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Classification of interior ideals in regular and intra regular semigroups

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Abstract. This paper delves into the classification of regular semigroups and their various subclassifications based on their interior ideals. By introducing and examining concepts such as strongly prime, prime, semiprime, strongly irreducible, and irreducible interior ideals, we present a new framework for understanding regular semigroups. In addition, we investigate the relationships among these different types of ideals, offering a thorough exploration of their interconnections.

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1. Introduction and Preliminaries

The significance of ideals in semigroup theory differs notably from their role in ring theory. Unlike ring ideals, semigroup ideals exhibit unique characteristics that present distinct challenges. These differences have led semigroup theorists to explore alternative ideal concepts to characterize semigroups, among which interior ideals have gained prominence. Introduced by S. Lajos [13], the study of interior ideals has been furthered by the contributions of Szasz [15, 16].

In a semigroup $(T, \cdot)$, a left (right) ideal $A \neq \phi$ is defined as $TA \subseteq A$ ($AT \subseteq A$). Lajos [12] defined a subset $M \subseteq T$ as a bi-ideal if M is a subsemigroup of T and $MTM \subseteq M$, and a quasi-ideal if $MT \cap TM \subseteq M$. An interior ideal of T, as introduced by Szasz [16], is a subsemigroup I of T such that $TIT \subseteq I$. The set of all interior ideals of a semigroup T is denoted by $\mathcal{I}(T)$. A semigroup T is termed regular [5] if for every element $a \in T$, there exists $b \in T$ such that $a = aba$. If, for each $a \in T$, $a \in Ta^2T$, then T is called intra regular. Due to Szasz [15] $\text{IN}(a) = a \cup a^2 \cup TaT$ is the intersection of all interior ideals of T containing a. For an in-depth understanding of semigroup theory, Howie's book [5] is an essential resource. Additional references [2, 4, 7, 8] discuss notions such as $L(a)$, $R(a)$, and $I(a)$.

We begin our discussion with the following lemma.

Lemma 1.1. [10] *Every ideal is an interior ideal of a semigroup.*

However, the converse does not hold, as demonstrated in the following example.

Example 1.2. *Consider the following semigroup $T = \{m, p, r, t\}$ together with the binary operation '$\cdot$' given below:*

.	m	p	r	t
m	m	r	p	t
p	r	t	t	t
r	p	t	t	t
t	t	t	t	t

In this context, $\{p, t\}$ and $\{r, t\}$ emerge as interior ideals of T, yet neither qualify as left nor right ideals within T since, $m \cdot p = r \notin \{p, t\}$ also $p \cdot m = r \notin \{p, t\}$, similarly we can check for $\{r, t\}$.

Despite the significant advancements in the theory of ideals over the past five decades, the exploration of interior ideals remains intriguing. While the concept itself is not novel, its profound implications have been underscored by various authors, notably Szasz [15, 16]. The contributions of Szasz prompted a fresh examination of interior ideals, leading to novel approaches in their study. Our research offers new insights, revealing compelling portrayals of semigroups through the lens of their interior ideals.

Definition 1.3. *Let $(T, \cdot)$ be a semigroup and K be an interior ideal of T. K is called semiprime if $I^2 \subseteq K$ gives $I \subseteq K$, for every interior ideal I of T. If for every $a \in T$, $a^2 \in K$ gives $a \in K$, we call K completely semiprime. K is called a prime(strongly prime) interior ideal of T if for any two interior ideals K_1, K_2 of T, $K_1K_2 \subseteq K(K_1K_2 \cap K_2K_1 \subseteq K)$ implies $K_1 \subseteq K$ or $K_2 \subseteq K$. K is called an irreducible (strongly irreducible) interior* *ideal if for any two interior ideals K_1, K_2 of T, $K_1 \cap K_2 = K(K_1 \cap K_2 \subseteq K)$ implies $K_1 = K$ or $K_2 = K(K_1 \subseteq K$ or $K_2 \subseteq K)$.*

Each strongly prime interior ideal within T also qualifies as a prime interior ideal. Moreover, every prime interior ideal is inherently a semiprime interior ideal of T. However, it's essential to note that while a prime interior ideal doesn't always imply strong primality, and a semiprime interior ideal doesn't always entail primality.

