

Examining Grade 9 Students' Engagement in Mathematical Problem-Solving (MPS) When Working as Individuals and in a Small Group Setting

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Abstract

Problem-solving is central to mathematics education across the world. The National Curriculum for Mathematics (Grade I-XII) in Pakistan emphasizes the importance of problem-solving in developing students' mathematical knowledge and understanding. Informed by Polya's heuristics that guides the problem-solving process, this study examined Grade 9 students' mathematical problem-solving (MPS) when working as individuals and in a small group setting. Data were triangulated through multiple methods including semi-structured interviews, observations of episodes of students solving problems in a small group, written responses to the problems, and focus group interview. The findings reveal that the participants demonstrated variations in the emphasis given to each stage as well as the manner in which problem-solving was operationalized at an individual and a group level. Moreover, students both individually and while working in a small group go back and forth among different stages of Polya's heuristics. These findings have implications for the teaching and learning of MPS in academically less advanced countries similar to Pakistan.

Keywords: Polya's heuristics, Mathematical problem-solving, Word problems, Individual vs. group

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Introduction

Mathematics is one of the core curriculum subjects taught at all levels of primary, middle and secondary schools in Pakistan (Government of Pakistan, 2006). Although a range of components and processes are emphasized in mathematics curricula, problem-solving has remained a fundamental constituent in the National Curriculum for Mathematics (Grade I – XII). Group work, development of relational understanding, mathematical thinking, reasoning, negotiation and communication are major emphases for students outlined in the national curriculum. Consistently, teachers are encouraged to shift their role from 'dispensing information' to "planning investigative tasks, managing a cooperative learning environment and supporting students' creativity in developing relational understanding of the concepts of Mathematics" (p. 2).

Contrary to the expectations of the curriculum (Government of Pakistan, 2006), a detailed analysis of the first national assessment data of middle school students in Pakistan indicates extremely low levels of achievement in mathematics (Tayyaba, 2010). In particular, students performed poorly on demanding items that required them to use complex cognitive skills such as reasoning and problem solving. Overall, the results displayed marked disparities between the expectations of the curriculum and the actual achievement of students.

Given the emphases in the national curriculum, and the huge deficits demonstrated in students' performance in mathematics and problem-solving, researchers have paid close attention to the effectiveness of teaching problem-solving skills (Ali, Hukamdad, Akhter, & Khan, 2010; Perveen, 2010). For example, Perveen examined the effect of heuristics oriented problem-solving instruction on secondary school students' mathematical achievement and problem-solving skills by conducting an experimental study. The results of the study demonstrated that students who were taught by the problem-solving heuristics outscored students who were taught by expository instruction on a mathematics achievement test. Ali et al. (2010) reported similar trends with elementary and high school students.

While the effectiveness of teaching students problem-solving heuristics in mathematics has largely been acknowledged (Karatas & Baki, 2017; Khan, 2012; Nieuwoudt, 2015), questions regarding how students actually engage in such heuristics while solving word problems on their own and/or in a group setting remain unanswered. However, such knowledge is crucial not only to apprehend and organize students' successful engagement in MPS but also to structure effective lessons for imparting problem-solving heuristics in classrooms. Based on this, the study investigates the following question:

How do Grade 9 students employ Polya's problem-solving heuristics during mathematical problem-solving (MPS) while working alone versus when they are in a small group?

Theoretical background

A mathematical word problem is a textual statement or proposition that presents a situation whose solution is not readily available (Stylianides & Stylianides, 2014; Yee & Bostic, 2014; Government of Pakistan, 2006). Learners are required to attempt a number of routes in order to reach the solution. These processes may require different algebraic, geometric or mathematical procedures as well as a consideration of contextual/situational aspects of the stated problem (Reusser & Stebler, 1997). This is because the task related (cognitive) and socio-contextual (situational) factors are inseparable. Overall, MPS is viewed as a sophisticated, socio-cognitive process that requires learners to look for solutions in settings that highlight the social and cultural format in which problem-solving occurs (Reusser & Stebler, 1997; Wathall, 2016). The social environment further provides the opportunity to observe how information and knowledge are exchanged and negotiated between learners.

