Bilinear Methods and Integrability

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Preface

Please eamil me: djzhang@staff.shu.edu.cn if you find any misprints and missing and uncorrect references. I appreciate that very much.

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Chapter 1

Hirota's Bilinear Integrability

1.1 Bilinear operator

Definition 1.1.1. *For C^∞ differential functions $f(x,y)$ and $g(x,y)$ defined on $\mathbb{R}^2$, their **bilinear derivatives** are defined as:*

$$D_x^m D_y^n f(x,y)\cdot g(x,y) = (\partial_x - \partial_{x'})^m(\partial_y - \partial_{y'})^n f(x,y)g(x',y')|_{x'=x,y'=y}. \tag{1.1.1}$$

*where D is called **Hirota's bilinear operator**. Here $\partial_x = \frac{\partial}{\partial x}$.*

(1.1.1) can also be defined as follows:

$$e^{\epsilon D_x + \kappa D_y} f(x,y)\cdot g(x,y) = f(x+\epsilon, y+\kappa)g(x-\epsilon, y-\kappa). \tag{1.1.2}$$

In fact, expanding both sides at $(\epsilon,\kappa) = (0,0)$ and comparing coefficients of power $\epsilon^m\kappa^n$ we get definition (1.1.1). As examples, let us see

$$\begin{aligned}
&D_x f\cdot g = f_x g - f g_x,\\
&D_x^2 f\cdot g = f_{xx} g - 2 f_x g_x + f g_{xx},\\
&D_x^3 f\cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx},\\
&D_x D_y f\cdot g = f_{xy} g - f_x g_y - f_y g_x + f g_{xy},\\
&D_x^m f\cdot g = \sum_{j=0}^{m} (-1)^j \mathrm{C}_m^j f^{(m-j)}(x) g^{(j)}(x),
\end{aligned}$$

where $\mathrm{C}_m^j = \frac{m!}{(m-j)!j!}$, $f^{(j)} = \partial_x^j f(x)$. From these examples one can see the difference between the bilinear derivatives of f and g and Leibniz's rule of the m-th order derivative of the product fg.

Bilinear derivatives admit symmetric and bilinear properties:

$$\begin{aligned}
&D_x^m f\cdot g=(-1)^m D_x^m g\cdot f,\\
&(aD_x^m+bD_y^n)f\cdot g=aD_x^m f\cdot g+bD_y^n f\cdot g,\\
&D_x^m D_y^n(af+bg)\cdot h=aD_x^m D_y^n f\cdot h+bD_x^m D_y^n g\cdot h,
\end{aligned}$$

where $a,b\in\mathbb{C}$, and particularly,

$$D_x^m f\cdot 1=\partial_x^m f(x).$$

For linear exponential functions like $e^{kx+\omega t}$ $(k,\omega\in\mathbb{C})$, there is a simple formula for their bilinear derivatives:

$$D_x^m D_y^n e^{\eta_1}\cdot e^{\eta_2}=(k_1-k_2)^m(\omega_1-\omega_2)^n e^{\eta_1+\eta_2}, \tag{1.1.3}$$

where

$$\eta_i=k_i x+\omega_i y+\eta_i^{(0)},\quad k_i,\omega_i,\eta_i^{(0)}\in\mathbb{C}. \tag{1.1.4}$$

The above definition and properties can be extended to arbitrary dimensions. Suppose that $\mathbf{t}=(t_1,t_2,\cdots,t_s)$, $\mathbf{p}=(p_1,p_2,\cdots,p_s)$, $\mathbf{q}=(q_1,q_2,\cdots,q_s)$ are vectors in $\mathbb{R}^s$, define inner product $\mathbf{p}\cdot\mathbf{t}=\sum_{i=1}^s p_i t_i$, and denote $D_{\mathbf{t}}=(D_{t_1},D_{t_2},\cdots,D_{t_s})$. Then we can define

$$e^{\mathbf{p}\cdot D_{\mathbf{t}}}f(\mathbf{t})\cdot g(\mathbf{t})=f(\mathbf{t}+\mathbf{p})g(\mathbf{t}-\mathbf{p}). \tag{1.1.5}$$

Suppose $P(\mathbf{t})$ is a polynomial of $\mathbf{t}$, and introduce $P(D_{\mathbf{t}})$. For example, if $P(\mathbf{t})=3t_1^2t_2+2t_2t_3$, then $P(D_{\mathbf{t}})=3D_{t_1}^2D_{t_2}+2D_{t_2}D_{t_3}$. For a general $P(D_{\mathbf{t}})$, it is not difficult to find

$$P(D_{\mathbf{t}})e^{\mathbf{p}\cdot\mathbf{t}}\cdot e^{\mathbf{q}\cdot\mathbf{t}}=P(\mathbf{p}-\mathbf{q})e^{(\mathbf{p}+\mathbf{q})\cdot\mathbf{t}}, \tag{1.1.6}$$

$$P(D_{\mathbf{t}})e^{\mathbf{p}\cdot\mathbf{t}}\cdot 1=P(\mathbf{p})e^{\mathbf{p}\cdot\mathbf{t}}=P(\partial_{\mathbf{p}})e^{\mathbf{p}\cdot\mathbf{t}}. \tag{1.1.7}$$

Besides, one can prove that

$$\begin{aligned}
&D_x^r D_y^s(e^{\eta_1}f(x,y))\cdot(e^{\eta_2}g(x,y))\\
=\ &e^{\eta_1+\eta_2}(D_x+k_1-k_2)^r(D_y+\omega_1-\omega_2)^s f(x,y)\cdot g(x,y),
\end{aligned}\tag{1.1.8}$$

and particularly, when $\eta_1=\eta_2$, one has

$$D_x^r D_y^s(e^{\eta_1}f)\cdot(e^{\eta_1}g)=e^{2\eta_1}D_x^r D_y^s f\cdot g, \tag{1.1.9}$$

which is called the gauge property of bilinear derivatives. A more general case is

$$P(D_{\mathbf{t}})(e^{\mathbf{p}\cdot\mathbf{t}}f(\mathbf{t}))\cdot(e^{\mathbf{q}\cdot\mathbf{t}}g(\mathbf{t}))=e^{(\mathbf{p}+\mathbf{q})\cdot\mathbf{t}}P(D_{\mathbf{t}}+\mathbf{p}-\mathbf{q})f(\mathbf{t})\cdot g(\mathbf{t}), \tag{1.1.10}$$

and

$$P(D_{\mathbf{t}})(e^{\mathbf{p}\cdot\mathbf{t}}f)\cdot(e^{\mathbf{p}\cdot\mathbf{t}}g)=e^{2\mathbf{p}\cdot\mathbf{t}}P(D_{\mathbf{t}})f\cdot g\,. \tag{1.1.11}$$

1.2 N soliton solutions

In this section we show that how Hirota's method works in finding NSS. KdV equation and KP equation will serve as examples.

1.2.1 NSS of the KdV equation

The KdV equation reads

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2.1}$$

Note that its coefficients can be arbitrary. Usually we consider potential form of the equation:

$$w_t + 3(w_x)^2 + w_{xxx} = 0, \quad (u = w_x). \tag{1.2.2}$$

Under transformation

$$u = 2(\ln f)_{xx}, \quad \text{i.e. } w = 2(\ln f)_x, \tag{1.2.3}$$

Eq.(1.2.2) can be written as

$$f_{xt}f - f_x f_t + f_{xxxx}f - 4f_{xxx}f_x + 3(f_{xx})^2 = 0, \tag{1.2.4}$$

which is

$$(D_x D_t + D_x^4) f \cdot f = 0 \tag{1.2.5}$$

if we employ the notation of bilinear operator D given in (1.1.1). The above equation is called the bilinear form of the KdV equation (1.2.1), also called the bilinear KdV equation; the solution f is usually called τ function of the KdV equation; once we have f, we can get back a solution of the KdV equation through the transformation (1.2.3).

Note: In 1971 Ryogo Hirota first introduced bilinear methods to derive NSS for the KdV equation [20]. At that time the bilinear form of the KdV equation was (1.2.4)*; the bilinear operator D as defined in Definition1.1.1 was formally introduced by Hirota in 1974 [22, 53].*

It is easy to check

$$f = 1 + e^{\eta}, \quad \eta = kx - k^3 t + \eta^{(0)}, \quad k, \eta^{(0)} \in \mathbb{R} \tag{1.2.6}$$

satisfies the bilinear KdV equation (1.2.4). For achieving more solutions, we can (perturbatively) expand

$$f = 1 + \sum_{i=1}^{\infty} f^{(i)} \varepsilon^i, \tag{1.2.7}$$

where subscript (i) is for numbering coefficients. Substituting the above into (1.2.5) and comparing coefficients of each power of ε, we reach an equation system

$$\varepsilon^1: \quad (\partial_{xt}+\partial_x^4)f^{(1)}=0, \tag{1.2.8a}$$

$$\varepsilon^2: \quad (\partial_{xt}+\partial_x^4)f^{(2)}=-\frac{1}{2}(D_xD_t+D_x^4)f^{(1)}\cdot f^{(1)}, \tag{1.2.8b}$$

$$\varepsilon^3: \quad (\partial_{xt}+\partial_x^4)f^{(3)}=-(D_xD_t+D_x^4)f^{(1)}\cdot f^{(2)}, \tag{1.2.8c}$$

$$\varepsilon^4: \quad (\partial_{xt}+\partial_x^4)f^{(4)}=-(D_xD_t+D_x^4)(f^{(1)}\cdot f^{(3)}+\frac{1}{2}f^{(2)}\cdot f^{(2)}), \tag{1.2.8d}$$

$$\cdots\cdots.$$

For (1.2.8a) we can take $f^{(1)}=e^{\eta_i}$, where

$$\eta_i=k_ix-k_i^3t+\eta_i^{(0)}, \quad k_i,\eta_i^{(0)}\in\mathbb{R}. \tag{1.2.9}$$

Since (1.2.8a) is a homogeneous linear equation, for any positive integer N

$$f^{(1)}=\sum_{i=1}^{N}e^{\eta_i} \tag{1.2.10}$$

gives a solution to (1.2.8a), where η_i is defined by(1.2.9).

Now, let us look at equation system (1.2.8) with (1.2.10) in more details. When $N=1$, $f^{(1)}=e^{\eta_1}$ satisfies (1.2.8a); meanwhile, when $f^{(2)}=f^{(3)}=\cdots=0$, the rest equations in (1.2.8) hold. Thus,

$$f=1+\varepsilon e^{\eta_1} \tag{1.2.11}$$

provides a solution to the bilinear KdV equation (1.2.5). This means the "perturbation" formula (1.2.7) can be truncated to (1.2.11), and therefore (1.2.11) is independent of ε. Taking $\varepsilon=1$ in (1.2.11) and through (1.2.3) we have

$$u=2(\ln f)_{xx}=2[\ln(1+e^{\eta_1})]_{xx} \tag{1.2.12}$$

which is an 1SS of the KdV equation.

When $N=2$, from (1.2.10) we have $f^{(1)}=e^{\eta_1}+e^{\eta_2}$, and then (1.2.8b) we find

$$f^{(2)}=A_{12}e^{\eta_1+\eta_2}, \quad A_{12}=\Big(\frac{k_1-k_2}{k_1+k_2}\Big)^2. \tag{1.2.13}$$

Meanwhile, $f^{(j)}=0$ $(j=3,4,\cdots)$ solve the rest of equations in (1.2.8). Thus,

$$f=1+\varepsilon(e^{\eta_1}+e^{\eta_2})+\varepsilon^2A_{12}\,e^{\eta_1+\eta_2} \tag{1.2.14}$$

provides a second solution to the bilinear KdV equation (1.2.5), which will lead to a 2SS for the KdV equation via (1.2.3).

When $N = 3$, from (1.2.10) we can find a solution for (1.2.8):

$$\begin{aligned} f =& 1 + \varepsilon(e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) \\ &+ \varepsilon^2(A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3}) \\ &+ \varepsilon^3 A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}, \quad A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2. \end{aligned} \tag{1.2.15}$$

Again, through (1.2.3) it gives a 3SS to the KdV equation.

For a general number N, Hirota gave the following compact form:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1\le i<j}^{N} \mu_i \mu_j a_{ij}\right), \tag{1.2.16}$$

where η_j is defined as in (1.2.9), $e^{a_{ij}} = A_{ij}$, and the summation of μ means to take all possible $\mu_j = \{0, 1\}$ $(j = 1, 2, \cdots, N)$. NSS of the KdV equation is given by (1.2.3). (1.2.16) is a truncated form of (1.2.7) (we have taken $\varepsilon = 1$ since it is independent of ε). A proof of (1.2.16) satisfying (1.2.5) can be found in [20] and [2].