Example 1.4. *Consider $T = \{p, q, r\}$. Define a binary operation '$\cdot$' on T as follows:*

$.$	p	q	r
p	p	p	p
q	p	q	p
r	p	p	r

Thus $(T, \cdot)$ is a semigroup. Interior ideals of T are $\{p\}, \{p, q\}, \{p, r\}$ and $\{p, q, r\}$.

Here $\{p, q, r\}$ is strongly prime interior ideal hence prime and semiprime ideal but $\{p\}$ is not strongly prime. Infact $\{p, q\}\{p, r\} \cap \{p, r\}\{p, q\} \subseteq \{p\}$ gives neither $\{p, q\} \subseteq \{p\}$ nor $\{p, r\} \subseteq \{p\}$.

For the convenience of the general reader and the necessity of this paper, we present a series of results as follows:

Lemma 1.5. [11] *A semigroup T is regular if and only if $R \cap L = RL$, for every right ideal R and left ideal L of T.*

Lemma 1.6. [12] *A semigroup T is regular if and only if $NTN = N$, for each quasi-ideal N of T.*

Lemma 1.7. [12] *A semigroup T is regular if and only if $MTM = M$, for each bi-ideal M of T.*

Lemma 1.8. [3] *Every right(left) ideal of a semigroup T is a quasi-ideal of T.*

Theorem 1.9. [12] *Every bi-ideal of a regular semigroup T is a quasi-ideal, and conversely.*

Theorem 1.10. [10][14] *In a regular semigroup T, every interior ideal is an ideal, and conversely.*

Theorem 1.11. [12] *In a regular duo semigroup T, every bi-ideal of T is an ideal of T.*

Theorem 1.12. [12] *Every regular duo semigroup is a completely regular inverse semigroup and conversely.*

Theorem 1.13. [1] *Every completely regular inverse semigroup is a Clifford semigroup and conversely.*

Although regularity ensures that many ideal-theoretic properties hold, the structure of interior ideals in such semigroups remains an area of further exploration. Previous works [9, 12, 15, 16] have examined various ideal classifications. however, several characterizations, such as the minimality of interior ideals and the structure of $\mathcal{I}(T)$, are absent in the existing literature. This study addresses this gap by systematically classifying interior ideals in regular and intra regular semigroups. We establish their fundamental properties, and analyze their interrelations with other ideal types.

2. Interior ideals in regular and intra regular semigroup

This section examines the properties of interior ideals in regular and intra regular semigroups. We explore their structural significance and relationships with other ideal concepts, providing key characterizations under regularity conditions.

Theorem 2.1. *The following statements are true in a semigroup T:*

1. *The intersection of arbitrary interior ideals (if non-empty) of a semigroup T is an interior ideal of T.*
2. *For a subsemigroup X of T, $K \cap X (\neq \emptyset)$ is an interior ideal of X, for every interior ideal K of T.*
3. *If T is regular, then for every interior ideal K of T, $K = TKT$.*

Proof. (1): Let $\{K_i\}_{i\in\Delta}$ (Δ denotes any indexing set) be a family of interior ideals of a semigroup T. Say $C = \cap_{i\in\Delta} K_i$. Suppose $C \neq \emptyset$. So C is a subsemigroup of T. Evidently $TK_iT \subseteq K_i$, for $i \in \Delta$. We have to show that $TCT \subseteq C$. Take $x \in TCT$ so that $x = t_1 y t_2$ where $t_1, t_2 \in T$ and $y \in C = \cap_{i\in\Delta} K_i \subseteq K_i$, for all $i \in \Delta$ which yields that $x \in TK_iT$ for all $i \in \Delta$. Since $TK_iT \subseteq K_i$, for all $i \in \Delta$ we have $x \in \cap_{i\in\Delta} K_i = C$ thus $TCT \subseteq C$. Hence $\cap_{i\in\Delta} K_i$ is an interior ideal of T.