Polya's (1973) four-stage problem-solving heuristics have often been utilized in classrooms to help students and teachers organize mathematical problem-solving (e.g., O'Shea & Leavy, 2013). The four stages are: understand the problem (S1), devise a plan (S2), carry out the plan (S3), and look back and reflect (S4). According to Polya, it is critical that learners clearly understand the problem by attending to the unknown, relevant data and underlying conditions. Here, learners may choose to write data, draw a figure or separate out various parts of the problem. The next stage involves devising a plan to solve the given problem. As learners plan possible solutions, they may restate the problem, go back to basic definitions/concepts, tease out the connections between given data and the unknown, consider related problems, use alternative methods and procedures, and communicate, discuss and revise proposed plans. Once plans are formulated, the next stage is to implement these plans. This involves carrying out the proposed plan, applying rules to produce a specified outcome, examining details, ensuring that different steps are correctly implemented, and utilizing and manipulating learning materials. Students are encouraged to derive problem solutions in different ways at this stage. The final stage is to check the results, create multiple representations, review completed solutions and make mathematical connections. For example, learners may develop connections between symbols and procedures or they may discern different mathematical concepts and how they relate to each other. Graphic organizers, reflective notes, discussions and other means of communication (verbal and visual) are particularly helpful at this stage (Wathall, 2016).

Polya's (1973) heuristics for problem-solving provides a useful framework to examine MPS processes that are utilized by students as individuals and/or in a small group setting. It encourages the use of open-ended problems that provoke planning, discussions, negotiations, interrogation and revision of proposed solutions rather than merely achieving the correct answer (O'Shea & Leavy, 2013). This emphasis also coincides with the standards laid out in the National Curriculum for Mathematics (Grade I-XII) (Government of Pakistan, 2006), thus providing a suitable framework to situate the study.

Research Methodology

Participants, settings and procedures

The participants of the study involved five Grade 9, male students (identified as M1, M2, M3, M4 and M5); who are situated in a public-sector, secondary school in Pakistan. The medium of instruction in the school is English, which is also the instructional language used in curriculum documents and mathematics textbook. Students knew each other as classmates, speak Urdu as their first language and come from low socio-economic to lower-middle backgrounds. The average age of the participants was 14 years. They had limited experience of collaborative work during learning in general, and mathematics learning in particular. The group comprised five members and was heterogenous in terms of students' abilities and achievement in mathematics.

Formal permission from relevant gatekeepers (e.g., principal, teacher) were obtained before approaching the students and inviting them to participate in the study. The purpose of the study, nature of participation, risks and benefits for participation, and issues related to confidentiality of information with key stakeholders (i.e., principal, teacher, and students) were discussed before obtaining their consent. Only those students who agreed to participate in the study were included in the sample.

Data collection stages and sources

Stage 1 – Semi-structured interviews (I)

Data from multiple sources were collected in three stages. The first stage involved conducting semi-structured interviews (I) with individual students. The interviews provided an opportunity to understand the underlying processes and strategies that were employed by students while working as individuals during MPS. The first author conducted the semi-structured interviews and asked different questions, for example: 'How do you solve word problems in mathematics? What is the best way to solve word problems in mathematics? What are the strategies that you most commonly use while solving word problems?' The interviews were recorded on a digital audio-recorder and later transcribed for meaning. Each interview lasted for about four to six minutes.

Stage 2 – Observation of MPS (O)

The second stage of data collection comprised observation (O) of students solving mathematical problems in a small a group comprising five members and generation of written responses to the given problems. The participants were presented with four, open-ended word problems in English language that were taken from a source other than their textbook; they were asked to generate possible solutions in a small group setting and hand in written responses. The decision to include four problems was informed by the fact that middle and high school students may solve up to three to four mathematical problems at one point in time while maintaining focus (Yee & Bostic, 2014).

Three aspects were considered in the selection of word problems. These were: a) compliance with the content present in the Grade 9 Mathematics textbook (Government of Punjab, 2016); b) compliance with the standards laid out in the National Curriculum for Mathematics (Grades I-XII) (Government of Pakistan, 2006); and c) the potential to evoke discussions, as well as multiple and visual representations (Yee & Bostic, 2014). For example, one problem (WP4) stated: A rectangular garden in Ms. Ayesha's house has a length of 100 meters and a width of 50 meters. A square swimming pool is to be constructed inside the garden. Find the length of one side of the swimming pool if the remaining area (not occupied by the pool) is equal to one half the area of the rectangular garden. The problems were also reviewed by two experts in the field (one school and one university teacher) to ensure that these were developmentally appropriate for the participants.

Keeping in view the difficulties in reading, understanding, translating and precisely documenting the information given in the problem statement, particularly for students who solve problems in a second language (Alvi et al., 2016; Sepeng & Madzorera, 2014), the participants were provided with a chart of simple translations for phrases that were frequently used in mathematical problems, for example, "Addition Words" included descriptions like: all together or altogether, combined, how many in all, increased by, and total. The intent was to provide participants with a tool that might help them to simplify the meaning of the problem statements and to transfer this understanding to a mathematical model, which is known to improve students' mathematical comprehension (Ilany & Margolin, 2010). They were also provided with the textbook, pens and A4 sheets.