1.2.2 NSS of the KP(II) equation

The KP equation reads

$$(4u_t + 6uu_x + u_{xxx})_x + 3\sigma u_{yy} = 0, \quad (\sigma = \pm 1), \tag{1.2.17}$$

where when $\sigma = 1$ it is the KP(II) equation and when $\sigma = -1$ it is the KP(I) equation. We solve the KP(II) equation. By the transformation

$$u = 2(\ln f)_{xx}, \tag{1.2.18}$$

the KP(II) equation is bilinearized as

$$(4D_xD_t + D_x^4 + 3D_y^2)f \cdot f = 0. \tag{1.2.19}$$

With the expension

$$f = 1 + \sum_{i=1}^{\infty} f^{(i)}\varepsilon^i \tag{1.2.20}$$

the bilinear KP(II) equation (1.2.19) yields

$$\varepsilon^1: \quad (4\partial_{xt} + \partial_x^4 + 3\partial_y^3) f^{(1)} = 0, \tag{1.2.21a}$$

$$\varepsilon^2: \quad (4\partial_{xt} + \partial_x^4 + 3\partial_y^3) f^{(2)} = -\frac{1}{2}(4D_x D_t + D_x^4 + 3D_y^2) f^{(1)} \cdot f^{(1)}, \tag{1.2.21b}$$

$$\cdots\cdots .$$

For (1.2.21a) we can take $f^{(1)} = e^{kx+hy+\omega t}$, and k, h, ω satisfy

$$4k\omega + k^4 + 3h^2 = 0. \tag{1.2.22}$$

Compared with the KdV equation, here we have two free parameters. For a better parametrisation, we introduce $h = ak$, by which we have $\omega = -k(k^2 + 3a^2)/4$. here a is an arbitrary parameter, as free as k. In practice we take $k = p - q,\ a = p + q$, and it follows that

$$k = p - q, \quad h = p^2 - q^2, \quad \omega = -(p^3 - q^3). \tag{1.2.23}$$

Thus, we have

$$f^{(1)} = e^{\eta_1}, \tag{1.2.24}$$

where

$$\eta_i = (p_i - q_i)x + (p_i^2 - q_i^2)y - (p_i^3 - q_i^3)t + \eta_i^{(0)}, \quad p_i, q_i, \eta_i^{(0)} \in \mathbb{R}. \tag{1.2.25}$$

Then 1SS of the KP(II) equation can be given as (taking $\varepsilon = 1$)

$$u = 2(\ln f)_{xx} = \frac{(p_1 - q_1)^2}{2} \operatorname{sech}^2 \frac{\eta_1}{2}. \tag{1.2.26}$$

Similar to the KdV equation, we take

$$f^{(1)} = \sum_{i=1}^{N} e^{\eta_i}, \tag{1.2.27}$$

where η_i is defined by (1.2.25). When $N = 2$, corresponding to 2SS, we find (taking $\varepsilon = 1$)

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1 + \eta_2}, \tag{1.2.28a}$$

solves (1.2.19), where

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}. \tag{1.2.28b}$$

Continuing this procedure, for 3SS, f has the same structure as (1.2.15). For NSS, there is

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1\le i<j}^{N} \mu_i \mu_j a_{ij}\right), \tag{1.2.29}$$

where η_j is defined as (1.2.25), $e^{a_{ij}} = A_{ij}$ which is defined as (1.2.28b), and the summation of μ means to take all possible $\mu_j = \{0, 1\}$ $(j = 1, 2, \cdots, N)$. Then, NSS of the KP(II) equation is given via (1.2.18).

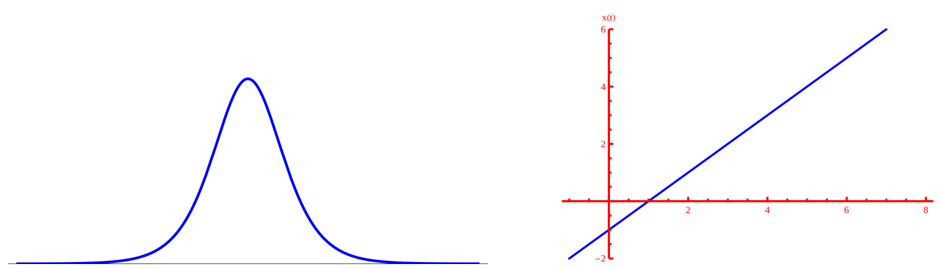
1.3 Asymptotic analysis of 2SS

1.3.1 The KdV equation

First, let us look at 1SS (1.2.12) of the KdV equation, i.e.

$$u = \frac{k_1^2}{2}\mathrm{sech}^2\eta_1, \quad \eta_1 = k_1 x - k_1^3 t + \eta_1^{(0)}, \tag{1.3.1}$$

which describes a solitary wave as depicted in Figure1.1(a). The maximum of the



(a) (b)

Figure 1.1: (a) 1SS of the KdV equation. (b) trajectory of the vortex of 1SS: $\eta_1 = 0$, where $k_1 = 1$, $\eta_1^{(0)} = 0$.

wave, i.e. amplitude, which is $\frac{k_1^2}{2}$, occurs when $\eta_1 = 0$; $\eta_1 = 0$, i.e.

$$x(t) = k_1^2 t - \frac{\eta_1^{(0)}}{k_1}, \tag{1.3.2}$$

is a straight line depicted in Figure1.1(b) for trajectory of the vortex of the wave; $x'(t) = k_1^2$ stands for velocity of the wave. One can see that the velocity is always positive, which means the solitary wave described by the KdV equation is of single direction.

2SS of the KdV equation exhibits scattering behavior, as depicted in Figture1.2 and 1.3. The 2SS is

$$u = 2(\ln f)_{xx}, \tag{1.3.3a}$$
$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1+\eta_2}, \tag{1.3.3b}$$

where

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, \quad A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{1.3.3c}$$

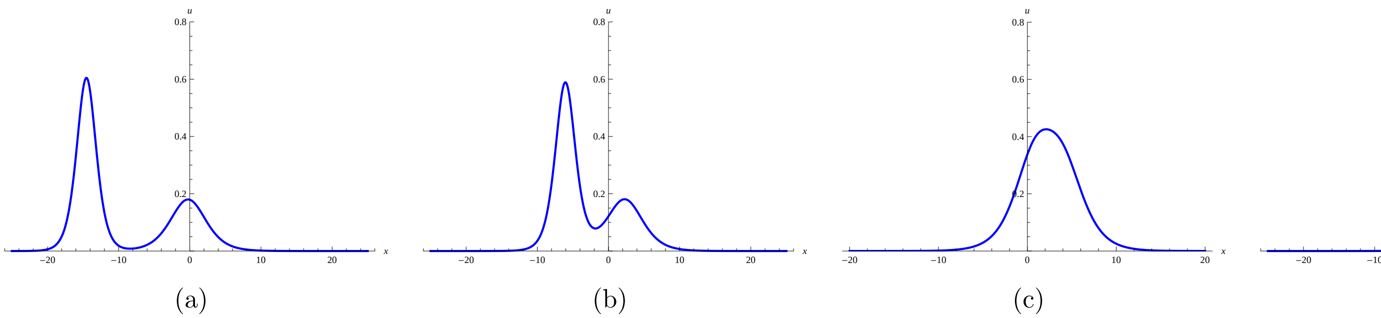


Figure 1.2: 2SS (1.3.3) of the KdV equation ($k_1 = 0.6,\ k_2 = 1.1,\ \eta_1^{(0)} = \eta_2^{(0)} = 0$). (a) $t = -12$, (b) $t = -5$, (c) $t = 1$, (d) $t = 10$.

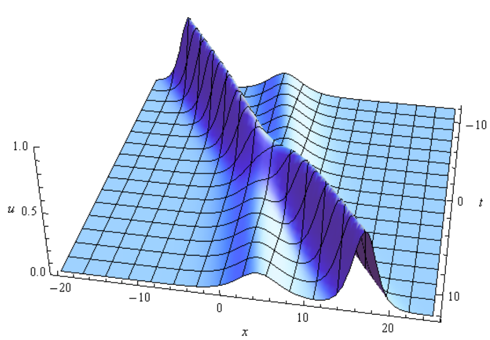


Figure 1.3: 2SS (1.3.3) of the KdV equation ($k_1 = 0.6,\ k_2 = 1.1,\ \eta_1^{(0)} = \eta_2^{(0)} = 0$).

Such a scattering behavior can be understood mathematically using asymptotic analysis, of which the generic procedure is the following.

We consider f in (1.3.3b) and assume $k_1 > k_2 > 0$ without loss of generality. Suppose $\eta_1 = c$ (c is certain constant) so that we can observe the 2SS along the straight line $\eta_1 = c$. To do that, we rewrite (1.3.3b) in the new coordinate system (η_1, t):

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1+\eta_2}, \tag{1.3.4a}$$

where

$$e^{\eta_2} = \exp\Big[\frac{k_2}{k_1}\eta_1 + k_2(k_1^2 - k_2^2)t + \eta_2^{(0)} - \frac{k_2}{k_1}\eta_1^{(0)}\Big]. \tag{1.3.4b}$$

Because of $k_1 > k_2 > 0$, we find

$$e^{\eta_2} \sim \begin{cases} 0, & t \to -\infty, \\ +\infty, & t \to +\infty. \end{cases}$$

Therefore in (η_1, t) we have

$$f \sim \begin{cases} 1 + e^{\eta_1}, & t \to -\infty, \\ e^{\eta_2}(1 + A_{12}e^{\eta_1}), & t \to +\infty. \end{cases} \tag{1.3.5}$$

Due to the gauge property of bilinear derivatives, the factor e^{η_2} in the above does not change solutions of the bilinear KdV equation (1.2.5), and does not change $u = 2(\ln f)_{xx}$ either. Thus, if we observe 2SS along the straight line $\eta_1 = c$, when $t \to -\infty$ we only see the 1SS

$$u = 2[\ln(1 + e^{\eta_1})]_{xx}; \tag{1.3.6a}$$

and when $t \to +\infty$, we see

$$u = 2[\ln(1 + A_{12}e^{\eta_1})]_{xx}. \tag{1.3.6b}$$

This is still the original 1SS (1.3.6a) (after interaction it has same amplitude and velocity as before) but there is a phase shift $-\frac{2}{k_1}\ln\big(\frac{k_1-k_2}{k_1+k_2}\big)$.

Similarly, in the coordinate system (η_2, t) we can see

$$e^{\eta_1} \sim \begin{cases} +\infty, & t \to -\infty, \\ 0, & t \to +\infty. \end{cases}$$

Then we have

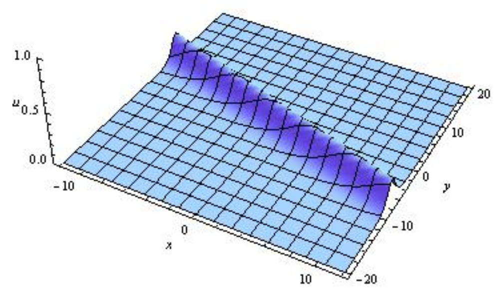
$$f \sim \begin{cases} e^{\eta_1}(1 + A_{12}e^{\eta_2}), & t \to -\infty, \\ 1 + e^{\eta_2}, & t \to +\infty. \end{cases} \tag{1.3.7}$$

Thus we can see that the soliton determined by k_2 keeps its amplitude and velocity before and after interaction but gains a phase shift $\frac{2}{k_2}\ln\big(\frac{k_1-k_2}{k_1+k_2}\big)$. Such phase shifts due to interaction can be seen in Figure 1.3.

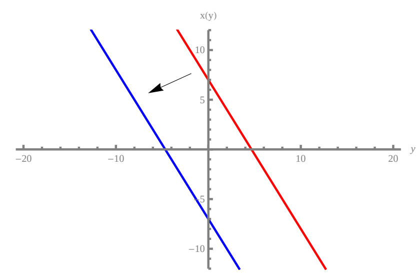
1.3.2 The KP(II) equation

The 1SS (1.2.26) of the KP(II) equation is depicted in Figure 1.4(a). At any given time t it exhibits like a straight line on (x, y) plane, with amplitude $(p_1 - q_1)^2/2$. The straight line is given by $\eta_1 = 0$ and it also provides the velocity by which the line is traveling on (x, y) plane:

$$(x'(t), y'(t)) = -(p^2 + pq + q^2)\Big(1, \frac{1}{p+q}\Big).$$



(a)



(b)

Figure 1.4: (a) 1SS of the KP(II). It is given by (1.2.26) with $(p_1, q_1, \eta_1^{(0)}) = (0.5, 1, 0), t = 0$. (b) Trajectory of the line soliton in (a): red line is for $t = -4$ and blue for $t = 4$.