(2): Since $K \cap X \neq \phi$, it is a subsemigroup of T. Now $X(K \cap X)X \subseteq XKX \cap XXX \subseteq TKT \cap XXX \subseteq K \cap X$. Hence $K \cap X$ is an interior ideal of X.

(3): Suppose T is regular. Consider an interior ideal K of T then $TKT \subseteq K$. Take $x \in K$. Then $x \in xTx \subseteq xTxTx \subseteq TKT$ which gives $K \subseteq TKT$. Hence for every interior ideal K of T, $K = TKT$.

□

Even though it has previously been covered in the literature review but we have provided alternative of some results that offers some different aspect of interior ideals and explore the interactions between interior ideals and various other significant ideals in a given semigroup.

Theorem 2.2. *These conditions are coequal for any bi-ideal M and any quasi-ideal N in a semigroup T:*

1. *T is regular.*
2. *$N \cap J = NJN$, where J is an ideal of T.*
3. *$N \cap K = NKN$, where K is an interior ideal of T.*
4. *$K \cap M = MKM$, where K is an interior ideal of T.*

Proof. (1) $\Rightarrow$ (2): Consider an ideal J of T. Now $NJN \subseteq NTN \subseteq NT \cap TN \subseteq N$. Again $NJN \subseteq TJT$. As T is regular, by Theorem 2.1 and Lemma 1.1 we have $NJN \subseteq TJT = J$, thus $NJN \subseteq N \cap J$. Also let $z \in N \cap J$, $z \in N$ and $z \in J$. By Lemma 2.1 and Lemma 1.6, regularity of T gives $z \in zTz \subseteq (zTz)(TzT)z \subseteq (NTN)(TJT)N \subseteq NJN$ thus $N \cap J \subseteq NJN$. Hence $N \cap J = NJN$.

(2) $\Rightarrow$ (3): This implication follows from Lemma 1.1.

(3) $\Rightarrow$ (1): Let N be a quasi-ideal of T so that $NTN = N \cap T = N$. Consequently Lemma 1.6 shows that T is regular.

$(1) \Rightarrow (4)$: Choose a bi-ideal M and an interior ideal K of T. Now $MKM \subseteq MTM \subseteq M$ and $MKM \subseteq TKT \subseteq K$. Hence $MKM \subseteq M \cap K$. Also for $a \in M \cap K$, $a \subseteq (aTa)(TaT)a \subseteq (MTM)(TKT)M \subseteq MKM$. Thus $M \cap K = MKM$.

$(4) \Rightarrow (1)$: This implication is obtained from Lemma 1.7. □

Theorem 2.3. *The following assertions are coequal for any bi-ideal M, left ideal L, right ideal R and any quasi-ideal N in a semigroup T:*

(1) *T is regular.*
(2) *In T, $M \cap K \cap L \subseteq MKL$, for any interior-ideal K.*
(3) *In T, $N \cap K \cap L \subseteq NKL$, for any interior-ideal K.*
(4) *In T, $M \cap K \cap R \subseteq RKM$, for any interior-ideal K.*
(5) *In T, $N \cap K \cap R \subseteq RKN$, for any interior-ideal K.*

Proof. Here we prove the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ and $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: Let $t \in M \cap K \cap L$. Then $t \in tTt$. Now $tTt \subseteq (tTt)(TtT)t \subseteq MKL$.

$(2) \Rightarrow (3)$: This is explicit.

$(3) \Rightarrow (1)$: Let R and L be right ideal and left ideal of T. Then by condition (3) it is evident that $R \cap T \cap L \subseteq RTL \subseteq RL$ which gives $R \cap L \subseteq RL$ and $RL \subseteq R \cap L$. Thus we get $R \cap L = RL$. Therefore by Lemma 1.5, T is regular.

