Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol.47(1)(2015) pp.00.00

New Operational Matrix For Shifted Legendre Polynomials and Fractional Differential Equations With Variable Coefficients

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Received: 17 February, 2015 / Accepted: 15 April, 2015 / Published online: 23 April, 2015

Abstract. This paper is devoted to study a computation scheme to approximate solution of fractional differential equations(FDEs) and coupled system of FDEs with variable coefficients. We study some interesting properties of shifted Legendre polynomials and develop a new operational matrix. The new matrix is used along with some previously derived results to provide a theoretical treatment to approximate the solution of a generalized class of FDEs with variable coefficients. The new method have ability to convert fractional order differential equations having variable coefficients to system of easily solvable algebraic equations. We gave some details to show the convergence of the scheme. The efficiency and applicability of the method is shown by solving some test problems. To show high accuracy of proposed method we compare out results with some other results available in the literature. The proposed method is computer oriented. We use $MatLab$ to carry out necessary calculations.

AMS (MOS) Subject Classification Codes: 65N35; 65M70; 35C11
Key Words: Legendre polynomials; Approximation theory; Fractional differential equations.

1. INTRODUCTION

In recent years considerable interest in fractional differential equations (FDE) has been stimulated due to their numerous applications in the areas of physics and engineering see for example [36, 33, 21, 28]. After the discovery of fractional calculus (derivative and integral of non integer order) it is shown that fractional order differential equations (FDEs) can provide a more real insight in the phenomena as compared to the ordinary differential equations (see for example [23]). The exact analytic solution of FDEs is available only for a considerable small class of FDEs. Some time it is very difficult to obtain the exact analytic solutions of fractional differential equations. In some cases it becomes impossible to arrive at the exact analytic solution. The reason of this difficulty is the great computational complexities of fractional calculus involving in these equations.

This paper deals with the approximate solution of generalized classes of multi term fractional differential equations with variable coefficients of the form

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = \sum_{i=0}^{a} \phi_i(t) \frac{\partial^i U(t)}{\partial t^i} + f(t), \tag{1. 1}$$

with initial conditions

$$U^i(0) = u_i, \, i = 0, 1, \cdots a.$$

Where u_i are all real constants, $a < \sigma \leq a + 1$, $t \in [0, \tau]$, $U(t)$ is the unknown solution to be determined, $f(t)$ is the given source term and $\phi_i(t)$ for $i = 0, 1 \cdots a$ are coefficients depends on t and are well defined on $[0, \tau]$.

$$\begin{aligned} \frac{\partial^\sigma U(t)}{\partial t^\sigma} &= \sum_{i=0}^{n} \phi_i(t) \frac{\partial^i U(t)}{\partial t^i} + \sum_{i=0}^{n} \psi_i(t) \frac{\partial^i V(t)}{\partial t^i} + f(t), \\ \frac{\partial^\sigma V(t))}{\partial t^\sigma} &= \sum_{i=0}^{n} \varphi_i(t) \frac{\partial^i U(t)}{\partial t^i} + \sum_{i=0}^{n} \varrho_i(t) \frac{\partial^i V(t)}{\partial t^i} + g(t), \end{aligned} \tag{1. 2}$$

with initial conditions

$$\begin{aligned} U^i(0) &= u_i, \, i = 0, 1, 2 \cdots n, \\ V^i(0) &= v_i, \, i = 0, 1, 2 \cdots n. \end{aligned}$$

Here $n < \sigma \leq n + 1$, u_i and v_i are real constants, $\phi_i(t),\, \psi_i(t),\, \varphi_i(t)\, and\, \varrho_i(t)$ are given variable coefficients and are continuously differentiable and well defined on $[0, \tau]$. $U(t)$ and $V(t)$ are the unknown solutions to be determined while $f(t)$ and $g(t)$ are the given source terms.

Many important differential equations which are of basic importance especially in modeling the real world problems belong generalized class of differential equations defined in (1. 1) see for example [28, 22, 34]. In [23] A. Monje, use such type of differential equations to model behavior of immersed plate in fluid. In [28] $PD^\nu-$controler is modeled using such types of equations and it is shown that using fractional order differential equations we can get a more real insight in the phenomena as compared to the ordinary case.

Many authors existence of solution of such problems. Among others Yi Chen et al [2] study the existence results of the solution of the problem and provide sufficient conditions under which the solution of the problem exists. They use Leggett-Williams fixed point theorem to prove the existence of positive solutions of the corresponding problems. In this paper we assume that $U(t)$ satisfy all the necessary conditions for existence of unique solution.

Various attempts were made to approximate solution of such type of problems. Yildiray Keskin [10] proposed a new technique based of generalized Taylor polynomials for the numerical solution of such types of equations. In [25, 38] an approach is made to approximate solution of such problems, their approach is based on rationalized Chebeshev polynomials combined with tau method. In [37] the author use collocation approach to solve such type of problems. In [29] the author study improve Chebeshev collocation method for solution of such problems. More recently J. Liu et al [20] use the Legendre spectral Tau method to obtain the solution of fractional order partial differential equations with variable coefficients. Kilbas et al [17], investigate solutions around an ordinary point for linear homogeneous Caputo fractional differential equations with sequential fractional derivatives of order $k\alpha(0 < \alpha = 1)$ having variable coefficients. In [18], the authors studied on explicit representations of Greens function for linear (Riemann-Liouvilles) fractional differential operators with variable coefficients continuous in $[0, \infty)$ and applied it to obtain explicit representations for solution of non-homogeneous fractional differential equation with variable coefficients of general type.

Recently the operational matrix method got attention of many mathematicians. The reason is high simplicity and efficiency of the method. Different kinds of differential and partial differential equations are efficiently solved using this method see for example [**?**, 35, 26, 3, 4, 5, **?**, **?**, 11, 12, 13] and the references quoted there. The operational matrix method is based on various orthogonal polynomials and wavelets. A deep insight in the method shows that the method is really very simple and accurate. But up to know to the best of our knowledge the method is only used to find the approximate solution of differential equations, partial differential equation (including fractional order) only with constant coefficients. Up to know this method is not able to solve fractional differential equations with variable coefficients, due to non availability of necessary operational matrices.

We generalize the operational matrix method to solve fractional differential equations with variable coefficients. We develop some new operational matrices. These new matrices are used to convert the generalize class of FDEs to a system of easily solvable algebraic equations. We also apply the new matrices to solve coupled system of FDEs with variable coefficients. For detailed study on spectral approximation and fractional calculus we refer the reader to studies [14, 7, 19].

The rest of the article is organized as follows : In section 2, we provide some preliminaries of fractional calculus, Legendre polynomials and some basic results from approximation theory. In section 3, we recall some other operational matrices of integration and differentiation for shifted Legendre polynomials and derive some new operational matrices. In section 4 operational matrices are used to establish a new scheme for solution of a generalized class of FDEs with variable coefficients and coupled FDEs with variable coefficients. In section 5 the we derive some relation for convergence of proposed method and obtain some upper bound for the error of approximate solution. In section 6, some numerical experiments is performed to show the efficiency of the new technique. And finally in section 7 a short conclusions is made.

2. Preliminaries

In this section we recall some basic definitions and concepts from open literature which are of basic importance in further development in this paper.

Definition 1: [27, 16]According to Riemann-Liouville the fractional order integral of order $\alpha \in \mathbf{R}_+$ of a function $\phi \in (L^1[a,b], \mathbf{R})$ on interval $[a,b] \subset \mathbf{R}$, is defined by

$$\mathcal{I}^{\alpha}_{a+}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1}\phi(s)ds, \tag{2. 3}$$

provided that the integral on right hand side exists.

Definition 2: For a given function $\phi(t) \in C^n[a,b]$, the Caputo fractional order derivative of order α is defined as

$$D^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\phi^{(n)}(s)}{(x-s)^{\alpha+1-n}}ds, \; n-1 \leq \alpha < n \;,\; n \in N, \tag{2. 4}$$

provided that the right side is point wise defined on (a,∞), where $n = [\alpha]+1$.