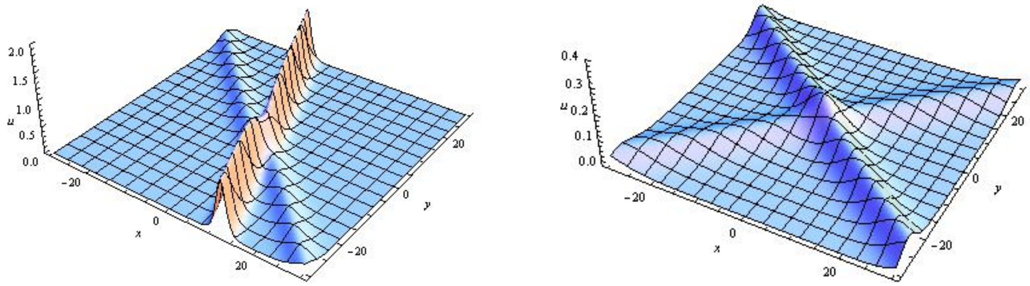
For the 2SS given by (1.2.18) with (1.2.28), at a given time t it behaves like two lines crossed in Figure 1.5. When t is fixed we can consider it as a constant and analysis asymptotic behaviors when $y \to \pm\infty$. When $A_{12} \neq 0$, the procedure is similar to the KdV case in §1.3.1, and here we skip it.

Let us consider the case of $A_{12} = 0$. Recalling in §1.3.1 for the KdV equation its 1SS (1.3.1) is completely determined by the k_1; in 2SS (1.3.3) if $k_1 = k_2$ then $A_{12} = 0$ and the 2SS degenerates to 1SS. However, for the KP(II) equation, its 1SS is determined by two parameters, p_1 and q_1. Particularly, on the basis of the special form of A_{12}, i.e. (1.2.28b),

$$A_{12} = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}, \tag{1.3.8}$$

for example, when $p_1 \neq p_2$ but $q_1 = q_2$, we have $A_{12} = 0$ and (1.2.28a) degenerates to

$$f = 1 + e^{\eta_1} + e^{\eta_2}, \tag{1.3.9}$$



(a) (b)

Figure 1.5: 2SS of the KP(II) equation. It is given by (1.2.18) with (1.2.28): (a) $(p_1, q_1, \eta_1^{(0)}) = (0.8, 0.2, 0)$, $(p_2, q_2, \eta_2^{(0)}) = (-0.5, 0.9, 0)$, $t = 0$; (b) $(p_1, q_1, \eta_1^{(0)}) = (-0.4, 0.7, 0)$, $(p_2, q_2, \eta_2^{(0)}) = (0.8, 0.4, 0)$, $t = 0$.

where η_i is given as (1.2.25). In this case, the 2SS does not degenerate to 1SS but exhibits resonance of two line solitons, as described in Figure 1.6.

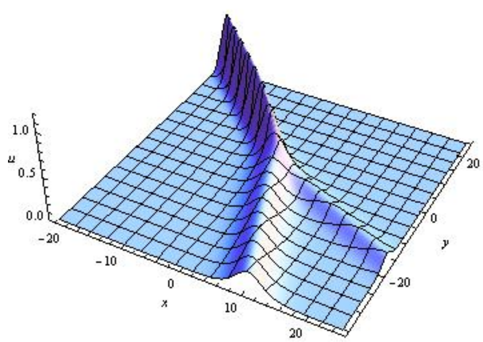


Figure 1.6: 2SS resonance of the KP(II) equation. It is given by (1.2.18) with (1.2.28), where $p_1 = 1.0, p_2 = -0.2$, $q_1 = q_2 = 0.5$, $\eta_1^{(0)} = \eta_2^{(0)} = 0$, $t = 0$.

Resonance of solitons can be understood as a behavior occurring when some parameters tend to be same ($q_1 = q_2$ can be considered as a result of $q_2 \to q_1$). Such a phenomena of solitary waves was studied by J.W. Miles [39], K. Ohkuma, M. Wadadi [47], etc.

Now let us give an asymptotic analysis for the resonance described in Figure1.6. Consider simplified case in which we take $t = 0$ and $\eta_i^{(0)} = 0$. Thus in (1.3.9),

$$\eta_i = k_i x + h_i y, \quad k_i = p_i - q_i, \; h_i = p_i^2 - q_i^2.$$

We rewrite η_2 in the coordinate frame $(\eta_1 = c, y)$:

$$\eta_2 = \frac{k_2}{k_1}\eta_1 + \frac{1}{k_1}(k_1h_2 - k_2h_1)y.$$

Using the data in Figure1.6 (noticing that $k_1 > 0,\ k_1h_2 - k_2h_1 > 0$), we find

$$e^{\eta_2} \sim \begin{cases} 0, & y \to -\infty, \\ +\infty, & y \to +\infty. \end{cases}$$

Thus, in the coordinate system (η_1, y) we have

$$f \sim \begin{cases} 1 + e^{\eta_1}, & y \to -\infty, \\ e^{\eta_2}, & y \to +\infty, \end{cases}$$

and for u in (η_1, y) we can see the following,

$$u \sim \begin{cases} \frac{k_1^2}{2}\mathrm{sech}^2\eta_1, & y \to -\infty, \\ 0, & y \to +\infty. \end{cases}$$

In a similar way, if in the coordinate system (η_2, t) we can see that

$$u \sim \begin{cases} \frac{k_2^2}{2}\mathrm{sech}^2\eta_2, & y \to -\infty, \\ 0, & y \to +\infty. \end{cases}$$

These can explain the two "legs" in Figure1.6 and there are no solitons along the same directions when $y \to +\infty$.

Next, let us find the soliton when $y \to +\infty$. In the coordinate system $(\eta_1 - \eta_2 = c, y)$ we rewrite η_i and find (noticing that with the date in Figure1.6 we have $k_1 - k_2 > 0,\ k_1h_2 - k_2h_1 > 0$)

$$e^{\eta_i} = e^{\frac{k_i}{k_1-k_2}(\eta_1-\eta_2)+\frac{k_1h_2-k_2h_1}{k_1-k_2}y} \sim \begin{cases} 0, & y \to -\infty, \\ +\infty, & y \to +\infty. \end{cases}$$

Then, in the coordinate system $(\eta_1 - \eta_2, y)$ there is

$$f = 1 + e^{\eta_2}(1 + e^{\eta_1-\eta_2}) \sim \begin{cases} 0, & y \to -\infty, \\ e^{\eta_2}(1 + e^{\eta_1-\eta_2}), & y \to +\infty. \end{cases}$$

Thus, for u in $(\eta_1 - \eta_2, y)$ we can see that

$$u \sim \begin{cases} 0, & y \to -\infty, \\ \frac{(k_1-k_2)^2}{2}\mathrm{sech}^2(\eta_1 - \eta_2), & y \to +\infty. \end{cases}$$

Asymptotic analysis is helpful to understand interaction of solitons and explain special scattering behaviors. For more examples of multi-soliton interactions and their asymptotic analysis, one can refer to the review paper [17] written by Jarmo Hietarinta. For the variety of resonance of line solitons of the KP(II) equation and their related interesting mathematical structures and applications, one can refer to [26–28] by Yuji Kodama, et al.

1.4 2SS of bilinear equations

It is amazing that many bilinear equations automatically admit 1SS and 2SS.

1.4.1 Bilinear equations of the KdV-type

Consider the following bilinear equation

$$P(D_{\mathbf{t}})f \cdot f = 0, \tag{1.4.1}$$

where P is an even polynomial, i.e. $P(\mathbf{t}) = P(-\mathbf{t})$ and satisfying $P(\mathbf{0}) = 0$. Such a bilinear equation (1.4.1) is call a bilinear equation of the KdV-type [13,23]. Assume that

$$f = 1 + e^{\eta_1}, \tag{1.4.2a}$$

where

$$\eta_1 = \mathbf{p_1} \cdot \mathbf{t} + \eta_1^{(0)}. \tag{1.4.2b}$$

It follows that

$$f \cdot f = 1 \cdot 1 + 1 \cdot e^{\eta_1} + e^{\eta_1} \cdot 1 + e^{\eta_1} \cdot e^{\eta_1}.$$

Since the terms $1 \cdot 1$ and $e^{\eta_1} \cdot e^{\eta_1}$ varnish under the action of $P(D_{\mathbf{t}})$, we have

$$P(D_{\mathbf{t}})f \cdot f = 2P(D_{\mathbf{t}})e^{\eta_1} \cdot 1 = 2P(\partial_{\mathbf{t}})e^{\eta_1} = 2P(\mathbf{p_1})e^{\eta_1}.$$

Thus, once $P(\mathbf{p_1}) = 0$, (1.4.2) is a solution of (1.4.1). We call $P(\mathbf{p_1}) = 0$ to be dispersion relation (DR) of the bilinear equation (1.4.1).

Consider

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}, \tag{1.4.3a}$$

where A_{12} is a constant to be determined,

$$\eta_i = \mathbf{p_i} \cdot \mathbf{t} + \eta_i^{(0)} \tag{1.4.3b}$$

satisfying DR

$$P(\mathbf{p_i}) = 0. \tag{1.4.3c}$$

Substitute (1.4.3a) into the equation (1.4.1) and making use of the DR (1.4.3c), it is easy to see that (1.4.3a) satisfies the equation (1.4.1) provided

$$A_{12} = -\frac{P(\mathbf{p_1} - \mathbf{p_2})}{P(\mathbf{p_1} + \mathbf{p_2})}. \tag{1.4.3d}$$

It is Hirota who first found this fact [23]. Here we note that "automatically" existing 2SS means there is no extra condition on $\mathbf{p_i}$ beyond the DR (1.4.3c).

1.4.2 Other cases

As an example let us look at the following bilinear equation system [15]

$$B(D_\mathbf{t})G \cdot F = 0, \tag{1.4.4a}$$

$$A(D_\mathbf{t})(F \cdot F + \epsilon G \cdot G) = 0, \tag{1.4.4b}$$

where A is an even polynomial and $\epsilon = \pm 1$. The above equation system admits 1SS;

$$F = 1, \quad G = e^{\eta_1}, \quad \eta_1 = \mathbf{p_1} \cdot \mathbf{t} + \eta_1^{(0)},$$

where η_1 satisfies DR: $B(\mathbf{p_1}) = 0$. One type of 2SS of (1.4.4) is:

$$F = 1 - A_{12} e^{\eta_1 + \eta_2}, \quad G = e^{\eta_1} + e^{\eta_2}, \tag{1.4.5a}$$

where $\eta_i = \mathbf{p_i} \cdot \mathbf{t} + \eta_i^{(0)}$ satisfies DR $B(\mathbf{p_i}) = 0$, and

$$A_{12} = -\epsilon \, \frac{A(\mathbf{p_1} - \mathbf{p_2})}{A(\mathbf{p_1} + \mathbf{p_2})}. \tag{1.4.5b}$$

Not any arbitrary bilinear equation (system) will automatically admit a 2SS. For example, the following bilinear equation system

$$B(D_\mathbf{t})G \cdot F = 0, \tag{1.4.6a}$$

$$A(D_\mathbf{t})F \cdot F = 2\epsilon |G|^2, \tag{1.4.6b}$$

where A is an even polynomial, $F \in \mathbb{R}(\mathbf{t})$ and $G \in \mathbb{C}(\mathbf{t})$. It has 1SS:

$$F = 1 + a\, e^{\eta_1 + \eta_1^*}, \quad G = e^{\eta_1}$$

where $\eta_1 = \mathbf{p_1} \cdot \mathbf{t} + \eta_1^{(0)}$, $\mathbf{p} \in \mathbb{C}^s$, $\eta_1^{(0)} \in \mathbb{C}$, $B(\mathbf{p_1}) = 0$, $a = \frac{-\epsilon}{A(\mathbf{p_1} + \mathbf{p_1}^*)}$, $*$ stands for complex conjugate, $|G|^2 = GG^*$. However, its 2SS does not exist automatically [16].

