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The Role of Combined Quinary Refinement Scheme in Geometric Modeling

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Abstract. In this paper, we introduce a novel combined 5-point quinary refinement scheme that incorporates five parameters to enhance its adaptability and resilience. The development of the scheme stands on translating a 3-point approximating quinary refinement scheme to the new location by introducing the displacement vectors. We also demonstrate that the combined scheme generates numerous sub-schemes within a given set of parameters. The combined scheme has the modest support with C^3 degree of smoothness that attains better shape control and flexibility. Additionally, we discuss the important characteristic like polynomial generation, reproduction and approximation order. Moreover, we analyze that the proposed scheme exhibits complex behavior under specific parameter configuration. Further, we present some visual depiction based on polygons to verify the outcomes of the scheme. Lastly, to showcase the efficiency of the scheme the comparison with the classical and recently proposed schemes is presented.

AMS (MOS) Subject Classification Codes: 65D17; 65D10; 65D07; 65D05
Key Words: Refinement scheme; Combined scheme; Convergence; Polynomial generation/reproduction, Fractal.

1. INTRODUCTION

The refinement scheme is an iterative procedure that refines a given set of control points or mesh to accurately define smooth curves and surfaces. These techniques are extensively used in wavelets, computer visions, computer aided geometric design and animation. Refinement schemes are mainly divided into two categories: *interpolating* and *approximating*. For *interpolating* refinement schemes, original vertices are unaffected [1, 3]. For *approximating* refinement schemes, the original vertices are moved [4, 6].

The work on the approximating and interpolating schemes shows that the approximating schemes produce smooth curves with higher orders, but they cannot generate curves with diverse shapes. On the other hand, interpolating schemes can control the shape of the limit curve but have low smoothness order. So, the researchers find interest in combining various approximating and interpolating schemes using specific parameter choices to make it possible to create limit curves with both desired shape control and high smoothness order. A combined scheme may be regarded as the approximating and interpolating one. A combined scheme adjusts the shape according to the requirement without modifying the appearance of a smooth curve.

Levin [7] was the first to propose a combined refinement scheme that specifically maps the problem of interpolating nets of curves whose algorithm depends on the Catmull-Clark refinement scheme. Pan et al. [9] worked on developing a ternary combined refinement scheme that provides C^2 continuity. Rehan and Sabri [11] proposed the combined ternary 4-point refinement scheme. Novara and Romani [8] examined a combined ternary refinement scheme involving three parameters. Zhang [19] introduced a combined approximating and interpolating ternary 4-point refinement scheme holding essential properties such as support, polynomial generation and polynomial reproduction. The fractal property of the scheme is also analyzed. Tariq et al. [14] proposed a unified class of combined refinement scheme with two shape parameters, achieving optimal smoothness. They also introduced a 3-point relaxed non-symmetric approximating stationary quaternary refinement scheme with two parameters, generating C^3 continuous limit curves.

1.1. **Fractal generation.**

A fractal is a dynamic structure with the same degree of distortion at all scales or infinite patterns. In other words, a fractal is a shape that maintains its appearance even when a component is infinitely magnified. The branching of trees, the weathering of rocks, the movement of rivers, and the complex structure of the human body are all examples of fractal patterns. Refinement scheme is an effective method for efficiently generating fractal curves. Zheng et al. [17, 18] studied how certain refinement schemes can create fractal patterns. They examined two types of refinement schemes: 4-point binary and 3-point ternary interpolating and approximating schemes respectively. Yao et al. [16] examined the quaternary 4-point scheme with their fractal generation property. The previous work on fractals is available in [13, 15, 10].

1.2. **Motivation and contribution.**

In the literature, lower-arity combined schemes have been extensively studied, and higher-arity schemes have been relatively neglected. Higher-arity schemes offer several advantages, including smooth limit functions with minimal support and faster convergence rates. Additionally, they can reduce computational cost as the arity increases. The convergence rate of quinary scheme is faster than the convergence rate

of lower arity counter parts. Because at each iteration the new sequence of newly inserted control points has five times as many points as the previous sequence of old control points. In other words, the computational cost decreases by increasing the arity of the scheme. Despite these benefits, previous research has primarily focused on fractal generation using interpolating and approximating schemes. The potential of combined schemes for fractal generation remains unexplored. This paper addresses these gaps by investigating a combined quinary refinement scheme with a symmetric mask. Inspired by the work of Zhang et al. [8] on ternary combined schemes, we explore the use of a parameterized approximating scheme to generate a combined scheme and stunning fractal curves.
The paper is structured as follows: In Section 2, some basic definitions and results are recalled. Section 3 introduces the newly developed combined quinary refinement scheme (CQ_5-*scheme*) with several sub-schemes. Sections 4 and 5 delve into the analysis of the remarkable features CQ_5-*scheme* like continuity, support, polynomial generation and reproduction. Section 6 presents the fractal generation property with pictorial presentation. Section 7 explores the graphical efficiency and comparative analysis of the CQ_5-*scheme*. Lastly, in Section 8 the conclusion of this work is presented.

2. Notations and Preliminaries

In this section, some vital results and definitions are reviewed to formulate the basis of this paper. Initially, begin with the set of initial control points $f^0 = f_i^0 \in R, i \in \mathbb{Z}$. Then, after kth refinement level newly generated control points $f^{k+1} = f_i^{k+1}, i \in \mathbb{Z}$ by the quinary subdivision scheme can be defined as

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} \omega_{i-5j} f_j^k, k \in \mathbb{Z}_+, \tag{2. 1}$$

where the collection $\omega = \omega_i, i \in \mathbb{Z}$ is refereed as the mask of the scheme. When the geometric rules of the scheme satisfy the conditions

$$\sum_{j\in\mathbb{Z}} \omega_{5j} = \sum_{j\in\mathbb{Z}} \omega_{5j+1} = \sum_{j\in\mathbb{Z}} \omega_{5j+2} = \sum_{j\in\mathbb{Z}} \omega_{5j+3} = \sum_{j\in\mathbb{Z}} \omega_{5j+4} = 1. \tag{2. 2}$$

Then, the scheme shows uniform convergence.
To analyze the essential features of the scheme like convergence and smoothness, the term Laurent polynomial is introduced which is given by

$$\omega(z) = \sum_{i\in\mathbb{Z}} \omega_i z^i. \tag{2. 3}$$

The term norm of the subdivision scheme is given by

$$||S||_\infty = \max\left\{\sum_{j\in\mathbb{Z}} |\omega_{5j+P}|, P = 0, 1, 2, 3, 4\right\}, \tag{2. 4}$$

$$\left\|\left(\frac{1}{5}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}} |\omega^{[n,L]}_{i+5^L_j}| : i = 0, 1, ..., 5^L - 1\right\}. \tag{2. 5}$$

.

Definition 2.1. ***Support of the scheme:*** If the shape of the curve alternate in a particular area by moving a single point of the control polygon, then this area provides the support of the scheme.

Definition 2.2. ***Basic Limit Function:*** The basic limit function of the refinement scheme is the limit function of the CQ_5-*scheme* corresponds to the following data:

$$f_t^0 = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0\,. \end{cases} \tag{2. 6}$$

Cardinal data is used for computing the basic limit function.

3. Construction of the scheme

This section details the development of a novel combined refinement scheme. We deduce our combined quinary refinement scheme by performing operations on the geometric rules of the approximating scheme. For this purpose, we consider the quinary approximating subdivision scheme. We merge the displacement matrix and parameter matrix with the approximating scheme to achieve a new combined quinary refinement scheme. Moving fixed points to the new location based on displacement vectors yields the new points. The presented scheme offers higher smoothness and modest support concerning the scheme in [4].

Given the set of initial control points $f^0 = \left\{f_i^0 \in \mathbb{R}\right\}_{i=-1}^{n+1}$. Let $f^k = \left\{f_i^k\right\}_{i=-1}^{5^k n+1}$ be the set of control points at level k $(k \geq 0, k \in \mathbb{Z})$, and $f^k = \left\{f_i^k\right\}_{i=-1}^{5^k n+1}$ satisfy the following rules respectively. Then, the 3-point quinary approximating subdivision scheme presented in [4] is given by

$$\begin{cases} \bar{f}_{5i-2}^{k+1} = (u_0 + \frac{2}{5})f_{i-1}^k + (\frac{3}{5} - 2u_0)f_i^k + u_0 f_{i+1}^k, \\ \bar{f}_{5i-1}^{k+1} = (u_0 + \frac{6}{25})f_{i-1}^k + (\frac{18}{25} - 2u_0)f_i^k + (u_0 + \frac{1}{25})f_{i+1}^k, \\ \bar{f}_{5i}^{k+1} = (u_0 + \frac{3}{25})f_{i-1}^k + (\frac{19}{25} - 2u_0)f_i^k + (u_0 + \frac{3}{25})f_{i+1}^k, \\ \bar{f}_{5i+1}^{k+1} = (u_0 + \frac{1}{25})f_{i-1}^k + (\frac{18}{25} - 2u_0)f_i^k + (u_0 + \frac{6}{25})f_{i+1}^k, \\ \bar{f}_{5i+2}^{k+1} = u_0 f_{i-1}^k + (\frac{3}{5} - 2u_0)f_i^k + (u_0 + \frac{2}{5})f_{i+1}^k. \end{cases} \tag{3. 7}$$