From (2. 3),(2. 4) it is easily deduced that

$$D^{\alpha}x^k = \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}x^{k-\alpha}, \; I^{\alpha}x^k = \frac{\Gamma(1+k)}{\Gamma(1+k+\alpha)}x^{k+\alpha} \text{ and } D^{\alpha}C = 0, \text{ for a constant } C. \tag{2. 5}$$

2.1. **The shifted Legendre polynomials:** The Legendre polynomials defined on $[-1,1]$ are given by the following recurrence relation (see [13])

$$L_{i+1}(z) = \frac{2i+1}{i+1}zL_i(z) - \frac{i}{i+1}L_{i-1}(z), \; i = 1,2.... \text{ where } L_0(z) = 0, \; L_1(z) = z.$$

The transformation $t = \frac{\tau(z+1)}{2}$ transforms the interval $[-1,1]$ to $[0,\tau]$ and the shifted Legendre polynomials are given by

$$L^{\tau}_i(t) = \sum_{k=0}^{i} \daleth_{i,k}t^k, i = 0,1,2,3..., \tag{2. 6}$$

$$\daleth_{(i,k)} = \frac{(-1)^{i+k}(i+k)!}{(i-k)!\tau^k(k!)^2}. \tag{2. 7}$$

These polynomials are orthogonal and the orthogonality condition is

$$\int_0^{\tau} L^{\tau}_i(t)L^{\tau}_j(t)dt = \begin{cases} \frac{\tau}{2i+1}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2. 8}$$

By the use of orthogonality condition (2. 8) any $f(t) \in C([0,\tau])$ can be approximated with Legendre polynomials ie

$$f(t) \approx \sum_{l=0}^{m} c_l L^{\tau}_l(t), \text{ where } \qquad c_l = \frac{(2l+1)}{\tau}\int_0^{\tau} f(t)L^{\tau}_l(t)dt. \tag{2. 9}$$

As $l \to \infty$ the approximation becomes equal to the exact function.

In vector notation, we write

$$f(t) \approx C_M^T \Lambda_M^\tau(t), \tag{2. 10}$$

where

$$\Lambda_M^\tau(t) = \begin{bmatrix} L_0^\tau(t) & L_1^\tau(t) & \cdots & L_i^\tau(t) & \cdots & L_m^\tau(t) \end{bmatrix}^T \tag{2. 11}$$

and

$$C_M = \begin{bmatrix} c_0 & c_1 & \cdots & c_i & \cdots & c_m \end{bmatrix}^T. \tag{2. 12}$$

$M = m + 1$ is the scale level of the approximation.

The following lemma is very important for our further analysis.

LEMMA 2.1. *The definite integral of product of any three Legendre polynomials on the domain* $[0, \tau]$ *is a constant and the value of that constant is* $\heartsuit_{(l,m,n)}^{(i,j,k)}$ *ie*

$$\int_0^\tau L_i^\tau(t) L_j^\tau(t) L_k^\tau(t) dt = \heartsuit_{(l,m,n)}^{(i,j,k)}, \tag{2. 13}$$

where

$$\heartsuit_{(l,m,n)}^{(i,j,k)} = \sum_{l=0}^{i} \sum_{m=0}^{j} \sum_{n=0}^{k} \daleth_{(i,l)} \daleth_{(i,m)} \daleth_{(i,n)} \Upsilon_{(l,m,n)}.$$

$\daleth_{(.,.)}$ *are as defined in* (2. 7) *and*

$$\Upsilon_{(l,m,n)} = \frac{\tau^{(l+m+n+1)}}{(l+m+n+1)}.$$

Proof. The proof of this lemma is straight forward.Consider

$$\int_0^\tau L_i^\tau(t) L_j^\tau(t) L_k^\tau(t) dt = \sum_{l=0}^{i} \daleth_{(i,l)} \sum_{m=0}^{j} \daleth_{(j,m)} \sum_{n=0}^{k} \daleth_{(k,n)} \int_0^\tau t^{(l+m+n)} dt. \tag{2. 14}$$

The integral in the above equation is equal to

$$\int_0^\tau t^{(l+m+n)} dt = \frac{\tau^{(l+m+n+1)}}{(l+m+n+1)}. \tag{2. 15}$$

Consider $\Upsilon_{(l,m,n)} = \frac{\tau^{(l+m+n+1)}}{(l+m+n+1)}$. Then

$$\int_0^\tau L_i^\tau(t) L_j^\tau(t) L_k^\tau(t) dt = \sum_{l=0}^{i} \sum_{m=0}^{j} \sum_{n=0}^{k} \daleth_{(i,l)} \daleth_{(j,m)} \daleth_{(k,n)} \Upsilon_{(l,m,n)}. \tag{2. 16}$$

Let suppose

$$\heartsuit_{(l,m,n)}^{(i,j,k)} = \sum_{l=0}^{i} \sum_{m=0}^{j} \sum_{n=0}^{k} \daleth_{(i,l)} \daleth_{(j,m)} \daleth_{(k,n)} \Upsilon_{(l,m,n)}. \tag{2. 17}$$

And hence the proof is complete. □

The constant defined in the above lemma plays (very)important role in the development of the new matrix.

The following theorem is important for the convergence of the scheme.

THEOREM 2.1. *Let $\prod_M$ be the space of M terms Legendre polynomials and let $u(t) \in C^m[0,1]$, then $u_m(t)$ is in space $\prod_M$. Then we have*

$$u(t) = \sum_{i=0}^{m} c_i L_i^{\tau}(t), \tag{2. 18}$$

then

$$c_k \simeq \frac{C}{\lambda_k}^{m} \|u^{(m)}\|. \tag{2. 19}$$

and

$$\|u(t) - \sum_{i=0}^{m} c_i L_i^{\tau}(t)\|^2 \leq \sum_{k=m+1}^{\infty} \lambda_k c_k^2, \tag{2. 20}$$

where

$$c_k = \frac{2k+1}{\tau} \int_0^{\tau} u(t) L_k^{\tau}(t) dt, \qquad \lambda_k = k(k+1). \tag{2. 21}$$

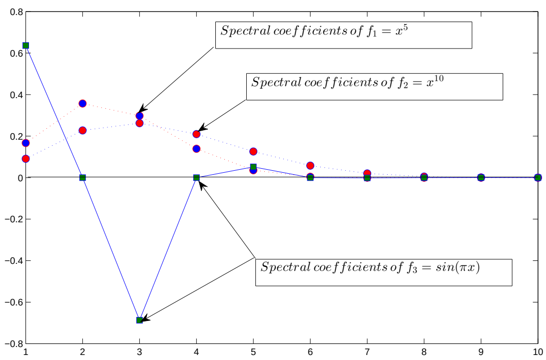
C is constant and can be chosen in such a way that $u^{(2m)}$ belong to $\prod_M$. Where $u^{(m)}$ is defined as

$$u^{(m)} = \mathbf{L}(u^{(m-1)}) = \mathbf{L}^m(u^{(0)}) \tag{2. 22}$$

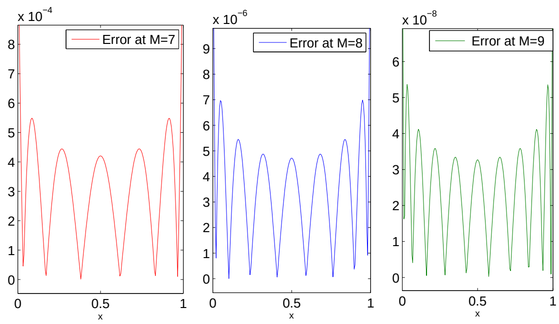
where $\mathbf{L}$ is storm livoli operator of legendre polynomials with $u^{(0)} = u(t)$.

Proof. Proof of this theorem is analogous as in [13] and references therein. □

2.1.1. *Numerical verification of Convergence.* We approximate three test functions $f_1 = x^5$,$f_2 = x^{10}$ and $f_3 = sin(\pi x)$ and observe the value of spectral coefficients.We observe that for first two function the spectral coefficients with index greater than the degree of the polynomials are zero.For the third function the coefficients also decay to zero.The spectral coefficients of all the three functions is given in Fig (1).We approximate the absolute error for the third function at different scale level and observe that the error of approximation decreases as the scale level increases.The results are displayed in Fig (2).



Fig(1):Spectral decay of the Legendre coefficients of three test functions .



Fig(2):Absolute amount of error for $f_3 = sin(\pi x)$ at different scale level.

It is concluded that the Legendre polynomials provide a good approximation to continuous function.The error of approximation decreases rapidly with the increase of scale level.For example (Fig (2))the error is less than 10^{-4} at scale level M=7 and as we increase the scale level the error is less than 10^{-6} and 10^{-8}.

3. Main Result:New Operational Matrix

First of all we recall some existent results which are of basic importance in the formulation of our result.