As participants solved problems in a group, the first author encouraged them to use related tools and materials (e.g., textbook, chart, pens, A4 sheets), project multiple representations and solutions, and discuss their ideas aloud. Group discussion was recorded on a digital audio recorder and clarification questions were asked if required. Overall, the first author restricted herself to observing the participants without interfering. The group took approximately 30 minutes to solve the given problems. However, time and gender were not a consideration for the study.

Stage 3 - Focus group interview (FG)

The first author conducted a focus group interview (FG) with the participants right after they finished working with the word problems. The interview was recorded on a digital audio recorder and later transcribed for meaning. At this point, the participants were questioned about their collective views, and decisions regarding, different aspects of problem-solving in a group setting. Several questions to understand the underlying processes were posed. Examples include: How do you determine what is relevant in a problem? Did you attempt different ways of solving problems, why, why not? How do you compare working in a group setting to solve word problems with working on your own?

The multiple sources of information (I, O, FG) yielded a rich corpus of data that enabled us to develop a detailed understanding of students' MPS when working as individuals and in a small group setting. Data from different sources were stored, organized, and analysed in NVivo 10, a software package for qualitative data analysis.

Data analysis

Interview transcripts from different sources (I, FG) served as a primary source of data that were used to examine participants' engagement in MPS when working as individuals and in small group setting. The preliminary analysis at this stage involved an inductive, open coding approach that allowed us to conceptualize data at a micro level and generate several concepts directly from the data (Saldaña, 2009). These included, for example, reading, translating, and attending to the unknown. Informed by the stages (S1 to S4) of Polya's heuristics (1973), the codes were re-organized after open coding. For example, codes like reading, translating, and attending to the unknown were organized under S1 (understanding the problem) of the heuristics. This process is referred to as axial coding. It allowed us to see relationships between different parts of the data, develop some initial patterns as well as a group narrative. The ground work for analysing participants' group discussion during problem-solving involved similar procedures of data handling and coding.

Data from multiple sources were triangulated to modify and sharpen categories and generate a comprehensive understanding of the phenomenon. For example, the analysis of semi-structured interviews revealed that students mainly referred to reading, translating, writing data, and attending to the unknown to develop an "understanding [of] the problem" (S1). However, this list was expanded as data from the discussion during group work was coded. New codes such as explaining, separating various parts of the problem, and making connections between different parts of the data were also added to the category.

Overall, Polya's (1973) heuristics (S1-S4) informed the analysis by providing a theoretical framework to make sense of data and interpret underlying processes. For example, data from different sources (I, O, FG) were woven around Polya's heuristics to explain how participants engaged in mathematical problem-solving when working as individuals and in a small group setting. The researchers engaged in iterative cycles of data analysis and established the strength of evidence for claims by adding layers of methodological and data triangulation (Barbour, 2008).

Results

An in-depth analysis of individual and social (O, FG) sources of data generated different yet interesting patterns of students' engagement in MPS at an individual and at a group level. As these patterns were organized around Polya's problem-solving heuristics, it was apparent that Polya's first stage of the heuristics (i.e., understand the problem) was the most frequently referred to process used by the participants when working as individuals. On the other hand, the analysis of social sources of data (O, FG) suggests that the third stage (S3), that is, "carry out the plan" was emphasized more than S1 when students worked in a small group setting. Figure 1 presents the different trends of students' engagement in MPS while working alone versus when they attempted problem-solving in a small group.

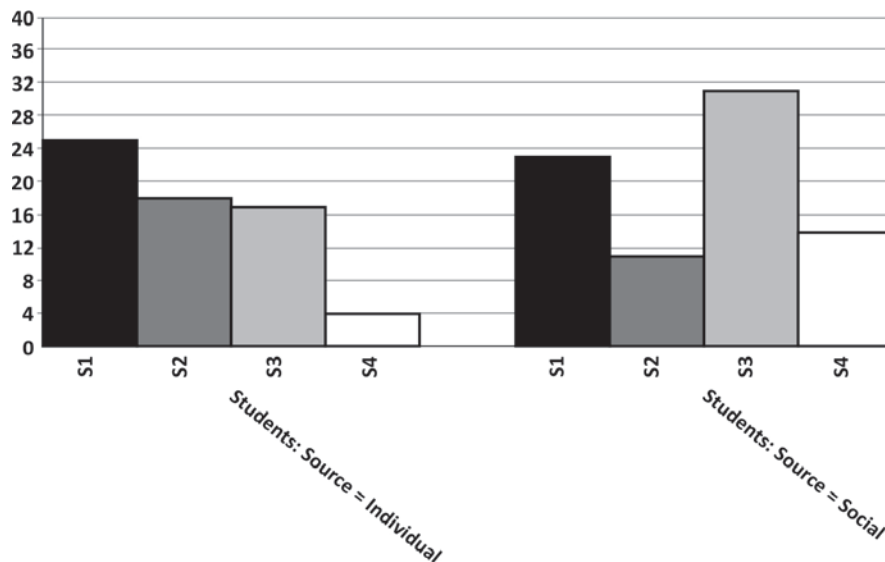


Figure 1. A comparison of coding on Polya's problem-solving heuristics from individual (I) and social (O, FG) sources of data. Vertical axis represents number of coding reference counts.

