^\pm, \ldots, \mathcal{P}_{0,N-1}^\pm)^T, \quad \mathcal{P}_{0,s}^\pm = \frac{1}{s!}\partial_{k_{12}}^s e^{\pm\xi_1}, \tag{3.26a}$$

$$\xi_1 = \lambda_1 x - 4\lambda_1^3 t + \xi_1^{(0)}, \quad \lambda_1 = \sqrt{k_{11}^2 + k_{12}^2}. \tag{3.26b}$$

Next, we substitute (3.25) together with (3.26) into (2.19b) so that we identify $\mathcal{A}, \mathcal{B}$ for (2.19b). This substitution yields

$$\mathcal{A}\mathcal{P}_{0,x}^+ + \mathcal{B}\mathcal{P}_{0,x}^- = \mathbb{B}(\bar{\mathcal{A}}\mathcal{P}_0^+ + \bar{\mathcal{B}}\mathcal{P}_0^-) = \bar{\mathcal{A}}\mathbb{B}\mathcal{P}_0^+ + \bar{\mathcal{B}}\mathbb{B}\mathcal{P}_0^-, \tag{3.27}$$

where we have made use of the commutative property of $\widetilde{G}_N(\mathbb{C})$. Then, noting that

$$\mathcal{P}_{0,x}^+ = W\mathcal{P}_0^+, \quad \mathcal{P}_{0,x}^- = -W\mathcal{P}_0^-, \tag{3.28}$$

where

$$W = \begin{pmatrix} w_0 & 0 & 0 & \cdots & 0 & 0 \\ w_1 & w_0 & 0 & \cdots & 0 & 0 \\ w_2 & w_1 & w_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{N-1} & w_{N-2} & w_{N-3} & \cdots & w_1 & w_0 \end{pmatrix}_N, \quad w_j = \frac{1}{j!}\partial_{k_{12}}^j \lambda_1, \tag{3.29}$$

and substituting (3.28) into (3.27), we obtain the relation

$$\mathcal{A}W = \bar{\mathcal{A}}\mathbb{B}, \quad -\mathcal{B}W = \bar{\mathcal{B}}\mathbb{B}. \tag{3.30}$$

To have a clearer result, we write

$$\mathcal{A} = \mathcal{A}_1 + i\mathcal{A}_2, \quad \mathcal{B} = \mathcal{B}_1 + i\mathcal{B}_2, \quad \mathbb{B} = \mathbb{B}_1 + i\mathbb{B}_2, \tag{3.31}$$

where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ are in $\widetilde{G}_N(\mathbb{R})$ and

$$\mathbb{B}_1 = \begin{pmatrix} k_{12} & & & 0 \\ 1 & k_{12} & & \\ & \ddots & \ddots & \\ 0 & & 1 & k_{12} \end{pmatrix}, \quad \mathbb{B}_2 = \begin{pmatrix} -k_{11} & & & 0 \\ 0 & -k_{11} & & \\ & \ddots & \ddots & \\ 0 & & 0 & -k_{11} \end{pmatrix}. \tag{3.32}$$

Then (3.30) yields

$$\mathcal{A}_1(W - \mathbb{B}_1) - \mathcal{A}_2\mathbb{B}_2 = 0, \quad \mathcal{A}_1\mathbb{B}_2 - \mathcal{A}_2(\mathbb{B}_1 + W) = 0, \tag{3.33a}$$

$$\mathcal{B}_1(W + \mathbb{B}_1) + \mathcal{B}_2\mathbb{B}_2 = 0, \quad \mathcal{B}_1\mathbb{B}_2 - \mathcal{B}_2(\mathbb{B}_1 - W) = 0, \tag{3.33b}$$

[b]Here we define $\mathcal{P}_{0,s}^\pm$ by taking derivative with respect to k_{12} rather than k_{11} because in this case we always have $k_{12} \neq 0$.

which provides an equation set for $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$. We note that since all the elements in (3.33) are in $\widetilde{G}_N(\mathbb{R})$, we can treat (3.33) as an ordinary linear equation set. Let us look at (3.33a). First, it indicates

$$\mathcal{A}_2 = \mathbb{B}_2^{-1}(W - \mathbb{B}_1)\mathcal{A}_1, \tag{3.34}$$

and further, by eliminating $\mathcal{A}_2$ we have

$$(\mathbb{B}_1^2 + \mathbb{B}_2^2 - W^2)\mathcal{A}_1 = 0. \tag{3.35}$$

We note that

$$\mathbb{B}_1^2 + \mathbb{B}_2^2 - W^2 = 0. \tag{3.36}$$

In fact, by calculation we find

$$(W^2)_{i,j} = \begin{cases} 0, & i < j, \\ \dfrac{1}{(i-j)!}\partial_{k_{12}}^{i-j}(k_{11}^2 + k_{12}^2), & i \geq j, \end{cases}$$

from which it is easy to verify (3.36) holds. Thus, it turns out that $\mathcal{A}_1$ can be an arbitrary element in $\widetilde{G}_N(\mathbb{R})$ and then from (3.34) and (3.31) we have

$$\mathcal{A} = \mathcal{A}_1\mathcal{M}, \quad \mathcal{M} = I_N + i\mathbb{B}_2^{-1}(W - \mathbb{B}_1), \tag{3.37}$$

where I_N is the Nth-order unit matrix. Similarly, we can find

$$\mathcal{B} = i\mathcal{B}_2\mathcal{M}, \quad \mathcal{B}_2 \in \widetilde{G}_N(\mathbb{R}).$$

Thus we conclude that the solution to the CES (2.13) can be given by

$$\varphi = \mathcal{M}(\mathcal{A}^+\mathcal{P}_0^+ + i\mathcal{A}^-\mathcal{P}_0^-), \quad \mathcal{A}^\pm \in \widetilde{G}_N(\mathbb{R}), \tag{3.38}$$

where $\mathcal{M}$ is defined in (3.37). Obviously, the matrix $\mathcal{M}$ contributes nothing through the transformation (2.9) to the mKdV equation. Therefore in practice we may remove $\mathcal{M}$ and use the effective part

$$\varphi = \mathcal{A}^+\mathcal{P}_0^+ + i\mathcal{A}^-\mathcal{P}_0^-, \quad \mathcal{A}^\pm \in \widetilde{G}_N(\mathbb{R}), \tag{3.39}$$

which is as same as (3.21).

3.2. *Breathers*

Case III. Breathers: When $\mathbb{A}$ has distinct complex (conjugate-pair) eigenvalues, we may have breather solutions. Let us consider a $2N$th order matrix,

$$\mathbb{A} = \mathrm{Diag}(k_1^2, \bar{k}_1^2, \ldots, k_N^2, \bar{k}_N^2)_{2N}, \quad k_j \neq 0, \tag{3.40}$$

where $k_j \in \mathbb{C}$, $j = 1, 2, \ldots, N$. The matrix $\mathbb{B}$ that generates breathers is

$$\mathbb{B} = \mathrm{Diag}(\Theta_1, \Theta_2, \ldots, \Theta_N)_{2N}, \quad \Theta_j = \begin{pmatrix} 0 & k_j \\ \bar{k}_j & 0 \end{pmatrix}. \tag{3.41}$$

For this case, the Wronskian entry vector, i.e. the solution to the CES (2.19), can be taken as

$$\varphi = (\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{N1}, \varphi_{N2})^T, \tag{3.42a}$$

where

$$\begin{aligned} \varphi_{j1} &= a_j e^{\xi_j} + b_j e^{-\xi_j}, \quad \varphi_{j2} = \bar{a}_j e^{\bar{\xi}_j} - \bar{b}_j e^{-\bar{\xi}_j}, \\ \xi_j &= k_j x - 4k_j^3 t + \xi_j^{(0)}, \quad a_j, b_j, \xi_j^{(0)} \in \mathbb{C}. \end{aligned} \tag{3.42b}$$

With the above φ as a basic entry vector, the Wronskian $f(\varphi)$ will provide breather solutions for the mKdV equation. For non-trivial solutions we need $k_{j1} \neq 0$, (see Sec. 5.3).

Case IV. Limit Solutions of Breathers: In this case let us consider the following block matrix,

$$\mathbb{A} = \begin{pmatrix} \mathcal{K} & 0 & 0 & \dots & 0 & 0 & 0 \\ \widetilde{\mathcal{K}} & \mathcal{K} & 0 & \dots & 0 & 0 & 0 \\ I_2 & \widetilde{\mathcal{K}} & \mathcal{K} & \dots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & I_2 & \widetilde{\mathcal{K}} & \mathcal{K} \end{pmatrix}_{2N}, \tag{3.43a}$$

where

$$\mathcal{K} = \begin{pmatrix} k_1^2 & 0 \\ 0 & \bar{k}_1^2 \end{pmatrix}, \quad \widetilde{\mathcal{K}} = \begin{pmatrix} 2k_1 & 0 \\ 0 & 2\bar{k}_1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.43b}$$

The matrix $\mathbb{B}$ satisfying (2.20) can be taken as

$$\mathbb{B} = \begin{pmatrix} \widetilde{B} & 0 & \dots & 0 & 0 \\ \widetilde{I}_2 & \widetilde{B} & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \dots & \widetilde{I}_2 & \widetilde{B} \end{pmatrix}_{2N}, \tag{3.44a}$$

where

$$\widetilde{I}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 & k_1 \\ \bar{k}_1 & 0 \end{pmatrix}. \tag{3.44b}$$

The Wronskian vector of this case is in the form of

$$\varphi = (\varphi_{1,1}, \varphi_{1,2}, \varphi_{2,1}, \varphi_{2,2}, \dots, \varphi_{N,1}, \varphi_{N,2})^T. \tag{3.45}$$

For convenience we set

$$\varphi^+ = (\varphi_{1,1}, \varphi_{2,1}, \dots, \varphi_{N,1})^T, \quad \varphi^- = (\varphi_{1,2}, \varphi_{2,2}, \dots, \varphi_{N,2})^T. \tag{3.46}$$

Substituting (3.45) into (2.19) with the above $\mathbb{A}$ and $\mathbb{B}$, i.e. (3.43a) and (3.44a), we find

$$\varphi^+_{xx} = \mathbb{A}_N \varphi^+, \quad \varphi^-_{xx} = \bar{\mathbb{A}}_N \varphi^-, \tag{3.47a}$$

$$\varphi_x^+ = \mathbb{B}_N \bar{\varphi}^-, \quad \varphi_x^- = \bar{\mathbb{B}}_N \bar{\varphi}^+, \tag{3.47b}$$

$$\varphi_t^\pm = -4\varphi_{xxx}^\pm, \tag{3.47c}$$

where

$$\mathbb{A}_N = \begin{pmatrix} k_1^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2k_1 & k_1^2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2k_1 & k_1^2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 2k_1 & k_1^2 \end{pmatrix}_N, \tag{3.48a}$$

$$\mathbb{B}_N = \begin{pmatrix} k_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & k_1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & k_1 \end{pmatrix}_N. \tag{3.48b}$$

From (3.47a), (3.47c) and (3.48a), one first has

$$\varphi^+ = \mathcal{A}\mathcal{Q}_0^+ + \mathcal{B}\mathcal{Q}_0^-, \quad \varphi^- = \mathcal{C}\bar{\mathcal{Q}}_0^+ + \mathcal{D}\bar{\mathcal{Q}}_0^-, \quad \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D} \in \widetilde{G}_N(\mathbb{C}), \tag{3.49}$$

where $\mathcal{Q}_0^\pm$ are defined by (3.15) together with ξ_1 defined in (3.42b). Then substituting (3.49) into (3.47b) and making using of (3.18), one finds

$$\mathcal{C} = \bar{\mathcal{A}}, \quad \mathcal{D} = -\bar{\mathcal{B}}. \tag{3.50}$$

Thus, (3.49) reads

$$\varphi^+ = \mathcal{A}\mathcal{Q}_0^+ + \mathcal{B}\mathcal{Q}_0^-, \quad \varphi^- = \bar{\mathcal{A}}\bar{\mathcal{Q}}_0^+ - \bar{\mathcal{B}}\bar{\mathcal{Q}}_0^-, \quad \mathcal{A},\mathcal{B} \in \widetilde{G}_N(\mathbb{C}). \tag{3.51}$$

The Wronskian $f(\varphi)$ with φ defined by (3.45) with (3.51) will provide a limit solution of breathers for the mKdV equation. In fact, quite similar to the procedure we described in Sec. 3.1 for the limit solutions of solitons, here the Wronskian $f(\varphi)$ with (3.45) is related to the N-breather solution by taking the limit $k_j \to k_1$ successively for $j = 2, 3, \ldots, N$.