LEMMA 3.1. *let $\Lambda_M^{\tau}(t)$ be the function vector as defined in* $(2.\ 11\ )$ *then the integration of order α of $\Lambda_M^{\tau}(t)$ is generalized as*

$$I^{\alpha}(\Lambda_M^{\tau}(t)) \simeq H_{M\times M}^{\tau,\alpha}\Lambda_M^{\tau}(t), \tag{3. 23}$$

where $H_{M\times M}^{\tau,\alpha}$ is the operational matrix of integration of order α and is defined as

$$H_{M\times M}^{\tau,\alpha} = \begin{bmatrix} \Theta_{0,0,\tau} & \Theta_{0,1,\tau} & \cdots & \Theta_{0,j,\tau} & \cdots & \Theta_{0,m,\tau} \\ \Theta_{1,0,\tau} & \Theta_{1,1,\tau} & \cdots & \Theta_{1,j,\tau} & \cdots & \Theta_{1,m,\tau} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{i,0,\tau} & \Theta_{i,1,\tau} & \cdots & \Theta_{i,j,\tau} & \cdots & \Theta_{i,m,\tau} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{m,0,\tau} & \Theta_{m,1,\tau} & \cdots & \Theta_{m,j,\tau} & \cdots & \Theta_{m,m,\tau} \end{bmatrix}, \tag{3. 24}$$

where

$$\Theta_{i,j,\tau} = \sum_{k=0}^{i} s_{k,j}\daleth_{i,k}\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}. \tag{3. 25}$$

Also $\daleth_{i,k}$ is similar as defined in $(2.\ 7\ )$ *and*

$$s_{k,j} = \frac{(2j+1)}{\tau}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!(\tau)^{k+l+\alpha+1}}{(\tau^{l})(j-l)(l!)^2(k+l+\alpha+1)}. \tag{3. 26}$$

Proof. Using (2. 5) along with (2. 6) we have

$$\begin{aligned} I^{\alpha} L_i^{\tau}(t) &= \sum_{k=0}^{i} \daleth_{i,k} I^{\alpha} t^k \\ &= \sum_{k=0}^{i} \daleth_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} t^{k+\alpha}. \end{aligned} \tag{3. 27}$$

Approximating $t^{k+\alpha}$ with $m+1$ terms of Legendre polynomial we get

$$t^{k+\alpha} \simeq \sum_{j=0}^{m} s_{k,j} L_j^{\tau}(t), \tag{3. 28}$$

we can easily calculate the value of $s_{k,j}$ by using the orthogonality condition ie

$$s_{k,j} = \frac{(2j+1)}{\tau} \sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!(\tau)^{k+l+\alpha+1}}{(\tau^l)(j-l)(l!)^2(k+l+\alpha+1)}. \tag{3. 29}$$

Employing (3. 27), and (3. 29) we get

$$I^{\alpha} L_i^{\tau}(t) = \sum_{j=0}^{m} \sum_{k=0}^{i} s_{k,j} \daleth_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} L_j^{\tau}(t).$$

Setting

$$\Theta_{i,j,\tau} = \sum_{k=0}^{i} s_{k,j} \daleth_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}, \tag{3. 30}$$

we get

$$I^{\alpha} L_i^{\tau}(t) = \sum_{j=0}^{m} \Theta_{i,j,\tau} L_j^{\tau}(t), \tag{3. 31}$$

or evaluating for different i we get the desired result. □

Corollary 1: The error $|E_M| = |I^{\alpha} U(t) - K_M^T H_{M\times M}^{\tau,\alpha} \Lambda_M^{\tau}(t)|$ in approximating $I^{\alpha} U(t)$ with operational matrix of fractional integration is bounded by the following.

$$|E_M| \leq | \sum_{k=m+1}^{\infty} c_k \{ \sum_{i=0}^{m} \Theta_{i,j,\tau} \}|. \tag{3. 32}$$

Proof. Consider

$$U(t) = \sum_{k=0}^{\infty} c_k L_k^{\tau}(t). \tag{3. 33}$$

Then using relation (3. 31) we get

$$I^{\alpha} U(t) = \sum_{k=0}^{\infty} c_k \sum_{j=0}^{m} \Theta_{k,j,\tau} L_j^{\tau}(t). \tag{3. 34}$$

Truncating the sum and writing in modified form we get

$$I^{\alpha} U(t) - \sum_{k=0}^{m} c_k \sum_{j=0}^{m} \Theta_{k,j,\tau} L_j^{\tau}(t) = \sum_{k=m+1}^{\infty} c_k \sum_{j=0}^{m} \Theta_{k,j,\tau} L_j^{\tau}(t). \tag{3. 35}$$

We can also write it in matrix form as

$$I^{\alpha}U(t) - K_M^T H_{M\times M}^{\tau,\alpha}\Lambda_M^{\tau}(t) = \sum_{k=m+1}^{\infty} c_k \sum_{j=0}^{m}\Theta_{k,j,\tau}L_j^{\tau}(t). \quad (3.\ 36)$$

But $L_j^{\tau}(t) \leq 1$ for $t \in [0,1]$ therefore we can write

$$|I^{\alpha}U(t) - K_M^T H_{M\times M}^{\tau,\alpha}\Lambda_M^{\tau}(t)| \leq |\sum_{k=m+1}^{\infty} c_k \sum_{j=0}^{m}\Theta_{k,j,\tau}|. \quad (3.\ 37)$$

And hence the proof is complete. □

LEMMA 3.2. *let $\Lambda_M^{\tau}(t)$ be the function vector as defined in* (2. 11) *then the derivative of order α of $\Lambda_M^{\tau}(t)$ is generalized as*

$$D^{\alpha}(\Lambda_M^{\tau}(t)) \simeq G_{M\times M}^{\tau,\alpha}\Lambda_M^{\tau}(t), \quad (3.\ 38)$$

where $G_{M\times M}^{\tau,\alpha}$ is the operational matrix of derivatives of order α and is defined as

$$G_{M\times M}^{\tau,\alpha} = \begin{bmatrix} \Phi_{0,0,\tau} & \Phi_{0,1,\tau} & \cdots & \Phi_{0,j,\tau} & \cdots & \Phi_{0,m,\tau} \\ \Phi_{1,0,\tau} & \Phi_{1,1,\tau} & \cdots & \Phi_{1,j,\tau} & \cdots & \Phi_{1,m,\tau} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{i,0,\tau} & \Phi_{i,1,\tau} & \cdots & \Phi_{i,j,\tau} & \cdots & \Phi_{i,m,\tau} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{m,0,\tau} & \Phi_{m,1,\tau} & \cdots & \Phi_{m,j,\tau} & \cdots & \Phi_{m,m,\tau} \end{bmatrix}. \quad (3.\ 39)$$

Where

$$\Phi_{i,j,\tau} = \sum_{k=\lceil\alpha\rceil}^{i} s_{k,j}\daleth_{i,k}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}, \quad (3.\ 40)$$

with $\Phi_{i,j,\tau} = 0$ if $i < \lceil\alpha\rceil$. Also $\daleth_{i,k}$ is similar as defined in (2. 7) *and*

$$s_{k,j} = \frac{(2j+1)}{\tau}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!(\tau)^{k+l-\alpha+1}}{(\tau^l)(j-l)(l!)^2(k+l-\alpha+1)}. \quad (3.\ 41)$$

Proof. The proof of this lemma is similar as above Lemma. □

Corollary 2: The error $|E_M| = |D^{\alpha}U(t) - K_M^T G_{M\times M}^{\tau,\alpha}\Lambda_M^{\tau}(t)|$ in approximating $D^{\alpha}U(t)$ with operational matrix of derivative is bounded by the following.

$$|E_M| \leq |\sum_{k=m+1}^{\infty} c_k\{\sum_{i=\lceil\alpha\rceil}^{m}\Phi_{i,j,\tau}\}| \quad (3.\ 42)$$

Proof. The proof of this corollary is similar as Corollary 1. □

LEMMA 3.3. *Let $U(t)$ and $\phi_n(t)$ be any function defined on $[0,\tau]$. Then*

$$\phi_n(t)\frac{\partial^{\sigma}U(t)}{\partial t^{\sigma}} = W_M^T Q_{\phi_n}^{\sigma}\Lambda_M^{\tau}(t). \quad (3.\ 43)$$

Where W_M^T is the Legendre coefficient vector of $U(t)$ as defined in (2. 10) *and*

$$Q_{\phi_n}^{\sigma} = G_{M\times M}^{\tau,\sigma}R_{M\times M}^{\tau,\phi_n}. \quad (3.\ 44)$$

The matrix $G_{M\times M}^{\tau,\sigma}$ (defined in Lemma 3.2) is the operational matrix of derivative of order σ and

$$R^{\tau,\phi_n}_{M\times M} = \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \cdots & \Theta_{0,s} & \cdots & \Theta_{0,m} \\ \Theta_{1,0} & \Theta_{1,1} & \cdots & \Theta_{1,s} & \cdots & \Theta_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{r,0} & \Theta_{r,1} & \cdots & \Theta_{r,s} & \cdots & \Theta_{r,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{m,0} & \Theta_{m,1} & \cdots & \Theta_{m,s} & \cdots & \Theta_{m,m} \end{bmatrix}. \tag{3. 45}$$