1.5 Hirota's integrability and 3SS condition

Take the KdV-type bilinear equation (1.4.1) as an example. For such an equation

$$P(D_{\mathbf{t}})f \cdot f = 0, \tag{1.5.1}$$

it is said to be Hirota-integrable if for all positive integers N, it has NSS of the form

$$f = 1 + \varepsilon \sum_{i=1}^{N} e^{\eta_i} + \{\text{finite number of higher-order terms in } \varepsilon\}, \tag{1.5.2}$$

without any further conditions on the parameters $\mathbf{p_i}$ beyond DR

$$P(\mathbf{p_i}) = 0. \tag{1.5.3}$$

In general, for a bilinear equation (system), when it has 1SS which is described by e^{η_i} ($\eta_i = \mathbf{p_i} \cdot \mathbf{t} + \eta_i^{(0)}$), there is a condition on e^{η_i}, for example, DR, which we call 1SS-condition. If the bilinear equation (system) allows a solution which is described by a polynomial of arbitrarily many $\{e^{\eta_i}\}$ (where each e^{η_i} describes one single soliton), and there is no extra condition on each e^{η_i} besides 1SS-condition, we say the bilinear equation (system) is **Hirota integrable**.

Hirota presented the following form for NSS of the KdV-type bilinear equation (1.5.1) [23]:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1\le i<j}^{N} \mu_i \mu_j a_{ij}\right), \tag{1.5.4a}$$

where $\eta_j = \mathbf{p_j} \cdot \mathbf{t} + \eta_j^{(0)}$ are defined as (1.4.3b),

$$P(\mathbf{p_i}) = 0, \quad e^{a_{ij}} = A_{ij} = -\frac{P(\mathbf{p_i} - \mathbf{p_j})}{P(\mathbf{p_i} + \mathbf{p_j})}, \tag{1.5.4b}$$

the summation of μ means to take all possible $\mu_j = \{0, 1\}$ ($j = 1, 2, \cdots, N$); and further than that, the following condition is needed:

$$\sum_{\sigma=\pm 1} \left[P(\sum_{j=1}^{N} \sigma_j \mathbf{p_j}) \times \Big(\prod_{1\le i<j\le N} \sigma_i \sigma_j P(\sigma_i \mathbf{p_i} - \sigma_j \mathbf{p_j}) \Big) \right] = 0, \tag{1.5.5}$$

the summation of σ means to take all possible $\sigma_j = \{1, -1\}$ ($j = 1, 2, \cdots, N$). This condition holds automatically for the $N = 2$ case.

Those bilinear equations (systems) that automatically have 2SS may not have 3SS; even when they have 3SS, they might not Hirota integrable (there may be extra condition on $\mathbf{p_i}$). One famous example is the (2+1)-dimensional sine-Gordon equation (see [21])

$$\varphi_{xx} + \varphi_{yy} - \varphi_{tt} = \sin\varphi, \tag{1.5.6}$$

which, by the transformation

$$\varphi = 4\arctan\frac{g}{f}, \tag{1.5.7}$$

is transformed into bilinear form

$$(D_x^2 + D_y^2 - D_t^2)g\cdot f = gf, \tag{1.5.8a}$$
$$(D_x^2 + D_y^2 - D_t^2)(f\cdot f - g\cdot g) = 0. \tag{1.5.8b}$$

The above equations have 2SS automatically (see (1.4.4) and (1.4.5)), and its 3SS reads

$$f = 1 + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3}, \tag{1.5.9a}$$
$$g = e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}, \tag{1.5.9b}$$

where

$$\eta_i = a_i x + b_i y - c_i t + \eta_i^{(0)}, \tag{1.5.9c}$$
$$\mathrm{DR}:\ a_i^2 + b_i^2 - c_i^2 = 1, \tag{1.5.9d}$$
$$A_{ij} = \frac{(a_i-a_j)^2 + (b_i-b_j)^2 - (c_i-c_j)^2}{(a_i+a_j)^2 + (b_i+b_j)^2 - (c_i+c_j)^2}, \tag{1.5.9e}$$

and an extra condition is needed:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0. \tag{1.5.9f}$$

In the above results, for the 2SS which exists automatically, the only condition on $\{a_i, b_i, c_i\}$ is the DR (1.5.9d); however, if 3SS exists, in addition to the DR, the extra condition (1.5.9f) is rquired. Thus, the equation system (1.5.8) is an bilinear system that possesses 3SS but is not integrable in Hirota's sense. This is an famous example. Hirota once proposed such a question in [23] for the KdV-type bilinear equations: "Under what conditions does P satisfy the identity (1.5.5)?"

3SS-condition is specially referred to the case: a bilinear equation (system) has a 3SS, and the condition on each η_i is nothing beyond the 1SS-condition. In general, it is conjectured that for a bilinear equation (system) the 3SS-condition is equivalent to the Hirota's integrability.

In the following, let us take the KdV-type bilinear equation as an example to see how the form of 3SS is determined by the elastic scattering behavior of multi-solitons.

The property of elastic scattering of multi-solitons requires the following: "removing a soliton from NSS, the left (N-1) solitons keep the elastic scattering structure of (N-1)SS". Removing a soliton means the soliton is far from others —— which, mathematically, can be done through two ways: either $e^{\eta_k} \to 0$ or $e^{\eta_k} \to \infty$ (refer to the asymptotic analysis of 2SS in §1.3.1). For the KdV-type bilinear equation (1.5.1), it automatically has 2SS (see (1.4.3)):

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}, \tag{1.5.10a}$$

where

$$P(\mathbf{p_i}) = 0, \tag{1.5.10b}$$

$$A_{ij} = -\frac{P(\mathbf{p_i} - \mathbf{p_j})}{P(\mathbf{p_i} + \mathbf{p_j})}. \tag{1.5.10c}$$

With the requirement of elastic scattering, if there is no further condition on $\mathbf{p_i}$ beyond (1.5.10b), after analysis we can find 3SS of equation (1.5.10) (if it has) can only be in the following form

$$\begin{aligned} f =& 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \\ &+ A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3} + A_{23} e^{\eta_3+\eta_3} \\ &+ A_{12} A_{13} A_{23} e^{\eta_1+\eta_2+\eta_3}, \end{aligned} \tag{1.5.11}$$

and A_{ij} must be defined as (1.5.10c). If we start from the 3SS (1.5.11) and using the requirement of elastic scattering once again, we can reach a form for 4SS. Continuing such a procedure one can obtain 5SS, 6SS, $\cdots\cdots$.

Thus, for the KdV-type bilinear equation (1.5.1), if we only require the DR (1.5.10b) and elastic scattering property, its NSS (if it has) can only be the form (1.5.4), which is the same as the NSS given in (1.2.16).

Now, if (1.5.4) provides a solution to (1.5.1), then (1.5.1) is Hirota's integrable. However, not all the bilinear equations have their 3SS which is only built on DR: $P(\mathbf{p_i}) = 0$. In general, it is conjectured that for a bilinear equation (system) the 3SS-condition is equivalent to the Hirota's integrability. In 1987 Jarmo Hietarinta found that the KdV-type bilinear equations that satisfy 3SS-condition are:

$$
\begin{aligned}
&(D_x^4 - 4D_xD_t + 3D_y^2)f \cdot f = 0,\\
&(D_x^3D_t + aD_x^2 + D_tD_y)f \cdot f = 0,\\
&[D_xD_t(D_x^2 + \sqrt{3}D_xD_t + D_t^2) + aD_x^2 + bD_xD_t + cD_t^2]f \cdot f = 0,\\
&(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y)f \cdot f = 0,
\end{aligned}
$$

etc., where, a, b, c are arbitrary constants. For more results one can refer to [13–16] and [18].

Chapter 2

Bilinearity and Transformations

2.1 Bilinear identities

To study bilinear Bäcklund transformation, we introduce some bilinear identities. First, we introduce a way to generate a large class of bilinear identities.

Property 2.1.1. *The following equality holds:*

$$\begin{aligned} & e^{D_1}(e^{D_2}a\cdot b)\cdot(e^{D_3}c\cdot d) \\ = \ & e^{\frac{1}{2}(D_2-D_3)}(e^{\frac{1}{2}(D_2+D_3)+D_1}a\cdot d)\cdot(e^{\frac{1}{2}(D_2+D_3)-D_1}c\cdot b), \end{aligned} \tag{2.1.1}$$

where $D_i=\varepsilon_i D_x+\delta_i D_t$, $\varepsilon_i,\delta_i\in\mathbb{R}$, and a,b,c,d are sufficiently smooth functions of (x,t).

It can be verified directly.

In the following we give examples to show how we use the formula (2.1.1) works in generating bilinear identities.

Example 2.1.1: Taking $D_2=D_3$ in (2.1.1) yields

$$e^{D_1}(e^{D_2}a\cdot b)\cdot(e^{D_2}c\cdot d)=(e^{D_2+D_1}a\cdot d)\cdot(e^{D_2-D_1}c\cdot b). \tag{2.1.2}$$

Next, taking $D_1=\delta D_x,\ D_2=\varepsilon D_x$ in (2.1.2) and expanding the exponential functions of both sides, we have

$$\begin{aligned} & (1+\delta D_x+\cdots)[(1+\varepsilon D_x+\cdots)a\cdot b)]\cdot[(1+\varepsilon D_x+\cdots)c\cdot d)] \\ = \ & [(1+(\varepsilon+\delta)D_x+\frac{1}{2}(\varepsilon+\delta)^2D_x^2+\cdots)a\cdot d)] \\ & \quad \times[(1+(\varepsilon-\delta)D_x+\frac{1}{2}(\varepsilon-\delta)^2D_x^2+\cdots)c\cdot b)]. \end{aligned}$$

The coefficient of the term $\varepsilon\delta$ leads to a bilinear identity

$$D_x[(D_x a \cdot b)\cdot(cd) - (D_x c\cdot d)\cdot(ab)] = (D_x^2 a\cdot d)bc - (D_x^2 c\cdot b)ad. \tag{2.1.3}$$

Example 2.1.2: Taking $D_2 = D_3, b = c, d = a$ in (2.1.1) yields

$$e^{D_1}(e^{D_2}a\cdot c)\cdot(e^{D_2}c\cdot a) = (e^{D_2+D_1}a\cdot a)\cdot(e^{D_2-D_1}c\cdot c). \tag{2.1.4}$$

Then we take $D_1 = \varepsilon D_x,\ D_2 = \delta D_t$, and from the coefficient of $\varepsilon\delta$ term in the expansion we find

$$2D_x(D_t a\cdot c)\cdot(ac) = (D_x D_t a\cdot a)c^2 - (D_x D_t c\cdot c)a^2; \tag{2.1.5}$$

from ε^4 term we find

$$2D_x[(D_x^3 a\cdot c)\cdot(ac) - 3(D_x^2 a\cdot c)\cdot(D_x a\cdot c)] = (D_x^4 a\cdot a)c^2 - (D_x^4 c\cdot c)a^2. \tag{2.1.6}$$

In general, by specially taking D_i in the formula (2.1.1) and comparing coefficients of $\{\varepsilon,\delta\}$, one can obtain variety of bilinear identities. They play important roles in constructing bilinear Bäcklund transformations and nonlinear superposition formulas from bilinear Bäcklund transformations.

2.2 Bilinear Bäcklund transformations

In this section we take the bilinear KdV equation (1.2.5), i.e.

$$(D_x D_t + D_x^4)f\cdot f = 0 \tag{2.2.1}$$

as an example, to explain how a bilinear Bäcklund transformation is constructed and how it works in finding solutions.

Suppose that g is also a solution of (2.2.1), i.e.

$$(D_x D_t + D_x^4)g\cdot g = 0. \tag{2.2.2}$$

Then the following holds

$$g^2(D_x D_t + D_x^4)f\cdot f - f^2(D_x D_t + D_x^4)g\cdot g = 0, \tag{2.2.3}$$

i.e.

$$[(D_x D_t f\cdot f)g^2 - (D_x D_t g\cdot g)f^2] + [(D_x^4 f\cdot f)g^2 - (D_x^4 g\cdot g)f^2] = 0. \tag{2.2.4}$$

Now, employing the identities (2.1.5) and (2.1.6), (taking $a = f, c = g$), one can rewrite (2.2.4) as

$$2D_x[(D_x^3 + D_t)f \cdot g] \cdot (fg) + 6D_x[(D_x f \cdot g) \cdot (D_x^2 f \cdot g)] = 0. \tag{2.2.5}$$

Next, introduce

$$D_x^2 f \cdot g = \lambda f g, \tag{2.2.6a}$$

where λ is a constant independent of x, by which (2.2.5) yields

$$2D_x[(D_x^3 + D_t + 3\lambda D_x)f \cdot g] \cdot (fg) = 0.$$

Then we can take

$$(D_x^3 + D_t + 3\lambda D_x)f \cdot g = 0. \tag{2.2.6b}$$

(2.2.6a,b) compose a bilinear equation system which provides a bilinear Bäcklund transformation of the KdV equation (1.2.1). In fact, from the above procedure we can see that when (2.2.6a) and (2.2.6b) hold, equation (2.2.3) holds; if f is a solution to (2.2.1), so is g to (2.2.2), due to (2.2.3).