$(1) \Rightarrow (4)$: For $t \in M \cap K \cap R$ it follows that $t \in tTt \subseteq t(TtT)(tTt) \subseteq RKM$. Hence $M \cap K \cap R \subseteq RKM$.

$(4) \Rightarrow (5)$: This is explicite.

$(5) \Rightarrow (1)$: By (5), it is evident that $L \cap T \cap R \subseteq RTL \subseteq RL$ produces $R \cap L \subseteq RL$. So $RL \subseteq R \cap L$ is always true. Hence $R \cap L = RL$. Therefore regularity of T follows from Lemma 1.5.

□

Verifying that in an intra regular semigroup, the concepts of an ideal and an interior ideal coincide is straightforward.

By referencing Theorem 1.12 and Theorem 1.13, we deduce that every regular duo semigroup is a Clifford semigroup, and vice versa. Subsequently, we proceed to establish several results that elucidate the relationship between interior ideals and other ideals within Clifford semigroups.

Theorem 2.4. *These assertions are true in a semigroup T:*

(1) *An interior ideal $K \subsetneq T$ is semiprime, if T is intra regular.*
(2) *If all interior ideals of T are completely semiprime then T is intra regular and conversely.*
(3) *If T is Clifford then all bi-ideals of T are interior ideals of T.*
(4) *If T is Clifford then all quasi-ideals of T are interior ideals of T.*

Proof. (1): First choose an interior ideal $A \subsetneq T$ with $A^2 \subseteq K$. Take $t \in A$, then $t \in Tt^2T \subseteq TKT \subseteq K$ and so $A \subseteq K$. Therefore K is a semiprime interior ideal of T.

(2): At first consider T as an intra regular semigroup and $K \subsetneq T$ be its interior ideal. Take $t \in T$ with $t^2 \in K$. Now $t \in Tt^2T \subseteq TKT \subseteq K$ implies $t \in K$. This proves the completely semiprimeness of the interior ideal K in T.

Conversely, make an hypothesis that the given condition is true in T. Choose $t \in T$. Say $K = Tt^2T$ then $TKT \subseteq K$. Thus K is an interior ideal. Now here the completely semiprimeness of K follows from our assumption. Now we have $t(t^2)t \in Tt^2T = K$ so that $(t^2)(t^2) \in K$ which implies that $t^2 \in K$. Since K is completely semiprime $t \in K$, that is, $t \in Tt^2T$. Hence T is intra regular.

(3): It follows from Theorem 1.11 and Lemma 1.1.

(4): It follows from (1) and Theorem of [12] .

□

Definition 2.5. *A semigroup T is defined as I-simple if it has no nontrivial interior ideal other than T itself.*

Theorem 2.6. *These assertions listed below are equivalent in a semigroup T:*

(1) *T is a I-simple semigroup.*
(2) *For every nonzero element $t \in T$, $TtT = T$.*
(3) *For every nonzero element $t \in T$, IN$(t) = T$.*

Proof. $(1) \Leftrightarrow (2)$: Assume that T is an I-simple semigroup. Take a nonzero $t \in T$. Then by assertion (1) together with the interior idealness of T it is evident that $TtT = T$.

For the converse consider an interior ideal K of T. Take a non zero element $b \in K$. Then $T = TbT \subseteq TKT \subseteq K$ so that $T = K$, this confirms the I-simplicity of T.

$(1) \Leftrightarrow (3)$: Let T be an I-simple semigroup. Take a non zero element $t \in T$, then IN$(t) = (t \cup t^2 \cup TtT)$, also the I-simplicity of T yields that $TtT = T$. Hence IN$(t) = (t \cup t^2 \cup TtT) = t \cup t^2 \cup T = T$.

For the reverse part consider an interior ideal K of T. Take a non zero element $t \in K$. Then IN$(t) = T$, by given assertion. So $T =$IN$(t) \subseteq K$. Therefore $K = T$ and hence I-simplicity of T follows.