4. Rational Solutions

4.1. *Backgrounds*

Following solution structures of the KdV equation [1], rational solutions should be led from zero eigenvalues of the coefficient matrix $\mathbb{A}$. For the mKdV equation this requires $|\mathbb{B}| = 0$ which disobeys the condition in (2.14a). However, even this is allowed, a trivial $\mathbb{B}$ will only bring a zero Wronskian in light of (A.1). Thus, it is clear that to get non-trivial rational solutions to the mKdV equation, we need a non-trivial matrix $\mathbb{B}$.

To have rational solutions, let us go back to the KdV-mKdV equation (1.4), i.e.

$$V_t + 12v_0 V V_X + 6V^2 V_X + V_{XXX} = 0, \tag{4.1}$$

which is related to the mKdV equation (1.3) through a Galilean transformation

$$v(x,t) = v_0 + V(X,t), \quad x = X + 6{v_0}^2 t, \tag{4.2}$$

where v_0 is a real parameter. We note that Eq. (4.1) admits non-trivial and non-singular rational solutions when $v_0 \neq 0$. Then using the transformation (4.2), rational solutions to the mKdV equation can be obtained. This fact has been realized via Bäcklund transformation (BT) [42] and Hirota method with a limiting procedure [41, 43], but in these literatures the presentation for high order rational solutions is complicated. In the following we will derive rational solutions in terms of Wronskian, which provides not only explicit but also impact forms for high order rational solutions.

4.2. *Rational solutions*

Still employing the same transformation as (2.9), i.e.

$$V = i\left(\ln \frac{\bar{f}}{f}\right)_X, \tag{4.3}$$

and after similar bilinearization procedure as for the mKdV equation in Sec. 2.2, the KdV-mKdV equation (4.1) can be written into the following bilinear form [44, 45]

$$(D_t + D_X^3)\bar{f}\cdot f = 0, \tag{4.4a}$$

$$(D_X^2 - 2iv_0 D_X)\bar{f}\cdot f = 0. \tag{4.4b}$$

For the solutions to (4.4) in Wronskian form, we have

Theorem 4.1. *The bilinear equation* (4.4) *admits Wronskian solution*

$$f = W(\phi) = |\widehat{N-1}|, \tag{4.5}$$

where the entry vector ϕ *satisfies*

$$i\phi_X = v_0\phi + B(t)\bar{\phi}, \tag{4.6a}$$

$$\phi_t = -4\phi_{XXX} + C(t)\phi, \tag{4.6b}$$

in which $B(t)$ *and* $C(t)$ *are two* $N\times N$ *matrices of* t *but independent of* x, *and satisfy*

$$|B(t)| \neq 0, \quad |B(t)^{-1}|_t \neq 0, \quad \mathrm{tr}(C(t)) \in \mathbb{R}(t), \tag{4.7a}$$

$$B_t(t) + B(t)\bar{C}(t) = C(t)B(t). \tag{4.7b}$$

The proof is similar to the one for Theorem 2.1, but (4.6a) results in complicated expression for $\bar{f}$ and its derivatives. We leave the proof in Appendix D.

As in Sec. 2.3, with the help of the auxiliary matrix $\mathbb{A}$ and auxiliary equation

$$\varphi_{XX} = \mathbb{A}\varphi, \tag{4.8}$$

one can simplify the CES (4.6) to

$$i\varphi_X = v_0\varphi + \mathbb{B}\bar{\varphi}, \tag{4.9a}$$

$$\varphi_t = -4\varphi_{XXX}, \tag{4.9b}$$

where both $\mathbb{A}$ and $\mathbb{B}$ are $N \times N$ complex constant matrices and related by

$$\mathbb{A} = \mathbb{B}\bar{\mathbb{B}} - v_0^2 I_N, \tag{4.10}$$

in which I_N is the Nth-order unit matrix. It might be possible that here we discuss all possible solutions according to the eigenvalues of $\mathbb{A}$, as we have done in the previous section. However, since we have had a clear description for solitons and breathers of the mKdV equation in the previous section, and the parameter v_0 will bring more complexity, in the following we neglect the discussion of $\mathbb{A}$ and let us only focus on rational solutions. In fact, rational solutions correspond to the zero eigenvalues of $\mathbb{A}$.

To derive rational solutions, let us first look at solitons. The N-soliton solution corresponds to

$$\mathbb{B} = \mathrm{diag}(-\sqrt{v_0^2 + k_1^2}, -\sqrt{v_0^2 + k_2^2}, \ldots, -\sqrt{v_0^2 + k_N^2}), \tag{4.11}$$

where we take k_j to be N distinct real positive numbers. In this case, a solution to the CES (4.9) is

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_N)^T, \tag{4.12}$$

with

$$\varphi_j = \sqrt{2v_0 + 2ik_j}e^{\eta_j} + \sqrt{2v_0 - 2ik_j}e^{-\eta_j}, \quad \eta_j = k_j X - 4k_j^3 t + \eta_j^{(0)}, \tag{4.13}$$

where $k_j, \eta_j^{(0)} \in \mathbb{R}$. This provides a N-soliton solution to the KdV-mKdV equation (4.1) through the transformation (4.3) with $f(\phi)$.

Now we consider φ_1, i.e.

$$\varphi_1 = \sqrt{2v_0 + 2ik_1}e^{\eta_1} + \sqrt{2v_0 - 2ik_1}e^{-\eta_1}, \quad \eta_1 = k_1 X - 4k_1^3 t, \tag{4.14}$$

which satisfies

$$i\varphi_{1,X} = v_0\varphi_1 - \sqrt{v_0^2 + k_1^2}\bar{\varphi}_1, \quad \varphi_{1,t} = -4\varphi_{1,XXX}, \tag{4.15}$$

where we have taken $\eta_1^{(0)} = 0$ so that φ_1 is an even function of k_1.[c] We do Taylor expansion for φ_1 with respect to k_1 at $k_1 = 0$, which gives

$$\varphi_1 = \sum_{j=0}^{N} \psi_{j+1} k_1^{2j}, \tag{4.16a}$$

[c]This guarantees ψ_j defined by (4.16b) to be a polynomial of order $2(j-1)$ with respect to X. If φ_1 is neither odd or even function of k_1, then ψ_j is a polynomial of order $j-1$ with respect to X, which leads to a constant Wronskian $f(\psi)$.

where

$$\psi_{j+1} = \frac{1}{(2j)!}\frac{\partial^{2j}}{\partial {k_1}^{2j}}\varphi_1\Big|_{k_1=0}, \quad (j = 0, 1, \ldots). \tag{4.16b}$$

Define

$$\psi = (\psi_1, \psi_2, \ldots, \psi_N)^T, \tag{4.17}$$

and substitute the expansion (4.16a) into (4.15). It then can be found that

$$i\psi_X = v_0\psi + \mathbb{B}\bar{\psi}, \tag{4.18a}$$

$$\psi_t = -4\psi_{XXX}, \tag{4.18b}$$

where

$$\mathbb{B} = \begin{pmatrix} \alpha_0 & & & \\ \alpha_1 & \alpha_0 & & \\ \vdots & \ddots & \ddots & \\ \alpha_{N-1} & \cdots & \alpha_1 & \alpha_0 \end{pmatrix}, \tag{4.19}$$

with

$$\alpha_j = -\frac{1}{(2j)!}\frac{\partial^{2j}}{\partial {k_1}^{2j}}\sqrt{v_0^2 + k_1^2}\big|_{k_1=0}, \quad (j = 0, 1, \ldots, N-1). \tag{4.20}$$

Obviously, $|B| \neq 0$.

The Wronskian with ψ as basic column vector, i.e.

$$f = W(\psi) = |\widehat{N-1}|, \tag{4.21}$$

provides non-singular rational solutions to the KdV-mKdV equation (4.1). A simplified form of these solutions is

$$V(X,t) = \frac{-2(F_{1,X}F_2 - F_1F_{2,X})}{F_1^2 + F_2^2}, \quad F_1 = \mathrm{Re}[f], \quad F_2 = \mathrm{Im}[f]. \tag{4.22}$$

We list the first three non-trivial f (for $N = 2, 3, 4$, respectively) in the following,

$$f = 4(2v_0X + i), \tag{4.23a}$$

$$f = \frac{16v_0\sqrt{2v_0}}{3}\left[X^3 + 12t - \frac{3X}{4v_0^2} + \frac{3i}{2v_0}\left(X^2 + \frac{1}{4v_0^2}\right)\right], \tag{4.23b}$$

$$f = -\frac{1}{v_0^4} - \frac{4X^2}{v_0^2} - \frac{16}{3}(12tX + X^4) - \frac{64}{45}v_0^2(720t^2 - 60tX^3 - X^6)$$
$$+ 4iv_0\left(-\frac{24t}{v_0^2} + \frac{X}{v_0^4} + 32tX^2 + \frac{16}{15}X^5\right). \tag{4.23c}$$

For the rational solutions to the mKdV equation (1.3), we have

Proposition 4.1. *The non-trivial rational solutions to the mKdV equation is given by*

$$v(x,t) = v_0 - \frac{2(F_{1,X}F_2 - F_1F_{2,X})}{F_1^2 + F_2^2}, \quad X = x - 6v_0^2 t, \tag{4.24}$$

where f is the Wronskian (4.21) *composed with* (4.17), $F_1 = \mathrm{Re}[f]$, $F_2 = \mathrm{Im}[f]$.

The first non-trivial rational solution to the mKdV equation reads

$$v = v_0 - \frac{4v_0}{4v_0^2X^2 + 1}, \quad X = x - 6v_0^2 t, \tag{4.25}$$

and the second one

$$v = v_0 - \frac{12v_0\left(X^4 + \frac{3}{2v_0^2}X^2 - \frac{3}{16v_0^4} - 24Xt\right)}{4v_0^2\left(X^3 + 12t - \frac{3X}{4v_0^2}\right)^2 + 9\left(X^2 + \frac{1}{4v_0^2}\right)^2}, \quad X = x - 6v_0^2 t. \tag{4.26}$$

We note that there is a bilinear BT [28] related to (4.4) and the BT admits multi-soliton solutions [28] and rational solutions [46] in Wronskian form. Besides, rational solutions of the mKdV equation (for both $\varepsilon = \pm 1$) may also be derived (through the Galilean transformed version) by using the long-wave-limit approach described in [47] (also see [29, 48]).