The matrix form of the scheme is given by

$$\begin{pmatrix} \bar{f}_{5i-2}^{k+1} \\ \bar{f}_{5i-1}^{k+1} \\ \bar{f}_{5i}^{k+1} \\ \bar{f}_{5i+1}^{k+1} \\ \bar{f}_{5i+2}^{k+1} \end{pmatrix} = \begin{pmatrix} u_0 + \frac{2}{5} & \frac{3}{5} - 2u_0 & u_0 \\ u_0 + \frac{6}{25} & \frac{18}{25} - 2u_0 & u_0 + \frac{1}{25} \\ u_0 + \frac{3}{25} & \frac{19}{25} - 2u_0 & u_0 + \frac{3}{25} \\ u_0 + \frac{1}{25} & \frac{18}{25} - 2u_0 & u_0 + \frac{6}{25} \\ u_0 & \frac{3}{5} - 2u_0 & u_0 + \frac{2}{5} \end{pmatrix} \begin{pmatrix} f_{i-1}^k \\ f_i^k \\ f_{i+1}^k \end{pmatrix}. \tag{3. 8}$$

$$F = \begin{pmatrix} f_{5i-2}^{k+1} \\ f_{5i-1}^{k+1} \\ f_{5i}^{k+1} \\ f_{5i+1}^{k+1} \\ f_{5i+2}^{k+1} \end{pmatrix}, \quad \bar{F} = \begin{pmatrix} \bar{f}_{5i-2}^{k+1} \\ \bar{f}_{5i-1}^{k+1} \\ \bar{f}_{5i}^{k+1} \\ \bar{f}_{5i+1}^{k+1} \\ \bar{f}_{5i+2}^{k+1} \end{pmatrix}, \quad P = \begin{pmatrix} c_0 & d_0 & 0 \\ d_0 & c_0 & 0 \\ a_0 & b_0 & a_0 \\ 0 & c_0 & d_0 \\ 0 & d_0 & c_0 \end{pmatrix}, \quad \text{and}$$

$$Q = \begin{pmatrix} \Delta f_{i-1}^k \\ \Delta f_i^k \\ \Delta f_{i+1}^k \end{pmatrix}.$$

Here, a_0, b_0, c_0, d_0 $(a_0, b_0, c_0, d_0 \in \mathbb{R})$ represent the shape parameters which control the size and direction of the displacements vectors. The displacement vectors Δf_{i-1}^k , Δf_i^k and Δf_{i+1}^k represent $-(f_{i-2}^k - 2f_{i-1}^k + f_i^k)$, $-(f_{i-1}^k - 2f_i^k + f_{i+1}^k)$ and $-(f_i^k - 2f_{i+1}^k + f_{i+2}^k)$ respectively. These displacement vectors represent the change in the position of the points. Now, the relation $F = \bar{F} + PQ$ is employed in the development of the combined scheme as follows:

$$\begin{pmatrix} f_{5i-2}^{k+1} \\ f_{5i-1}^{k+1} \\ f_{5i}^{k+1} \\ f_{5i+1}^{k+1} \\ f_{5i+2}^{k+1} \end{pmatrix} = \begin{pmatrix} \bar{f}_{5i-2}^{k+1} \\ \bar{f}_{5i-1}^{k+1} \\ \bar{f}_{5i}^{k+1} \\ \bar{f}_{5i+1}^{k+1} \\ \bar{f}_{5i+2}^{k+1} \end{pmatrix} + \begin{pmatrix} c_0 & d_0 & 0 \\ d_0 & c_0 & 0 \\ a_0 & b_0 & a_0 \\ 0 & c_0 & d_0 \\ 0 & d_0 & c_0 \end{pmatrix} \begin{pmatrix} \Delta f_{i-1}^k \\ \Delta f_i^k \\ \Delta f_{i+1}^k \end{pmatrix},$$

$$\begin{pmatrix} f_{5i-2}^{k+1} \\ f_{5i-1}^{k+1} \\ f_{5i}^{k+1} \\ f_{5i+1}^{k+1} \\ f_{5i+2}^{k+1} \end{pmatrix} = \begin{pmatrix} \bar{f}_{5i-2}^{k+1} \\ \bar{f}_{5i-1}^{k+1} \\ \bar{f}_{5i}^{k+1} \\ \bar{f}_{5i+1}^{k+1} \\ \bar{f}_{5i+2}^{k+1} \end{pmatrix} + \begin{pmatrix} c_0 \Delta f_{i-1}^k + d_0 \Delta f_i^k \\ d_0 \Delta f_{i-1}^k + c_0 \Delta f_i^k \\ a_0 \Delta f_{i-1}^k + b_0 \Delta f_i^k + a_0 \Delta f_{i+1}^k \\ c_0 \Delta f_i^k + d_0 \Delta f_{i+1}^k \\ d_0 \Delta f_i^k + c_0 \Delta f_{i+1}^k \end{pmatrix}.$$

Hence, the geometric rules of the newly generated combined refinement scheme named as CQ_5-*scheme* is given by

$$\begin{cases} f_{5i-2}^{k+1} = -c_0 f_{i-2}^k + (\frac{2}{5} + u_0 + 2c_0 - d_0) f_{i-1}^k + (\frac{3}{5} - 2u_0 - c_0 + 2d_0) f_i^k \\ \qquad +(u_0 - d_0) f_{i+1}^k, \\ f_{5i-1}^{k+1} = -d_0 f_{i-2}^k + (\frac{6}{25} + u_0 - c_0 + 2d_0) f_{i-1}^k + (\frac{18}{25} - 2u_0 + 2c_0 - d_0) f_i^k \\ \qquad +(\frac{1}{25} + u_0 - c_0) f_{i+1}^k, \\ f_{5i}^{k+1} = -a_0 f_{i-2}^k + (\frac{3}{25} + u_0 + 2a_0 - b_0) f_{i-1}^k + (\frac{19}{25} - 2u_0 - 2a_0 + 2b_0) f_i^k \\ \qquad +(\frac{3}{25} + u_0 + 2a_0 - b_0) f_{i+1}^k - a_0 f_{i+2}^k, \\ f_{5i+1}^{k+1} = (\frac{1}{25} + u_0 - c_0) f_{i-1}^k + (\frac{18}{25} - 2u_0 + 2c_0 - d_0) f_i^k + (\frac{6}{25} + u_0 - c_0 + 2d_0) \\ \qquad f_{i+1}^k - d_0 f_{i+2}^k, \\ f_{5i+2}^{k+1} = (u_0 - d_0) f_{i-1}^k + (\frac{3}{5} - 2u_0 - c_0 + 2d_0) f_i^k + (\frac{2}{5} + u_0 + 2c_0 - d_0) f_{i+1}^k \\ \qquad -c_0 f_{i+2}^k. \end{cases}$$

The corresponding Laurent polynomial of the CQ_5-*scheme* is as follows:

$$\begin{aligned} \omega(z) \quad = \quad & -a_0(z^{10} + z^{-10}) - d_0(z^9 + z^{-9}) - c_0(z^8 + z^{-8}) + (u_0 - d_0)(z^7 + z^{-7}) + \\ & \left(\frac{1}{25} + u_0 - c_0\right)(z^6 + z^{-6}) + \left(\frac{3}{25} + u_0 + 2a_0 - b_0\right)(z^5 + z^{-5}) + \left(\frac{6}{25} + \right. \\ & \left. u_0 - c_0 + 2d_0\right)(z^4 + z^{-4}) + \left(\frac{2}{5} + u_0 + 2c_0 - d_0\right)(z^3 + z^{-3}) + \left(\frac{3}{5} - 2u_0 \right. \\ & \left. -c_0 + 2d_0\right)(z^2 + z^{-2}) + \left(\frac{18}{25} - 2u_0 + 2c_0 - d_0\right)(z + z^{-1}) + \left(\frac{19}{25} - 2u_0 \right. \\ & \left. -2a_0 + 2b_0\right). \end{aligned}$$

Remark 1:- If $(a_0, b_0) = (0, u_0 + \frac{3}{25})$, then the CQ_5-*scheme* turns into an interpolating refinement scheme (IQ_5-*scheme*).

$$\begin{cases} f_{5i-2}^{k+1} = -c_0 f_{i-2}^k + (\frac{2}{5} + u_0 + 2c_0 - d_0) f_{i-1}^k + (\frac{3}{5} - 2u_0 - c_0 + 2d_0) f_i^k \\ \qquad\quad +(u_0 - d_0) f_{i+1}^k, \\ f_{5i-1}^{k+1} = -d_0 f_{i-2}^k + (\frac{6}{25} + u_0 - c_0 + 2d_0) f_{i-1}^k + (\frac{18}{25} - 2u_0 + 2c_0 - d_0) f_i^k \\ \qquad\quad +(u_0 + \frac{1}{25} - c_0) f_{i+1}^k, \\ f_{5i}^{k+1} = f_i^k, \\ f_{5i+1}^{k+1} = (\frac{1}{25} + u_0 - c_0) f_{i-1}^k + (\frac{18}{25} - 2u_0 + 2c_0 - d_0) f_i^k + (\frac{6}{25} + u_0 - c_0 \\ \qquad\quad +2d_0) f_{i+1}^k - d_0 f_{i+2}^k, \\ f_{5i+2}^{k+1} = (u_0 - d_0) f_{i-1}^k + (\frac{3}{5} - 2u_0 - c_0 + 2d_0) f_i^k + (\frac{2}{5} + u_0 + 2c_0 - d_0) f_{i+1}^k \\ \qquad\quad -c_0 f_{i+2}^k. \end{cases} \tag{3. 9}$$

3.1. **Sub-schemes of the CQ_5-*scheme*.** Considering the different values of parameters, CQ_5-*scheme* generates several sub-schemes. Table 1 illustrates the four different numerical sub-schemes of the CQ_5-*scheme*. The type of the scheme (*App.* or *int.*) indicates whether it is approximating or interpolating.