In many cases, using a bilinear Bäcklund transformation to calculate soliton solutions is not as convenient as directly using the orginal bilinear equation. Now we show some details. First, taking $\lambda = \frac{k_1^2}{4}$ and $f = 1$ (noticing that $f = 1$ is a solution to (2.2.1)), and substituting them into (2.2.6), one can find g needs to satisfy the following

$$\begin{aligned} g_{xx} &= \frac{k_1^2}{4} g, \\ g_t + g_{xxx} &+ \frac{3}{4} k_1^2 g_x = 0. \end{aligned}$$

As a solution we find

$$g = g_1 = e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}},$$

where

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, \quad k_i, \eta_i \in \mathbb{R}. \tag{2.2.7}$$

Thus, 1SS of the KdV equation (1.2.1) can be expressed by $u = 2(\ln g_1)_{xx}$.

Next, taking $f = g_1,\ \lambda = \frac{k_2^2}{4}$ and substituting them into (2.2.6), we have

$$(D_x^2 - \frac{k_2^2}{4}) g \cdot (e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}) = 0, \tag{2.2.8a}$$

$$(D_x^3 + D_t + \frac{3}{4} k_2^2 D_x) g \cdot (e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}) = 0. \tag{2.2.8b}$$

To get a solution we assume g to be the following form:

$$g = g_2 = \alpha(e^{\frac{\eta_1+\eta_2}{2}} + e^{-\frac{\eta_1+\eta_2}{2}}) + \beta(e^{\frac{\eta_1-\eta_2}{2}} + e^{-\frac{\eta_1-\eta_2}{2}}),$$

where η_i is defined as (2.2.7) and α, β are undetermined constants. Substituting the above g into (2.2.8) we find: when $\alpha = k_1 - k_2$ and $\beta = -(k_1 + k_2)$, g satisfies (2.2.8).

Next, we can take $f = g_2,\ \lambda = \frac{k_3^2}{4}$, and from (2.2.6) we solve out solution $g = g_3$. However, it is obviously the above procedure is not as "mechanized" as the one we used in §1.2 to calculate NSS from the bilinear KdV equation (1.2.5). In general we can successively take $\lambda = \frac{k_j^2}{4},\ j = 1, 2, \cdots, N$ to calculate higher order solutions; for generic N, g has the following expression:

$$g_N = \sum_{\varepsilon=\pm 1} \left[\prod_{1\leq j<l}^{N} \varepsilon_l(\varepsilon_j k_j - \varepsilon_l k_l) e^{\frac{1}{2}\sum_{j=1}^{N} \varepsilon_j \eta_j} \right], \tag{2.2.9}$$

where η_j is defined as (2.2.7), the summation over ε means to take all possible $\varepsilon_j = \{1, -1\}$ $(j = 1, 2, \cdots, N)$.

A proof that (2.2.9) satisfies the bilinear KdV equation (1.2.5) will be given in Chapter **??** by making use of Wronskians.

2.3 Deformations of bilinear BTs

We have already seen that using bilinear Bäcklund transformation (2.2.6) to calculate soliton solutions is not as convenient as using the original bilinear equation (1.2.5). The reason is $f = 1,\ g = 1$ are not a solution pair to (2.2.6). To change such a situation, we try deforming bilinear Bäcklund transformations.

In (2.2.6) we replace f and g with $e^{\xi_1} f$ and $e^{\xi_2} g$, respectively, i.e.

$$f \to e^{\xi_1} f, \quad g \to e^{\xi_2} g, \tag{2.3.1}$$

where $\xi_i = p_i x + q_i t + \xi_i^{(0)},\ p_i, q_i, \xi_i^{(0)} \in \mathbb{R}$. Noticing that solutions of the KdV equation can be expressed through $u = 2(\ln f)_{xx}$ or $u = 2(\ln g)_{xx}$, such a replacement does not change solutions; of course, due to gauge property (1.1.11) of bilinear equations, such a replacement does not change (2.2.1) and (2.2.2), either. Using identity (1.1.8), we have

$$
\begin{aligned}
D_x(e^{\xi_1}f)\cdot(e^{\xi_2}g) &= e^{\xi_1+\xi_2}[(p_1-p_2)fg+D_xf\cdot g],\\
D_x^2(e^{\xi_1}f)\cdot(e^{\xi_2}g) &= e^{\xi_1+\xi_2}[(p_1-p_2)^2fg+2(p_1-p_2)D_xf\cdot g+D_x^2f\cdot g],\\
D_x^3(e^{\xi_1}f)\cdot(e^{\xi_2}g) &= e^{\xi_1+\xi_2}[(p_1-p_2)^3fg+3(p_1-p_2)^2D_xf\cdot g\\
&\qquad\qquad\qquad +3(p_1-p_2)D_x^2f\cdot g+D_x^3f\cdot g],\\
D_t(e^{\xi_1}f)\cdot(e^{\xi_2}g) &= e^{\xi_1+\xi_2}[(q_1-q_2)fg+D_xf\cdot g].
\end{aligned}
$$

By means of them we rewrite (2.2.6) into

$$
[D_x^2+2(p_1-p_2)D_x]f\cdot g=[\lambda-(p_1-p_2)^2]fg, \tag{2.3.2a}
$$

$$
\begin{aligned}
&\{D_t+D_x^3+3(p_1-p_2)D_x^2+3[\lambda+(p_1-p_2)^2]D_x\}f\cdot g\\
&\qquad\qquad = -[(q_1-q_2)+(p_1-p_2)^3+3\lambda(p_1-p_2)^2]fg.
\end{aligned} \tag{2.3.2b}
$$

Introducing

$$
2(p_1-p_2)=\lambda',\quad \lambda=(p_1-p_2)^2,\quad (q_1-q_2)+4(p_1-p_2)^3=0,
$$

so that we can simplify (2.3.2a) to

$$
(D_x^2+\lambda' D_x)f\cdot g=0, \tag{2.3.3a}
$$

and further, eliminating D_x^2 term in (2.3.2b), we reach

$$
(D_t+D_x^3)f\cdot g=0. \tag{2.3.3b}
$$

(2.3.3a,b) compose a deformed bilinear Bäcklund transformation of the KdV equation (1.2.1).

Compared with (2.2.6), (2.3.3) admits solutions $f=g=1$, which bring convenience in calculation: it allows to calculate f and g by a perturbation expansion. Assuming

$$
f=1+\sum_{i=1}^{\infty}f^{(i)}\varepsilon^i,\quad g=1+\sum_{i=1}^{\infty}g^{(i)}\varepsilon^i \tag{2.3.4}
$$

and substituting them into (2.3.3) yield

$$(\partial_x^2+\lambda'\partial_x)(f^{(1)}-g^{(1)})=0, \tag{2.3.5a}$$
$$(\partial_x^2+\lambda'\partial_x)(f^{(2)}-g^{(2)})=-(D_x^2+\lambda' D_x)f^{(1)}\cdot g^{(1)}, \tag{2.3.5b}$$
$$(\partial_x^2+\lambda'\partial_x)(f^{(3)}-g^{(3)})=-(D_x^2+\lambda' D_x)(f^{(1)}\cdot g^{(2)}+f^{(2)}\cdot g^{(1)}), \tag{2.3.5c}$$
$$\cdots\cdots;$$
$$(\partial_t+\partial_x^3)(f^{(1)}-g^{(1)})=0, \tag{2.3.5d}$$
$$(\partial_t+\partial_x^3)(f^{(2)}-g^{(2)})=-(D_t+D_x^3)f^{(1)}\cdot g^{(1)}, \tag{2.3.5e}$$
$$(\partial_t+\partial_x^3)(f^{(3)}-g^{(3)})=-(D_t+D_x^3)(f^{(1)}\cdot g^{(2)}+f^{(2)}\cdot g^{(1)}), \tag{2.3.5f}$$
$$\cdots\cdots.$$

To get solutions, first we can take $g^{(j)}=0,\ (j\geq 1)$, which means we take zero as a seed solution in the transformation. Taking $\lambda'=-k_1$, from (2.3.5a,d) we find

$$f^{(1)}=e^{\eta_1},\quad \eta_i=k_i x-k_i^3 t+\eta_i^{(0)},\ k_i,\eta_i^{(0)}\in\mathbb{R}, \tag{2.3.6}$$

where η_i is defined as (1.2.9). For those $f^{(j)},\ (j\geq 2)$ we can trivially take them to be 0. Thus, 1SS is obtained as $u=2[\ln(1+f^{(1)})]_{xx}$.

Next, still considering g as a new seed solution and taking $g^{(1)}=e^{\eta_1},\ g^{(j)}=0,\ (j\geq 2)$, in the following we will see that the deformed bilinear Bäcklund transformation does bring some new aspects.

Case 1: Taking $\lambda'=-k_2$ and assuming

$$f^{(1)}=ae^{\eta_1}+be^{\eta_2},$$

where η_i is defined as (2.3.6), from (2.3.5a,d) we find $a=-\frac{k_1+k_2}{k_1-k_2}$ and b can be an arbitrary constant. Next, from (2.3.5b,e) we find

$$f^{(2)}=-\frac{b}{a}e^{\eta_1+\eta_2}.$$

For $f^{(j)},\ (j\geq 3)$, we can take them to be 0. Thus, 2SS is obtained as $u=2(\ln f)_{xx}$, where

$$f=1+ae^{\eta_1}+be^{\eta_2}-\frac{b}{a}e^{\eta_1+\eta_2},\quad a=-\frac{k_1+k_2}{k_1-k_2}. \tag{2.3.7}$$

Case 2: Still take $\lambda'=-k_1$ (as in getting 1SS). In this case we have

$$f^{(1)}=\zeta_1 e^{\eta_1},\quad f^{(2)}=e^{2\eta_1},$$

where η_1 is defined as (2.3.6), $\zeta_1 = 2k_1(x-3k_1^2 t)+\zeta^{(0)}$, $\zeta^{(0)} \in \mathbb{R}$, and $f^{(j)}=0$, $(j \geq 3)$. The obtained solution is $u = 2(\ln f)_{xx}$, where

$$f = 1 - \zeta_1 e^{\eta_1} + e^{2\eta_1}, \tag{2.3.8}$$

The solution is called double-pole solution (corresponding to the case $k_2 \to k_1$).

The above are two examples. If in (2.3.7) we scale a, b to be 1 by redefining the constants $\eta_i^{(0)}$, f can be written as

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1+\eta_2}, \quad A_{12} = \Big(\frac{k_1-k_2}{k_1+k_2}\Big)^2,$$

which is the standard Hirota's form for 2SS of the KdV equation. If in (2.3.7) we take $b = -a$ and the limit $k_2 \to k_1$, we can obtain (2.3.8). Double-pole solutions correspond to the case that the transmission coefficient $T(k)$ in the Inverse Scattering Transform has a double pole (note that simple poles leads to solitons). Double-pole solutions can be obtained from several ways (e.g. [9, 31, 55]).

The deformed bilinear Bäcklund transformations have advantages in allowing more freedom in calculations and obtaining more kinds of solutions. If we keep taking $\lambda' = 0$ in each step of the transformations, we may obtain high order rational solutions. Besides, rational solutions can also be derived directly from bilinear equations [23], or from determinantal approach.For more examples on deformed bilinear Bäcklund transformations, one can refer to [5, 8], etc.

2.4 Bäcklund transformations and Lax pairs

A bilinear Bäcklund transformation appears as an equation system and actually requires compatibility among these equations. This fact brings Bäcklund transformations and Lax pairs together.

For the KdV equation (1.2.1), i.e.

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.4.1}$$

its Lax pair reads (see Appendix A.1)

$$\phi_{xx} + u\phi = \lambda\phi, \tag{2.4.2a}$$
$$\phi_t = \phi_{xxx} + 3(\lambda + u)\phi_x. \tag{2.4.2b}$$

Noticing that in the bilinear Bäcklund transformation (2.2.6), f and g correspond to two solutions of the KdV equation: $u = 2(\ln f)_{xx}$ and $\widetilde{u} = 2(\ln g)_{xx}$, we put them together and we have

$$\widetilde{u} = u + 2(\ln \phi)_{xx}, \quad \phi = \frac{g}{f}, \tag{2.4.3}$$

which is as same as the relation of two solutions of the KdV equation obtained from the Darboux transformation [38].