In next section, we will discuss dynamics of solutions including these rational solutions.

5. Dynamics Analysis

In this section, we investigate dynamics of two-soliton solutions, limit solutions, breathers and rational solutions. To describe the relationship between two-soliton solution and the simplest limit solution, let us start from the asymptotic behaviors of two-soliton interactions.

5.1. *Dynamics of solitons*

Soliton solutions to the mKdV equation (1.3) can be described by

$$v(x,t) = -\frac{2(F_{1,x}F_2 - F_1F_{2,x})}{F_1^2 + F_2^2}, \quad F_1 = \mathrm{Re}[f], \quad F_2 = \mathrm{Im}[f], \tag{5.1}$$

where $f = f(\varphi) = |\widehat{N-1}|$ is the Wronskian composed by the basic column vector φ which is defined by (3.5), or equivalently, by either (3.9a) or (3.11) in Case I. We note that from the transformation (2.9) the solution to the mKdV equation (1.3) can also be written as

$$v = -2\left(\arctan\frac{F_1}{F_2}\right)_x = 2\left(\arctan\frac{F_2}{F_1}\right)_x, \quad F_1 = \mathrm{Re}[f], \quad F_2 = \mathrm{Im}[f], \tag{5.2}$$

while (5.1) gives a more explicit form.

In the following we investigate 1- and 2-soliton solutions, which are, respectively, corresponding to

$$f = f_1 = f(\varphi_1) = \varphi_1, \tag{5.3a}$$

$$f = f_2 = f((\varphi_1, \varphi_2)^T) = \begin{vmatrix} \varphi_1 & \varphi_{1,x} \\ \varphi_2 & \varphi_{2,x} \end{vmatrix}, \tag{5.3b}$$

with φ_j defined by (3.5b), i.e.

$$\varphi_j = a_j^+ e^{\xi_j} + i a_j^- e^{-\xi_j}, \quad \xi_j = k_j x - 4k_j^3 t + \xi_j^{(0)}, \quad a_j^\pm, k_j, \xi_j^{(0)} \in \mathbb{R}. \tag{5.4}$$

Then, one-soliton solution to the mKdV equation (1.3) reads

$$v = -2 \cdot \mathrm{sgn}\left[\frac{a_1^-}{a_1^+}\right] \cdot k_1 \cdot \mathrm{sech}\left(2k_1 x - 8k_1^3 t - \ln\left|\frac{a_1^-}{a_1^+}\right|\right), \tag{5.5}$$

as depicted in Fig. 1, where for convenience, we call (a) soliton and (b) anti-soliton due to the signs of their amplitudes.

Obviously, this soliton is identified by the amplitude

$$\mathrm{Amp} = -2 \cdot \mathrm{sgn}\left[\frac{a_1^-}{a_1^+}\right] \cdot k_1$$

and top trace (trajectory)

$$x(t) = 4k_1^2 t + \frac{1}{2k_1} \ln\left|\frac{a_1^-}{a_1^+}\right|, \tag{5.6}$$

or velocity $4k_1^2$. Obviously, solitons of the mKdV equation are single-direction waves.

Next, let us look at two-soliton solution. The two-soliton solution of the mKdV (1.3) can be expressed by (5.1) where from (5.3b)

$$F_1 = (k_2 - k_1)(a_1^- a_2^- e^{4(k_1^3+k_2^3)t-(k_1+k_2)x} + a_1^+ a_2^+ e^{-4(k_1^3+k_2^3)t+(k_1+k_2)x}), \tag{5.7a}$$

$$F_2 = (k_2 + k_1)(a_1^- a_2^+ e^{4(k_1^3-k_2^3)t-(k_1-k_2)x} - a_1^+ a_2^- e^{4(-k_1^3+k_2^3)t+(k_1-k_2)x}). \tag{5.7b}$$

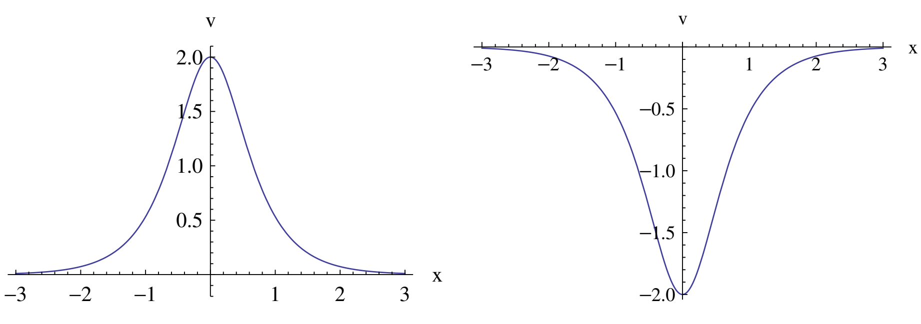


Fig. 1. (Color online) The shape of one-soliton solution given by (5.5) at $t = 0$. (a) Soliton for $a_1^+ = a_1^- = 1$, $k_1 = -1$ and $\xi_1^{(0)} = 0$. (b) Anti-soliton for $a_1^+ = a_1^- = 1$, $k_1 = 1$ and $\xi_1^{(0)} = 0$.

We assume that $a_j^{\pm} \neq 0$ and set $\text{sgn}[\frac{a_1^-}{a_1^+}] = \text{sgn}[\frac{a_2^-}{a_2^+}]$, so that F_1, F_2 are not zero at same time. Without loss of generality, we also take $0 < |k_2| < |k_1|$. For the analysis of asymptotic behaviors, it is convenient to use the following expression

$$v = 2\left(\arctan \frac{F_2}{F_1}\right)_x. \tag{5.8}$$

There are two types of 2-soliton interactions, soliton–soliton (or anti-soliton–anti-soliton) interaction and soliton–anti-soliton interaction, as shown in Fig. 2.

To investigate asymptotic behaviors of the two solitons involved in interaction, we first name them k_1-soliton and k_2-soliton, respectively. Then we rewrite the two-soliton solution (5.8) in terms of the following coordinates,

$$(X_1 = x - 4k_1^2 t, t), \tag{5.9}$$

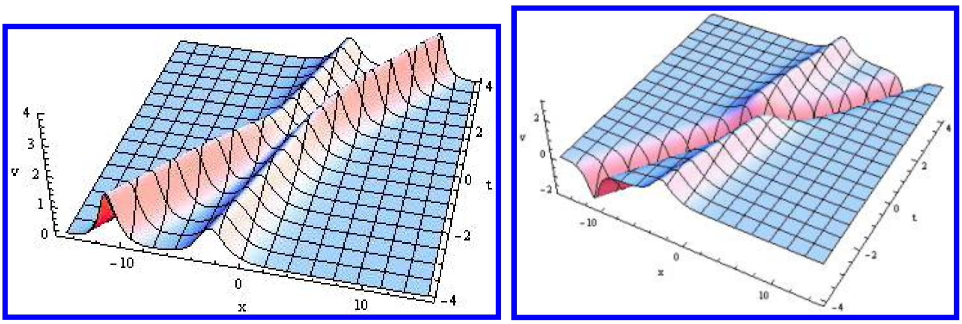
which then gives

$$v = 2\left(\arctan \frac{(a_1^+ a_2^- e^{2k_1 X_1} - a_2^+ a_1^- e^{8k_2(k_1^2-k_2^2)t+2k_2 X_1})(k_1+k_2)}{(a_1^- a_2^- + a_1^+ a_2^+ e^{8k_2(k_1^2-k_2^2)t+2(k_1+k_2)X_1})(k_1-k_2)}\right)_{X_1}. \tag{5.10}$$

Noting that for any $0 < |k_2| < |k_1|$ it is always valid that $k_1^2 - k_2^2 > 0$ and $\frac{k_1+k_2}{k_1-k_2} > 0$, we can keep X_1 to be constant and let t go to infinity. Then we can find there is only k_1-soliton remained along the line $X_1 = \text{const.}$ and also find how the k_1-soliton is asymptotically identified by its top trace and amplitude, for both $t \to \pm\infty$.

In details, when $k_2 > 0$, $t \to -\infty$ or $k_2 < 0$, $t \to +\infty$, i.e. $\text{sgn}[k_2] \cdot t \to -\infty$, the solution (5.10) becomes

$$\begin{aligned} v &= 2\left(\arctan \frac{a_1^+(k_1+k_2)e^{2k_1 X_1}}{a_1^-(k_1-k_2)}\right)_{X_1} \\ &= 2 \cdot \text{sgn}\left[\frac{a_1^-}{a_1^+}\right] \cdot k_1 \cdot \text{sech}\left(2k_1 X_1 - \ln\left|\frac{a_1^-}{a_1^+}\right| + \ln \frac{k_1+k_2}{k_1-k_2}\right); \end{aligned} \tag{5.11}$$



(a) (b)

Fig. 2. (Color online) Two-soliton interactions. (a) Soliton–soliton for $a_1^+ = a_1^- = a_2 = 1$, $b_2 = -1$, $k_1 = 0.8$, $k_2 = 0.5$ and $\xi_1^{(0)} = \xi_2^{(0)} = 0$. (b) Soliton–anti-soliton for $a_1^+ = a_1^- = a_2 = 1$, $b_2 = -1$, $k_1 = -0.8$, $k_2 = 0.5$ and $\xi_1^{(0)} = \xi_2^{(0)} = 0$.

and when $k_2 > 0$, $t \to +\infty$ or $k_2 < 0$, $t \to -\infty$, i.e. $\mathrm{sgn}[k_2] \cdot t \to +\infty$, (5.10) becomes

$$\begin{aligned} v &= 2\left(\arctan \frac{-a_1^-(k_1+k_2)}{a_1^+ e^{2k_1X}(k_1-k_2)}\right)_{X_1} \\ &= 2\cdot \mathrm{sgn}\left[\frac{a_1^-}{a_1^+}\right]\cdot k_1 \cdot \mathrm{sech}\left(2k_1X_1 - \ln\left|\frac{a_1^-}{a_1^+}\right| - \ln\frac{k_1+k_2}{k_1-k_2}\right). \end{aligned} \tag{5.12}$$

We can also rewrite the two-soliton solution (5.8) in terms of the coordinates

$$(X_2 = x - 4k_2^2 t, t), \tag{5.13}$$

and do a similar asymptotic analysis for the k_2-soliton. Finally, we reach the following.

Theorem 5.1. *Suppose that* $\mathrm{sgn}[\frac{a_1^-}{a_1^+}] = \mathrm{sgn}[\frac{a_2^-}{a_2^+}]$, $a_j^\pm \neq 0$ *and* $0 < |k_2| < |k_1|$ *in* (5.10)*. Then, when* $\mathrm{sgn}[k_2]\cdot t \to \pm\infty$, *the* k_1*-soliton asymptotically follows*

$$\textit{top trace:}\quad x(t) = 4k_1^2 t + \frac{1}{2k_1}\ln\left|\frac{a_1^-}{a_1^+}\right| \pm \frac{1}{2k_1}\ln\frac{k_1+k_2}{k_1-k_2}, \tag{5.14a}$$

$$\textit{amplitude:}\quad 2\cdot\mathrm{sgn}\left[\frac{a_1^-}{a_1^+}\right]\cdot k_1, \tag{5.14b}$$

and when $\mathrm{sgn}[k_1]\cdot t \to \pm\infty$, *the* k_2*-soliton asymptotically follows*

$$\textit{top trace:}\quad x(t) = 4k_2^2 t + \frac{1}{2k_2}\ln\left|\frac{a_2^-}{a_2^+}\right| \pm \frac{1}{2k_2}\ln\frac{k_1+k_2}{k_1-k_2}, \tag{5.15a}$$

$$\textit{amplitude:}\quad -2\cdot\mathrm{sgn}\left[\frac{a_2^-}{a_2^+}\right]\cdot k_2. \tag{5.15b}$$

The phase shift for the k_j*-soliton after interactions is* $\frac{1}{k_j}\ln\frac{k_2+k_1}{k_1-k_2}$.