- $\hat{C}Q_5$-*scheme* is a 5-point approximating scheme. The parameters for the scheme are $(a_0, b_0, c_0, d_0, u_0) = (\frac{3}{25}, \frac{21}{25}, \frac{7}{25}, 1, 0)$.

$$\begin{cases} f_{5i-2}^{k+1} = -\frac{7}{25} f_{i-2}^k - \frac{1}{25} f_{i-1}^k + \frac{58}{25} f_i^k - f_{i+1}^k, \\ f_{5i-1}^{k+1} = -f_{i-2}^k + \frac{49}{25} f_{i-1}^k + \frac{7}{25} f_i^k - \frac{6}{25} f_{i+1}^k, \\ f_{5i}^{k+1} = -\frac{3}{25} f_{i-2}^k - \frac{12}{25} f_{i-1}^k + \frac{11}{5} f_i^k - \frac{12}{25} f_{i+1}^k - \frac{3}{25} f_{i+2}^k, \\ f_{5i+1}^{k+1} = -\frac{6}{25} f_{i-1}^k + \frac{7}{25} f_i^k + \frac{49}{25} f_{i+1}^k - f_{i+2}^k, \\ f_{5i+2}^{k+1} = -f_{i-1}^k + \frac{58}{25} f_i^k - \frac{1}{25} f_{i+1}^k - \frac{7}{25} f_{i+2}^k. \end{cases}$$

- $\tilde{C}Q_4$-*scheme* is a 4-point interpolating scheme. The parameters for the scheme are $(a_0, b_0, c_0, d_0, u_0) = (0, -\frac{2}{25}, \frac{1}{5}, \frac{3}{5}, -\frac{1}{5})$.

$$\begin{cases} f_{5i-2}^{k+1} = -\frac{1}{5} f_{i-2}^k + 2 f_i^k - \frac{4}{5} f_{i+1}^k, \\ f_{5i-1}^{k+1} = -\frac{3}{5} f_{i-2}^k + \frac{26}{25} f_{i-1}^k + \frac{23}{25} f_i^k - \frac{9}{25} f_{i+1}^k, \\ f_{5i}^{k+1} = f_i^k, \\ f_{5i+1}^{k+1} = -\frac{9}{25} f_{i-1}^k + \frac{23}{25} f_i^k + \frac{26}{25} f_{i+1}^k - \frac{3}{5} f_{i+2}^k, \\ f_{5i+2}^{k+1} = -\frac{4}{5} f_{i-1}^k + 2 f_i^k - \frac{1}{5} f_{i+2}^k. \end{cases}$$

- $\breve{C}Q_5$-*scheme* is a 5-point approximating scheme. The parameters for the scheme are $(a_0, b_0, c_0, d_0, u_0) = (-\frac{6}{125}, \frac{108}{125}, \frac{27}{125}, -\frac{4}{125}, 1)$.

$$\begin{cases} f_{5i-2}^{k+1} = -\frac{27}{125} f_{i-2}^k + \frac{233}{125} f_{i-1}^k - \frac{42}{25} f_i^k + \frac{129}{125} f_{i+1}^k, \\ f_{5i-1}^{k+1} = \frac{4}{125} f_{i-2}^k + \frac{24}{25} f_{i-1}^k - \frac{102}{125} f_i^k + \frac{103}{125} f_{i+1}^k, \\ f_{5i}^{k+1} = \frac{6}{125} f_{i-2}^k + \frac{4}{25} f_{i-1}^k + \frac{73}{125} f_i^k + \frac{4}{25} f_{i+1}^k + \frac{6}{125} f_{i+2}^k, \\ f_{5i+1}^{k+1} = \frac{103}{125} f_{i-1}^k - \frac{102}{125} f_i^k + \frac{24}{25} f_{i+1}^k + \frac{4}{125} f_{i+2}^k, \\ f_{5i+2}^{k+1} = \frac{129}{125} f_{i-1}^k - \frac{42}{25} f_i^k + \frac{233}{125} f_{i+1}^k - \frac{27}{125} f_{i+2}^k. \end{cases}$$

- $\grave{C}Q_4$-*scheme* is a 4-point interpolating scheme. The parameters for the scheme are $(a_0, b_0, c_0, d_0, u_0) = (0, \frac{3}{25}, -\omega_1, -\frac{8}{25} + \omega_1, 0)$. The parameter ω_1 in this scheme

TABLE 1. *Sub-schemes of the CQ_5-scheme along with their respective type.*

Parameter	Point	Proposed scheme	Type
$(a_0, b_0, c_0, d_0, u_0) = (\frac{3}{25}, \frac{21}{25}, \frac{7}{25}, 1, 0)$	5-point	$\hat{C}Q_5$-*scheme*	*App.*
$(a_0, b_0, c_0, d_0, u_0) = (0, -\frac{2}{25}, \frac{1}{5}, \frac{3}{5}, -\frac{1}{5})$	4-point	$\tilde{C}Q_4$-*scheme*	*Int.*
$(a_0, b_0, c_0, d_0, u_0) = (-\frac{6}{125}, \frac{108}{125}, \frac{27}{125}, -\frac{4}{125}, 1)$	5-point	$\breve{C}Q_5$-*scheme*	*App.*
$(a_0, b_0, c_0, d_0, u_0) = (0, \frac{3}{25}, -\omega_1, -\frac{8}{25} + \omega_1, 0)$	4-point	$\grave{C}Q_4$-*scheme*	*Int.*

is a free parameter that can be chosen to improve the accuracy of the scheme.

$$\begin{cases} f_{5i-2}^{k+1} = \omega_1 f_{i-2}^k + (\frac{18}{25} - 3\omega_1) f_{i-1}^k + (-\frac{1}{25} + 3\omega_1) f_i^k + (\frac{8}{25} - \omega_1) f_{i+1}^k, \\ f_{5i-1}^{k+1} = (-\frac{8}{25} + \omega_1) f_{i-2}^k + (-\frac{2}{5} + 3\omega_1) f_{i-1}^k + (\frac{26}{25} - 3\omega_1) f_i^k + (\frac{1}{25} + \omega_1) f_{i+1}^k, \\ f_{5i}^{k+1} = f_i^k, \\ f_{5i+1}^{k+1} = (\frac{1}{25} + \omega_1) f_{i-1}^k + (\frac{26}{25} - 3\omega_1) f_i^k + (-\frac{2}{5} + 3\omega_1) f_{i+1}^k + (-\frac{8}{25} + \omega_1) f_{i+2}^k, \\ f_{5i+2}^{k+1} = (\frac{8}{25} - \omega_1) f_{i-1}^k + (-\frac{1}{25} + 3\omega_1) f_i^k + (\frac{18}{25} - 3\omega_1) f_{i+1}^k + \omega_1 f_{i+2}^k. \end{cases}$$

3.2. **The geometrical interpretation of parameters.** $\bar{f}_{5i-2}^{k+1}$, $\bar{f}_{5i-1}^{k+1}$, $\bar{f}_{5i}^{k+1}$, $\bar{f}_{5i+1}^{k+1}$, $\bar{f}_{5i+2}^{k+1}$ are the points of the quinary approximating refinement scheme presented in (3. 7). These points are moved to the new positions f_{5i-2}^{k+1}, f_{5i-1}^{k+1}, f_{5i}^{k+1}, f_{5i+1}^{k+1} and f_{5i+2}^{k+1} according to the displacement $c_0 \Delta f_{i-1}^k + d_0 \Delta f_i^k$, $d_0 \Delta f_{i-1}^k + c_0 \Delta f_i^k$, $a_0 \Delta f_{i-1}^k + b_0 \Delta f_i^k + a_0 \Delta f_{i+1}^k$, $c_0 \Delta f_i^k + d_0 \Delta f_{i+1}^k$ and $d_0 \Delta f_i^k + c_0 \Delta f_{i+1}^k$ respectively to obtain a new combined quinary refinement scheme. We use an example to illustrate the geometric description of the system to make it intelligible.
Assume f_{i-2}^k, f_{i-1}^k, f_i^k, f_{i+1}^k and f_{i+2}^k indicates the control points at the kth level of refinement.
Step 1 : $\bar{f}_{5i+1}^{k+1}$ is provided by quinary approximating scheme with $\bar{f}_{5i+1}^k = (u_0 + \frac{1}{25}) f_{i-1}^k + (\frac{18}{25} - 2u_0) f_i^k + (u_0 + \frac{6}{25}) f_{i+1}^k$.
Step 2 : Δf_i^k is a vector whose size is equal to the length of the diagonal of the parallelogram $O_i^k f_{i-1}^k f_i^k f_{i+1}^k$ with its direction pointing to f_i^k.
Step 3 : Δf_{i+1}^k is a vector whose size is equal to the length of the diagonal of the parallelogram $O_{i+1}^k f_i^k f_{i+1}^k f_{i+2}^k$ with its direction to f_{i+1}^k.
Step 4 : Relocate $\bar{f}_{5i+1}^{k+1}$ in accordance with the displacement $c_0 \Delta f_i^k + d_0 \Delta f_{i+1}^k$.
The new positions of f_{5i-2}^{k+1}, f_{5i-1}^{k+1}, f_{5i}^{k+1}, and f_{5i+2}^{k+1} may be obtained in the same way. When $(a_0, b_0) = (0, u_0 + \frac{3}{25})$, the limit curve interpolates the vertices of the beginning polygon. When some are specified, limit curves get closer to the relevant polygons while the remaining parameters are reduced.