Other patterns generated during the analysis of data indicate that students both individually and while working in groups go back and forth among the four stages of Polya's problem-solving heuristics which emphasize a cyclical and blended rather than a linear tracking of students' engagement in MPS. The episode presented below is a rich description of these patterns. It includes the problem statement and corroborates evidence of individual and group engagement in MPS from multiple sources of data including I, O and FG.

Problem-solving episode

WP4 - problem statement: A rectangular garden in Ms. Ayesha's house has a length of 100 meters and a width of 50 meters. A square swimming pool is to be constructed inside the garden. Find the length of one side of the swimming pool if the remaining area (not occupied by the pool) is equal to one half the area of the rectangular garden.

As the group read the problem together, the problem-solving process marked students' engagement in S1.

1. M3: Rectangle?
2. M4: They asked for [the] perimeter!
3. M1: Just a minute. Let's read from the start. . .

M4 made a hasty judgment about what is unknown, however M1 reminded the group to read from the beginning. It seems as if students were aware of their tendency to rush through the problem-solving process, as M5 mentioned during the interview (I): "the first thing that [we] should do, is to try to read the problem *patiently* . . . Our rush leads us to an incorrect answer and we lose four or five marks [during tests]." M3 ignored S2 and prompted the group to engage in S3 and carry out a mathematical operation (division), only to be corrected by M2:

4. M3: Divide [the given numbers] with each other.
5. M2: This is [not the way] to find the area. Do you know how to find [area]?
Area is length into width!

The group re-read the problem several times. Re-reading was a frequently reported strategy during the individual interviews, for example, "read the problem three, four times; so, you understand a little bit in the first time, then a bit more, and then more" (M4, I); and "if the question is not correctly solved, then I re-read the problem from the beginning, read through the data, and if there is some mistake, it is identified" (M1, I). M1 re-stated the problem and argued that one side of the garden would be 100 meters but M4 did not agree:

6. M1: Here, the length of the garden is 100 meters, and this is square, so each side of the garden is 100 meters, so one side would be 100 meters too.
7. M4: No, this is not the way, it would never [solve the problem]. Because the problem states that the rectangular area in Ms. Ayesha's house, its length is 100 meters and width is 50 meters. So, we have to construct a square swimming pool in[side] the [rectangular] garden, so its [square's] one side is – If the remaining area that is not occupied by the pool is equal to the half area [of the rectangular garden] . . . means to find the area of a rectangle that two into length plus width [2 (l+w)].

The group carried out the calculation as suggested by M4. Their comments during the interviews (I) also suggest that they focus on procedural, mechanical and rule-driven computation to reach the solution of the problem instead of making a concrete plan. For example, M3 said: "You have to organize the data given in the problem statement, [you] have to understand that which formula is applicable, [decide] whether to solve [the problem] through equation, proportion or ratio." Similarly, M3 calculated the area to be 300[m²] and M4 explained that it would be 150[m²] since "[we] write its half." After the discussion and mutual consensus over the solution, M1 prompted the group to generate a written response to the problem. As they started to write down the data, M3 insisted on reading the problem once again. M5 described this behaviour as:

. . . when we are half way down into solving the problem, in the middle, so we don't understand if we need to divide or multiply, we get a problem. It wastes our time, . . . we read the problem once again, look at it again, solve it again, so that we may understand it.

As the group re-read the problem, M4 realized that the formula that was used to find the area of a rectangle was not correct. Although students emphasized the need to memorize formulae during semi-structured interviews, they admitted that, "we forget formulae quite often" (M3, I). However, when probed whether it was possible to solve a problem without applying the formula, M1 responded: "[That] could be done. . . through different ways, but I don't know."

8. M1: Start solving . . .
9. M3: First write length of the rectangle . . . One minute, [Let] me [read the problem]
10. M1: He is reading, wait [addressing other members]

11. M4: Oh, the formula to find area is not this $[2(l+w)]$. It's the other one. Length into width $[l \times w]$. This is the one!
12. M1: Yes. This is the one!
13. M4: Length into width. That one is [to find] perimeter $[2(l+w)]$. . .
14. M3: So, we would get the answer 5000. We would divide 5000 by two, so we would get 2500.
15. M4: Then we would divide 2500 by four, to get [the length of one side of the pool]
16. M1: Let's do it, we will see. . .
17. M1: We have to write here, length of four sides. . .
18. M4: Not length!
19. M3: Rectangular area, friend! . . .
20. M4: We have not found it yet. We said that the length . . . Let's read it once again . . . Just a minute! [Reading the problem]
21. M3: Is equal to, remaining area is equal to?
22. M4: Its half! You know why? Because this is a rectangle. If we draw a line here [pointing to and drawing an imaginary line on the rectangular table upon which they were working]
23. M1: It would become a square [looking at the table]
24. M4: Yes, it would become a square, so they have asked [the length of one side] of the remaining area [square]
25. M3: So, divide it by four. . .
26. M4: No, that would be 625. First, we would find the remaining area
27. M3: I can't understand
28. M4: First, [we] will find the remaining area
29. M3: First, [we] will divide by four
30. M4: Look at this table! Let's draw a square here. It is half of the full [table].