Where

$$\Theta_{r,s} = \frac{2s+1}{\tau}\sum_{i=0}^{m} c_i \heartsuit^{(i,r,s)}_{(l,m',n)}. \tag{3. 46}$$

Where $c_i = \int_0^\tau \phi_n(t)L^\tau_i(t)dt$*, and* $\heartsuit^{(i,r,s)}_{(l,m',n)}$ *is as defined in Lemma 2.2.1.*

Proof. Consider $U(t) \simeq W^T_M \Lambda^\tau_M(t)$, then by use of Lemma 3.2 we can easily write

$$\phi_n(t)\frac{\partial^\sigma U(t)}{\partial t^\sigma} = \phi_n(t) W^T_M G^{\tau,\sigma}_{M\times M}\Lambda^\tau_M(t). \tag{3. 47}$$

The above equation can also be written in the following form

$$\phi_n(t)\frac{\partial^\sigma U(t)}{\partial t^\sigma} = W^T_M G^{\tau,\sigma}_{M\times M}\overbrace{\Lambda^\tau_M(t)}. \tag{3. 48}$$

Where

$$\overbrace{\Lambda^\tau_M(t)} = \begin{bmatrix} \phi_n(t)L^\tau_0(t) & \phi_n(t)L^\tau_1(t) & \cdots & \phi_n(t)L^\tau_i(t) & \cdots & \phi_n(t)L^\tau_m(t) \end{bmatrix}^T. \tag{3. 49}$$

Approximating $\phi_n(t)$ with M terms of Legendre polynomials we get

$$\phi_n(t) \simeq \sum_{i=0}^{m} c_i L^\tau_i(t). \tag{3. 50}$$

Using (3. 50) in (3. 49) we can get

$$\overbrace{\Lambda^\tau_M(t)} = \begin{bmatrix} \aleph_0(t) & \aleph_1(t) & \cdots & \aleph_r(t) & \cdots & \aleph_m(t) \end{bmatrix}^T, \tag{3. 51}$$

where

$$\aleph_r(t) = \sum_{i=0}^{m} c_i L^\tau_i(t) L^\tau_r(t)\,, r = 0, 1\cdots m. \tag{3. 52}$$

We can approximate the general term $\aleph_r(t)$ with M terms of Legendre polynomials as follows

$$\aleph_r(t) = \sum_{s=0}^{m} h^r_s L^\tau_s(t), \tag{3. 53}$$

where

$$h^r_s = \frac{2s+1}{\tau}\int_0^\tau \aleph_r(t)L^\tau_s(t)dt. \tag{3. 54}$$

Using (3. 52) in (3. 54) we get

$$h^r_s = \frac{2s+1}{\tau}\sum_{i=0}^{m} c_i \int_0^\tau L^\tau_i(t)L^\tau_r(t)L^\tau_s(t)dt. \tag{3. 55}$$

Know using lemma 2.1 and (3. 55) we get

$$h_s^r = \frac{2s+1}{\tau}\sum_{i=0}^{m} c_i \heartsuit_{(l,m',n)}^{(i,r,s)}. \tag{3. 56}$$

Let suppose

$$\Theta_{r,s} = \frac{2s+1}{\tau}\sum_{i=0}^{m} c_i \heartsuit_{(l,m',n)}^{(i,r,s)}. \tag{3. 57}$$

Then repeating the procedure for $r = 0, 1, \cdots m$ and $s = 0, 1, \cdots m$ we can write

$$\begin{bmatrix} \aleph_0(t) \\ \aleph_1(t) \\ \cdots \\ \aleph_r(t) \\ \cdots \\ \aleph_m(t) \end{bmatrix} = \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \cdots & \Theta_{0,s} & \cdots & \Theta_{0,m} \\ \Theta_{1,0} & \Theta_{1,1} & \cdots & \Theta_{1,s} & \cdots & \Theta_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{r,0} & \Theta_{r,1} & \cdots & \Theta_{r,s} & \cdots & \Theta_{r,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{m,0} & \Theta_{m,1} & \cdots & \Theta_{m,s} & \cdots & \Theta_{m,m} \end{bmatrix} \begin{bmatrix} L_0^\tau(t) \\ L_1^\tau(t) \\ \vdots \\ L_s^\tau(t) \\ \vdots \\ L_m^\tau(t) \end{bmatrix}. \tag{3. 58}$$

We may write the above equation as

$$\overbrace{\Lambda_M^\tau(t)} = R_{M\times M}^{\tau,\phi_n}\Lambda_M^\tau(t). \tag{3. 59}$$

Using (3. 59) in (3. 48) we get

$$\phi_n(t)\frac{\partial^\sigma U(t)}{\partial t^\sigma} = W_M^T G_{M\times M}^{\tau,\sigma} R_{M\times M}^{\tau,\phi_n}\Lambda_M^\tau(t). \tag{3. 60}$$

Let $G_{M\times M}^{\tau,\sigma} R_{M\times M}^{\tau,\phi_n} = Q_{\phi_n}^\sigma$. Then we have

$$\phi_n(t)\frac{\partial^\sigma U(t)}{\partial t^\sigma} = W_M^T Q_{\phi_n}^\sigma \Lambda_M^\tau(t). \tag{3. 61}$$

And hence the proof is complete. □

4. Application of the new matrix

In this section we apply the new matrix to approximate the solution of fractional order differential equations.

4.1. **FDEs with variable equations.** Consider the following generalized class of FDEs with variable coefficients

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = \sum_{i=0}^{n}\phi_i(t)\frac{\partial^i U(t)}{\partial t^i} + f(t), \tag{4. 62}$$

subject to initial conditions

$$U^i(0) = u_i,\ i = 0, 1, \cdots n.$$

Where u_i are all real constant, $n < \sigma \leq n+1$, $t \in [0, \tau]$, $U(t)$ is the unknown solution to be determined, $f(t)$ is the given source term and $\phi_i(t)$ for $i = 0, 1 \cdots n$ are coefficients depends on t and are well defined on $[0, \tau]$. The solution of the above problem can be written in terms of shifted Legendre series such that

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = W_M^T \Lambda_M^\tau(t). \tag{4. 63}$$

Applying fractional integral of order σ and by using the given initial conditions we get

$$U(t) - \sum_{j=0}^{n} t^j u_j = W_M^T H_{M\times M}^{\tau,\sigma} \Lambda_M^\tau(t). \quad (4.\ 64)$$

Which can be simplified as

$$U(t) = W_M^T H_{M\times M}^{\tau,\sigma} \Lambda_M^\tau(t) + F_1^T \Lambda_M^\tau(t), \quad (4.\ 65)$$

where $F_1^T \Lambda_M^\tau(t) = \sum_{j=0}^{n} t^j u_j$. We can also write it as

$$U(t) = \hat{W}_M^T \Lambda_M^\tau(t), \quad (4.\ 66)$$

where

$$\hat{W}_M^T = W_M^T H_{M\times M}^{\eta,\sigma} + F_1^T. \quad (4.\ 67)$$

Using (4. 66) along with Lemma 3.3 we can write

$$\phi_i(t)\frac{\partial^i U(t)}{\partial t^i} = \hat{W}_M^T Q_{\phi_i}^i \Lambda_M^\tau(t). \quad (4.\ 68)$$

Approximating $f(t) = F_2\Lambda_M^\tau(t)$ and using (4. 68) in (4. 62) we get

$$W_M^T \Lambda_M^\tau(t) = \sum_{i=0}^{n} \hat{W}_M^T Q_{\phi_i}^i \Lambda_M^\tau(t) + F_2\Lambda_M^\tau(t). \quad (4.\ 69)$$

On further simplification we get

$$\{W_M^T - \hat{W}_M^T \sum_{i=0}^{n} Q_{\phi_i}^i - F_2\}\Lambda_M^\tau(t) = 0. \quad (4.\ 70)$$

Or

$$\{W_M^T - \hat{W}_M^T \sum_{i=0}^{n} Q_{\phi_i}^i - F_2\} = 0. \quad (4.\ 71)$$

Now using (4. 67) we can write the above equation as

$$\{W_M^T - W_M^T \sum_{i=0}^{n} H_{M\times M}^{\eta,\sigma} Q_{\phi_i}^i - \sum_{i=0}^{n} F_1 Q_{\phi_i}^i - F_2\} = 0. \quad (4.\ 72)$$

Equation (4. 72) is easily solvable algebraic equation and can be easily solved for the unknown coefficient vector W_M^T.Using the value of W_M^T in equation (4. 65) will arrive us to the approximate solution of the problem.