4. CONTINUITY OF THE CQ_5-*scheme*

Laurent polynomial method is an effective method to demonstrate the smoothness of the scheme. To find the continuity of the scheme we utilize the Laurent polynomial method. The straightforward implementation of continuity and the determination of parameter values are attained with the assistance of the mathematical tool called Maple.

Theorem 4.1. *Consider the scheme specified in (3. 9). Then,*
(i) C^0 *limit curve is produced when*

$$a_0,b_0,c_0,d_0,u_0\in\{a_0,b_0,c_0,d_0,u_0\in\mathbb{R},\max\left\{|-a_0|+\left|\left(\frac{2}{25}+a_0-b_0+c_0\right)\right|+\right.$$
$$\left|\left(\frac{3}{25}-a_0+b_0-c_0+d_0\right)\right|+|(a_0-d_0)|,|(-c_0+d_0)|+\left|\left(\frac{4}{25}+2c_0-2d_0\right)\right|+$$
$$\left.\left|\left(\frac{1}{25}-c_0+d_0\right)\right|,2|(u_0+c_0-d_0)|+\left|\left(\frac{1}{5}-2u_0-2c_0+2d_0\right)\right|\right\}<1\};$$

(ii) C^1 *limit curve is produced when*

$$a_0,b_0,c_0,d_0,u_0\in\{a_0,b_0,c_0,d_0,u_0\in\mathbb{R},\max\left\{|-5a_0|+\left|\left(\frac{1}{5}-5b_0+10c_0-5d_0\right)\right|+\right.$$
$$|(-5a_0-5c_0+10d_0)|,2|(10a_0-5d_0)|+\left|\left(\frac{1}{5}+10b_0-10c_0\right)\right|,|(5u_0+10c_0-10d_0)|$$
$$\left.+\left|\left(\frac{1}{5}-5u_0-10c_0+10d_0\right)\right|\right\}<1;$$

(iii) C^2 *limit curve is produced when*

$$a_0,b_0,c_0,d_0,u_0\in\{a_0,b_0,c_0,d_0,u_0\in\mathbb{R},b=-2a_0+d_0+c_0,\max\{|-25a_0|+$$
$$|(25u_0+25a_0-100d_0+75c_0)|,|(75a_0-25d_0)|+|(-75a_0+75d_0-25c_0)|,$$
$$|(1-50u_0+100d_0-100c_0)|\}<1\};$$

(iv) C^3 *limit curve is produced when*

$$a_0,b_0,c_0,d_0,u_0\in\{a_0,b_0,c_0,d_0,u_0\in\mathbb{R},b_0=-\frac{1}{125}-2a_0+3d_0,c_0=-\frac{1}{125}+2d_0,$$
$$u_0=\frac{4}{125}-2d_0,\max\{125|a_0|,|(500a_0-125d_0)|,|(1-750a_0+250d_0)|\}<1\}.$$

Proof. *The scheme (3. 9) in the form of parameter* ω *can be written as*

$$\left\{\begin{array}{ll} f_{5i-2}^{k+1}=\omega_{-8}f_{i-2}^k+\omega_{-3}f_{i-1}^k+\omega_2 f_i^k+\omega_7 f_{i+1}^k,\\ f_{5i-1}^{k+1}=\omega_{-9}f_{i-2}^k+\omega_{-4}f_{i-1}^k+\omega_1 f_i^k+\omega_6 f_{i+1}^k,\\ f_{5i}^{k+1}=\omega_{-10}f_{i-2}^k+\omega_{-5}f_{i-1}^k+\omega_0 f_i^k+\omega_5 f_{i+1}^k+\omega_{10}f_{i+2}^k,\\ f_{5i+1}^{k+1}=\omega_{-6}f_{i-1}^k+\omega_{-1}f_i^k+\omega_4 f_{i+1}^k+\omega_9 f_{i+2}^k,\\ f_{5i+2}^{k+1}=\omega_{-7}f_{i-1}^k+\omega_{-2}f_i^k+\omega_3 f_{i+1}^k+\omega_8 f_{i+2}^k.\end{array}\right. \quad (4.\ 10)$$

Here,
$\omega_0=\frac{19}{25}-2u_0-2a_0+2b_0,\quad \omega_1=\omega_{-1}=\frac{18}{25}-2u_0+2c_0-d_0,$
$\omega_2=\omega_{-2}=\frac{3}{5}-2u_0-c_0+2d_0,\quad \omega_3=\omega_{-3}=\frac{2}{5}+u_0+2c_0-d_0,$
$\omega_4=\omega_{-4}=\frac{6}{25}+u_0-c_0+2d_0,\quad \omega_5=\omega_{-5}=\frac{3}{25}+u_0+2a_0-b_0,$
$\omega_6=\omega_{-6}=\frac{1}{25}+u_0-c_0,\quad \omega_7=\omega_{-7}=u_0-d_0,\omega_8=\omega_{-8}=-c_0,$
$\omega_9=\omega_{-9}=-d_0,\quad \omega_{10}=\omega_{-10}=-a_0.$

The CQ_5-scheme satisfies the necessary condition of the basic sum rules for its convergence.

$$\sum_{i\in\mathbb{Z}} \omega_{5i+r} = 1, \quad r = 0, \ldots, 4. \tag{4. 11}$$

The generating function of the CQ_5-scheme exhibits the following sequence of coefficients.
$\omega = \{\omega\}_{i\in z} = \{\ldots, \omega_{-10}, \omega_{-9}, \omega_{-8}, \ldots, \omega_8, \omega_9, \omega_{10}, \ldots\}$
The Laurent polynomial of (4. 10) can be written as

$$\begin{aligned}\omega(z) &= \omega_{-10}z^{-10} + \omega_{-9}z^{-9} + \omega_{-8}z^{-8} + \omega_{-7}z^{-7} + \omega_{-6}z^{-6} + \omega_{-5}z^{-5} + \omega_{-4}z^{-4} \\ &\quad + \omega_{-3}z^{-3} + \omega_{-2}z^{-2} + \omega_{-1}z^{-1} + \omega_0 z^0 + \omega_1 z^1 + \omega_2 z^2 + \omega_3 z^3 + \omega_4 z^4 \\ &\quad + \omega_5 z^5 + \omega_6 z^6 + \omega_7 z^7 + \omega_8 z^8 + \omega_9 z^9 + \omega_{10} z^{10}.\end{aligned}$$

Now, use the formula

$$\begin{aligned}\omega^{(n)}(z) &= \left(\frac{5z^4}{1+z+z^2+z^3+z^4}\right)\omega^{(n-1)}(z) \\ &= \left(\frac{5z^4}{1+z+z^2+z^3+z^4}\right)^{(n)}\omega(z),\end{aligned} \tag{4. 12}$$

where $z \in \mathbb{C}\backslash 0$.
(i) *Fix* $n = 1$ *in (4. 12), we have*

$$\frac{1}{5}\omega^{(1)}(z) = \frac{1}{5}\left(\frac{5z^4}{1+z+z^2+z^3+z^4}\right)\omega(z) = \sum \eta_i z^i, \qquad i = -6, \ldots, 10,$$

after simplification, we have
$\eta_{-6} = \eta_{10} = -a_0, \quad \eta_{-5} = \eta_9 = a_0 - d_0, \quad \eta_{-4} = \eta_8 = -c_0 + d_0,$
$\eta_{-3} = \eta_7 = c_0 - d_0 + u_0, \quad \eta_{-2} = \eta_6 = \frac{1}{25} - c_0 + d_0, \quad \eta_{-1} = \eta_5 = \frac{2}{25} + a_0 - b_0 + c_0,$
$\eta_0 = \eta_4 = \frac{3}{25} - a_0 + b_0 - c_0 + d_0, \quad \eta_1 = \eta_3 = \frac{4}{25} + 2c_0 - 2d_0, \quad \eta_2 = \frac{1}{5} - 2u_0 - 2c_0 + 2d_0.$
When

$$a_0, b_0, c_0, d_0, u_0 \in \{a_0, b_0, c_0, d_0, u_0 \in \mathbb{R}, \max\left\{|-a_0| + \left|\left(\frac{2}{25} + a_0 - b_0 + c_0\right)\right| + \left|\left(\frac{3}{25} - a_0 + b_0 - c_0 + d_0\right)\right| + |(a_0 - d_0)|, |(-c_0 + d_0)| + \left|\left(\frac{4}{25} + 2c_0 - 2d_0\right)\right| + \left|\left(\frac{1}{25} - c_0 + d_0\right)\right|, 2|(u_0 + c_0 - d_0)| + \left|\left(\frac{1}{5} - 2u_0 - 2c_0 + 2d_0\right)\right|\right\} < 1\},$$

we have

$$\left\|\frac{1}{5}S_1\right\|_\infty = \frac{1}{5}\max\left\{\sum_{i\in z}|\omega^{(1)}_{5i+j}| : j = -2, \ldots, 2\right\} < 1.$$