In the bilinear Bäcklund transformation (2.2.6), taking

$$u = 2(\ln f)_{xx}, \quad \phi = \frac{g}{f}, \tag{2.4.4}$$

and rewriting (2.2.6) in terms of u and ϕ directly yield the Lax pair (2.4.2) for the KdV equation. In the reverse direction, substituting (2.4.4) into the Lax pair (2.4.2), one can find the bilinear Bäcklund transformation (2.2.6).

In general, once we have a bilinear equation we can construct its bilinear Bäcklund transformation, and with suitable substitutions from the bilinear Bäcklund transformation one can obtained a Lax pair for the original nonlinear equation.

There is a nonlinear Bäcklund transformation which was given by Wahlquist and Estabrook in 1973 [57]:

$$(\widetilde{w} + w)_x = 2\lambda - \frac{1}{2}(\widetilde{w} - w)^2, \tag{2.4.5a}$$

$$(\widetilde{w} - w)_t = \frac{1}{2}[(\widetilde{w} - w)^3]_x - 6\lambda(\widetilde{w} - w)_x - (\widetilde{w} - w)_{xxx}, \tag{2.4.5b}$$

where w satisfies the potential KdV equation (1.2.2), i.e. $u = w_x$ satisfying the KdV equation (1.2.1). Once $\widetilde{w}$ is solved out from (2.4.5), $\widetilde{u} = \widetilde{w}_x$ provides a new solution to the KdV equation. For the derivation of the Bäcklund transformation (2.4.5), one can refer to Appendix A.2.

The Bäcklund transformation (2.4.5) can be bilinearized [23]. Taking

$$w = 2(\ln f)_x, \quad \widetilde{w} = 2(\ln g)_x, \tag{2.4.6}$$

and substituting them into (2.4.5), one can obtain the Bäcklund transformation (2.2.6).

With (2.4.6) and $\phi = g/f$, which yields $\widetilde{w} - w = 2(\ln \phi)_x$, from (2.4.5) one can directly obtain the Lax pair (2.4.2) of the KdV equation.

Note: Lax pair, bilinear Bäcklund transformation and nonlinear Bäcklund transformation provide different forms for a same thing. Around the year 1974, many researchers considered relations between Bäcklund transformations and Lax pairs, *see [6,22,32,42,56], etc. Besides, Lambert and Springael also discussed the relations between bilinear Bäcklund transformations and Lax pairs in "Bilinear Integrable Systems: From Classical to Quatum, Continuous to Discrete" [34].*

2.5 BTs and superposition formulas

2.5.1 Nonlinear BTs and superposition formulas

How can we use a Bäcklund transformation to generate solutions? In the nonlinear Bäcklund transformation (2.4.5), taking $w = 0$ and $\lambda = k_1^2$, we have

$$\widetilde{w}_x = 2k_1^2 - \frac{1}{2}\widetilde{w}^2, \tag{2.5.1a}$$

$$\widetilde{w}_t = \frac{1}{2}(\widetilde{w}^3)_x - 6k_1^2\widetilde{w}_x - \widetilde{w}_{xxx}. \tag{2.5.1b}$$

From (2.5.1a) we can assume

$$\widetilde{w} = 2k_1 \tanh(kx + c(t)),$$

where $c(t)$ is undetermined. Substituting it into (2.5.1b) we find $c(t) = 4k_1^3 t + \eta_1^{(0)}$. Thus,

$$\widetilde{w} = 2k_1 \tanh \eta_1, \quad \eta_i = k_i x + 4k_i^3 t + \eta_i^{(0)}, \quad (k_i, \eta_i^{(0)} \in \mathbb{R}) \tag{2.5.2}$$

provides a solution to the potential KdV equation (1.2.2) and $u = \widetilde{w}_x$ is an 1SS of the KdV equation (1.2.1).

Next, taking w to be (2.5.2) and substituting it into the Bäcklund transformation (2.4.5) to solve $\widetilde{w}$, (taking $\lambda = k_2^2$), one will get a 2SS for the KdV equation. However, we have to make use of quadrature and obviously this is not as convenient as using bilinear Bäcklund transformation.

Once we get the first few solutions from the Bäcklund transformation (2.4.5), a recursive relation of these solutions can be built, from which new solutions can easily be generated. Such a recursive relation is referred to as a nonlinear superposition formula of solutions. Next, let us derive the nonlinear superposition formula for the KdV equation. We only use equation (2.4.5a) in the Bäcklund transformation.

Next, we start from (2.4.5a) with a seed solution w, denote $\widetilde{w} = w_1$ when taking $\lambda = \lambda_1$ and $\widetilde{w} = w_2$ when taking $\lambda = \lambda_2$, i.e.

$$(w_1 + w)_x = 2\lambda_1 - \frac{1}{2}(w_1 - w)^2, \tag{2.5.3a}$$

$$(w_2 + w)_x = 2\lambda_2 - \frac{1}{2}(w_2 - w)^2. \tag{2.5.3b}$$

Then, using (2.4.5a) with a new seed $w = w_1$ and $\lambda = \lambda_2$, the obtained solution being denoted by $\widetilde{w} = w_{12}$, we have

$$(w_{12} + w_1)_x = 2\lambda_2 - \frac{1}{2}(w_{12} - w_1)^2; \tag{2.5.4a}$$

and taking $w = w_2,\ \lambda = \lambda_1$ in (2.4.5a) and denoting $\widetilde{w} = w_{21}$, yields

$$(w_{21} + w_2)_x = 2\lambda_1 - \frac{1}{2}(w_{21} - w_2)^2. \tag{2.5.4b}$$

The above procedure can be described as in Figure 2.1. The question is whether

$$\begin{tikzcd}
 & w_1 \arrow[dr, dash, "\lambda_2"] & \\
w \arrow[ur, dash, "\lambda_1"] \arrow[dr, dash, "\lambda_2"'] & & w_{12} = w_{21}? \\
 & w_2 \arrow[ur, dash, "\lambda_1"'] &
\end{tikzcd}$$

Figure 2.1: Permutability property of solutions based on Bäcklund transformation.

w_{12} and w_{21} are same. To answer this question, we eliminate $w_{1,x}$ from (2.5.3a) and (2.5.4a) and we reach

$$w_1 = \frac{1}{2}(w_{12} + w) + \frac{2(\lambda_1 - \lambda_2)}{w_{12} - w} + [\ln(w_{12} - w)]_x.$$

Substituting it into (2.5.4a) we find

$$\begin{aligned}\lambda_1 + \lambda_2 =& (w_{12} + w)_x + [\ln(w_{12} - w)]_{xx} + \frac{1}{2}[\ln(w_{12} - w)]_x^2 \\ &+ \frac{1}{8}(w_{12} - w)^2 + \frac{2(\lambda_1 - \lambda_2)^2}{(w_{12} - w)^2}.\end{aligned} \tag{2.5.5}$$

This can be viewed as an ODE for both w_{12} and because it is invariant if switching λ_1 and λ_2 in the equation. Thus, once we impose same initial condition on w_{12} and w_{21}, we will get $w_{12} = w_{21}$.

To obtain a neat form of the recursive relation for the solutions, from (2.5.3a) and (2.5.3b) we have

$$(w_1 - w_2)_x = 2(\lambda_1 - \lambda_2) - \frac{1}{2}(w_1 - w_2)(w_1 + w_2 - 2w);$$

and from (2.5.4a) and (2.5.4b) we have (noticing that $w_{12} = w_{21}$)

$$(w_1 - w_2)_x = -2(\lambda_1 - \lambda_2) - \frac{1}{2}(w_1 - w_2)(w_1 + w_2 - 2w_{12}).$$

Eliminating derivative terms from them yields

$$4(\lambda_1-\lambda_2)=(w_1-w_2)(w_{12}-w), \tag{2.5.6}$$

which is referred to as the nonlinear superposition formula of solutions of the (potential) KdV equation, also known as the Bianchi identity[1], also called the discrete potential KdV equation [19,45]. As a discrete equation, (2.5.6) is usually written as

$$(w_{n+1,m}-w_{n,m+1})(w_{n,m}-w_{n+1,m+1})=q^2-p^2, \tag{2.5.7}$$

in which p and q are spacing parameters of n- and m- direction, respectively.

It is remarkable that the derivation of the nonlinear superposition formula is only based on x-part in the Bäcklund transformation, which implies the superposition formula is valid for the solutions of the whole KdV hierarchy. The formula is so simple and neat. In addition to the KdV equation, some equations such as the modified KdV(mKdV) equation, sine-Gordon equation, an so on, have superposition formulas in simple forms (cf. [6,29,31,32,54]). In fact, the mKdV equation and sine-Gordon equation share a same superposition formula. Besides, when these nonlinear superposition formulas are treated as 2D discrete equations, they exhibit beautiful 3D consistency [3,19].

2.5.2 Bilinear BTs and superposition formulas

Using bilinear Bäcklund transformations one can derive nonlinear superposition formulas for bilinear equations, which was first found by Hirota and Satsuma in 1978 [24]. Still we take the KdV equation as an example to show the construction procedure.

We start from the x-part of the bilinear Bäcklund transformation (2.2.6) of the KdV equation, i.e.

$$D_x^2 f\cdot\widetilde{f}=\lambda f\widetilde{f}, \tag{2.5.8}$$

where for convenience we replaced g with $\widetilde{f}$. We will construct a relation as shown in Figure2.2. Of course, first we need to investigate the possibility $f_{12}=f_{21}$. Similar to §2.5.1, from (2.5.8) we have

$$(D_x^2-\lambda_1)f\cdot f_1=0, \tag{2.5.9a}$$
$$(D_x^2-\lambda_2)f\cdot f_2=0, \tag{2.5.9b}$$
$$(D_x^2-\lambda_2)f_1\cdot f_{12}=0, \tag{2.5.9c}$$
$$(D_x^2-\lambda_1)f_2\cdot f_{21}=0. \tag{2.5.9d}$$

[1] It is Bianchi who first derived a nonlinear superposition formula of solutions of the sine-Gordon equation and first proved permutation property of solutions [4].

$$
\begin{tikzcd}
 & f_1 \arrow[dr, dash, "\lambda_2"] & \\
f \arrow[ur, dash, "\lambda_1"] \arrow[dr, dash, "\lambda_2"'] & & f_{12}=f_{21} \\
 & f_2 \arrow[ur, dash, "\lambda_1"'] &
\end{tikzcd}
$$

Figure 2.2: Permutability property of bilinear Bäcklund transformations.

Introducing $w = 2(\ln f)_x$, we write (2.5.9a) as

$$(w + w_1)_x = 2\lambda_1 - \frac{1}{2}(w - w_1)^2,$$

which is the same as (2.5.3a). (2.5.9b,c,d) can also written as (2.5.3b) and (2.5.4a,b) in terms of w.[2] Thus, both $w_{12} = 2(\ln f_{12})_x$ and $w_{21} = 2(\ln f_{21})_x$ will satisfy the same ODE (2.5.5), and f_{12} and f_{21} can be same provided that enjoy same initial conditions.

Now, $f_2 f_{12} \times$ (2.5.9a) $- f f_1 \times$ (2.5.9d) yields

$$(D_x^2 f \cdot f_1) f_2 f_{12} - (D_x^2 f_2 \cdot f_{12}) f f_1 = 0; \tag{2.5.10}$$

meanwhile, using bilinear identity (2.1.3) we have

$$(D_x^2 f \cdot f_1) f_2 f_{12} - (D_x^2 f_2 \cdot f_{12}) f f_1 = D_x[(D_x f \cdot f_{12}) \cdot (f_1 f_2) - D_x(f_2 \cdot f_1) \cdot (f f_{12})]. \tag{2.5.11}$$

By a comparison we immediately find

$$D_x[(D_x f \cdot f_{12}) \cdot (f_1 f_2) - (D_x f_2 \cdot f_1) \cdot (f f_{12})] = 0;$$

Switching indices: $1 \leftrightarrow 2$, yields

$$D_x[(D_x f \cdot f_{12}) \cdot (f_1 f_2) - (D_x f_1 \cdot f_2) \cdot (f f_{12})] = 0.$$

For the above two equations, adding and subtracting each other yield, respectively

$$D_x(D_x f \cdot f_{12}) \cdot (f_1 f_2) = 0,$$
$$D_x(D_x f_1 \cdot f_2) \cdot (f f_{12}) = 0.$$

Then, noticing the property $D_x g \cdot g = 0$ we can take

$$D_x f \cdot f_{12} = \alpha f_1 f_2, \tag{2.5.12a}$$
$$D_x f_2 \cdot f_1 = \beta f f_{12}, \tag{2.5.12b}$$

[2]This shows the coincidence between bilinear and nonlinear Bäcklund transformations and Lax pairs.

where $\alpha, \beta \in \mathbb{R}$. These two equations together compose a nonlinear superposition formula for the bilinear KdV equation (1.2.5).