4.2. **Coupled system of FDEs with variable equations.** The Q-Matrix is also helpful when we want to approximate the solution of coupled system of fractional order differential equations having variable coefficients. Consider the generalize class of system

$$\begin{aligned} \frac{\partial^\sigma U(t)}{\partial t^\sigma} &= \sum_{i=0}^{n} \phi_i(t)\frac{\partial^i U(t)}{\partial t^i} + \sum_{i=0}^{n} \psi_i(t)\frac{\partial^i V(t)}{\partial t^i} + f(t), \\ \frac{\partial^\sigma V(t))}{\partial t^\sigma} &= \sum_{i=0}^{n} \varphi_i(t)\frac{\partial^i U(t)}{\partial t^i} + \sum_{i=0}^{n} \varrho_i(t)\frac{\partial^i V(t)}{\partial t^i} + g(t), \end{aligned} \quad (4.\ 73)$$

with initial conditions

$$U^i(0) = u_i,\ i = 0, 1, 2\cdots n,$$
$$V^i(0) = v_i,\ i = 0, 1, 2\cdots n.$$

Here $n < \sigma \leq n+1$, u_i and v_i are real constants, $\phi_i(t),\ \psi_i(t),\ \varphi_i(t)\, and\, \varrho_i(t)$ are given variable coefficients and are continuously differentiable and well defined on $[0,\tau]$. $U(t)$ and $V(t)$ are the unknown solutions to be determined while $f(t)$ and $g(t)$ are the given source terms. We seek the solutions of above equations in terms of shifted Legendre polynomials such that

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = W_M^T \Lambda_M^\tau(t), \qquad \frac{\partial^\sigma Z(x)}{\partial x^\sigma} = E_M^T \Lambda_M^\tau(t). \tag{4. 74}$$

By the application of fractional integration of order σ on both equations and using initial conditions allows us to write (4. 74) as

$$U(t) = \hat{W}_M^T \Lambda_M^\tau(t), \qquad V(t) = \hat{E}_M^T \Lambda_M^\tau(t). \tag{4. 75}$$

Where

$$\hat{W}_M^T = W_M^T H_{M\times M}^{\tau,\sigma} + F_1^T, \qquad \hat{E}_M^T = E_M^T H_{M\times M}^{\tau,\sigma} + F_2^T. \tag{4. 76}$$

Note here that $F_1^T \Lambda_M^\tau(t) = \sum_{i=0}^n u_i t^i$ and $F_2^T \Lambda_M^\tau(t) = \sum_{i=0}^n v_i t^i$. Using equation (4. 75) along with Lemma 3.3 we may write

$$\begin{aligned}
\sum_{i=0}^n \phi_i(t)\frac{\partial^i U(t)}{\partial t^i} &= \hat{W}_M^T \sum_{i=0}^n Q_{\phi_i}^i \Lambda_M^\tau(t), \qquad \sum_{i=0}^n \psi_i(t)\frac{\partial^i V(t)}{\partial t^i} = \hat{E}_M^T \sum_{i=0}^n Q_{\psi_i}^i \Lambda_M^\tau(t),\\
\sum_{i=0}^n \varphi_i(t)\frac{\partial^i U(t)}{\partial t^i} &= \hat{W}_M^T \sum_{i=0}^n Q_{\varphi_i}^i \Lambda_M^\tau(t), \qquad \sum_{i=0}^n \varrho_i(t)\frac{\partial^i V(t)}{\partial t^i} = \hat{E}_M^T \sum_{i=0}^n Q_{\varrho_i}^i \Lambda_M^\tau(t).
\end{aligned} \tag{4. 77}$$

Approximating the source terms $f(t)$ and $g(t)$ with shifted Legendre polynomials and using (4. 77),(4. 74) in (4. 73) and writing in vector notation we get

$$\left[\begin{array}{c} W_M^T \Lambda_M^\tau(t) \\ E_M^T \Lambda_M^\tau(t) \end{array}\right] = \left[\begin{array}{c} \hat{W}_M^T \sum_{i=0}^n Q_{\phi_i}^i \Lambda_M^\tau(t) \\ \hat{E}_M^T \sum_{i=0}^n Q_{\varrho_i}^i \Lambda_M^\tau(t) \end{array}\right] + \left[\begin{array}{c} \hat{E}_M^T \sum_{i=0}^n Q_{\psi_i}^i \Lambda_M^\tau(t) \\ \hat{W}_M^T \sum_{i=0}^n Q_{\varphi_i}^i \Lambda_M^\tau(t) \end{array}\right] + \left[\begin{array}{c} \hat{F}_M \Lambda_M^\tau(t) \\ \hat{G}_M \Lambda_M^\tau(t) \end{array}\right]. \tag{4. 78}$$

By taking the transpose of the (4. 78) and after a short manipulation we get

$$\begin{aligned}
\left[\begin{array}{cc} W_M^T & E_M^T \end{array}\right] A &= \left[\begin{array}{cc} \hat{W}_M^T & \hat{E}_M^T \end{array}\right] \left[\begin{array}{cc} \sum_{i=0}^n Q_{\phi_i}^i & O_{M\times M} \\ O_{M\times M} & \sum_{i=0}^n Q_{\varrho_i}^i \end{array}\right] A \\
&+ \left[\begin{array}{cc} \hat{W}_M^T & \hat{E}_M^T \end{array}\right] \left[\begin{array}{cc} O_{M\times M} & \sum_{i=0}^n Q_{\varphi_i}^i \\ \sum_{i=0}^n Q_{\psi_i}^i & O_{M\times M} \end{array}\right] A + \left[\begin{array}{cc} \hat{F}_M & \hat{G}_M \end{array}\right] A.
\end{aligned} \tag{4. 79}$$

Where $A = \left[\begin{array}{cc} \Lambda_M^\tau(t) & O_M \\ O_M & \Lambda_M^\tau(t)) \end{array}\right]$, O_M and $O_{M\times M}$ is zero vector and zero matrix of order M respectively. Canceling out the common terms and after a short simplification we get

$$\left[\begin{array}{cc} W_M^T & E_M^T \end{array}\right] - \left[\begin{array}{cc} \hat{W}_M^T & \hat{E}_M^T \end{array}\right] \left[\begin{array}{cc} \sum_{i=0}^n Q_{\phi_i}^i & \sum_{i=0}^n Q_{\varphi_i}^i \\ \sum_{i=0}^n Q_{\psi_i}^i & \sum_{i=0}^n Q_{\varrho_i}^i \end{array}\right] - \left[\begin{array}{cc} \hat{F}_M & \hat{G}_M \end{array}\right] = 0. \tag{4. 80}$$

Using (4. 76) in (4. 80) we get

$$\begin{bmatrix} W_M^T & E_M^T \end{bmatrix} - \begin{bmatrix} W_M^T & E_M^T \end{bmatrix} \begin{bmatrix} H^{\tau,\sigma}_{M\times M} \sum_{i=0}^{n} Q^i_{\phi_i} & H^{\tau,\sigma}_{M\times M} \sum_{i=0}^{n} Q^i_{\varphi_i} \\ H^{\tau,\sigma}_{M\times M} \sum_{i=0}^{n} Q^i_{\psi_i} & H^{\tau,\sigma}_{M\times M} \sum_{i=0}^{n} Q^i_{\varrho_i} \end{bmatrix}$$
$$- \begin{bmatrix} F_1^T & F_2^T \end{bmatrix} \begin{bmatrix} \sum_{i=0}^{n} Q^i_{\phi_i} & \sum_{i=0}^{n} Q^i_{\varphi_i} \\ \sum_{i=0}^{n} Q^i_{\psi_i} & \sum_{i=0}^{n} Q^i_{\varrho_i} \end{bmatrix} - \begin{bmatrix} \hat{F}_M & \hat{G}_M \end{bmatrix} = 0.$$

Which is easily solvable generalized sylvester type matrix equation and can be easily solved for the unknown $\begin{bmatrix} W_M^T & E_M^T \end{bmatrix}$. Using these values in (4. 75) along with (4. 76) will lead us to the approximate solutions to the problem.

5. Error bound of the approximate solution

In this section we calculate a bound for error of approximation of solution with the proposed method. From Lemma 2.2 we conclude that Legendre polynomials are well suited to approximate a sufficiently continuous function on the bounded domain. We can also see that $c_k \rightarrow 0$ faster than any algebraic sequence of λ_k. Which means that as the scale level increase coefficients decreases and approaches to zero. Consider the following fractional differential equation.