The scheme (4. 10) has C^0 continuity.
(ii) *Fix* $n = 2$ *in (4. 12), we have*

$$\frac{1}{5}\omega^{(2)}(z) = \frac{1}{5}\left(\frac{5z^4}{1+z+z^2+z^3+z^4}\right)^2\omega(z) = \sum \eta_i z^i, \qquad i = -2, \ldots, 10,$$

after simplification, we have
$\eta_{-2}=\eta_{10}=-5a_0, \quad \eta_{-1}=\eta_9=10a_0-5d_0, \quad \eta_0=\eta_8=-5a_0-5c_0+10d_0,$
$\eta_1=\eta_7=-5u_0+10c_0-10d_0, \quad \eta_2=\eta_6=\frac{1}{5}-5u_0-10c_0+10d_0, \quad \eta_3=\eta_5=\frac{1}{5}-5b_0+10c_0-5d_0, \eta_4=\frac{1}{5}+10b_0-10c_0.$
When

$$a_0,b_0,c_0,d_0,u_0\in\{a_0,b_0,c_0,d_0,u_0\in\mathbb{R},\max\left\{|-5a_0|+\left|\left(\frac{1}{5}-5b_0+10c_0-5d_0\right)\right|+\right.$$
$$|(-5a_0-5c_0+10d_0)|,2|(10a_0-5d_0)|+\left|\left(\frac{1}{5}+10b_0-10c_0\right)\right|,|(5u_0+10c_0-10d_0)|+$$
$$\left.\left|\left(\frac{1}{5}-5u_0-10c_0+10d_0\right)\right|\right\}<1,$$

we have

$$\left\|\frac{1}{5}S_2\right\|_\infty = \frac{1}{5}\max\left\{\sum_{i\in z}|\omega^{(2)}_{5i+j}|:j=-2,\ldots,2\right\}<1.$$

The scheme (4. 10) has C^1 continuity.
(iii) Fix $n=3$ in (4. 12), if $b=-2a+c+d$

$$\frac{1}{5}\omega^{(3)}(z) = \frac{1}{5}\left(\frac{5z^4}{1+z+z^2+z^3+z^4}\right)^3\omega(z)=\sum\eta_i z^i, \qquad i=2,\ldots,10,$$

after simplification, we have
$\eta_2=\eta_{10}=-25a_0, \quad \eta_3=\eta_9=75a_0-25d_0, \quad \eta_4=\eta_8=-75a_0-75d_0-25c_0,$
$\eta_5=\eta_7=25u_0+25a_0-100d_0+75c_0, \quad \eta_6=1-50u_0+100d_0-100c_0.$
When

$$a_0,b_0,c_0,d_0,u_0\in\{a_0,b_0,c_0,d_0,u_0\in\mathbb{R},b_0=-2a_0+c_0+d_0,\max\{|-25a_0|+$$
$$|(25u_0+25a_0-100d_0+75c_0)|,|(75a_0-25d_0)||(-75a_0+75d_0-25c_0)|,$$
$$|(1-50u_0+100d_0-100c_0)|\}<1\},$$

we have

$$\left\|\frac{1}{5}S_3\right\|_\infty = \frac{1}{5}\max\left\{\sum_{i\in z}|\omega^{(3)}_{5i+j}|:j=-2,\ldots,2\right\}<1.$$

The scheme (4. 10) has C^2 continuity.
(iv) Fix $n=4$ in (4. 12), if $b_0=-\frac{1}{125}-2a_0+3d_0$, $c_0=2d_0-\frac{1}{125}$, and $u_0=\frac{4}{125}-2d_0$,

$$\frac{1}{5}\omega^{(4)}(z) = \frac{1}{5}\left(\frac{5z^4}{1+z+z^2+z^3+z^4}\right)^4\omega(z)=\sum\eta_i z^i, \qquad i=6,\ldots,10,$$

after simplification, we have
$\eta_6=\eta_{10}=-125a_0, \quad \eta_7=\eta_9=500a_0-125d_0, \quad \eta_4=\eta_8=1-750a_0+250d_0.$

(a) C^0 limit curve (b) C^1 limit curve

(c) C^2 limit curve (d) C^3 limit curve

FIGURE 1. *Curves with variations in continuity orders. The black solid lines and black boxes represent the initial control polygon and the control points respectively.*

When

$$a_0, b_0, c_0, d_0, u_0 \in \{a_0, b_0, c_0, d_0, u_0 \in \mathbb{R}, b_0 = -\frac{1}{125} - 2a_0 + 3d_0,$$
$$c_0 = -\frac{1}{125} + 2d_0, u_0 = \frac{4}{125} - 2d_0, \max\{125|a_0|, |(500a_0 - 125d_0)|,$$
$$|(1 - 750a_0 + 250d_0)|\} < 1\},$$

we have

$$\left\|\frac{1}{5}S_4\right\|_\infty = \frac{1}{5}\max\left\{\sum_{i\in z}|\omega^{(4)}_{5i+j}| : j = -2, \ldots, 2\right\} < 1.$$

*Hence, the CQ_5-*scheme *has C^3 continuity.* □

The behavior of the CQ_5-*scheme* on various continuity orders with the same control polygon is explained in detail in Figure 1. Table 2 provides sufficient conditions for the continuity of the CQ_5-*scheme*. Specifically, the condition requires that the maximum of the expressions is less than 1.

5. PROPERTIES OF THE CQ_5-*scheme*

In this section, some remarkable properties of the combined scheme such as support, polynomial generation, polynomial reproduction and approximation order are discussed.

5.1. Support of the CQ_5-*scheme*.

Support width measures how far one vertex is extensively affected by its neighboring points. For this purpose, it is necessary to assess the vertices at the far left and far right of the CQ_5-*scheme*. Upon completing the k-steps of subdivision, we measure the gap between the leftmost and rightmost vertices.

Lemma 5.2. *By applying the CQ_5-*scheme *on the data*

$$f_t^0 = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0 \,. \end{cases}$$

The non-zero points after first refinement step are $f^1_{-10}, f^1_{-9}, f^1_{-8}, \ldots, f^1_8, f^1_9, f^1_{10}$, the non-zero points after second refinement step are $f^2_{-60}, f^2_{-59}, f^2_{-58}, \ldots, f^2_{58}, f^2_{59}, f^2_{60}$ and the non-zero points after third refinement step are $f^3_{-310}, f^3_{-309}, f^3_{-308}, \ldots, f^3_{308}, f^3_{309}, f^3_{310}$.

Theorem 5.3. *The basic limit function ψ of the CQ_5-*scheme *has support size 5 which suggests that it disappears outside of the designated interval $\left[-\frac{5}{2}, \frac{5}{2}\right]$.*

TABLE 2. *Parameter values and respective continuity order.*

Parameter values	Continuity
$a_0,b_0,c_0,d_0,u_0 \in \{a_0,b_0,c_0,d_0,u_0 \in \mathbb{R},$ $\max\{\lvert -a_0\rvert + \lvert(\frac{2}{25}+a_0-b_0+c_0)\rvert + \lvert(a_0-d_0)\rvert$ $+\lvert(\frac{3}{25}-a_0+b_0-c_0+d_0)\rvert, \lvert(-c_0+d_0)\rvert$ $+\lvert(\frac{4}{25}+2c_0-2d_0)\rvert + \lvert(\frac{1}{25}-c_0+d_0)\rvert, 2\lvert(u_0+c_0-d_0)\rvert$ $+\lvert(\frac{1}{5}-2u_0-2c_0+2d_0)\rvert\} < 1\}$	C^0
$a_0,b_0,c_0,d_0,u_0 \in \{a_0,b_0,c_0,d_0,u_0 \in \mathbb{R},$ $\max\{\lvert -5a_0\rvert + \lvert(\frac{1}{5}-5b_0+10c_0-5d_0)\rvert + \lvert(-5a_0-5c_0+10d_0)\rvert,$ $2\lvert(10a_0-5d_0)\rvert + \lvert(\frac{1}{5}+10b_0-10c_0)\rvert, \lvert(5u_0+10c_0-10d_0)\rvert$ $+\lvert(\frac{1}{5}-5u_0-10c_0+10d_0)\rvert\} < 1\}$	C^1
$a_0,b_0,c_0,d_0,u_0 \in \{a_0,b_0,c_0,d_0,u_0 \in \mathbb{R},$ $b_0=-2a_0+c_0+d_0, \max\{\lvert -25a_0\rvert + \lvert(25u_0+25a_0+75c_0-100d_0)\rvert,$ $\lvert(75a_0-25d_0)\rvert + 2\lvert(-75a_0-25c_0+75d_0)\rvert,$ $\lvert(1-50u_0+100d_0-100c_0)\rvert\} < 1\}$	C^2
$a_0,b_0,c_0,d_0,u_0 \in \{a_0,b_0,c_0,d_0,u_0 \in \mathbb{R},$ $b_0=-\frac{1}{125}-2a_0+3d_0, c_0=-\frac{1}{125}+2d_0, u_0=\frac{4}{125}-2d_0,$ $\max\{125\lvert a_0\rvert, \lvert(500a_0-125d_0)\rvert, \lvert(1-750a_0+250d_0)\rvert\} < 1\}$	C^3