One can derive the nonlinear superposition formula (2.5.6) from (2.5.12). In fact, multiplying each other in (2.5.12) yields

$$(D_x f \cdot f_{12}) \times (D_x f_2 \cdot f_1) = \alpha\beta f f_1 f_2 f_{12}.$$

Then introducing $w = 2(\ln f)_x$, we reach

$$(w - w_{12})(w_1 - w_2) = -4\alpha\beta, \tag{2.5.13}$$

which is the nonlinear superposition formula (2.5.6). Thus, two forms of the superposition formulas are unified.

2.6 Vertex operators

2.6.1 Vertex operator of the KdV equation

The function f defined by (1.2.16) corresponds to the NSS of the KdV equation (1.2.1). It is called τ function of the KdV equation, denoted by τ_N. The bilinear Bäcklund transformation (2.2.6) provides a transformation between τ_N and τ_{N+1}. Besides, there is an operator $X(k)$ (called vertex operator) which provides a more direct transformation between τ_N and τ_{N+1}:

$$\tau_{N+1} = e^{c_{N+1} X(k_{N+1})} \tau_N. \tag{2.6.1}$$

We will explain such an operator in this subsection.

We rewrite the KdV equation (by $t \to -4t$) as

$$4u_t - 6uu_x - u_{xxx} = 0. \tag{2.6.2}$$

Then, under transformation

$$u = 2(\ln \tau)_{xx}, \tag{2.6.3}$$

bilinear KdV equation is

$$(4D_x D_t - D_x^4)\tau \cdot \tau = 0. \tag{2.6.4}$$

For convenience we introduce $t_1 = x,\ t_3 = t$, with which the above bilinear equation reads

$$(4D_1 D_3 - D_1^4)\tau \cdot \tau = 0. \tag{2.6.5}$$

Its NSS is given by

$$\tau_N = \sum_{\mu=0,1} \exp\left(2\sum_{j=1}^{N} \mu_j(\zeta_j + \zeta_j^{(0)}) + \sum_{1\le i<j}^{N} \mu_i\mu_j a_{ij}\right), \tag{2.6.6a}$$

where

$$\zeta_j = \sum_{i=0}^{\infty} k_j^{2i+1} t_{2i+1}, \quad e^{a_{ij}} = A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \tag{2.6.6b}$$

$\zeta_j^{(0)} \in \mathbb{R}$ and the summation over μ is as same as in (1.2.16). Here we note that, to employ those notations in Sato's theory, we use infinite coordinates $(t_1, t_3, t_5, \cdots)$. In fact, for the KdV equation we can treat $(t_5, t_7, \cdots)$ (which do not appear in the equation) as parameters.

The above τ_N can be written as [43]

$$\tau_N = \sum_{J\subset I}\left[\left(\prod_{i\in J} c_i\right)\left(\prod_{\substack{i,j\in J\\ i<j}} A_{ij}\right)\exp\left(2\sum_{i\in J}\zeta_i\right)\right], \tag{2.6.7}$$

where $c_i \in \mathbb{R}$, I stands for the set $I = \{1, 2, \cdots, N\}$, J is a subset of I, and summation over $J \subset I$ means taking all possible subsets of I. In the above expression, $J = \varnothing$ corresponds to "1", $J = \{i\}$ corresponds to $c_i e^{2\zeta_i}$, and $J = \{1, 2\}$ corresponds to $c_1c_2A_{12}e^{2(\zeta_1+\zeta_2)}$, $\cdots\cdots$. Obviously, c_i takes the place of $e^{2\zeta_i^{(0)}}$ in (2.6.6). When $N = 2$, we have

$$\tau_2 = 1 + c_1e^{2\zeta_1} + c_2e^{2\zeta_2} + c_1c_2A_{12}\, e^{2(\zeta_1+\zeta_2)},$$

which is the same as (1.2.14) ($\varepsilon = 1$).

For the transformations between τ functions based on vertex operators, there is the following [10, 43].

Theorem 2.6.1. *For the τ function defined by* (2.6.7), *there is*

$$\tau_{N+1} = e^{c_{N+1}X(k_{N+1})}\tau_N, \tag{2.6.8}$$

where

$$X(k) = e^{2\zeta(\mathbf{t},k)}\, e^{-2\zeta(\widetilde{\partial},\, k^{-1})}, \tag{2.6.9a}$$

$$\zeta(\mathbf{t}, k) = \sum_{j=0}^{\infty} k^{2j+1} t_{2j+1}, \quad \mathbf{t} = (t_1, t_3, t_5, \cdots), \tag{2.6.9b}$$

$$\widetilde{\partial} = \left(\partial_1, \frac{\partial_3}{3}, \frac{\partial_5}{5}, \cdots\right), \quad \partial_j = \partial_{t_j}. \tag{2.6.9c}$$

(2.6.9a) *is called a vertex operator.*

The vertex operator (2.6.9a) was constructed by James Lepowsky and Robert Lee Wilson, who also found the operator is isomorphic to the affine Lie algebra $A_1^{(1)}$ [35]. Date et al [10] found the connection (2.6.8) between the operator and the τ function of the KdV equation, which led to a series of beautiful work on transformation groups and integrable systems.

We prove this theorem through some lemmas.

Lemma 2.6.1. *$\forall a, k \in \mathbb{R}$, there is*

$$e^{a\zeta(\widetilde{\partial},\, k^{-1})} f(\mathbf{t}) = f(\mathbf{t} + a\varepsilon(k)), \tag{2.6.10}$$

where

$$\varepsilon(k) = \Big(\frac{1}{k}, \frac{1}{3k^3}, \frac{1}{5k^5}, \cdots\Big).$$

The proof is obvious.

Lemma 2.6.2.

$$e^{-4\zeta(\varepsilon(p), q)} = \Big(\frac{p-q}{p+q}\Big)^2. \tag{2.6.11}$$

Proof.

$$\begin{aligned}
\ln\Big(\frac{p-q}{p+q}\Big) &= \ln(1 - q/p) - \ln(1 + q/p)\\
&= -\sum_{j=1}^{\infty} \frac{q^j}{j\, p^j} - \sum_{j=1}^{\infty} (-1)^{j+1} \frac{q^j}{j\, p^j}\\
&= -2\sum_{j=0}^{\infty} \frac{q^{2j+1}}{(2j+1)\, p^{2j+1}}\\
&= -2\zeta(\varepsilon(p), q),
\end{aligned}$$

which is equivalent to (2.6.11). □

Lemma 2.6.3.

$$e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} e^{2\zeta(\mathbf{t}, k_j)} = A_{ij} e^{2\zeta(\mathbf{t}, k_j)}\, e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})}. \tag{2.6.12}$$

Proof. For any sufficiently smooth function $f(\mathbf{t})$, successively using Lemma 2.6.1 and Lemma 2.6.2, we find

$$\begin{aligned}
&e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} e^{2\zeta(\mathbf{t}, k_j)} f(\mathbf{t})\\
=&e^{2\zeta(\mathbf{t} - 2\varepsilon(k_i), k_j)} f(\mathbf{t} - 2\varepsilon(k_i))\\
=&e^{2\zeta(\mathbf{t}, k_j)} e^{-4\zeta(\varepsilon(k_i), k_j)} e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} f(\mathbf{t})\\
=&A_{ij} e^{2\zeta(\mathbf{t}, k_j)}\, e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} f(\mathbf{t}).
\end{aligned}$$

The lemma is proved due to arbitrariness of $f(\mathbf{t})$. □

There is another expression for (2.6.12). If denoting $A=-2\zeta(\widetilde{\partial},\, k_i^{-1})$, $B=2\zeta(\mathbf{t},k_j)$, first we have

$$[A,B]=-4\zeta(\varepsilon(k_i),k_j)=\ln A_{ij}, \tag{2.6.13}$$

where $[A,B]=AB-BA$. In fact, because B is a linear function, $[A,B]$ must be a scalar. Noticing that

$$[\partial_{2r+1},\, t_{2s+1}]=\delta_{r,s},$$

we have

$$\begin{aligned}
[A,B]&=-4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{2s+1}\frac{k_j^{2r+1}}{k_i^{2s+1}}[\partial_{2s+1},\, t_{2r+1}]\\
&=-4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{2s+1}\frac{k_j^{2r+1}}{k_i^{2s+1}}\delta_{r,s}\\
&=-4\sum_{r=0}^{\infty}\frac{1}{2r+1}\frac{k_j^{2r+1}}{k_i^{2r+1}}=-4\zeta(\varepsilon(k_i),k_j).
\end{aligned}$$

Then, using (2.6.13), (2.6.12) can be written as

$$e^Ae^B=e^{[A,B]}e^Be^A. \tag{2.6.14}$$

Lemma 2.6.4. *For the vertex operator $X(k)$ defined by* (2.6.9a)*, there is*

$$X(k_i)X(k_j)=A_{ij}e^{2(\zeta(\mathbf{t},k_i)+\zeta(\mathbf{t},k_j))}e^{-2\zeta(\widetilde{\partial},\,k_i^{-1})}e^{-2\zeta(\widetilde{\partial},\,k_j^{-1})}. \tag{2.6.15}$$

Proof. Using formula (2.6.12), we find

$$\begin{aligned}
X(k_i)X(k_j)&=e^{2\zeta(\mathbf{t},k_i)}e^{-2\zeta(\widetilde{\partial},\,k_i^{-1})}e^{2\zeta(\mathbf{t},k_j))}e^{-2\zeta(\widetilde{\partial},\,k_j^{-1})}\\
&=e^{2\zeta(\mathbf{t},k_i)}A_{ij}\,e^{2\zeta(\mathbf{t},k_j))}e^{-2\zeta(\widetilde{\partial},\,k_i^{-1})}e^{-2\zeta(\widetilde{\partial},\,k_j^{-1})},
\end{aligned}$$

which is (2.6.15). □

Lemma 2.6.5. *(Can be considered as a corollary of Lemma 2.6.4) For the vertex operator $X(k)$ defined by* (2.6.9a)*, there are*

$$(X(k))^2=0, \tag{2.6.16}$$

$$e^{cX(k)}=1+cX(k), \tag{2.6.17}$$

$$\begin{aligned}
&X(k_s)\cdots X(k_2)X(k_1)\\
&=\left(\prod_{1\le i<j}^{s}A_{ij}\right)\exp\left(2\sum_{j=1}^{s}\zeta(\mathbf{t},k_j)\right)\exp\left(2\sum_{j=1}^{s}\zeta(\widetilde{\partial},k_j^{-1})\right);
\end{aligned} \tag{2.6.18}$$

and

$$X(k)\circ 1 = e^{2\zeta(\mathbf{t},k)}, \tag{2.6.19}$$

$$X(k_s)\cdots X(k_2)X(k_1)\circ 1 = \left(\prod_{1\le i<j}^{s} A_{ij}\right)\exp\left(2\sum_{j=1}^{s}\zeta(\mathbf{t},k_j)\right), \tag{2.6.20}$$

where "$\circ 1$" means an operator acting on "1".

Making use of Lemma 2.6.5, it is not difficult to calculate:

$$\begin{aligned}
\tau_1 &= e^{c_1 X(k_1)}\circ 1 = (1+c_1X(k_1))\circ 1 = 1 + c_1 e^{2\zeta(\mathbf{t},k_1)},\\
\tau_2 &= e^{c_2X(k_2)}e^{c_1X(k_1)}\circ 1 = (1+c_2X(k_2))(1+c_1X(k_1))\circ 1\\
&= 1 + c_1e^{2\zeta(\mathbf{t},k_1)} + c_2e^{2\zeta(\mathbf{t},k_2)} + c_1c_2A_{12}e^{2(\zeta(\mathbf{t},k_1)+\zeta(\mathbf{t},k_2))},\\
\tau_N &= e^{c_NX(k_N)}\cdots e^{c_2X(k_2)}e^{c_1X(k_1)}\circ 1\\
&= (1+c_NX(k_N))\cdots(1+c_2X(k_2))(1+c_1X(k_1))\circ 1\\
&= e^{c_NX(k_N)}\tau_{N-1}.
\end{aligned}$$

Thus we also finish the proof for Theorem 2.6.1.