Now it is completely clear how the two-soliton interactions are related to the parameters $\{k_j, a_j, b_j\}$. This will be helpful to understand the asymptotic behavior of limit solutions. Here we refer the reader to an elegant survey [49] by Hietarinta for asymptotic analysis of scattering of solitonary waves.

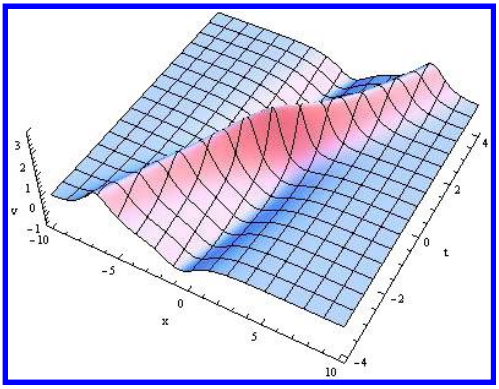
5.2. *Asymptotic behavior of limit solutions*

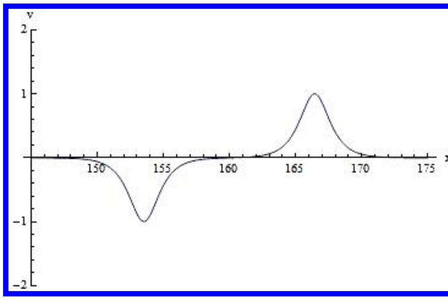
The simplest limit solution in Case II is

$$v = 2\left(\arctan\frac{a_1^+a_1^-(-48k_1^3t + 4k_1x)}{{a_1^-}^2 e^{8k_1^3t - 2k_1x} + {a_1^+}^2 e^{-8k_1^3t + 2k_1x}}\right)_x. \tag{5.16}$$

This is derived from (5.2) with

$$f = \begin{vmatrix} \varphi_1 & \varphi_{1,x} \\ \partial_{k_1}\varphi_1 & (\partial_{k_1}\varphi_1)_x \end{vmatrix}, \tag{5.17a}$$





(a) (b)

Fig. 3. (Color online) Limit solution given by (5.16) for $a_1^+ = a_1^- = 1$, $k_1 = 0.5$ and $\xi_1^{(0)} = 0$. (a) Shape and motion. (b) Asymmetric wave shape at $t = 160$.

where φ_1 is defined in (3.5b), i.e.

$$\varphi_1 = a_1^+ e^{\xi_1} + i a_1^- e^{-\xi_1}, \quad \xi_1 = k_1 x - 4k_1^3 t + \xi_1^{(0)}, \quad a_1^\pm, k_1, \xi_1^{(0)} \in \mathbb{R}. \tag{5.17b}$$

The solution (5.16) is depicted in Fig. 3. We characterize dynamics of the solution by the following two points, which are typically different from the interaction of two normal solitons that we described in the previous subsection. These two points are

- Soliton–anti-soliton interaction with (asymptotically) asymmetric wave shape,
- Top trace of each soliton is asymptotically governed by logarithm and linear functions.

The first point can be explained as follows. Recall the two-soliton interaction with $a_2^\pm = a_1^\pm \neq 0$ and $k_1 \cdot k_2 > 0$. According to Theorem 5.1, this is a soliton–anti-soliton interaction and the absolute of amplitude of each soliton is $2|k_j|$. Obviously, the asymmetric shape of limit solution (5.16) coincides well with the limit $k_2 \to k_1$.

To understand the second point, again we put the limit solution in the coordinate system

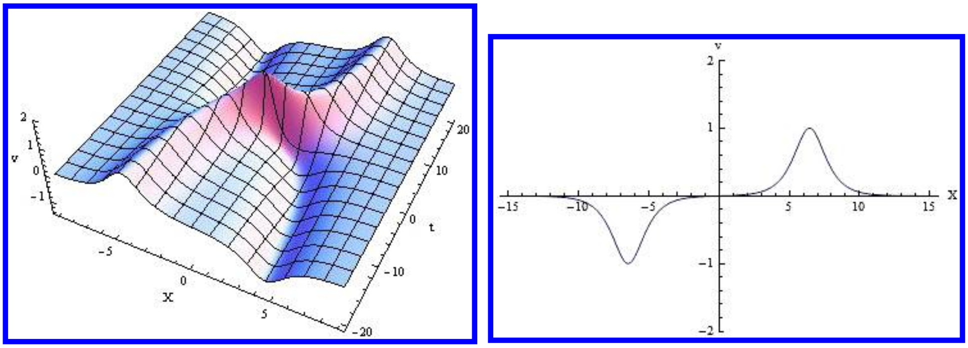
$$(Y = x - 4k_1^2 t, t), \tag{5.18}$$

and this gives

$$v = 2\left(\arctan \frac{4a_1^+ a_1^- e^{2k_1 Y} k_1 (Y - 8k_1^2 t)}{{a_1^-}^2 + {a_1^+}^2 e^{4k_1 Y}}\right)_Y, \tag{5.19}$$

which is described in Fig. 4.

For convenience here we suppose that $k_1 > 0$. Then by analyzing the leading terms as $t \to \pm\infty$ in the numerator and denominator in (5.19), we can conclude the asymptotic behaviors of the limit solution (5.19) as follows.



(a) (b)

Fig. 4. (Color online) Limit solution given by (5.19) for $a_1^+ = a_1^- = 1$ and $k_1 = 0.5$. (a) Shape and motion. (b) Asymmetric wave shape at $t = 160$.

Theorem 5.2. *Suppose that $a_1^\pm \neq 0$ and $k_1 > 0$ in (5.19). When $t \to -\infty$ there are two waves moving with amplitudes $\mp 2 \cdot \mathrm{sgn}[\frac{a_1^-}{a_1^+}] \cdot k_1$ and top traces governed by the logarithm functions*

$$Y_\pm = \frac{1}{2k_1}\left(\ln\left|\frac{a_1^-}{a_1^+}\right| \pm \ln(-32k_1^3 t)\right), \tag{5.20a}$$

where the subscript $\pm$ of Y stands for $Y \to \pm\infty$. When $t \to +\infty$, there are also two waves moving with amplitudes $\pm 2 \cdot \mathrm{sgn}[\frac{a_1^-}{a_1^+}] \cdot k_1$ and top traces

$$Y_\pm = \frac{1}{2k_1}\left(\ln\left|\frac{a_1^-}{a_1^+}\right| \pm \ln(32k_1^3 t)\right). \tag{5.20b}$$

Replacing Y by x by using (5.18) the top traces of the waves in Fig. 3 are then asymptotically governed by linear and logarithm functions of t.

Let us end up this subsection with the following remark. With regard to the limit solutions the top traces (or waves trajectories) are governed by (linear and) logarithm functions should be a typical characteristic, which differs from normal soliton interactions with straight line trajectories. (See [33–36] for more examples.)

5.3. *Breathers*

Wronskian entries in Case III provides breather solutions to the mKdV equation (1.3). The simplest one corresponds to

$$f = \begin{vmatrix} \varphi_{11} & \varphi_{11,x} \\ \varphi_{12} & \varphi_{12,x} \end{vmatrix} = F_1 + iF_2, \tag{5.21a}$$

where

$$\varphi_{11} = a_1 e^{\xi_1} + b_1 e^{-\xi_1}, \quad \varphi_{12} = \bar{a}_1 e^{\bar{\xi}_1} - \bar{b}_1 e^{-\bar{\xi}_1}, \tag{5.21b}$$

$$\xi_1 = k_1 x - 4k_1^3 t, \quad k_1, a_1, b_1 \in \mathbb{C}, \tag{5.21c}$$

$$F_1 = 4k_{11}(a_{11}b_{11} + a_{12}b_{12})\cos(24k_{11}^2k_{12}t - 8k_{12}^3t - 2k_{12}x) + 4k_{11}(a_{12}b_{11} - a_{11}b_{12})\sin(24k_{11}^2k_{12}t - 8k_{12}^3t - 2k_{12}x), \quad (5.21d)$$

$$F_2 = -2k_{12}e^{-2k_{11}(4k_{11}^2t+12k_{12}^2t+x)}[(b_{11}^2 + b_{12}^2)e^{16k_{11}^3t} + (a_{11}^2 + a_{12}^2)e^{4k_{11}(12k_{12}^2t+x)}], \quad (5.21e)$$

and we have written

$$k_1 = k_{11} + ik_{12}, \quad a_1 = a_{11} + ia_{12}, \quad b_1 = b_{11} + ib_{12}.$$

Then the breather solution is expressed by (5.1) with the above F_1, F_2. We further assume that

$$\sin\theta = \frac{a_{11}b_{11} + a_{12}b_{12}}{\alpha}, \quad \alpha = \sqrt{(a_{11}b_{11} + a_{12}b_{12})^2 + (a_{12}b_{11} - a_{11}b_{12})^2},$$

and then rewrite (5.1) as[d]

$$v = -2\left(\arctan\frac{P}{Q}\right)_x, \quad (5.22a)$$

where

$$P = 2k_{11}\alpha\sin[2k_{12}(x - 4t(3k_{11}^2 - k_{12}^2)) - \theta], \quad (5.22b)$$

$$Q = k_{12}[(a_{11}^2 + a_{12}^2)e^{2k_{11}(x+4t(3k_{12}^2-k_{11}^2))} + (b_{11}^2 + b_{12}^2)e^{-2k_{11}(x+4t(3k_{12}^2-k_{11}^2))}]. \quad (5.22c)$$

Such a breather is described in Fig. 5.

Figure 5(a) shows an oscillating wave moving along a straight line. The oscillation comes from the sine function and the frequency depends on both x and t. To understand more on the wave we use the coordinates

$$(Z = x + 4t(3k_{12}^2 - k_{11}^2), t) \quad (5.23)$$

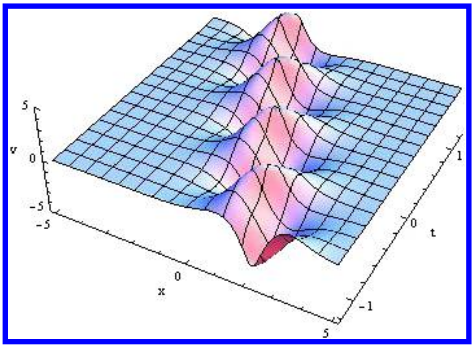
to rewrite the solution (5.22) as

$$v = -2\left(\arctan\frac{2k_{11}\alpha\sin(2k_{12}(Z - 8t(k_{11}^2 + k_{12}^2)) - \theta)}{k_{12}((a_{11}^2 + a_{12}^2)e^{2k_{11}Z} + (b_{11}^2 + b_{12}^2)e^{-2k_{11}Z})}\right)_Z, \quad (5.24)$$

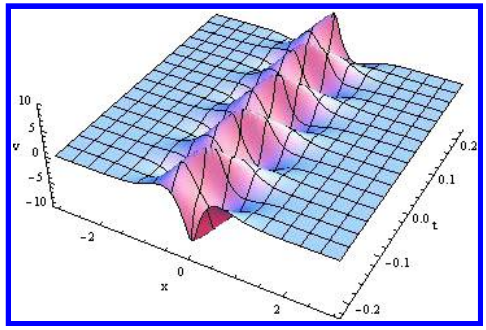
and then fix $Z = 0$, i.e. looking at the wave along the straight line $Z = 0$. Then it is clear that

- The breather travels along the straight line $Z = 0$, in other words, the wave speed is $4(3k_{12}^2 - k_{11}^2)$, which means it admits bi-direction traveling.
- A stationary breather appears when $3k_{12}^2 = k_{11}^2$, as depicted in Fig. 5(b).