Proof. *To identify the support size, we calculate the gap between the leftmost and rightmost non-zero vertices.*
Since

$$F_k = \left\{ \frac{t}{5^k} : \forall\, t \in \mathbb{Z} \right\}, \tag{5. 13}$$

so that

$$\psi\left(\frac{t}{5^k}\right) = f_t^k,\ \forall\, t \in \mathbb{Z}. \tag{5. 14}$$

With the aid of Lemma 5.2 and by substituting $k = 0$ *and* $i = -2, -1, 0, 1, 2$ *in (4. 10), the leftmost and rightmost non-zero vertices after first refinement step by mean of (5. 14) are* $f_{-10}^1 = \psi\left[\frac{-10}{5}\right]$ *and* $f_{10}^1 = \psi\left[\frac{10}{5}\right]$ *respectively. Similarly, after the second refinement step, by mean of Lemma 5.2, (4. 10) and (5. 14), the leftmost and rightmost non-zero vertices are* $f_{-60}^2 = \psi\left[\frac{-10(1+5)}{5^2}\right]$ *and* $f_{60}^2 = \psi\left[\frac{10(1+5)}{5^2}\right]$ *respectively and so on. After the k-step refinement the obtained leftmost and rightmost non-zero vertices are* $f_{-10}^k = \psi\left[\frac{-10(1+5+\ldots+5^{k-1})}{5^k}\right]$ *and* $f_{10}^k = \psi\left[\frac{10(1+5+\ldots+5^{k-1})}{5^k}\right]$ *respectively. Since,* ψ *of the scheme (3. 9) is symmetric. So, the difference between the leftmost and rightmost*

FIGURE 2. *Comparison of the* CQ_5-scheme *for the basis function at parametric values* $a_0 = 0, b_0 = \frac{11}{175}, c_0 = \frac{1}{27}, d_0 = -\frac{1}{625}, u_0 = -\frac{2}{35}$ *(purple),* $a_0 = \frac{4}{875}, b_0 = \frac{3}{250}, c_0 = \frac{2}{175}, d_0 = \frac{17}{1750}, u_0 = \frac{11}{875}$ *(blue),* $a_0 = \frac{3}{2375}, b_0 = -\frac{53}{4750}, c_0 = -\frac{4}{475}, d_0 = -\frac{1}{4750}, u_0 = \frac{77}{2375}$ *(green), and* $a_0 = -\frac{1}{375}, b_0 = -\frac{29}{750}, c_0 = -\frac{4}{125}, d_0 = -\frac{3}{250}, u_0 = \frac{7}{125}$ *(red).*

non-zero vertices at the k-step refinement is given by

$$\begin{aligned} d &= \left[\frac{10(1+5+\ldots+5^{k-1})}{5^k} - \frac{-10(1+5+\ldots+5^{k-1})}{5^k}\right] \\ &= \left[20\left(\frac{1}{5}+\frac{1}{5^2}\ldots+\frac{1}{5^{k-1}}\right)\right] \\ &= \frac{20}{5}\left(1+\frac{1}{5}+\ldots+\frac{1}{5^{k-1}}\right). \end{aligned}$$

The support width of the CQ_5-scheme *when* k *approaches to infinity is given by*

$$\lim_{k\to\infty}\left[\frac{20}{5}\left(1+\frac{1}{5}+\ldots+\frac{1}{5^{k-1}}\right)\right] = \frac{20}{5}\left(\sum_{i=0}^{k-1}\frac{1}{5^i}\right) = 5.$$

Hence, the support width is 5*, which implies that it disappears outside the designated interval* $\left[-\frac{5}{2}, \frac{5}{2}\right]$. □

Figure 2 elaborates the behavior of the basic function on different values of parameters for CQ_5-*scheme*.

5.4. **Polynomial generation.** Here, we discuss the generation of polynomials and its degree. Polynomial generation is a process of creating mathematical expressions known as polynomials that allows for the creation and manipulation of geometric shapes. The generation degree of a refinement scheme reveals the highest degree of polynomial functions that the scheme can accurately generate. To generate the C^n-continuous limit curve, polynomial generation of degree n is required. The polynomial generation property of the scheme can be obtained by using the symbol $\omega(z)$. The CQ_5-*scheme* can produce polynomials up to degree d if and only if

$$\omega(z) = \left(\frac{1+z+z^2+z^3+z^4}{5}\right)^{d+1} \delta_i(z), \quad z \in \mathbb{C}\backslash 0$$

Here, $\delta_i(z)$, $i = 1,2,3,4$ are Laurent polynomials and $\kappa(z) = \frac{1+z+z^2+z^3+z^4}{5}$, where $\kappa(z)$ is known as smoothing factor.
Table 3 presents a relationship between specific parameter values, the resulting degeneration of a symbol (denoted by $\kappa(z)$), and generation degree (G_d).

TABLE 3. *The relationship between parameter values, degeneration of symbol and generation degree.*

Parameter values	**Degeneration of symbol**	G_d
$\{a_0, b_0, c_0, d_0, u_0 \in \mathbb{R}\}$	$\omega(z) = \kappa(z)^2\delta_2(z)$	1
$\{a_0, b_0, c_0, d_0, u_0 \in \mathbb{R} : b_0 = -2a_0 + d_0 + c_0\}$	$\omega(z) = \kappa(z)^3\delta_3(z)$	2
$\{a_0, b_0, c_0, d_0, u_0 \in \mathbb{R} : b_0 = -\frac{1}{125} - 2a_0 + 3d_0,$ $c_0 = 2d_0 - \frac{1}{125}, u_0 = \frac{4}{125} - 2d_0\}$	$\omega(z) = \kappa(z)^4\delta_4(z)$	3

5.5. **Polynomial reproduction and approximation order.** Here, we discuss the degree of polynomial reproduction of the CQ_5-*scheme* and approximation order. Polynomial reproduction refers to produce the polynomials of certain degree. The feature of polynomial reproduction is useful since every convergent refinement scheme with approximation order $n+1$ may reproduce polynomials of degree n.

Definition 5.6. *(*[2]*). For any refinement scheme, denoted by $\tau = \frac{\omega'(1)}{5}$, the corresponding parametric shift and the data f_n^k for $n \in k$ to the parameter values are:*

$$x_n^k = x_0^k + \frac{n}{5^k} \quad with \quad x_0^k = x_0^{k-1} - \frac{\tau}{5^k}. \tag{5. 15}$$

Theorem 5.7. *(*[2]*). A convergent refinement scheme S with any arity $m \geq 2$ reproduces polynomials of degree $d \geq 1$ with respect to the parametrization in (5. 15) if and only if*

$$\sum_{j\in\mathbb{Z}} j^k \omega_{mj+i} = \left(\frac{\tau - i}{m}\right)^k, \; i = 0, 1, 2, \ldots, m-1, \; for \; k = 1, 2, \ldots, d, \tag{5. 16}$$

where

$$\tau = \frac{\omega'(1)}{m}. \tag{5. 17}$$

In the case of a CQ_5-*scheme* (where m=5), the (5. 16) takes the following form as

$$\begin{cases} \sum_{j\in\mathbb{Z}} j^k \omega_{5j} = \left(\frac{\tau}{5}\right)^k, \\ \sum_{j\in\mathbb{Z}} j^k \omega_{5j+1} = \left(\frac{\tau-1}{5}\right)^k, \\ \sum_{j\in\mathbb{Z}} j^k \omega_{5j+2} = \left(\frac{\tau-2}{5}\right)^k, \\ \sum_{j\in\mathbb{Z}} j^k \omega_{5j+3} = \left(\frac{\tau-3}{5}\right)^k, \\ \sum_{j\in\mathbb{Z}} j^k \omega_{5j+4} = \left(\frac{\tau-4}{5}\right)^k. \end{cases}$$

for $k = 1, \ldots, n$.