At that point, the researcher prompted M4 to use visualization tools and “draw it on a paper”. M4 drew a plain rectangle and explained to M3 that a square swimming pool would be made when a line is drawn in the middle. The group concluded that one side of the square was 625m. When they were asked to validate the answer, M3 explained:

. . . first, you find the area of the rectangular [garden] . . . then take its half to find the area of the square. A square has four sides, [we were asked] to find [the length of] one side, so we would divide [the area of the square] by four to get the answer. (M3, FG)

When being asked to support the answer with an alternative explanation, the group could not produce one. M1 justified their response:

We do not think of other ways [of solving a problem] because we know one method and we know that we will get 100% correct answer by using this method; if we devise another method of our own, what if it gets wrong? (M1, FG)

Discussion

Polya’s problem-solving heuristics (1973) has been instrumental in enabling teachers to understand students’ engagement in MPS when working as individuals and in a group setting. The participants in the study have no previous experience of using the heuristics; nor did they were prompted to use one for the purpose of the study. Nevertheless, Polya’s stages were observable in different ways in which they responded to, and engaged in MPS. However, different patterns emerged in the emphasis given to each of these stages as well as the manner in which the heuristics were employed at an individual and at a group level.

Students’ engagement in MPS – Individuals vs. Group

Understand the problem – S1

Students’ engagement in Polya’s heuristics when working as individuals as revealed through the analysis of interview (I) data suggests that participants stressed S1 (understand the problem) more than at any other stage of the heuristics (see Figure 1). This means that, as individuals, participants emphasized developing conceptual understanding as the most important route towards successful problem-solving and mainly reported (re)reading the problem as the main strategy (see M4 and M1’s comments). However, this was not the case when the participants approached MPS when working in a group setting, since they were engaged in S3 more than S1.

Nevertheless, S1 was emphasized in the group setting and the strategies employed during S1 were richer than those that participants reported as individuals. For example, besides (re)reading the problem, they also employed explanations, discussions (e.g., Turns, 6-7), and re-examining the basic concept/definition (e.g., Turns 5, 11-13) as means to develop conceptual understanding. This implies that students employed more complex strategies during S1 when they approached MPS in a group setting. Even the participants realized that their engagement in MPS when working in a group led them to better understanding of the problem because: "If we are [solving the problems] in groups, so every child [group member] will explain the problem one by one, so that if someone does not understand someone's [explanation], he can understand someone [else's] explanation" (M3, FG). M5 added that "[group work] allowed them to work [through] and understand [the problem] better" (FG).

However, students tended to decide too quickly what was being asked during S1 without being fully cognizant of the related facts. For example, M4 made an impulsive judgement about what was unknown by declaring before even reading the statement in full that "They asked for [the] perimeter!" (Turn 2). Similarly, M3 embarked upon mathematical operations (e.g., dividing, turn 4) without even understanding what was being asked. This inclination results from students' tendency to make hasty actions to reach the solution of the problem as revealed by M5 during the interview.

Similarly, students' ability to understand the problem was hampered by a lack of proficiency in mathematical concepts and relevance of information, for example, M4 stated that the formula to find the area of a rectangle is " $2(l + w)$ " while confronting M1's explanation (Turn 7). He re-explained the problem to the group and attended to the unknown. However, there are some conceptual flaws in the description of the problem and the solution provided by M4. He left out an important consideration (to find the length of one side of the swimming pool) while attending to the unknown ". . . so its [square's] one side is if the remaining area . . ." Nevertheless, the group build on his input and embarked upon computations without having seen the main connections between the given and the unknown, devising a plan, and questioning the accuracy of mathematical concepts. Student tendency to solve problems without sufficient understanding has been frequently reported in the literature (e.g., Ali, 2011; Reusser & Stebler, 1997; Sepeng & Sigola, 2013).