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = \sum_{i=0}^{n} \phi_i(t) \frac{\partial^i U(t)}{\partial t^i} + f(t), \tag{5. 81}$$

Our aim is to derive upper bound for proposed method. We have to calculate $|E_M^1|$ defined as

$$|E_M^1| = |\frac{\partial^\sigma U(t)}{\partial t^\sigma} - K_M^T \Lambda_M^\tau(t)|. \tag{5. 82}$$

As in previous section we initially assume the highest derivative in terms of legender polynomials and then we use operational matrices to convert differential equation to system of algebraic equations. In last we get the initial assumption we made and then using operational matrix of integration we get the approximate solution. Therefore to obtain the upper bound for approximate solution we follow the same route.

Consider the following generalized class of FDEs with variable coefficients

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = \sum_{i=0}^{n} \phi_i(t) \frac{\partial^i U(t)}{\partial t^i} + f(t), \tag{5. 83}$$

subject to initial conditions

$$U^i(0) = u_i,\ i = 0, 1, \cdots n.$$

The solution of the above problem can be written in terms of shifted Legendre series such that

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = W_M^T \Lambda_M^\tau(t) + \sum_{k=m+1}^{\infty} c_k L_k^\tau(t). \tag{5. 84}$$

Applying fractional integral of order σ, using operational matrix of integration and using corollary 1 we can write

$$U(t) - \sum_{j=0}^{n} t^j u_j = W_M^T H^{\tau,\sigma}_{M\times M} \Lambda_M^\tau(t) + \sum_{k=m+1}^{\infty} c_k \sum_{i=0}^{m} \Theta_{i,k,\tau} L_k^\tau(t). \tag{5. 85}$$

Which can be simplified as

$$U(t) = W_M^T H_{M\times M}^{\tau,\sigma}\Lambda_M^\tau(t) + F_1^T\Lambda_M^\tau(t) + \sum_{k=m+1}^{\infty} c_k \sum_{i=0}^{m}\Theta_{i,k,\tau}L_k^\tau(t), \tag{5. 86}$$

Assume as in section 4.1 $\hat{W} = W_M^T H_{M\times M}^{\tau,\sigma} + F_1^T$

Using Lemma 3.3 and Corollary 2 we may write we can write

$$\phi_i(t)\frac{\partial^i U(t)}{\partial t^i} = \hat{W}_M^T Q_{\phi_i}^i\Lambda_M^\tau(t) + \phi_i(t)\sum_{k=m+1}^{\infty} c_k\sum_{i=0}^{m}\sum_{i'=0}^{m}\Theta_{i,k,\tau}\Phi_{i',k,\tau}L_k^\tau(t). \tag{5. 87}$$

Approximating $f(t) = F_2\Lambda_M^\tau(t) + \sum_{k'=m+1}^{\infty} f_k L_k^\tau(t)$ and using (5. 87) in (5. 83) we get

$$W_M^T\Lambda_M^\tau(t) - \sum_{i=0}^{n}\hat{W}_M^T Q_{\phi_i}^i\Lambda_M^\tau(t) - F_2\Lambda_M^\tau(t) = R_M(t). \tag{5. 88}$$

Where $R_M(t)$ is defined by relation

$$\begin{aligned} R_M(t) = &\sum_{k=m+1}^{\infty} c_k\sum_{i=0}^{m}\sum_{i'=0}^{m}\Theta_{i,k,\tau}\Phi_{i',k,\tau}\sum_{s=0}^{n}\phi_s L_s^\tau(t)(t)L_k^\tau(t) \\ &- \sum_{k=m+1}^{\infty} c_k L_k^\tau(t) + \sum_{k'=m+1}^{\infty} f_k L_k^\tau(t) \end{aligned} \tag{5. 89}$$

The proposed scheme works under assumption (see section 4.1) that $R_M(t) = 0$. Now as we observe that $L_k^\tau(t) \leq 1$ for $t \in [0,\tau]$ therefore using this property of Legendre polynomials we get upper bound of approximate solution as

$$|R_M| \leq \sum_{k=m+1}^{\infty}|c_k\sum_{i=0}^{m}\sum_{i'=0}^{m}\sum_{s=0}^{n}\Theta_{i,k,\tau}\Phi_{i',k,\tau}\phi_s - \sum_{k=m+1}^{\infty} c_k + \sum_{k'=m+1}^{\infty} f_k| \tag{5. 90}$$

In view of Theorem 2.1 we see that c_k decays to zero, as the index of truncation increases. Therefore it is evident that proposed algorithm converges to the approximate solution as we increase the scale level. Using the similar procedure we may also obtain the upper bound for error of approximation of coupled system of fractional differential equations. As the proof is analogous therefore we skip it and proceed to further analysis.

6. Test Problems

To show the efficiency and applicability of proposed method, we solve some test problems. Where possible we compare our results with some of results available. For illustration purpose we show results graphically.

Example 1: Consider the following fractional differential equation

$$\frac{\partial^\sigma U(t)}{\partial t^\sigma} = e^t\frac{\partial U(t)}{\partial t} + \cos(t)U(t) + F(t), \tag{6. 91}$$

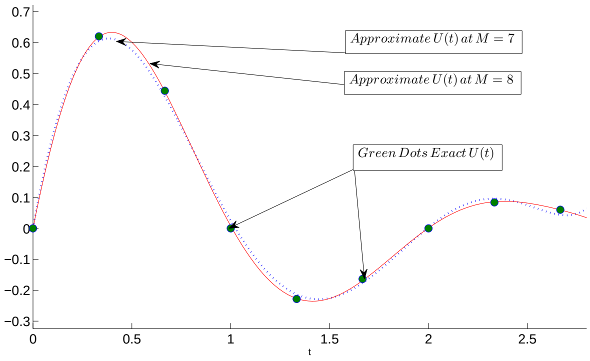
with initial condition $U(0) = 0$ and $U'(0) = \pi$.

Where $1 < \sigma \leq 2$, $t \in [0, 3]$ and the source term is defines as

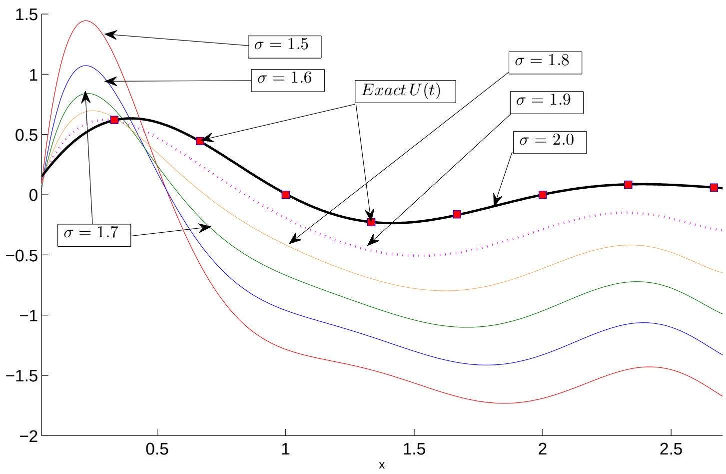
$$F(t) = e^{-t}sin(\pi t)(1-\pi^2) - e^{-t}cos(\pi t)(sin(\pi t) + 2\pi) + sin(\pi t) - \pi cos(\pi t).$$

The solution of this problem at $\sigma = 2$ is $Y(t) = e^{-t}sin(\pi t)$. However solution at fractional value of σ is not known. It is well known that solution of fractional differential equations approaches to solution of classical integer order differential equation as the order of derivative approaches from fractional to integer. Using this property of fractional differential equations we show that the solution of our problem approaches to solution at $\sigma = 1$ as $\sigma \longrightarrow 2$.

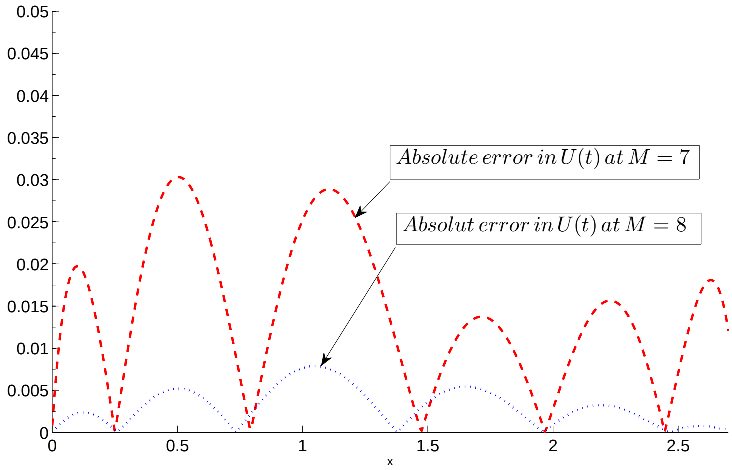
At first we fix $\sigma = 2$ and approximate the solution at different scale level. We observe that the accuracy of the solution depends solely on scale level. As scale level M increases the solution become more and more accurate. Fig (3) shows the comparison of exact solution with approximate solution at different scale level.One can easily see that at scale level $M = 8$, the approximate solution just matches the exact solution. One can see from Fig (5) that the absolute amount of error decrease as the scale level increases and at $M = 8$ the error is much more less that 10^{-2}. Which is much more acceptable number for such hard problems. We also approximate the solution at some fractional value of σ and observe that as $\sigma \longrightarrow 2$ (see Fig (4))the solution approaches to the exact solution at $\sigma = 2$. Which guarantees the accuracy of the scheme for fractional differential equations.