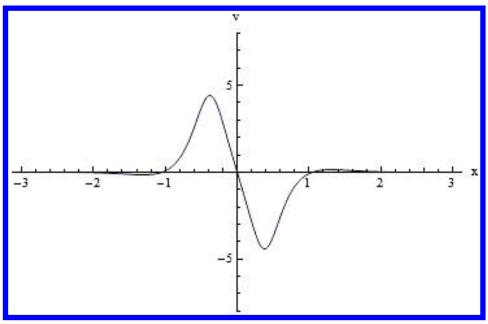
[d]From (5.22), we can see that $k_{1j} \neq 0$ is necessary for getting non-trivial breather solutions.



(a)



(b)



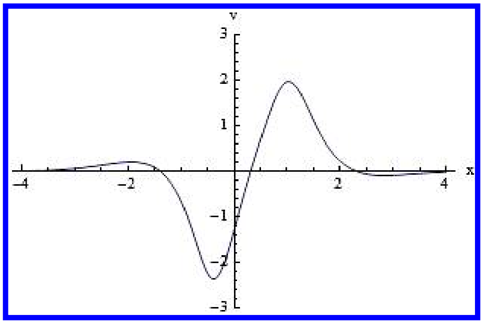
(c)

Fig. 5. (Color online) Breathers given by (5.22). (a) A moving breather with $a_1 = 1 + i$, $b_1 = 2 + 0.5i$, $k_1 = 0.8 - 0.6i$. (b) A stationary breather with $a_1 = b_1 = 1$, $k_1 = \sqrt{3} + i$. (c) A 2D-plot of (b) at $t = 0$.

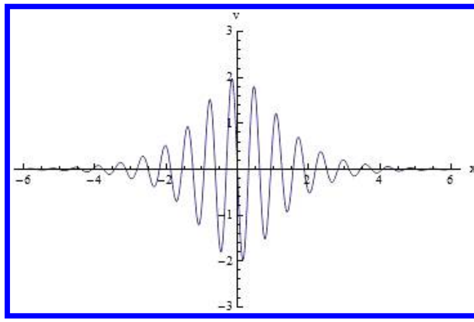
- Under the coordinate system (5.23) the frequency only depends on t and the period reads

$$T = \frac{\pi}{8k_{12}(k_{11}^2 + k_{12}^2)}. \tag{5.25}$$

We note that in two dimensions (fixing t) the breather is in fact a spindle-shape wave. Let us go back to the solution (5.22). If we fix time t, then the breather oscillates with frequency $\frac{|k_{12}|}{\pi}$ and its amplitude decays by the rate $e^{-2|xk_{11}|}$ as $x \to \pm\infty$. That means if $|k_{12}|$ is small and $|k_{11}|$ is relatively large so that the amplitude decay becomes the dominating factor, we get a "normal" breather as shown in Fig. 6(a); while if $|k_{11}|$ is small enough and $|k_{12}| \gg |k_{11}|$ so that the oscillation dominates, we will see a spindle-like wave shown in Fig. 6(b). In the latter case, the wave will travel with high speed and high oscillating frequency. This can lead to overlaps of "normal" oscillating waves (like Fig. 6(a)) during their traveling, which makes a spindle shape.



(a)



(b)

Fig. 6. (Color online) Shape of breathers given by (5.22). (a) "Normal" shape at $t = 0$ with $a_1 = 1 + i$, $b_1 = 2 + 0.5i$, $k_1 = 0.8 - 0.6i$. (b) Spindle-shape breather at $t = 0$ with $a_1 = 1 + i$, $b_1 = 1 + i$, $k_1 = 0.5 + 5i$.

Finally, in this subsection we list and depict two-breather solution and the simplest limit breather solution, without further asymptotic analysis. Both solutions can be given by (5.1) with f being a 4-by-4 Wronskian

$$f = |\varphi, \varphi_x, \varphi_{xx}, \varphi_{xxx}|, \tag{5.26}$$

where for the 2-breather solution

$$\varphi = (\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22})^T, \tag{5.27a}$$

$$\varphi_{j1} = a_j e^{\xi_j} + b_j e^{-\xi_j}, \quad \varphi_{j2} = \bar{a}_j e^{\bar{\xi}_j} - \bar{b}_j e^{-\bar{\xi}_j}, \quad \xi_j = k_j x - 4k_j^3 t + \xi_j^{(0)}, \tag{5.27b}$$

in which $a_j, b_j, k_j, \xi_j^{(0)} \in \mathbb{C}$, and for the limit breather

$$\varphi = (\varphi_{11}, \varphi_{12}, \partial_{k_1}\varphi_{11}, \partial_{\bar{k}_1}\varphi_{12})^T, \tag{5.28}$$

in which $\varphi_{11}, \varphi_{12}$ are defined by (5.27b).

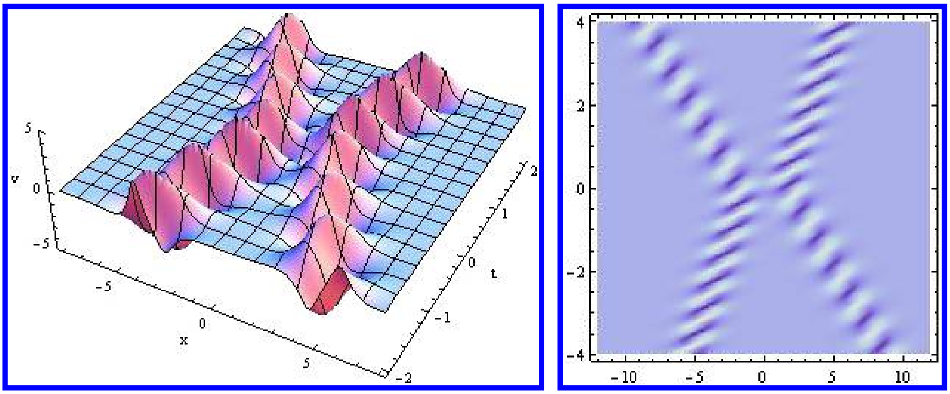
Figure 7 shows the two-breather interaction where from the density plot (b) one can clearly see that the two breathers are traveling along straight lines and a phase shift appears after interaction. Figure 8 shows the shape and motion of a limit breather solution, where from the density plot (b) one can clearly see that the breather trajectories are not any longer straight lines. Here we conjecture that they are governed by logarithm functions.

5.4. *Dynamics of rational solutions*

The first non-trivial rational solution to the mKdV equation is (4.25), i.e.

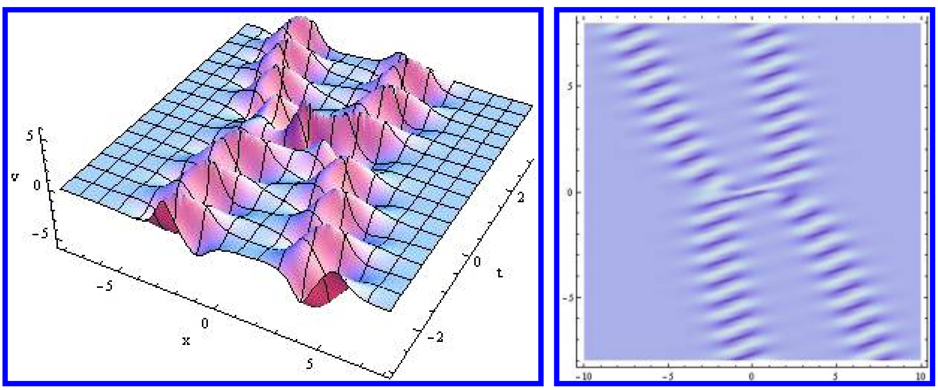
$$v = v_0 - \frac{4v_0}{4v_0^2 X^2 + 1}, \quad X = x - 6v_0^2 t. \tag{5.29}$$

This is a non-singular traveling wave moving with the constant speed $6v_0^2$, constant amplitude $-3v_0$ and asymptotic line $v = v_0$. It is depicted in Fig. 9.



(a) (b)

Fig. 7. (Color online) Shape and motion of two breather solution given by (5.1) with (5.26) and (5.27). (a) 3D-plot for $a_1 = b_1 = a_2 = b_2 = 1$, $k_1 = 1 + 0.5i$, $k_2 = 0.8 - 0.6i$ and $\xi_1^{(0)} = \xi_2^{(0)} = 0$. (b) Density plot of (a) for $x \in [-12, 12]$, $t \in [-4, 4]$.



(a) (b)

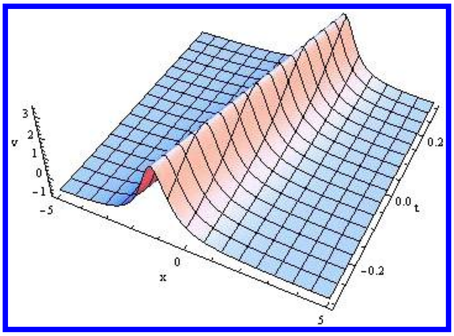
Fig. 8. (Color online) Shape and motion of the limit breather solution given by (5.1) with (5.26) and (5.28). (a) 3D-plot for $a_1 = b_1 = 1$, $k_1 = 0.8 + 0.5i$ and $\xi_1^{(0)} = 0$. (b) Density plot of (a) for $x \in [-10, 10]$, $t \in [-8, 8]$.

The next rational solution is given by (4.26), i.e.