□

Theorem 2.7. *In a semigroup T, any strongly irreducible semiprime interior ideal is also a strongly prime interior ideal.*

Proof. Consider K, a strongly irreducible semiprime interior ideal of a semigroup T such that $K_1K_2 \cap K_2K_1 \subseteq K$ for any two interior ideals K_1, K_2 of T. Now we have $(K_1 \cap K_2)^2 \subseteq K_1K_2 \cap K_2K_1 \subseteq K$. Since K is semiprime $(K_1 \cap K_2)^2 \subseteq K$ gives that $K_1 \cap K_2 \subseteq K$. This together with the strongly irreducibility of K yields that either $K_1 \subseteq K$ or $K_2 \subseteq K$. So K is strongly prime interior ideal. □

Theorem 2.8. *If for an interior ideal K of a semigroup T, $t \notin K$ for some $t \in T$, then an irreducible interior ideal B exists in T so that $K \subseteq B$ with $t \notin B$.*

Proof. Suppose that $\mathcal{L} = \{L \subseteq T : \ L$ is an interior ideal with $L \supseteq K$ and $t \notin L\}$. As $K \in \mathcal{L}$ evidently $\mathcal{L}$ is non empty. Also $(\mathcal{L}, \subseteq)$ is a partially ordered set. If $\mathcal{H}$ is a chain of $\mathcal{L}$ then $\cup_{J \in \mathcal{H}} J$ is interior ideal of T containing K. Therefore, according to Zorn's Lemma, a maximal element B exists in $\mathcal{L}$. We claim that B is an irreducible. For this consider P and Q as two interior ideals of T so that $B = P \cap Q$. If both P and Q properly contain B then for $t \in P$ and $t \in Q$ which infers that $t \in P \cap Q = B$. This is contrary to the fact that $t \notin B$. Therefore either $B = P$ or $B = Q$ and thus the result follows. □

As every interior ideal in a regular semigroup is an ideal and vice-versa, for any interior ideal K in a regular semigroup T is idempotent with respect to the semigroup product " $\cdot$ ". We use this fact to explore semi-primeness and semiprime irreducibility of these ideals. Here with respect to the semigroup product " $\cdot$ ", for any two ideals $A, B \in T$, $AB = \{ab : a \in A, b \in B\}$.

Theorem 2.9. *These assertions listed below are true in a regular semigroup T:*

(1) *If K be an interior ideal of T then $K^2 = K$.*
(2) *$K_1 \cap K_2 = K_1K_2 \cap K_2K_1$ for all interior ideals K_1 and K_2 of T.*
(3) *Every interior ideal within the semigroup T is semiprime.*
(4) *Every interior ideal K within the semigroup T is an intersection of all semiprime irreducible interior ideals of T that contains K.*

Proof. $(1) \Rightarrow (2)$: Let K_1 and K_2 be any two interior ideals of semigroup T, then by hypothesis, $K_1 \cap K_2 = (K_1 \cap K_2)^2 \subseteq K_1K_2$. Similarly, $K_1 \cap K_2 \subseteq K_2K_1$. Hence

$$K_1 \cap K_2 \subseteq K_1K_2 \cap K_2K_1. \tag{2. 1}$$

Now K_1K_2 and K_2K_1 are interior ideals being product of interior ideals in a regular semigroup. Also $K_1K_2 \cap K_2K_1$ is an interior ideal. Thus,

$$\begin{aligned} K_1K_2 \cap K_2K_1 = (K_1K_2 \cap K_2K_1)^2 &\subseteq K_1K_2K_2K_1 \\ &\subseteq TK_2T \\ &\subseteq K_2 \end{aligned}$$

Similarly, $K_1K_2 \cap K_2K_1 \subseteq K_1$. Hence

$$K_1K_2 \cap K_2K_1 \subseteq K_1 \cap K_2. \tag{2. 2}$$

Therefore from 2. 1 and 2. 2 , $K_1 \cap K_2 = K_1K_2 \cap K_2K_1$ for all interior ideals K_1 and K_2 of T.