Fig(3):Comparison of exact and approximate solution of example 1 *at scale level* $M = 7, 8$.*The Dots represents the exact solutions while the lines represents the approximate solutions.*



Fig(4):Approximate solution of example 1 at fractional value of σ and its comparison with the exact solution at $\sigma = 2$.The red dots represents the exact solution at $\sigma = 2$.



Fig(5):Absolute error in U(t) of example 1 at different scale level ie $M = 7$ and $M = 8$.

Example 2:

As a second example we solve the following integer order differential equation [29].

$$D^2U(t) + 2tDU(T) = 0,$$
$$U(0) = 0, \qquad U'(0) = \frac{2}{\sqrt{\pi}}. \tag{6. 92}$$

The exact solution of the problem is $U(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$. We approximate solution to this problem using different scale level. And observe that the approximate solution very accurate. The comparison of exact solution with the solution obtained with this method at different scale level is displayed in Table 1. This problem is also solved by other authors. We compare absolute error obtained using proposed method with the absolute error reported in [25, 38, 37, 29]. It is clear that solution obtained with the proposed method is more accurate than the error reported in previous references. The results are displayed in Table 2.

Example 3: Consider the following fractional differential equation

$$D^\sigma U(t) = (t^3 - t^2 + 2t + 1)DU(t) + (t^4 + t^2 - 4t + 2)U(t) + g(t), \tag{6. 93}$$

with initial conditions $U(0) = 0$ and $U'(0) = 0$.

Where the source term is defined as

$$g(t) = -e^{(t)}cos(t)\{2+t-8t^2+t^4-t^5+t^6\}-e^{(t)}sin(t)\{1-11t+8t^3-6t^4+2t^5+t^6+t^7\}.$$

The order of derivative $1 < \sigma \leq 2$. One can easily check that the exact solution of the problem at $\sigma = 2$ is

$$U(t) = -\mathrm{e}^t \sin(t) \left(t - t^3\right).$$

We approximate solution of this problem with proposed technique. For comparison we fix $\sigma = 2$, and approximate solution at different scale level. As expected we get a high accurate estimate of the solution. We compare the exact solution with the approximate solution obtained at different scale level (see Fig (6)). We observe Fig (6) that the accuracy of the solution increases with the increase of scale level. And the absolute amount of error

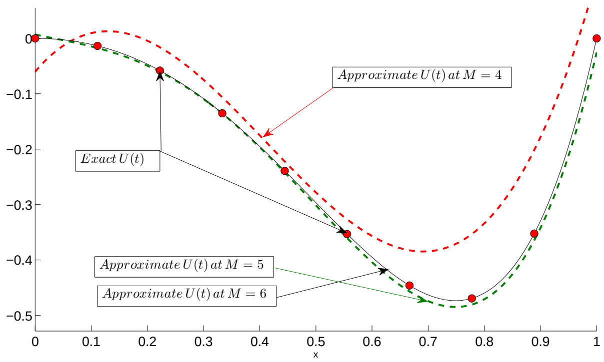
TABLE 1. Comparison of Exact and approximate solution of Example 2.

t \ M	Exact U(t)	M=8	M=9	M=10
t=0.0	0	-0.0000007347	-0.0000000295	0.0000000082
t=0.1	0.1124629159	0.1124629232	0.1124629114	0.1124629184
t=0.2	0.2227025889	0.2227024418	0.2227025889	0.2227025872
t=0.3	0.3286267591	0.3286269538	0.3286267673	0.3286267603
t=0.4	0.4283923546	0.4283923701	0.4283923466	0.4283923559
t=0.5	0.5204998773	0.5204996851	0.5204998761	0.5204998759
t=0.6	0.6038560903	0.6038561360	0.6038561011	0.6038560920
t=0.7	0.6778011933	0.6778013735	0.6778011869	0.6778011943
t=0.8	0.7421009642	0.7421008017	0.7421009616	0.7421009629
t=0.9	0.7969082119	0.7969082438	0.7969082215	0.7969082145

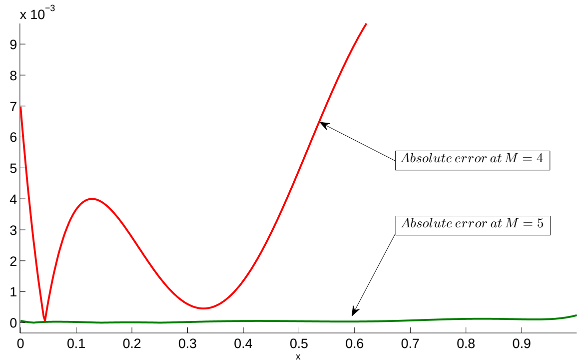
TABLE 2. Comparison of absolute error of Example 2 with other methods.

t	Tau [25]	Haar [38]	Collocation [37]	Chebyshev [29]	Present $M = 11$
0.1	84.09×10^{-9}	22.91×10^{-6}	24.31×10^{-6}	213×10^{-12}	275.24×10^{-12}
0.2	11.00×10^{-9}	22.58×10^{-6}	187.51×10^{-6}	5.70×10^{-9}	75.3×10^{-12}
0.3	140.85×10^{-9}	16.75×10^{-6}	61.35×10^{-6}	9.63×10^{-9}	452.20×10^{-12}
0.4	145.33×10^{-9}	22.35×10^{-6}	23.55×10^{-6}	12.05×10^{-9}	289.06×10^{-12}
0.5	77.37×10^{-9}	29.87×10^{-6}	76.37×10^{-6}	13.54×10^{-9}	427.22×10^{-12}
0.6	9.62×10^{-9}	16.09×10^{-6}	40.39×10^{-6}	32.87×10^{-9}	588.85×10^{-12}
0.7	6.64×10^{-9}	11.19×10^{-6}	129.99×10^{-6}	228.99×10^{-9}	324.38×10^{-12}
0.8	135.76×10^{-9}	20.96×10^{-6}	136.53×10^{-6}	1.07×10^{-6}	630.16×10^{-12}
0.9	288.02×10^{-9}	18.21×10^{-6}	87.11×10^{-6}	2.82×10^{-6}	257.18×10^{-12}

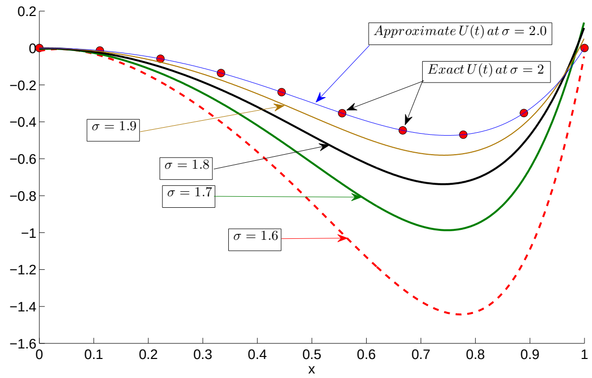
is less than 10^{-3} at $M = 5$. Fig (7) shows the absolute amount of error at different scale level. The same conclusion is made about the solution at fractional value of σ. The solution approaches uniformly to the exact solution as $\sigma \to 2$. Fig (8) shows this phenomena. Note that here we fix scale level $m = 5$.



Fig(6):Comparison of exact and approximate solution of example 3 *at scale level* $M = 5, 6$.*The Dots represents the exact solutions while the lines represents the approximate solutions.*



Fig(7):Absolute error in U(t) of example 3 at different scale level.



Fig(8):Approximate solution of example 3 at fractional value of σ and its comparison with the exact solution at $\sigma = 2$,$M = 5$.