Moreover, the participants did not employ effective ways to develop better understanding of the problem such as visualizing the problem by drawing a figure, separating various parts of the problem and making connections (Polya, 1973). It is important to note that they did not draw a diagram to visually represent the problem. It was only when M3 challenged M4 to justify the problem solution of WP4, did M4 refer

to the rectangular table as a visual tool (Turn 30). At this point, the first author encouraged them to draw a figure on the paper. Yet, the image drawn by M4 was blank; and carried no schematic, graphical, iconic, proportional or algebraic presentation(s) that might have helped to represent the problem concretely (Kapur, 2010). Alvi et al.(2016) reported a similar trend and argued that students have little or no prior, formal instructions in using visual representations as an effective problem-solving strategy.

Devise a plan – S2

While students emphasized devising a plan (S2) as an important problem-solving process during individual interviews (I), it was least employed when they attempted MPS in a group setting (O) (see Figure 1). The analysis of interviews further revealed that students relied on surface-level strategies such as memorizing and applying formulae, focusing on rules and procedures, and trial and error as their main tactics to devise a plan for MPS (M3's comments). Nevertheless, some of them indicated using sophisticated strategies such as examining related or similar problems. M4 explained: “. . . we [consult] the examples that precede the questions [in the book]. We match the problem statements, that [if] it is a similar problem, then we examine the method and [use the same method to] solve [our] problem” (M4, I). This shows that if students do not fully understand the problem, they may come up with an emergent, contextually dependent way of solving the problem (Schoenfeld, 1992).

In practice, students made little effort to examine how the various items were connected within the problem, and how the unknown was linked to the given data to mark their engagement in S2 as they solved problems in a group (O). For example, M3 suggested even before reading and understanding the problem (WP4) dividing the given numbers (Turn 4). In turn, M2 expressed the relationship between a mathematical concept and the given problem by correctly referring to the formula (Turn 5). Similarly, M4's plan to find out the area of the rectangle reflects insufficient mathematical knowledge and a lack of proficiency over primary mathematical concepts such as area and perimeter (Turn 7). Interestingly, none of the group members, including M2 (who knew the concept/formula, turn 5), questioned or corrected the plan to find out the area of the rectangle. Instead, the group embarked on mental calculations and developed a consensus over the solution (150[m²]). In between the attempts to make and implement a plan, M4 realized that the formula used to calculate the area was incorrect (Turns 8-12). The group then made a new plan (based on the correct formula, Turn 13). They calculated the area of the pool to be 2500 [m²] (Turn 14); however, they incorrectly implied that dividing the area by four would lead to the length of one side of the pool (Turn 15). They were unable to connect the concept of area to the length of one side of the square. It seems as if they did not comprehend that area is measured in “square” units and that they needed to multiply side with side to find the area of a square.

Overall, the path from S1 to S2 was ineffective since students carried out calculations without consideration. While a concrete plan may provide opportunities for learners to reflect collectively about the strategies employed in a group setting, participants in the study were unable to devise a tangible problem-solving approach and indeed started doing calculations without even formulating a plan. There could be different reasons for this such as students did not have a substantial plan due to little or no subject knowledge and conceptual understanding (e.g., WP4) (Polya, 1973; Wathall, 2016), students have had little or no training in mathematical thinking and problem-solving (Ali, 2011; Alvi et al., 2016), or students were already frustrated with the practice of developing a plan (Tabach & Schwarz, 2018).

Carry out the plan – S3

Although students acknowledged the importance of carrying out the plan (S3) in the interviews (I), it is during their engagement in MPS when they worked in a group, that they activated S3 more than any other stage of the heuristics (see Figure 1). Students' experiences related to the solution processes revealed during the interviews suggest that they tend to focus on rules, procedures, and methods while carrying out the plan (e.g., M1, M3's comments). This tendency might be due to the fact that the "teacher makes us do it, like [he] makes us understand every single step [of the procedure] . . . so, we understand for once. We also note it and whenever we forget, we consult it again" (M1, I).

Students' engagement in MPS when working in a group setting demonstrated that the execution stage (S3) was activated as soon as they started to explain the problem to each other (e.g., Turn 4). They actively employed arbitrary mathematical procedures such as division (Turns 4, 15, 25, 29) and application of formulae (Turn 7). Only a few of these actions, however, were informed by a well-thought-out plan. For example, students worked out the area of the rectangle by operating on an incorrect formula which showed a lack of mathematical knowledge. Nevertheless, they corrected their mistake (Turn 8) and figured out the required area. Interestingly, none of the group members questioned the incorrect application of the formula, and the correction was made by the same member (M4) who executed it incorrectly in the first place. The students at last calculated the area of the square; however, they executed another uninformed action (i.e., division) at this point to find out the length of the one side of the square that led them to an incorrect problem solution (625 m). Here, students reasoned that since a square has four equal sides, the length of one side can be determined by dividing the area of the square by four. Their reasoning lacks logic and complexity, which is a common weakness in mathematical problem-solving (Alvi et al., 2016; Tabach & Schwarz, 2018).