$(2) \Rightarrow (3)$: Take an be interior ideal K of T. To show K is semiprime, consider another interior ideal $K_1 \in T$ such that $K_1^2 \subseteq K$. By assertion (2) we have that $K_1 = K_1 \cap K_1 = K_1K_1 \cap K_1K_1 = K_1^2$, which implies that $K_1 \subseteq K$. Therefore assertion (3) follows.

$(3) \Rightarrow (4)$: First we take a interior ideal $K \subset T$. Then $K \subseteq \bigcap_{\alpha\in\Delta} K_\alpha$ where $\{K_\alpha : \alpha \in \Delta\}$ are all irreducible interior ideals of T with $K_\alpha \supseteq K$. Assume that $\cap_{\alpha\in\Delta} K_\alpha \nsubseteq K$, then there exists $a \in \cap_{\alpha\in\Delta} K_\alpha$ such that $a \notin K$. Hence by Theorem 2.8 it yields that there exists an irreducible interior ideal L of T which contains K but does not contains a which evidently implies that $a \notin \cap_{\alpha\in\Delta} K_\alpha$, a contradiction. Hence $K = \bigcap_{\alpha\in\Delta} K_\alpha$ such that $K_\alpha \supseteq K$. Since by assertion (3), all interior ideal in T are semiprime, assertion (4) follows.

$(4) \Rightarrow (1)$: Consider an interior ideal K of T. If $K^2 = T$ then clearly K is an idempotent, that is, $K^2 = K$. If $K^2 \neq T$, then K^2 is a proper interior ideal of T containing K^2 and so by hypothesis $K^2 = \cap_\alpha\{K_\alpha : K_\alpha$ is irreducible semiprime interior ideals of $T\}$. Since each K_α is irreducible semiprime interior ideal, $K \subseteq K_\alpha$, for all α. Hence $K \subseteq \cap_\alpha K_\alpha = K^2$. Hence each interior ideal in T is an idempotent.

□

We now examine the ordering of the set of interior ideals in a semigroup. The following theorem establishes key properties of this ordering.

Theorem 2.10. *In a semigroup T these assertions are true:*

(1) $(\mathcal{I}(T), \subseteq)$ *is a chain.*
(2) *All interior ideals of T is strongly irreducible.*
(3) *All interior ideals of T is irreducible.*

Proof. $(1) \Rightarrow (2)$: First we assume that $(\mathcal{I}(T), \subseteq)$ is a chain. Choose an arbitrary interior ideal K of T and K_1, K_2 be two interior ideals of T with $K_1 \cap K_2 \subseteq K$. As $(\mathcal{I}(T), \subseteq)$ is totally ordered, we have either $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. Thus either $K_1 \cap K_2 = K_1$ or $K_2 \cap K_1 = K_2$. Hence either $K_1 \subseteq K$ or $K_2 \subseteq K$, thus K is strongly irreducible.

$(2) \Rightarrow (3)$: Suppose K be an arbitrary interior ideal of T and K_1, K_2 two interior ideals of T with $K_1 \cap K_2 = K$. Then $K \subseteq K_1$ and $K \subseteq K_2$. By hypothesis, either $K_1 \subseteq K$ or $K_2 \subseteq K$. Hence either $K_1 = K$ or $K_2 = K$, showing that K is irreducible .

$(3) \Rightarrow (1)$: Let $K_1, K_2 \in \mathcal{I}(T)$. Then $K_1 \cap K_2$ is an interior ideal of T. Again $K_1 \cap K_2 = K_1 \cap K_2$. By our hypothesis $K_1 = K_1 \cap K_2$ or $K_2 = K_1 \cap K_2$, that is, either $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. Hence $(\mathcal{I}(T), \subseteq)$ is a chain.

□

For now, the remaining part of this article will focus on the study of the minimality of interior ideals.

Definition 2.11. *An interior ideal K of a semigroup T is minimal if no other interior ideal L of T exists such that L contained in K.*

The following theorem characterizes minimal interior ideals in a semigroup.