Example 4: Consider the following coupled system of FDEs

$$\begin{aligned} \frac{\partial^\sigma U(t)}{\partial t^\sigma} &= (2t^3 - t)\frac{\partial U(t)}{\partial t} + (3t^2 + 2t)\frac{\partial V(t)}{\partial t} + f_1(t) \\ \frac{\partial^\sigma V(t)}{\partial t^\sigma} &= (4t^2 + 1)\frac{\partial U(t)}{\partial t} + (t^3 + 4)\frac{\partial V(t)}{\partial t} + f_2(t), \end{aligned} \tag{6. 94}$$

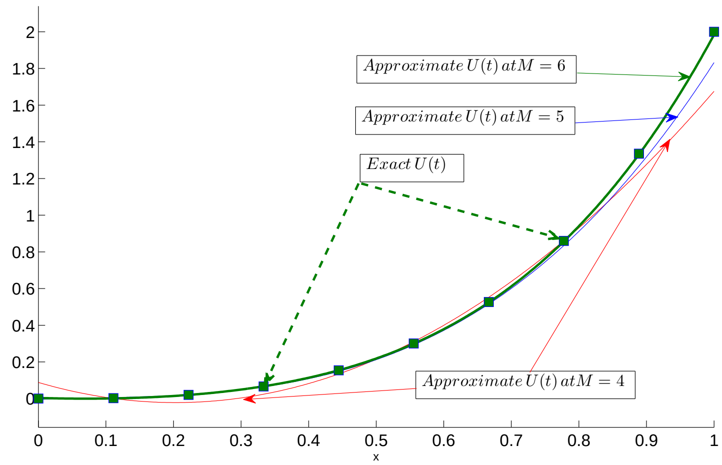
where the source terms are defined as

$$f_1(t) = 12t + (3t^2 + 2t)(-4t^3 + 3t^2 + 4t) + (-2t^3 + t)(5t^4 - 4t^3 + 6t^2) - 12t^2 + 20t^3,$$

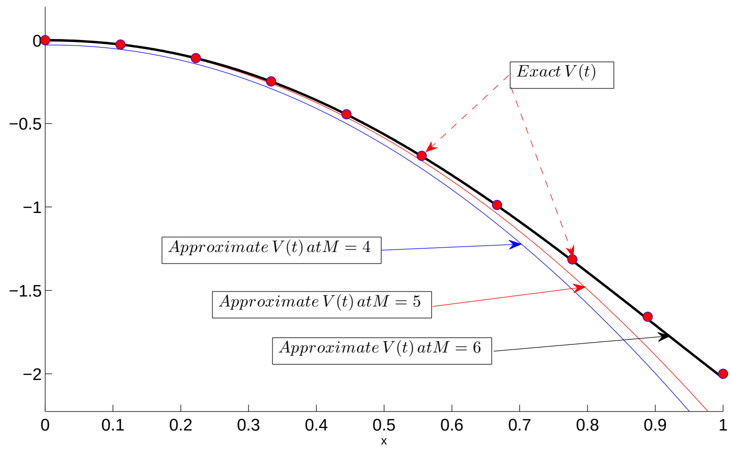
and

$$f_2(t) = (t^3 + 4)(-4t^3 + 3t^2 + 4t) - 6t + 12t^2 - (4t^2 + 1)(5t^4 - 4t^3 + 6t^2) - 4.$$

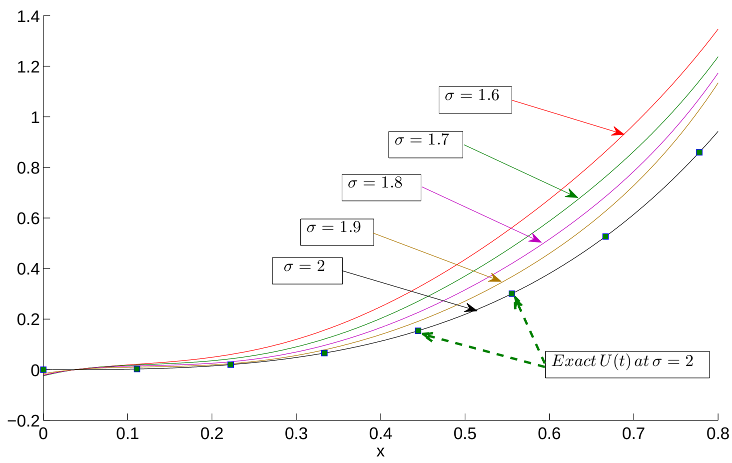
Note that $1 < \sigma \leq 2$ and $t \in [0, 1]$.The exact solutions of the coupled system at $\sigma = 2$ is $U(t) = (t^5 - t^4) + 2t^3$ and $V(t) = (t^4 - t^3) - 2t^2$. We analyze this problem with the new technique, and as expected we get the high accuracy of the solution. As the previous examples we first fix $\sigma = 2$ and simulate the algorithm at different scale level. The results are displayed in Fig (9) and Fig (10). In these figures we show comparison of exact solutions with approximate $U(t)$ and $V(t)$ respectively. One can easily note that the approximate solution becomes more and more accurate with the increase of scale level. At scale level $m = 6$ the absolute amount of error is less than 10^{-14}, unbelievable accuracy. Fig (13) shows absolute error in $U(t)$ and $V(t)$ at scale level $M = 6$. We also approximate the solution of the above problem at fractional value of σ and the same conclusion is made. See Fig (11) and Fig (12) for the approximate solution at fractional value of σ.



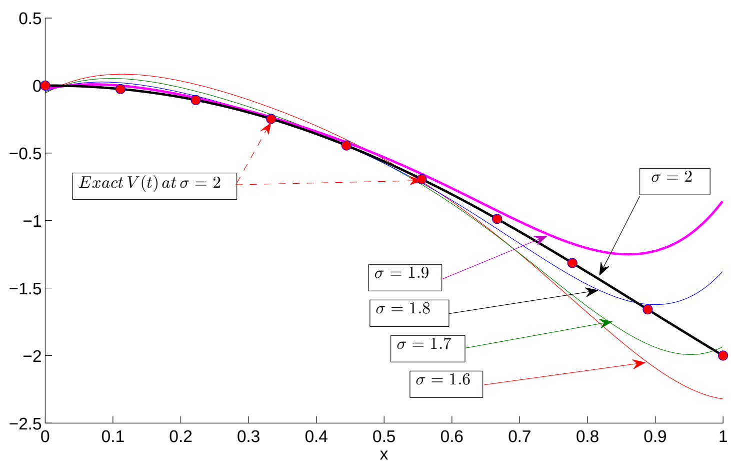
Fig(9):Comparison of exact and approximate U(t) of example 3 *at scale level* $M = 4, 5, 6$.*The Dots represents the exact solutions while the lines represents the approximate solutions.*



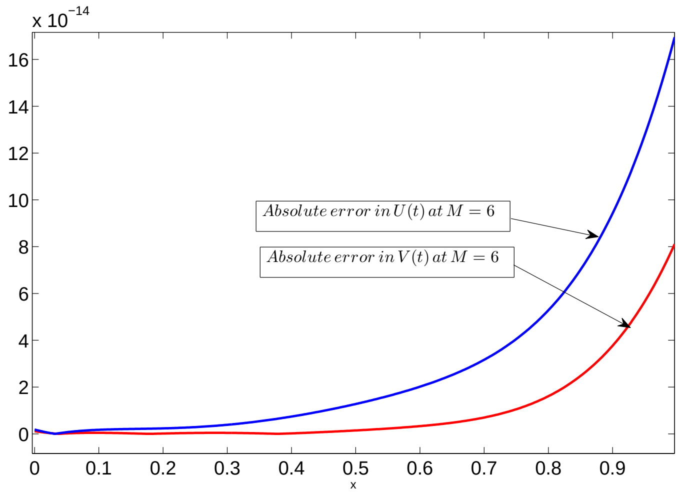
Fig(10):Comparison of exact and approximate V(t) of example 4 *at scale level* $M = 4, 5, 6$.*The Dots represents the exact solutions while the lines represents the approximate solutions.*



Fig(11):Approximate U(t) of example 4 at fractional value of σ, ie $\sigma = 1.6 : .1 : 2$ and its comparison with exact U(t) at $\sigma = 2$.



Fig(12):Approximate V(t)of example 4 at fractional value of σ, ie $\sigma = 1.6 : .1 : 2$ and its comparison with exact V(t) at $\sigma = 2$.



Fig(13):Absolute error in U(t) and V(t) at scale level $M = 6$.

7. Conclusion and Future Work

From analysis and experimental work we conclude that the proposed method works very well for approximating the solution of FDEs with variable coefficients. The results obtained are satisfactory. The method can be easily modified to solve some other types of FDEs under different types of boundary condition. It is also expected that the method yield more accurate solution by using some other orthogonal polynomials or wavelets. Our future work is related to the extension of the method to solve partial differential equations with variable coefficients.

Acknowledgments: The authors are extremely thankful to the reviewers for their useful comments that improves the quality of the manuscript.

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