Generally, learners are encouraged to utilize and manipulate learning tools, and derive results differently during the execution stage (S3) (Polya, 1973). However, the participants of the study made minimal use of the learning tools (i.e., textbook and chart) provided to them and rarely attempted different ways of solving the problems despite timely reminders by the researcher. They explained the reasons for not consulting tools as: “Questions [problems] were easy” (M5, FG), and “. . . [we] were [working] in a group. . . Had [you] given us the problems individually . . . we would have used the Table, we had to!” (M1, FG). Similarly, while the problems could be represented differently in words, drawings, symbols, geometric and algebraic forms, the participants did not attempt different ways of problem solution and representation because they were afraid of making mistakes and taking chances (see M1’s comments). M4 concurred and said: “because what the teacher has taught us and what is given in the book, is the same; then why should we take a second chance?”

Even though S3 turned out to be the most commonly employed stage during students’ engagement in MPS in a group setting, it was restricted to random acts of mathematical actions. Students’ execution of mathematical operations during this time was limited by a lack of mathematical knowledge, skills, logical thinking and reasoning, and interest in utilizing learning tools and materials.

Review and reflect – S4

While it is important to survey and scrutinize solutions to generate well-ordered, practical knowledge and skills needed for problem-solving in mathematics (Polya, 1973), data from the study suggest that students engaged in the review and reflect stage (S4) at a surface level only. Data from the interviews (I) revealed that students relied on basic mathematical procedures such as formula and calculations to check the correctness of the solutions as apparent in M5’s comments. While students rarely brought up their engagement in S4 during the individual interviews, they appeared more engaged when they approached MPS in a group setting (see Figure 1). However, this participation is limited to checking for the correctness of solution steps and seeking clarifications (turns 21-30) rather than extending, expanding and generalizing mathematical problems.

Students started to review the solutions as soon as these were proposed in the group. For example, M2 challenged M3’s solution to divide the given numbers in order to find the area of the rectangular garden (Turns 4-5) and further reiterated the correct formula. Similarly, M4 rejected M1’s reasoning that one side of the square would be 100 m by restating the problem and re-emphasizing the given data (“the length of the rectangular garden is 100 m”) (Turns 6-7). Students executed mathematical operations, questioned their solutions and corrected their mistakes (Turns 8-14). For example, M4

questioned M3's calculation that the area is $300\text{[m}^2\text{]}$ and explained to him that the area of the square would be a half of the rectangular garden, which is $150\text{[m}^2\text{]}$. While the formula used to calculate the area was incorrect, M4 corrected this error in response to the prompt by M1 to start writing the problem solution. This time, the group members returned to the basic concept of perimeter and differentiated it from that of the area. M3 once again raised his concern on the problem solution and sought clarification further along the process (Turn 27). At this point, M4's explanation included the table as a tool to visualize the rectangular garden (Turn 30). However, students did not try to verify solutions by considering alternative explanations. Neither did they attempt different ways to reach the solution because of their fixation on a single, sequential, step-by-step method to solve the problems (see M1's comments).

Overall, most of the reviewing done by students during S4 focused on challenging each other's ideas, seeking clarifications, questioning solutions, and correcting errors in group settings. No attempts were made to encompass the wider parameters of S4 by re-examining the agreed solution, considering alternative explanations, and extending and generalizing beyond the given problems to make mathematical connections. This later disposition, however, is not common among learners (O'Shea & Leavy, 2013); yet it is needed for mathematicians to deal with mathematical problems effectively (Leong, Toh, Tay, Quek, & Dindyal, 2012).

In sum, data from individual (I) and social sources (O, FG) suggest that students' engagement in Polya's first (S1) and third (S3) stages exceeded that exhibited during the second (S2) and fourth (S4) stages. This trend is also reported in the research literature. For example, Weber, Radu, Mueller, Powell and Maher (2010) conducted a longitudinal study spanning three years to investigate the extent to which middle school students participated in an increasingly wide range of problem-solving activity. Their findings suggest that students rarely engage in aspects such as projecting diverse arguments and approaches, evaluating the validity of arguments, and actively shaping the direction of mathematical inquiry under conventional circumstances.

Polya (1973) himself admitted that devising a plan to solve a mathematical problem is not an easy task. This is because an effective plan involves a number of factors such as previous knowledge, cognitive skills, concentration, personal entities, contextual circumstances, luck and creativity (Schoenfeld, 1992). On the other hand, carrying out the plan is a lot easier because, as Polya recognizes, "what we need is mainly patience" (p. 12). Similarly, the review and reflect stage of problem-solving involves "an examination of solutions with a view of deepening and enlarging one's conception of the given problem . . ." which is not easy for the average students (Leong, et al., 2012, pp. 367-368; O'Shea & Leavy, 2013).