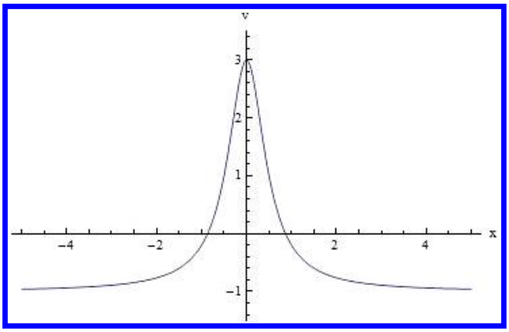
$$v = v_0 - \frac{12v_0\left(X^4 + \frac{3}{2v_0^2}X^2 - \frac{3}{16v_0^4} - 24Xt\right)}{4v_0^2\left(X^3 + 12t - \frac{3X}{4v_0^2}\right)^2 + 9\left(X^2 + \frac{1}{4v_0^2}\right)^2}, \quad X = x - 6v_0^2 t. \tag{5.30}$$

It can be viewed as a double-traveling wave solution from the following form

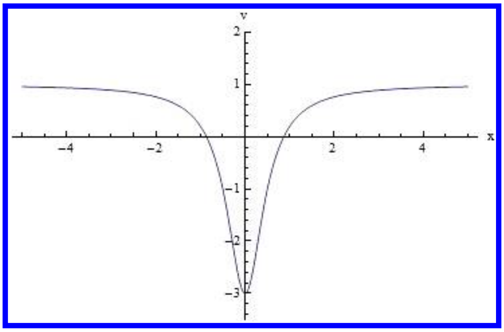
$$v = v_0 - \frac{12v_0\left(X^4 + \frac{3}{2v_0^2}XY - \frac{3}{16v_0^4}\right)}{4v_0^2\left(X^3 - \frac{3Y}{4v_0^2}\right)^2 + 9\left(X^2 + \frac{1}{4v_0^2}\right)^2}, \tag{5.31}$$



(a)



(b)



(c)

Fig. 9. (Color online) Shape and motion of the rational solution given by (5.29) for $v_0 = -0.8$ in (a), $v_0 = -1$ in (b) and $v_0 = 1$ in (c).

where

$$X = x - 6v_0^2 t, \quad Y = x - 22v_0^2 t. \tag{5.32}$$

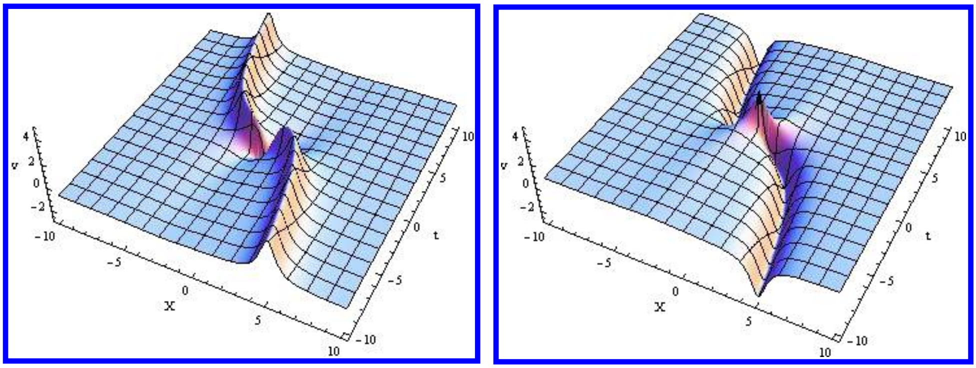
However, it does not show interactions of two single rational solutions. Only one wave is remained for large x, t (see Fig. 10). We re-depict Fig. 10(a) in Fig. 11 by a density plot so that we can see the wave top trace clearer.

To realize the asymptotic behavior analytically, we rewrite the solution (5.30) in the following coordinates system

$$\left(X, T = X^3 + 12t + \frac{3}{16v_0^4 X}\right) \tag{5.33}$$

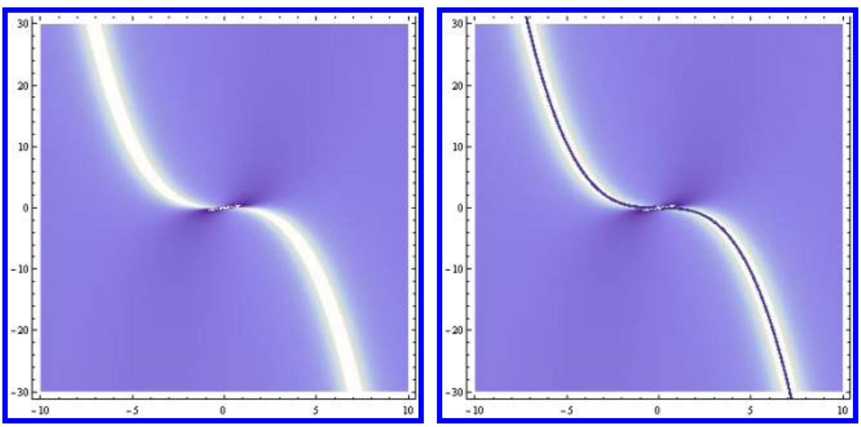
and this gives

$$v = v_0 - \frac{12v_0\left(3X^4 + \frac{3}{2v_0^2}X^2 - 2TX + \frac{3}{16v_0^4}\right)}{4v_0^2\left(T - \frac{3}{16v_0^4 X} - \frac{3X}{4v_0^2}\right)^2 + 9\left(X^2 + \frac{1}{4v_0^2}\right)^2}. \tag{5.34}$$



(a) (b)

Fig. 10. (Color online) Shape and motion of the rational solution given by (5.30) for $v_0 = -0.8$ in (a) and $v_0 = 0.8$ in (b).



(a) (b)

Fig. 11. (Color online) (a) Density plot of Fig. 10(a) with $X \in [-10, 10]$, $t \in [-30, 30]$ (b) (a) overlapped by the trajectory curve given by (5.36).

Then, by calculation it can be found that for given large X the wave (5.34) has a single stationary point at $T = 0$ where v gets a local extreme value

$$v = \frac{v_0(1 - 12v_0^2 X^2)}{1 + 4v_0^2 X^2}, \tag{5.35}$$

which goes to $-3v_0$ as $X \to \pm\infty$. Thus we can conclude that for large X, t the wave asymptotically travels along the curve

$$T = X^3 + 12t + \frac{3}{16v_0^4 X} = 0 \tag{5.36}$$

with amplitude $-3v_0$. Figure 11(b) displays a density plot overlapped by the above wave trajectory curve.

More details on the rational solutions to the mKdV equation can be found in [46] where the rational solutions are derived via bilinear Bäcklund transformation.

6. Conclusions

6.1. *Summary*

In the paper, we reviewed the Wronskian solutions to the mKdV equation (1.3) in terms of Wronskians. When a solution is expressed through the Wronskian

$$f = f(\varphi) = |\widehat{N-1}|, \tag{6.1}$$

one needs to solve the finalized CES (2.19) together with (2.20), i.e.

$$\varphi_{xx} = \mathbb{A}\varphi, \tag{6.2a}$$

$$\varphi_x = \mathbb{B}\bar{\varphi}, \tag{6.2b}$$

$$\varphi_t = -4\varphi_{xxx}, \tag{6.2c}$$

and

$$\mathbb{A} = \mathbb{B}\bar{\mathbb{B}}. \tag{6.3}$$

$\mathbb{A}$ is the auxiliary matrix that we introduced to deal with the complex operation in (6.2b) and it works in practice. As a result, with the help of $\mathbb{A}$ we solved the above CES and then categorized the solutions to the mKdV equation in terms of the canonical form of $\mathbb{A}$ (rather than the canonical form of $\mathbb{B}$). Solutions are categorized by solitons (together with their limit case) and breathers (together with their limit case). There is no rational solution arising from (6.2) because rational solutions correspond to zero eigenvalues of $\mathbb{A}$ while we need $|\mathbb{B}| \neq 0$ to finish Wronskian verification. To derive rational solutions for the mKdV equation (1.3), we employed the Galilean transformed equation, i.e. the KdV-mKdV equation (1.4) which admits rational solutions in Wronskian form. Then the rational solutions to the mKdV equation can be recovered through the inverse transformation. Dynamics of some obtained solutions was analyzed and illustrated. Single soliton is always moving along one direction but admits either positive or negative amplitude. Breather, as wave package, can be bidirectional or stationary. Here, particularly, we would like to sum up a typical characteristic of limit solitons: the wave trajectories asymptotically follow logarithm curves (combined with linear functions). This point is based on several examples we have examined [33–36, 50].

Obviously, through the Galilean transformation (4.2), all these obtained solutions of the mKdV equation (1.3) can easily be used for the KdV-mKdV equation (1.4) which often appears in physics contexts. In fact, in the paper we do not differ them from each other. In addition to the KdV-mKdV equation, our treatment to the complex operation in (6.2) can also be applied to the sine-Gordon equation.

There are Miura transformations between the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{6.4}$$

and the mKdV equation (for both $\varepsilon = \pm 1$). For the mKdV($\varepsilon = -1$) equation, the Miura transformation is

$$u = -v^2 \pm v_x, \tag{6.5}$$

which provides a real map between solutions of the KdV equation (6.4) and the mKdV equation ($\varepsilon = -1$) (cf. [26]). However, when $\varepsilon = 1$, i.e. for the mKdV equation (1.3), the Miura transformation between (6.4) and (1.3) is

$$u = v^2 \pm iv_x. \tag{6.6}$$

This means, for the real solution v of the mKdV equation (1.3), u defined by (6.6) is a complex-valued function, which provides a solution to the complex KdV equation (6.4). In other words, suppose that in (6.4)

$$u = u_1 + iu_2, \tag{6.7a}$$

then

$$u_1 = v^2, \quad u_2 = \pm v_x \tag{6.7b}$$

together with (6.7a) solves the complex KdV equation (6.4), where v is a real solution that we already obtained for the mKdV equation (1.3). Such a relation has been used to investigate solutions and dynamics of the complex KdV equation [51].

6.2. *List of solutions*

Let us list out the obtained solutions and their corresponding basic Wronskian vectors. Solutions to the mKdV equation (1.3) can be expressed as

$$v = 2\left(\arctan\frac{F_2}{F_1}\right)_x = \frac{-2(F_{1,x}F_2 - F_1F_{2,x})}{F_2^2 + F_1^2}, \tag{6.8a}$$

where

$$f = f(\varphi) = |\widehat{N-1}| = F_1 + iF_2, \quad F_1 = \mathrm{Re}[f], \quad F_2 = \mathrm{Im}[f]. \tag{6.8b}$$

The available Wronskian vectors are the following.

- **For soliton solutions:**

$$\varphi = \varphi_N^{[s]} = (\varphi_1, \varphi_2, \ldots, \varphi_N)^T, \tag{6.9a}$$

with

$$\varphi_j = a_j^+ e^{\xi_j} + ia_j^- e^{-\xi_j}, \quad \xi_j = k_j x - 4k_j^3 t + \xi_j^{(0)}, \quad a_j^+, a_j^-, k_j, \xi_j^{(0)} \in \mathbb{R}. \tag{6.9b}$$

- **For limit solutions of solitons:**

$$\varphi = \varphi_N^{[ls]}(k_1) = \mathcal{A}^+\mathcal{Q}_0^+ + i\mathcal{A}^-\mathcal{Q}_0^-, \quad \mathcal{A}^{\pm} \in \widetilde{G}_N(\mathbb{R}), \tag{6.10a}$$

with

$$\mathcal{Q}_0^{\pm} = (\mathcal{Q}_{0,0}^{\pm}, \mathcal{Q}_{0,1}^{\pm}, \ldots, \mathcal{Q}_{0,N-1}^{\pm})^T, \quad \mathcal{Q}_{0,s}^{\pm} = \frac{1}{s!}\partial_{k_1}^{s} e^{\pm\xi_1}, \tag{6.10b}$$

where ξ_1 is defined in (6.9b).

- **For breather solutions:**

$$\varphi = \varphi_{2N}^{[b]} = (\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{N1}, \varphi_{N2})^T, \tag{6.11a}$$

with

$$\varphi_{j1} = a_j e^{\xi_j} + b_j e^{-\xi_j}, \quad \varphi_{j2} = \bar{a}_j e^{\bar{\xi}_j} - \bar{b}_j e^{-\bar{\xi}_j}, \tag{6.11b}$$

$$\xi_j = k_j x - 4k_j^3 t + \xi_j^{(0)}, \quad k_j = k_{j1} + ik_{j2}, \quad a_j, b_j, \xi_j^{(0)} \in \mathbb{C}, \tag{6.11c}$$

and $k_{j1} \cdot k_{j2} \neq 0$.