Theorem 5.8. *The CQ_5*-scheme *reproduces linear polynomials.*

Proof. *To prove this theorem, we take the differential of (3. 9) with respect to z and set $z = 1$, we obtain $\omega'(z) = 1$. Thus, $\tau = \frac{\omega'(1)}{5} = \frac{1}{5}$. By putting the values of τ in (5. 18),*

(a) $a_0 = 0, b_0 = \frac{11}{175}, c_0 = \frac{1}{27}$, $d_0 = -\frac{1}{625}, u_0 = -\frac{2}{35}$ (b) $a_0 = -\frac{1}{375}, b_0 = -\frac{29}{750}, c_0 = -\frac{4}{125}$, $d_0 = -\frac{3}{250}, u_0 = \frac{7}{125}$

FIGURE 3. *Visual inspection of polynomial reproduction for* CQ_5-scheme*. Colored lines show the resulting curves.*

we have

$$\begin{cases} (-2)^k\omega_{-10} + (-1)^k\omega_{-5} + (0)^k\omega_0 + (1)^k\omega_5 + (2)^k\omega_{10} = \left(\frac{1}{25}\right)^k, \\ (-2)^k\omega_{-9} + (-1)^k\omega_{-4} + (0)^k\omega_1 + (1)^k\omega_6 + (2)^k\omega_{11} = \left(\frac{-4}{25}\right)^k, \\ (-2)^k\omega_{-8} + (-1)^k\omega_{-3} + (0)^k\omega_2 + (1)^k\omega_7 + (2)^k\omega_{12} = \left(-\frac{9}{25}\right)^k, \\ (-2)^k\omega_{-7} + (-1)^k\omega_{-2} + (0)^k\omega_3 + (1)^k\omega_8 + (2)^k\omega_{13} = \left(-\frac{14}{25}\right)^k, \\ (-2)^k\omega_{-6} + (-1)^k\omega_{-1} + (0)^k\omega_4 + (1)^k\omega_9 + (2)^k\omega_{14} = \left(-\frac{19}{25}\right)^k, \end{cases}$$

The system is valid for $k = 1$. *Hence, the* CQ_5-scheme *reproduces the degree* 1 *polynomial. No more reproduction of the system exists.* □

Corollary 5.9. *The* CQ_5-scheme *provided in (3. 9) has approximation order* 2.

Figure 3 illustrates the graphical behavior of the CQ_5-*scheme* for linear polynomials.

6. FRACTAL PROPERTY

Intricate fractal curves can be produced from the given set of parameters. For ease, we examine this property by adjusting the values of parameters $a_0 = 0, b_0 = u_0 + \frac{3}{25}, d_0 = 0$ and $u_0 = \frac{6}{125}$ in (3. 9). Then, we get an interpolating refinement scheme depending only on parameter c_0. The parameter c_0 in fractal curve generation significantly impacts the range within which the scheme can produce fractals.

$$\begin{cases} f_{5i-2}^{k+1} = -c_0 f_{i-2}^k + (\frac{56}{125} + 2c_0) f_{i-1}^k + (\frac{63}{125} - c_0) f_i^k + \frac{6}{125} f_{i+1}^k, \quad 0 \le i \le 5^k n, \\ f_{5i-1}^{k+1} = (\frac{36}{125} - c_0) f_{i-1}^k + (\frac{78}{125} + 2c_0) f_i^k + (\frac{11}{125} - c_0) f_{i+1}^k, \quad 0 \le i \le 5^k n, \\ f_{5i}^{k+1} = f_i^k, \quad 0 \le i \le 5^k n, \\ f_{5i+1}^{k+1} = (\frac{11}{125} - c_0) f_{i-1}^k + (\frac{78}{125} + 2c_0) f_i^k + (\frac{36}{125} - c_0) f_{i+1}^k, \quad 0 \le i \le 5^k n, \\ f_{5i+2}^{k+1} = \frac{6}{125} f_{i-1}^k + (\frac{63}{125} - c_0) f_i^k + (\frac{56}{125} + 2c_0) f_{i+1}^k - c_0 f_{i+2}^k. \quad 0 \le i \le 5^k n. \end{cases} \tag{6. 18}$$

The geometric rules in (6. 18) can be written as follows

$$\begin{cases} f_{5i-2}^{k+1} = \alpha_0 f_{i-2}^k + \alpha_1 f_{i-1}^k + \alpha_2 f_i^k + \alpha_3 f_{i+1}^k, \\ f_{5i-1}^{k+1} = \alpha_4 f_{i-1}^k + \alpha_5 f_i^k + \alpha_6 f_{i+1}^k, \\ f_{5i}^{k+1} = f_i^k, \\ f_{5i+1}^{k+1} = \alpha_6 f_{i-1}^k + \alpha_5 f_i^k + \alpha_4 f_{i+1}^k, \\ f_{5i+2}^{k+1} = \alpha_3 f_{i-1}^k + \alpha_2 f_i^k + \alpha_1 f_{i+1}^k + \alpha_0 f_{i+2}^k. \end{cases} \tag{6. 19}$$

Here

$$\alpha_0 = -c_0, \quad \alpha_1 = \frac{56}{125} + 2c_0, \quad \alpha_2 = \frac{63}{125} - c_0, \quad \alpha_3 = \frac{6}{125},$$
$$\alpha_4 = \frac{36}{125} - c_0, \quad \alpha_5 = \frac{78}{125} + 2c_0, \quad \alpha_6 = \frac{11}{125} - c_0.$$

The five edge vectors between f_0^0 and f_1^0 after k-step subdivision are obtained as follows:

$$V_k = f_1 - f_0, \quad S_k = f_2 - f_1, \quad R_k = f_3 - f_2, \quad L_k = f_4 - f_3, \quad T_k = f_5 - f_4.$$

Firstly, assume that

$$U_k = f_1 - f_{-1}, \quad W_k = f_0 - f_{-1}, \quad Z_k = f_2 - f_{-2}, \quad M_k = f_1 - f_{-2}, \quad N_k = f_3 - f_{-3}.$$

From (6. 19), we obtain

$$U_k = V_k + W_k, \quad Z_k = S_k + M_k.$$

Now, we find the solutions of edge vectors V_k, S_k, R_k, L_k and T_k as follows:
First of all, consider the vector U_{k+1}

$$U_{k+1} = f_1^{k+1} - f_{-1}^{k+1} = U_{k+1} - \frac{1}{5}U_k,$$

the general solution of U_k is given by

$$U_k = b_1 r_1^k = b_1 \left(\frac{1}{5}\right)^k, \quad \text{where} \quad b_1 = U_0 = f_1^0 - f_0^0. \tag{6. 20}$$

Now, consider the edge vector V_{k+1}

$$V_{k+1} = f_1^{k+1} - f_0^{k+1} = -\alpha_6 U_k - (1 - \alpha_5)V_k,$$
$$V_{k+1} - \left(\frac{47}{125} + 2c_0\right) V_k = \left(-\frac{11}{125} + 2c_0\right) U_k, \tag{6. 21}$$

Substitute the value of U_k from (6. 20) in (6. 21), the solution of V_k is given as

$$V_k = d_1 r_2^k + d_2 r_1^k, \quad \text{where} \quad d_1 = f_1^0 - f_0^0 + \frac{1}{2}b_1 \quad \text{and} \quad d_2 = -\frac{1}{2}b_1. \tag{6. 22}$$

Consider the vector S_{k+1}

$$S_{k+1} = f_2^{k+1} - f_1^{k+1} = -(\alpha_3 - \alpha_6)U_k - (-\alpha_2 - \alpha_5)V_k + \alpha_0 S_k,$$
$$S_{k+1} = -c_0 S_k + \left(\frac{1}{25} - c_0\right) U_k + \left(\frac{3}{25} - c_0\right) V_k, \tag{6. 23}$$

from (6. 20) and (6. 22), we get

$$S_{k+1} + c_0 S_k = \left(\frac{1}{25} - c_0\right)(b_1 r_1^k) + \left(\frac{3}{25} - c_0\right)(d_1 r_2^k + d_2 r_1^k), \tag{6. 24}$$

by splitting (6. 24) into two parts and after simplification, we obtain the general solution of (6. 23) in the following form

$$S_k = e_1(-c_0)^k + e_2 r_1^k + e_3 r_2^k, \tag{6. 25}$$

where

$$e_1 = \xi_2^0 - \xi_1^0 - \frac{1}{5}\left[\frac{3125c_0^2(b_1+d_2-1)+c_0(1050b_1+800d_2+1000)-(47b_1+141d_2+75)}{(-1+5c_0)(47+125c_0)}\right],$$

$$e_2 = \frac{1}{5}\frac{(25c_0-1)b_1+(25c_0-3)d_2}{5c_0-1},$$

$$e_3 = \frac{-5(25c_0-3)}{47+125c_0}.$$

Consider the vector Z_{k+1}

$$Z_{k+1} = f_2^{k+1} - f_{-2}^{k+1} = \alpha_0 Z_k + (\alpha_1 - \alpha_3)U_k, \tag{6. 26}$$

the corresponding non-homogeneous equation is

$$Z_{k+1} = -c_0 Z_k + \left(\frac{50}{125} + 2c_0\right) U_k. \tag{6. 27}$$

by using (6. 20) in (6. 27), the general solution of Z_k can be written as

$$Z_k = g_1\rho_2^k + g_2 r_1^k, \quad \text{where} \quad g_1 = \xi_2^0 - \xi_{-2}^0 - 2b_1 \quad \text{and} \quad g_2 = 2b_1. \tag{6. 28}$$

Similarly, the non-homogeneous equations and corresponding solutions for the other edge vectors can be determined as:
for edge vector R_k,

$$R_{k+1} = \frac{63}{125}U_k + \frac{6}{125}Z_k. \tag{6. 29}$$

The general solution of (6.29) is

$$R_k = h_1 r_1^k + h_2\rho_2^k, \quad \text{where} \quad h_1 = \frac{63}{125}b_1 + \frac{6}{25}g_2 \quad \text{and} \quad h_2 = -\frac{6}{125}g_1.$$