Nevertheless, the findings highlighted the variations in students' engagement in MPS when working as individuals and in a small group setting. It occurred that students actively engaged in S3 and S4 of Polya's heuristics (1973) when they solved problems in a group setting. The study demonstrated that solving mathematical problems in a group allows students to challenge each other's ideas, promotes discussions and different ways of approaching the problem. This allowed them to better comprehend the problem by attending to each other's perspectives and developing a shared understanding.

Another major pattern highlighted through the findings of the study emphasized that students both individually and while working in a small group go back and forth among the four stages (S1, S2, S3 and S4). It was apparent that participants repeatedly changed positions and engaged in iterative and cyclical phases that highlight the four stages of the heuristics. For example, as students moved from S1 to S3 while solving WP4, they came back to S1 several times to "read it once again" (Turns 8-10, 20). M5's comments also imply that students tend to move forward and backward between different stages while solving problems. Overall, as students proceeded with MPS, they frequently came back to the problem to develop new understandings by correcting errors, or to pose new or related procedures to work on. Leong et al. (2012) conducted a case study of a high-achieving student as he pushed beyond the review stage of Polya's heuristics (1973). The student described his movement between Polya's stages as: "During the planning part, I can already sort of think of what is going to happen when I carry out the plan" (p. 362). Even Polya approves a frequent shift in positions or views while making progress in the problem-solving process.

While Polya's heuristics has often been interpreted and employed as linear and distinct (Brijlall, 2015; O'Shea & Leavy, 2013), data from the study demonstrated that it is an iterative, cyclical and blended process. This interpretation is in line with the researchers who build upon Polya's work to describe MPS as a dynamic and cyclical process (Rott, 2012; Schoenfeld, 1992; Wilson, Fernandez, & Hadaway, 1993).

Conclusion

This study examined secondary school students' engagement in MPS when working as individuals and in a small group setting. The research revealed that although students have little to no prior experience of working in groups, they actively communicated ideas, shared strategies, sought clarifications, negotiated and adapted solutions, and reflected upon their experiences when they attempted MPS in a small group. These tendencies are often reported in the findings from previous research (e.g., O'Shea & Leavy, 2013).

Nevertheless, the study demonstrates that students rarely utilized and manipulated learning tools during group work. This is because they preferred seeking help from each other before manipulating other tools. Researchers argue that adaptive help seeking from peers involves clarifications and explanations instead of a request for a ready-made answer. Such help has educational benefits such as enhanced engagement in meta-cognitive processes and academic achievement (Shim, Kiefer, & Wang, 2013). As students seek help from each other, they enjoy the process of learning and benefit not only by receiving but by providing help as well (Alvi, Iqbal, Masood, & Batool, 2016). The social setting in the context of the present study allowed students to seek adaptive help from each other since none of them knew the correct answer. They worked together towards the problem solution, provided explanations in a language that was easily understandable, and created a supportive environment. M5 explained: "It is better for us to work in groups: one, it [helps] to develop unity between us; second, [we] get support from one another."

Overall, the study demonstrates that MPS is a cyclical and dynamic process in which students frequently move and shift positions between different stages/phases. The participants of the study employed often blended paths where they read the problem together, made plans (if any), acted out their strategies, re-read the problem, developed new understandings, made new plans, and looked back to confirm their strategies in a back and forth fashion. Working in a group allowed students to extend thinking from their individualized domains to include meta-cognitive, social, contextual and interpersonal domains as well.

Recommendations

The study has implications for understanding and organizing secondary school students' engagement in MPS. While the participants were not systematically trained to utilize Polya's heuristics, the knowledge generated through the study sheds light on their engagement in different stages of MPS when working as individuals and in a small group setting. It is recommended that teachers utilize this knowledge while training students to effectively employ problem-solving heuristics. This can be done in different ways. For example, students can be trained by teachers to shift their problem-solving disposition from impulsive thinking to mindful planning along Polya's stages. Teachers should also allow students to move back and forth among the different stages of MPS and encourage them to employ different strategies when engaged in MPS. Moreover, students' active engagement in S1 and S3 of Polya's heuristics, from both individual and social sources of data, sheds light on how they approach MPS. Although limited, students were nevertheless more engaged in S4 when they approached MPS in a group setting rather than as individuals. Consistently, it is recommended that MPS should be taught in group

settings that provide students with the opportunities to think, justify, explain, and review their actions. However, in order to fully develop the disposition of engagement with peers, students should be trained in complex skills such as reinterpreting the problem, interpreting the result, or posing a new problem after the given problem is solved.

As teachers utilize the knowledge generated from the study to engage students in mathematical problem-solving, they may get a step closer to transforming their teaching to align with the emphasis laid in the National Curriculum for Mathematics (Government of Pakistan, 2006).

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