2.6.2 Vertex operator of the KP(II) equation

Rewriting the bilinear KP(II) equation (1.2.19) as (2.6.5), we have

$$(4D_1D_3 - D_1^4 - 3D_2^2)\tau\cdot\tau = 0. \tag{2.6.21}$$

Its NSS(1.2.29) is

$$\tau_N = \sum_{J\subset I}\left[\left(\prod_{i\in J} c_i\right)\left(\prod_{\substack{i,j\in J\\ i<j}} A_{ij}\right)\exp\left(\sum_{i\in J}\xi_i\right)\right], \tag{2.6.22a}$$

where $c_i\in\mathbb{R}$,

$$\xi_j = \sum_{i=0}^{\infty}(p_j^i - q_j^i)t_i, \quad e^{a_{ij}} = A_{ij} = \frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}, \tag{2.6.22b}$$

I stands for the set $I=\{1,2,\cdots,N\}$, J is a subset of I, and summation over $J\subset I$ means taking all possible subsets of I.

For the vertex operator related to the KP(II) equation, we have the following [10, 43].

Theorem 2.6.2. *For the τ function defined by* (2.6.22a)*, there is*

$$\tau_{N+1} = e^{c_{N+1}X(p_{N+1},q_{N+1})}\tau_N, \tag{2.6.23}$$

where

$$X(p,q) = e^{\xi(\mathbf{t},p)-\xi(\mathbf{t},q)}\, e^{-(\xi(\widetilde{\partial},p^{-1})-\xi(\widetilde{\partial},q^{-1})}, \tag{2.6.24a}$$

$$\xi(\mathbf{t},k) = \sum_{j=0}^{\infty} k^j t_j, \quad \mathbf{t} = (t_1, t_2, t_3, \cdots), \tag{2.6.24b}$$

$$\widetilde{\partial} = \Big(\partial_1, \frac{\partial_2}{2}, \frac{\partial_3}{3}, \cdots\Big), \quad \partial_j = \partial_{t_j}. \tag{2.6.24c}$$

In fact, (similar to §2.6.1) one can prove that (will be given in Chapter**??**)

$$X(p_i,q_i)X(p_j,q_j) = A_{ij}\ :X(p_i,q_i)X(p_j,q_j):, \tag{2.6.25}$$

where

$$:X(p_i,q_i)X(p_j,q_j): = e^{\xi(\mathbf{t},p_i)-\xi(\mathbf{t},q_i)}e^{\xi(\mathbf{t},p_j)-\xi(\mathbf{t},q_j)}\, e^{-(\xi(\widetilde{\partial},p_i^{-1})-\xi(\widetilde{\partial},q_i^{-1}))}\, e^{-(\xi(\widetilde{\partial},p_j^{-1})-\xi(\widetilde{\partial},q_j^{-1}))}$$

is the normally arranged product of $X(p_i,q_i)X(p_j,q_j)$, (just moving all differential operators to the most right place).

Then we can results which are similar to Lemma 2.6.5:

$$(X(p,q))^2 = 0, \tag{2.6.26}$$

$$e^{cX(p,q)} = 1 + cX(p,q), \tag{2.6.27}$$

$$\begin{aligned} &X(p_s,q_s)\cdots X(p_2,q_2)X(p_1,q_1)\\ =&\left(\prod_{1\le i<j}^{s} A_{ij}\right) :X(p_s,q_s)\cdots X(p_2,q_2)X(p_1,q_1): ; \end{aligned} \tag{2.6.28}$$

and

$$X(p,q)\circ 1 = e^{\xi(\mathbf{t},p)-\xi(\mathbf{t},q)}, \tag{2.6.29}$$

$$\begin{aligned} &X(p_s,q_s)\cdots X(p_2,q_2)X(p_1,q_1)\circ 1\\ =&\left(\prod_{1\le i<j}^{s} A_{ij}\right)\exp\left(\sum_{j=1}^{s}(\xi(\mathbf{t},p_j)-\xi(\mathbf{t},q_j))\right). \end{aligned} \tag{2.6.30}$$

Then it is not difficult to reach Theorem 2.6.2.

Appendix A

Lax pair and BT of the KdV equation

A.1 KdV hierarchy

The KdV equation (1.2.1) can be expressed as a compatibility of the Schrödinger special problem

$$\phi_{xx} + u\phi = \lambda\phi \tag{A.1.1}$$

and

$$\phi_t = 4\phi_{xxx} + 6u\phi_x + 3u_x\phi, \tag{A.1.2}$$

where the spectral parameter λ is independent of t, i.e. $\lambda_t = 0$. On the basis of this fact, GGKM [12] developed the IST to derive NSS for the (nonlinear) KdV equation. It is P.D. Lax who first realized that not only the KdV equation, but also more nonlinear evolution equations can be expressed as compatibilities of some linear problems; in his celebrated paper [30] in 1968, he also introduced symmetries, conserved covariants and conserved quantities to investigate properties of these equations, which, in some sense, together with [12], triggered the modern theory of integrable systems.

In the following we start from the Schrödinger spectral problem (A.1.1) to derive the KdV hierarchy. Introduce a time evolution relation

$$\phi_t = A\phi + B\phi_x, \tag{A.1.3}$$

where A and B are undetermined functions of the potential u and spectral parameter λ, and $\lambda_t = 0$. From the compatibility condition $(\phi_{xx})_t = (\phi_t)_{xx}$ we have

$$(2A_x + B_{xx})\phi_x + [u_t + A_{xx} + 2(\lambda - u)B_x - u_xB]\phi = 0,$$

and then

$$2A_x + B_{xx} = 0, \tag{A.1.4a}$$
$$u_t = -A_{xx} - 2(\lambda - u)B_x + u_x B. \tag{A.1.4b}$$

Eliminating A yields

$$u_t = TB - 2\lambda B_x, \quad T = \frac{1}{2}\partial_x^3 + 2u\partial_x + u_x. \tag{A.1.5}$$

To construct the KdV hierarchy, we assume B is a polynomial of λ:

$$B = \sum_{j=0}^{n} b_j (2\lambda)^{n-j}. \tag{A.1.6}$$

Substituting it into (A.1.5) and comparing coefficients of each power of λ yield

$$u_t = Tb_n, \tag{A.1.7a}$$
$$b_{j+1,x} = Tb_j \quad (j = 0, 1, \cdots, n-1), \tag{A.1.7b}$$
$$b_{0,x} = 0. \tag{A.1.7c}$$

$\{b_j\}$ can be obtained by integrating successively. To achieve that we assume the integration constants to be[1]

$$B|_{u=0} = (4\lambda)^n.$$

Then, from (A.1.7) we can successively get $b_0 = 2^n,\ b_1 = 2^n u,\ b_2 = 2^{n-1}(u_{xx} + 3u^2)$,

$$b_{j+1} = 2^{n-j}\partial_x^{-1} L^j u_x, \quad j = 0, 1, \cdots, n-1,$$

where

$$L = 2T\partial_x^{-1} = \partial_x^2 + 4u + 2u_x\partial_x^{-1} \tag{A.1.8}$$

is called the recursion operator of the KdV hierarchy. By (A.1.7a), the KdV hierarchy can be expressed as:

$$u_{t_n} = L^n K_0, \quad K_0 = u_x, \quad (n = 0, 1, 2, \cdots). \tag{A.1.9}$$

When $n = 1, 2$, there are

$$u_t = K_1 = u_{xxx} + 6uu_x, \tag{A.1.10a}$$
$$u_t = K_2 = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x, \tag{A.1.10b}$$

[1] Integration operator ∂_x^{-1} satisfies $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$, and usually is defined as $\partial_x^{-1} = \frac{1}{2}(\int_{-\infty}^{x} - \int_{x}^{+\infty})$. Since for soliton solutions u satisfies $u \to 0$ ($x \to \pm\infty$), usually integration constants are given through $(\cdot)|_{u=0}$.

where (A.1.10a) is the KdV equation (1.2.1) (with $t \to -t$) and (A.1.10b) is called the 5th-order KdV equation.

For the KdV equation (A.1.10a), corresponding to $n = 1$, we have $b_0 = 2,\ b_1 = 2u$, i.e. $B = 4\lambda + 2u$; then from (A.1.4a) we have $A = -u_x$. Substituting them to (A.1.3) gives

$$\phi_t = -u_x\phi + (4\lambda + 2u)\phi_x. \tag{A.1.11}$$

Corresponding to the KdV equation (1.2.1), we need to switch $t \to -t$, and we have

$$\phi_t = u_x\phi - (4\lambda + 2u)\phi_x. \tag{A.1.12}$$

(A.1.1) and (A.1.12) compose the Lax pair of the KdV equation (1.2.1). If using (A.1.1) to eliminate λ in (A.1.12), we get (A.1.2). If we only eliminate only one λ in (A.1.12) and leave 3λ, we get

$$\phi_t = \phi_{xxx} + 3(\lambda + u)\phi_x. \tag{A.1.13}$$

(A.1.1) and (A.1.13) also present a Lax pair for the KdV equation.

A.2 BT of the KdV equation

In 1968 R.M. Miura [40] proposed famous Miura transformation

$$u = -v_x - v^2, \tag{A.2.1}$$

by which the KdV equation (1.2.1) and modified KdV (mKdV) equation

$$v_t - 6v^2v_x + v_{xxx} = 0 \tag{A.2.2}$$

are related together through

$$u_t + 6uu_x + u_{xxx} = -(2v + \partial_x)(v_t - 6v^2v_x + v_{xxx}).$$

Miura's transformation provides more profound links for the KdV and mKdV equation, which can be summarized as the following:

- Miura's transformation corresponds to the Schrödinger spectral problem of the KdV equation: Making use of the fact that the KdV equation is invariant under the Galilean transformation ($u(x,t) \to \lambda + u(x - 6\lambda t, t)$), one can introduce λ and rewrite the Miura transformation (A.2.1) by $u \to u - \lambda$, i.e.

  $$u = \lambda - v_x - v^2. \tag{A.2.3}$$

  Then, using the Cole-Hopf transformation $v = \phi_x/\phi$ it becomes the Schrödinger spectral problem (A.1.1) of the KdV equation. In fact, Miura found the transformation before the IST.

- The mKdV equation provides the time evolution part in the Lax pair of the KdV equation: Under the Miura transformation (A.2.1) and $v = \phi_x/\phi$, the mKdV (A.2.2) becomes (A.1.2).

Making use of Miura's transformation (A.2.3) one can construct nonlinear Bäcklund transformation for the KdV equation. First, noticing that the mKdV equation (A.2.2) is invariant under $v \to -v$, from (A.2.3) we have two relations:

$$\widetilde{u} = \lambda + v_x - v^2, \tag{A.2.4a}$$

$$u = \lambda - v_x - v^2. \tag{A.2.4b}$$

Obviously, when v solves the mKdV equation (A.2.2), $\widetilde{u}$ satisfies the KdV equation (1.2.1). For the relation between u and $\widetilde{u}$, it is easy to see that

$$\widetilde{u} + u = 2(\lambda - v^2), \tag{A.2.5a}$$

$$\widetilde{u} - u = 2v_x. \tag{A.2.5b}$$

Consider the potential form of the KdV equation, i.e. (1.2.2). After introducing $u = w_x$, $\widetilde{u} = \widetilde{w}_x$, from (A.2.5b) we have

$$\widetilde{w} - w = 2v; \tag{A.2.6}$$

substituting it into (A.2.5a) yields

$$(\widetilde{w} + w)_x = 2\lambda - \frac{1}{2}(\widetilde{w} - w)^2. \tag{A.2.7a}$$

Next, since both w and $\widetilde{w}$ satisfies (1.2.2), i.e.

$$w_t + 3(w^2)_x + w_{xxx} = 0,$$
$$\widetilde{w}_t + 3(\widetilde{w}^2)_x + \widetilde{w}_{xxx} = 0,$$

subtracting them each other yields[2]

$$(\widetilde{w} - w)_t = -3(\widetilde{w} - w)_x(\widetilde{w} + w)_x - (\widetilde{w} - w)_{xxx},$$

in which replacing $(\widetilde{w} + w)_x$ with (A.2.7a) we have

$$(\widetilde{w} - w)_t = \frac{1}{2}[(\widetilde{w} - w)^3]_x - 6\lambda(\widetilde{w} - w)_x - (\widetilde{w} - w)_{xxx}. \tag{A.2.7b}$$

[2]If adding them each other we will have

$$(\widetilde{w} + w)_t = -(\widetilde{w} - w)(\widetilde{w} - w)_{xx} + 2[(\widetilde{w}_x)^2 + \widetilde{w}_x w_x + (\widetilde{w}_x)^2],$$

which, coupled with (A.2.7a), can also compose a Bäcklund transformation for the KdV equation.

(A.2.7a) and (A.2.7b) compose a nonlinear Bäcklund transformation for the KdV equation [57].

Note: For the early history of Bäcklund's transformation and Bianchi's permutability theorem, please refer to [49] and [50].

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