- **For limit solutions of breathers:**

$$\varphi = \varphi_{2N}^{[lb]}(k_1) = (\varphi_{1,1}^{+}, \varphi_{1,2}^{-}, \varphi_{2,1}^{+}, \varphi_{2,2}^{-}, \ldots, \varphi_{N,1}^{+}, \varphi_{N,2}^{-})^T, \tag{6.12a}$$

and the elements are given through

$$\varphi^{+} = (\varphi_{1,1}^{+}, \varphi_{2,1}^{+}, \ldots, \varphi_{N,1}^{+})^T = \mathcal{A}\mathcal{Q}_0^{+} + \mathcal{B}\mathcal{Q}_0^{-}, \tag{6.12b}$$

$$\varphi^{-} = (\varphi_{1,2}^{-}, \varphi_{2,2}^{-}, \ldots, \varphi_{N,2}^{-})^T = \bar{\mathcal{A}}\bar{\mathcal{Q}}_0^{+} - \bar{\mathcal{B}}\bar{\mathcal{Q}}_0^{-}, \tag{6.12c}$$

where $\mathcal{A}, \mathcal{B} \in \widetilde{G}_N(\mathbb{C})$,

$$\mathcal{Q}_0^{\pm} = (\mathcal{Q}_{0,0}^{\pm}, \mathcal{Q}_{0,1}^{\pm}, \ldots, \mathcal{Q}_{0,N-1}^{\pm})^T, \quad \mathcal{Q}_{0,s}^{\pm} = \frac{1}{s!}\partial_{k_1}^{s} e^{\pm\xi_1}, \tag{6.12d}$$

and ξ_1 is defined in (6.11c).

We note that, due to the linear property of the CES (2.19), one may also get mixed solutions by arbitrarily combining the above vectors to be a new Wronskian vector. For example, take

$$\varphi = \begin{pmatrix} \varphi_{N_1}^{[s]} \\ \varphi_{N_2}^{[ls]}(k_{N_1+1}) \end{pmatrix}. \tag{6.13}$$

The related solution corresponds to the interaction between N_1-soliton and a $(N_2 - 1)$-order limit-soliton solutions.

Finally, for the rational solution, it is expressed as

$$v(x,t) = v_0 - \frac{2(F_{1,X}F_2 - F_1F_{2,X})}{F_2^2 + F_1^2}, \quad X = x - 6v_0^2 t, \quad v_0 \neq 0 \in \mathbb{R}, \tag{6.14a}$$

where still

$$f = f(\psi) = |\widehat{N-1}| = F_1 + iF_2, \quad F_1 = \mathrm{Re}[f], \quad F_2 = \mathrm{Im}[f], \tag{6.14b}$$

and the Wronskian is composed by

$$\psi = (\psi_1, \psi_2, \ldots, \psi_N)^T, \tag{6.15a}$$

with

$$\psi_{j+1} = \frac{1}{(2j)!}\frac{\partial^{2j}}{\partial {k_1}^{2j}}\varphi_1\bigg|_{k_1=0}, \quad (j = 0, 1, \ldots, N-1), \tag{6.15b}$$

and

$$\varphi_1 = \sqrt{2v_0 + 2ik_1}e^{\eta_1} + \sqrt{2v_0 - 2ik_1}e^{-\eta_1}, \quad \eta_1 = k_1 X - 4k_1^3 t, \;\; k_1 \in \mathbb{R}. \tag{6.15c}$$

Acknowledgments

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Appendix A. Proof of Theorem 2.1

Proof. The compatibility of (2.13a) and (2.13b), i.e. $\phi_{xt} = \phi_{tx}$, yields (2.14b). (2.13a) implies $\bar{\phi} = \bar{B}(t)\partial_x^{-1}\phi$ or $\bar{\phi} = B(t)^{-1}\phi_x$. Using the former we obtain the complex conjugate form of f as

$$\bar{f} = |\bar{B}(t)||-1, \widehat{N-2}|. \tag{A.1}$$

If we use $\bar{\phi} = B(t)^{-1}\phi_x$, we have

$$\bar{f} = |B(t)|^{-1}|\widetilde{N}|. \tag{A.2}$$

Both of them can be used as the expression of $\bar{f}$ to implement Wronskian verification. Here it is natural that we require $|B(t)| \neq 0$. Although it is allowed $|B(t)| = 0$ in (A.1), this brings $f = 0$ which is trivial to the bilinear equations (2.10).

Then the necessary derivatives of f and $\bar{f}$ are presented as the following,

$$f_x = |\widehat{N-2}, N|, \tag{A.3a}$$

$$f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{A.3b}$$

$$\begin{aligned} f_{xxx} &= |\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2| \\ &\quad + 2|\widehat{N-3}, N-1, N+1|, \end{aligned} \tag{A.3c}$$

$$\begin{aligned} f_t &= -4(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| \\ &\quad + |\widehat{N-2}, N+2|) + \mathrm{tr}(C(t))|\widehat{N-1}|, \end{aligned} \tag{A.3d}$$

and (using the expression (A.1))

$$\bar{f}_x = |\bar{B}(t)||-1, \widehat{N-3}, N-1|, \tag{A.4a}$$

$$\bar{f}_{xx} = |\bar{B}(t)|(|-1, \widehat{N-4}, N-2, N-1| + |-1, \widehat{N-3}, N|), \tag{A.4b}$$

$$\begin{aligned}\bar{f}_{xxx} &= |\bar{B}(t)|(|-1, \widehat{N-3}, N+1| + 2|-1, \widehat{N-4}, N-2, N| \\ &\quad + |-1, \widehat{N-5}, N-3, N-2, N-1|),\end{aligned} \tag{A.4c}$$

$$\begin{aligned}\bar{f}_t &= -4|\bar{B}(t)|(|-1, \widehat{N-5}, N-3, N-2, N-1| \\ &\quad - |-1, \widehat{N-4}, N-2, N| + |-1, \widehat{N-3}, N+1|) \\ &\quad + |\bar{B}(t)|_t|-1, \widehat{N-2}| + \mathrm{tr}(C(t))|\bar{B}(t)||-1, \widehat{N-2}|,\end{aligned} \tag{A.4d}$$

where $\mathrm{tr}(C(t))$ means the trace of $C(t)$. Using $\bar{\phi} = \bar{B}(t)\partial_x^{-1}\phi$ which is implied from (2.13a), the complex conjugate of (A.3d) is

$$\begin{aligned}\bar{f}_t = -4|\bar{B}(t)|(|-1, \widehat{N-5}, N-3, N-2, N-1| - |-1, \widehat{N-4}, N-2, N| \\ + |-1, \widehat{N-3}, N+1|) + \mathrm{tr}(\bar{C}(t))|\bar{B}(t)||-1, \widehat{N-2}|,\end{aligned} \tag{A.5}$$

which should be the same as (A.4d). This requires

$$|B(t)|_t = 0, \quad \mathrm{tr}(C(t)) \in \mathbb{R}(t), \tag{A.6}$$

i.e. the condition in (2.14a).

Noting that $\phi_{xx} = B(t)\bar{B}(t)\phi$ and using Proposition 2.1 with $\Omega_{j,s} = \partial_x^2$, we find

$$\begin{aligned}\mathrm{tr}(B(t)\bar{B}(t))|-1, \widehat{N-3}, N-1| &= -|-1, \widehat{N-5}, N-3, N-2, N-1| \\ &\quad + |-1, \widehat{N-3}, N+1|,\end{aligned} \tag{A.7}$$

$$\begin{aligned}\mathrm{tr}(B(t)\bar{B}(t))|\widehat{N-2}, N| &= -|\widehat{N-4}, N-2, N-1, N| \\ &\quad + |\widehat{N-2}, N+2|,\end{aligned} \tag{A.8}$$

$$\begin{aligned}\mathrm{tr}(B(t)\bar{B}(t))|-1, \widehat{N-2}| &= -|-1, \widehat{N-4}, N-2, N-1| \\ &\quad + |-1, \widehat{N-3}, N|,\end{aligned} \tag{A.9}$$

$$\mathrm{tr}(B(t)\bar{B}(t))|\widehat{N-1}| = -|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|. \tag{A.10}$$

Then, substituting (A.3) and (A.4) (with the condition (A.6)) into (2.10a) and making use of (A.10), we have

$$\begin{aligned}&\bar{f}_t f - \bar{f} f_t + \bar{f}_{xxx} f - 3\bar{f}_{xx} f_x + 3\bar{f}_x f_{xx} - \bar{f} f_{xxx} \\ &\quad = 6|\bar{B}(t)|(-|-1, \widehat{N-3}, N+1||\widehat{N-1}| \\ &\qquad - |-1, \widehat{N-4}, N-2, N-1||\widehat{N-2}, N|\end{aligned}$$

$$+ |\widehat{N-4}, N-2, N-1, N||-1, \widehat{N-2}| - |\widehat{N-3}, N-1, N+1||-1, \widehat{N-2}|$$
$$+ |\widehat{N-2}, N+1||-1, \widehat{N-3}, N-1| - |-1, \widehat{N-4}, N-2, N||\widehat{N-1}|),$$

which is zero in the light of Proposition 2.2. Similarly, one can prove (2.10b). Thus the proof is completed. □

Appendix B. Eigen-Polynomial of $\mathbb{A} = \mathbb{B}\bar{\mathbb{B}}$

We prove the Proposition 3.1 through the following two lemmas.

Lemma B.1. *For two arbitrary Nth-order complex matrices A and B,*

$$\det(\lambda I_N - AB) = \det(\lambda I_N - BA), \tag{B.1}$$

where I_N is the Nth-order unit matrix.