Theorem 2.12. *For a non zero interior ideal K of a semigroup T the following assertions are same:*

(1) *K is minimal.*
(2) *For each non zero $g \in K$, $K = TgT$.*
(3) *For each non zero $g \in K$, K =IN(g).*

Proof. Here we proof $(1) \Leftrightarrow (2)$ and $(1) \Leftrightarrow (3)$.

$(1) \Leftrightarrow (2)$: Choose $g \in K \setminus \{0\}$ so that $TgT \subseteq TKT \subseteq K$ and the ideal TgT is interior in T. Then by minimality of K we conclude that $K = TgT$.

For the converse part let us consider an arbitrary interior ideal L of T with $L \subseteq K$. Now for any $(0 \neq)g \in L \subseteq K$, we have $K = TgT \subseteq TLT \subseteq L$. Evidently $K = L$ and so the assertion (1) follows.

$(1) \Leftrightarrow (3)$: First choose a non zero element $a \in K$. Now IN$(a) = (a \cup a^2 \cup TaT) \subseteq K$. But K is minimal so that K=IN(a).

To prove the reverse we consider an interior ideal L contained in K. Choose $a \in L \setminus \{0\}$. This implies $a \in K$, so that assertion (3) gives K =IN$(a) \subseteq L$. So $K = L$ showing the minimality of the interior ideal K in T. □

Theorem 2.13. *An interior ideal K of a semigroup T is minimal if and only if for every $(0 \neq)a, (0 \neq)b \in K$, IN$(a)$=IN$(b)$.*

Proof. Consider that a K is minimal in T. Let $(0 \neq)a, (0 \neq)b \in K$. By Theorem 2.12, IN$(a) = K$ =IN(b).

On the other hand, consider that given condition holds. Choose an interior ideal L of T such that $L \subseteq K$. Let $l \in L$ where $l \neq 0$. From the given condition it yields that, for any $k \in K$ where $k \neq 0$, IN(k) = IN(l). Since $k \in$ IN(K), $k \in$ IN$(l) \subseteq L$. Therefore, $K \subseteq L$, which concludes that K is a minimal in T.

□

Following [5], we define a relation $\mathcal{I}$ on a semigroup T by: For any $a, b \in T$, $a\mathcal{I}b$ if and only if IN(a)=IN(b). It is evident that $\mathcal{J} \subseteq \mathcal{I}$.

Theorem 2.14. *In a semigroup T, if an interior ideal K is an $\mathcal{I}$-class, then K is minimal and conversely.*

Proof. Let K be an interior ideal of T. Assume that K is a minimal interior ideal of T. Take any $(0 \neq)a, (0 \neq)b \in K$. Thus by Theorem 2.12, K=IN(a) and K=IN(b). Hence IN(a)=IN(b) implies $a\mathcal{I}b$. Thus K is an $\mathcal{I}$-class.

Conversely assume that K is an $\mathcal{I}$-class. Then for all $a, b \in K$, IN(a)=IN(b). Hence by Theorem 2.13, K is a minimal interior ideal of S.

□

The next result has an obvious proof, that is why it is left out.

Theorem 2.15. *Let T be a regular and intra regular semigroup then $\mathcal{J} = \mathcal{I}$.*

Theorem 2.16. *Let K be an interior ideal of a regular semigroup T. If K is an $\mathcal{J}$-class then K is a minimal and conversely.*

Proof. This follows from Theorem 2.14 and 2.15, which establish that in a regular semigroup, every J-class is an interior ideal. Consequently, minimal interior ideals coincide with J-classes, proving the result. □

Conclusion

This paper classifies interior ideals in regular and intra regular semigroups, examining their structural properties and interrelations. We establish key results on prime, semi-prime, and irreducible interior ideals, highlighting their role in semigroup theory. Our findings extend classical results and provide new insights into the minimality and simplicity of interior ideals, contributing to a deeper understanding of ideal theory in algebraic structures.

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