Proof. Assuming that $\operatorname{rank}(A) = r$, then there exist Nth-order non-singular matrices P and Q such that

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q. \tag{B.2}$$

Thus

$$\begin{aligned}
\det(\lambda I_N - AB) &= \det\left(\lambda I_N - P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} QB\right) \\
&= \det\left(P^{-1}\left(\lambda I_N - P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} QB\right) P\right) \\
&= \det\left(\lambda I_N - P^{-1}\left(P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} QB\right) P\right) \\
&= \det\left(\lambda I_N - \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} QBP\right).
\end{aligned}$$

In a similar way we have

$$\det(\lambda I_N - BA) = \det\left(\lambda I_N - QBP \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\right). \tag{B.3}$$

If we rewrite the matrix QBP into the following block matrix form with same structure as $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, i.e.

$$QBP = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{B.4}$$

then we have

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} QBP = \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix}, \quad QBP \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ B_{21} & 0 \end{pmatrix}, \tag{B.5}$$

which further means

$$\det(\lambda I_N - AB) = \lambda^{N-r}\det(\lambda I_r - B_{11}) = \det(\lambda I_N - BA). \tag{B.6}$$

We complete the proof. □

Lemma B.2. *Assuming that B is an arbitrary Nth-order complex matrix and $\bar{B}$ is its complex conjugate, then* $\det(\lambda I - \bar{B}B)$ *is a polynomial of λ with real coefficients.*

Proof. Write

$$f(\lambda) = \det(\lambda I_N - \bar{B}B) = a_N\lambda^N + \cdots + a_1\lambda + a_0. \tag{B.7}$$

Then using Lemma B.1 with $A = \bar{B}$ we have

$$f(\lambda) = \det(\lambda I_N - \bar{B}B) = \det(\lambda I_N - B\bar{B}) = \overline{\det(\bar{\lambda} I_N - \bar{B}B)} = \overline{f(\bar{\lambda})}, \tag{B.8}$$

which means all the coefficients $\{a_j\}$ are real. □

Appendix C. Discussions on the Trivial Solutions to the CES (2.19)

Let the square matrices $\mathbb{A}, \mathbb{B}$ follow the relation

$$\mathbb{A} = \mathbb{B}\bar{\mathbb{B}}. \tag{C.1}$$

We start from the 2×2 case. Noting that the product of all the eigenvalues of $\mathbb{A}$ is non-negative, in the following we first look at

$$\mathbb{A} = \begin{pmatrix} -k_1^2 & 0 \\ 0 & -k_2^2 \end{pmatrix}, \quad k_1 \neq k_2 \neq 0, \ \ k_1, k_2 \in \mathbb{R}, \tag{C.2}$$

and suppose

$$\mathbb{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{C.3}$$

with undetermined $a, b, c, d \in \mathbb{C}$. However, in this case it can be found that the matrix relation (C.1) does not have any solutions unless $k_1^2 = k_2^2$. So next we turn to consider

$$\mathbb{A} = \begin{pmatrix} -k^2 & 0 \\ 0 & -k^2 \end{pmatrix}, \quad k \in \mathbb{R}. \tag{C.4}$$

In this case, Eq. (C.1) admits a non-diagonal matrix solution $\mathbb{B}$ as

$$\mathbb{B} = \begin{pmatrix} -d & -\dfrac{d^2 + k^2}{c} \\ c & d \end{pmatrix} e^{i\theta}, \quad c \neq 0, \ \ c, d, \theta \in \mathbb{R}. \tag{C.5}$$

We note that such a $\mathbb{B}$ does not lead to any non-trivial solutions to the mKdV equation. In fact, in the CES (2.19), the general solution to the equation set (2.19a) and (2.19c) is

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = H \begin{pmatrix} e^{i\xi} \\ e^{-i\xi} \end{pmatrix} \tag{C.6a}$$

with arbitrary matrix $H \in \mathbb{C}_{2\times 2}$ and

$$\xi = kx + 4k^3 t + \xi^{(0)}, \quad k, \xi^{(0)} \in \mathbb{R}. \tag{C.6b}$$

However, no matter what condition the matrix H should satisfy under the equation (2.19b), the Wronskian

$$f(\varphi) = |H| f((e^{i\xi}, e^{-i\xi})^T)$$

is always a constant, which leads to a trivial solution to the mKdV equation. In the case of the $N \times N$ matrix (N is even)

$$A = \mathrm{diag}(-k^2, -k^2, \ldots, -k^2), \quad k \in \mathbb{R},$$

similar to (C.6), the general solution to (2.19a) and (2.19c) is

$$\phi = H \times (e^{i\xi}, e^{-i\xi}, 0, 0, \ldots, 0)^T, \quad H \in \mathbb{C}_{N\times N},$$

which leads to a zero Wronskian $f(\varphi)$.

With these discussions we can conclude that for the CES (2.19) with an $N \times N$ matrix

$$\mathbb{A} = \mathrm{diag}(\alpha, \alpha, \ldots, \alpha), \quad \alpha \in \mathbb{R}, \tag{C.7}$$

the possible solution φ to (2.19) composes a trivial Wronskian $f(\varphi)$.

Appendix D. Proof for Theorem 4.1

Proof. The compatibility of (4.6a) and (4.6b), i.e. $(\phi_X)_t = (\phi_t)_X$, yields the condition (4.7b). Besides, we will use $\bar{\phi} = iB(t)^{-1}\phi_X - v_0 B(t)^{-1}\phi$ to calculate $\bar{f}$. Then $|B(t)| \neq 0$ is required.

The parameter v_0 will lead to complicated expressions for $\bar{f}$ and its derivatives. For simplification let us introduce the notation $|\cdot|_j$ where the subscript j indicates the absence of the $\phi^{(j)}$ column [28], for example,

$$|\widehat{N}|_j = |\phi^{(0)}, \ldots, \phi^{(j-1)}, \phi^{(j+1)}, \ldots, \phi^{(N)}|.$$

Derivatives of f have already given in (A.3). For $\bar{f}$, using the CES (4.6) we can reach

$$\bar{f} = |B^{-1}(t)| \sum_{j=0}^{N} (-v_0)^j i^{N-j} |\widehat{N}|_j,$$

$$\bar{f}_X = |B^{-1}(t)| \left(\sum_{j=0}^{N-1} (-v_0)^{j+1} i^{N-j-1} |\widehat{N}|_j + \sum_{j=0}^{N-1} (-v_0)^j i^{N-j} |\widehat{N-1}, N+1|_j \right),$$

$$\bar{f}_{XX} = |B^{-1}(t)| \Bigg((-v_0)^N |\widehat{N-2}, N+1| + \sum_{j=0}^{N-1} (-v_0)^j i^{N-j} |\widehat{N-1}, N+2|_j$$

$$+ \sum_{j=0}^{N-2} (-v_0)^j i^{N-j} |\widehat{N-2}, N, N+1|_j + \sum_{j=0}^{N-2} (-v_0)^{j+2} i^{N-j-2} |\widehat{N}|_j$$

$$+ 2 \sum_{j=0}^{N-2} (-v_0)^{j+1} i^{N-j-1} |\widehat{N-1}, N+1|_j \Bigg),$$

$$\bar{f}_{XXX} = |B^{-1}(t)| \Bigg(3 \sum_{j=0}^{N-3} (-v_0)^{j+2} i^{N-j-2} |\widehat{N-1}, N+1|_j + (-v_0)^N |\widehat{N-2}, N+2|$$

$$+ 3 \sum_{j=0}^{N-2} (-v_0)^{j+1} i^{N-j-1} |\widehat{N-1}, N+2|_j + 2(-v_0)^N |\widehat{N-3}, N-1, N+1|$$

$$+ \sum_{j=0}^{N-1} (-v_0)^j i^{N-j} |\widehat{N-1}, N+3|_j + 2 \sum_{j=0}^{N-2} (-v_0)^j i^{N-j} |\widehat{N-2}, N, N+2|_j$$

$$+ 3 \sum_{j=0}^{N-3} (-v_0)^{j+1} i^{N-j-1} |\widehat{N-2}, N, N+1|_j + 2(-v_0)^{N-1} i |\widehat{N-3}, N, N+1|$$

$$+ \sum_{j=0}^{N-3} (-v_0)^j i^{N-j} |\widehat{N-3}, N-1, N, N+1|_j + \sum_{j=0}^{N-3} (-v_0)^{j+3} i^{N-j-3} |\widehat{N}|_j \Bigg),$$

$$\bar{f}_t = -4|B^{-1}(t)| \Bigg(\sum_{j=0}^{N-1} (-v_0)^j i^{N-j} |\widehat{N-1}, N+3|_j + \sum_{j=0}^{N-3} (-v_0)^{j+3} i^{N-j-3} |\widehat{N}|_j$$

$$+ \sum_{j=0}^{N-3} (-v_0)^j i^{N-j} |\widehat{N-3}, N-1, N, N+1|_j + (-v_0)^N |\widehat{N-2}, N+2|$$

$$- \sum_{j=0}^{N-2} (-v_0)^j i^{N-j} |\widehat{N-2}, N, N+2|_j - (-v_0)^{N-1} i |\widehat{N-3}, N, N+1|$$

$$- (-v_0)^N |\widehat{N-3}, N-1, N+1| \Bigg) + |B(t)^{-1}|_t \sum_{j=0}^{N} (-v_0)^j i^{N-j} |\widehat{N}|_j$$

$$+ \operatorname{tr}(C(t)) |B^{-1}(t)| \sum_{j=0}^{N} (-v_0)^j i^{N-j} |\widehat{N}|_j.$$

Using $\bar{\phi} = iB(t)^{-1}\phi_X - v_0 B(t)^{-1}\phi$, the complex conjugate of f_t given in (A.3d) is

$$\begin{aligned}\bar{f}_t = -4|B^{-1}(t)| \Bigg(&\sum_{j=0}^{N-1} (-v_0)^j i^{N-j} |\widehat{N-1}, N+3|_j + \sum_{j=0}^{N-3} (-v_0)^{j+3} i^{N-j-3} |\widehat{N}|_j \\ &+ \sum_{j=0}^{N-3} (-v_0)^j i^{N-j} |\widehat{N-3}, N-1, N, N+1|_j + (-v_0)^N |\widehat{N-2}, N+2| \\ &- \sum_{j=0}^{N-2} (-v_0)^j i^{N-j} |\widehat{N-2}, N, N+2|_j - (-v_0)^{N-1} i |\widehat{N-3}, N, N+1| \\ &- (-v_0)^N |\widehat{N-3}, N-1, N+1| \Bigg) + \mathrm{tr}(\bar{C}(t)) |B^{-1}(t)| \sum_{j=0}^{N} (-v_0)^j i^{N-j} |\widehat{N}|_j.\end{aligned}$$

This should be the same as the $\bar{f}_t$ that we previously derived from $\bar{f}$, which requires $|B(t)^{-1}|_t = 0$ and $\mathrm{tr}(C(t)) \in \mathbb{R}(t)$, i.e. the condition in (4.7a).

Besides, noting that $\phi_{XX} = (B(t)\bar{B}(t) - v_0^2 I_N)\phi$ and using Proposition 2.1 with $\Omega_{j,s} = \partial_X^2$, we can have the following identities:

$$\begin{aligned}\mathrm{tr}(B(t)\bar{B}(t) - v_0^2 I_N)|\widehat{N}|_j &= |\widehat{N-1}, N+2|_j - |\widehat{N-2}, N, N+1|_j \\ &\quad - |\widehat{N}|_{j-2}, \quad (j = 0, 1, \ldots, N-2), \\ \mathrm{tr}(B(t)\bar{B}(t) - v_0^2 I_N)|\widehat{N-1}, N+1|_j &= |\widehat{N-1}, N+3|_j - |\widehat{N-3}, N-1, N, N+1|_j \\ &\quad - |\widehat{N-1}, N+1|_{j-2}, \quad (j = 0, 1, \ldots, N-3), \\ \mathrm{tr}(B(t)\bar{B}(t) - v_0^2 I_N)|\widehat{N-1}| &= -|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \\ \mathrm{tr}(B(t)\bar{B}(t) - v_0^2 I_N)|\widehat{N-2}, N| &= -|\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2|.\end{aligned}$$

With these results and (A.3) in hand, for (4.4a), we have

$$\begin{aligned}&\bar{f}_t f - \bar{f} f_t + \bar{f}_{XXX} f - 3\bar{f}_{XX} f_X + 3\bar{f}_X f_{XX} - \bar{f} f_{XXX} \\ &\quad = 6(-v_0)^{N-1} i (|\widehat{N-3}, N, N+1||\widehat{N-1}| - |\widehat{N-2}, N||\widehat{N-3}, N-1, N+1| \\ &\qquad + |\widehat{N-2}, N+1||\widehat{N-3}, N-1, N|) \\ &\quad = 0.\end{aligned}$$

Similarly, one can prove (4.4b). ◻

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