For edge vector L_k, the non-homogeneous equation is

$$L_{k+1} = \left(\frac{4}{25} + 3c_0\right)V_k + \left(\frac{1}{25} - c_0\right)S_k - c_0U_k. \tag{6. 30}$$

The solution of (6. 30) is

$$L_k = I_1 r_1^k + I_2 r_2^k + I_3\rho_1^k,$$

where

$$I_1 = \left(\frac{4}{5} + 5c_0\right)d_2 + \left(\frac{1}{5} - 5c_0\right)e_2 - 5c_0b_1,$$

$$I_2 = \frac{5\left(-1+25c_0\right)}{47+250c_0}e_3,$$

$$I_3 = \left(-\frac{1}{25c_0} + 1\right)e_1.$$

For edge vector T_{k+1}

$$T_{k+1} = \left(-\frac{11}{125} + c_0\right)S_k + \left(\frac{36}{125} - c_0\right)V_k. \tag{6. 31}$$

The solution of (6. 31) is determined as:

$$T_k = j_1\rho_1^k + j_2 r_1^k + j_3 r_2^k,$$

where

$$\begin{aligned} j_1 &= \left(\frac{11}{125c_0} - 1\right) e_1, \\ j_2 &= \left(-\frac{11}{25} + 5c_0\right) e_2 + \left(\frac{36}{25} - 5c_0\right) d_2, \\ j_3 &= \left(\frac{-11+25c_0}{47+250c_0}\right) e_3 - \left(\frac{-36+125c_0}{47+250c_0}\right) d_1. \end{aligned}$$

Theorem 6.1. *For $c_0 \in \left(-\frac{7}{25}, -\frac{1}{5}\right)$, the limit curves produced by the scheme (6. 18) are fractal curves.*

Proof. The edge vectors between $f_0^0 - f_1^0$ after k-step refinement can be expressed as

$$Y_t = f_t^k - f_{t-1}^k = \sigma_{1,t}\rho_1^k + \sigma_{2,t} r_1^k + \sigma_{3,t} r_2^k,$$

where $\sigma_{s,t} \neq 0$, $s = 1, 2, 3$, *and* $t = 1, 2, 3, \ldots, 5^k$.
When $-\frac{7}{25} < c_0 < -\frac{1}{5}$, *we have* $|r_2| < |r_1|$ *and* $|r_2| < |\rho_1|$.
Let $Y_{t,0}^k = \min |Y_t^k|$, *we get*

$$\sum_{t=1}^{5^k} |Y_t^k| \geq 5^k |Y_{t,0}^k| = 5^k \left| \sigma_{1,t}(-c_0)^k + (\frac{1}{5})^k \sigma_{2,t} + (\frac{47}{125} + 2c_0)^k \sigma_{3,t} \right| \to \infty (k \to \infty).$$

So, the limit curves of the interpolating refinement scheme are fractal curves for $c_0 \in \left(-\frac{7}{25}, -\frac{1}{5}\right)$. □

6.2. **Graphical presentation of fractals.** In this section, variety of fractal curves is produced by the CQ_5-*scheme*. The performance of the scheme is evaluated for various values of parameter c_0 within the range $-\frac{7}{25} < c_0 < -\frac{1}{5}$. Additionally, when one or more parameters are provided, the fractal characteristic can be further investigated. The results display the excellent performance for generating stunning fractals by the CQ_5-*scheme*.

Example 6.3. In this example, the fractals produced by the CQ_5-*scheme* at $(a_0, b_0, c_0, d_0, u_0)$=$(0, \frac{21}{125}, c_0, 0, \frac{6}{125})$ for $c_0 = -\frac{1}{10}$ and $c_0 = -\frac{4}{25}$ is shown in Figures 4(a) and 4(c). Figures 4(b) and 4(d) are with zoom-in on a small part of these fractals to highlight their self-similarity. These images indicate that the smaller details of the fractal look like the larger structure repeated recursively at different scales.

Example 6.4. In this example, the fractal generation ability of the CQ_5-*scheme* to generate intricate fractals curves, as demonstrated in Figure 5 for parametric values $(a_0, b_0, c_0, d_0, u_0)$=$(0, \frac{21}{125}, -\frac{11}{100}, 0, \frac{6}{125})$ on the sketch of tree at different refinement levels is visualized. This representation identifies that the scheme produces dense fractals even at low iteration levels, showcasing the potential for creating complex patterns with minimal computational effort.

(a) $c_0 = -\frac{1}{10}$ (b) Amplifying element

(c) $c_0 = -\frac{11}{100}$ (d) Amplifying element

FIGURE 4. *Fractals generation under the specific range of parameter c_0 with their amplifying element.*

(a) First refinement level (b) Second refinement level (c) Third refinement level

FIGURE 5. *Fractals generation under the parameter values $(a_0, b_0, c_0, d_0, u_0)$=$(0, \frac{21}{125}, -\frac{11}{100}, 0, \frac{6}{125})$ on different iteration levels.*

(a) (b) (c)

FIGURE 6. *Fractals generation for appropriate choices of parameters: (a) $(a_0, b_0, c_0, d_0, u_0) = (0, \frac{12}{125}, -\frac{4}{25}, \frac{1}{25}, \frac{2}{25})$ (b) $(a_0, b_0, c_0, d_0, u_0) = (0, \frac{19}{125}, -\frac{1}{10}, \frac{1}{100}, -\frac{7}{50})$ and (c) $(a_0, b_0, c_0, d_0, u_0) = (0, \frac{21}{125}, -\frac{11}{100}, \frac{3}{100}, -\frac{3}{20})$ at second refinement level.*

Example 6.5. In this example, the fractal curves for suitable selection of parameters are observed. Figure 6(a) showcase the fractal curves for $a_0 = 0, b_0 = -\frac{2}{25}, c_0 = -\frac{4}{25}, d_0 = -\frac{1}{25}, u_0 = \frac{6}{125}$, Figure 6(b) showcase the fractal curves for $a_0 = 0, b_0 = \frac{17}{125}, c_0 = -\frac{2}{25}, d_0 = 0, u_0 = \frac{4}{125}$, and Figure 6(c) represents the fractal curves for $a_0 = 0, b_0 = \frac{19}{125}, c_0 = -\frac{1}{10}, d_0 = \frac{1}{100}, u_0 = -\frac{7}{50}$ on same refinement level. These fractal images showcase a variety of shapes, from web-like to organic and chaotic, influenced by specific parameter values. So, by carefully manipulating the parameters, we can achieve a wide variety of complex and aesthetically pleasing fractal structures.

Example 6.6. In this example, we display the comparison of fractal curves produced by the CQ_5-*scheme* with the existing schemes. Figure 7(a) showcases the fractal curves produced by the binary interpolating scheme [17] for $\omega = 0.2$. Figure 7(b) showcases the fractal curves produced by the ternary interpolating scheme [10] for $\alpha = 0.00792$. Figure 7(c) represents the fractal curves produced by the quaternary interpolating scheme [16] for $\omega = -\frac{7}{23}$. Figure 7(d) showcases the fractal curves produced by our proposed scheme $c_0 = -\frac{4}{25}$. All of these images are designed at the second iteration level. This visual comparison demonstrates that the combined scheme generates stunning fractals faster than the existing schemes at less computational cost.

(a) (b)

(c) (d)

FIGURE 7. *Comparison of fractal curves with existing schemes after two iterations: (a) Binary scheme* [17] *for* $\omega = 0.2$ *(b) Ternary scheme* [10] *for* $\alpha = 0.00792$ *(c) Quaternary scheme* [16] *for* $\omega = -\frac{7}{23}$ *and (d) our proposed scheme for* $c_0 = -\frac{4}{25}$.

7. GRAPHICAL EFFICIENCY AND COMPARATIVE ANALYSIS

A control polygon is a series of points that provides a simple and intuitive way to manipulate the shape by adjusting the positions of its points. In this section, we illustrate the graphical behavior of the scheme CQ_5-*scheme* on several polygons. The beauty of the CQ_5-*scheme* is that it is a combined scheme, which offers the greatest flexibility in generating C^2 and C^3 interpolating and approximating limit curves respectively. Therefore, we demonstrate that for appropriate choices of parameters a_0, b_0, c_0, d_0, u_0 the resulting limit curves display interproximal behavior.

Example 7.1. In this example, the deformation in the candy sketch by disturbing a single control point is visualized. The control points for the sketch of candy in Figure 8(a) are $(9,12),(10,13),(11,13),(11,12),(12,12),(12,11),(13,11),(13,10),(12,9),(10,9),$ $(10,7),(9,5),(7,4),(5,4),(5,2),(4,1),(3,1),(3,2),(2,2),(2,3),(1,3),(1,4),(2,5),$ $(4,5),(4,7),(5,9),(7,10),(9,10)$ and $(9,12)$ respectively. The another sketch is drawn in Figure 8(b) by displacing the point from $(5,9)$ to $(5,13)$. From these models, we conclude that if a single point is disturbed from its original position only a small portion of the curve is deformed while the rest remains the same.

Example 7.2. In this example, the behavior of the CQ_5-*scheme* on different refinement levels is observed. To observe how the scheme governs the resulting curves for $(a_0,b_0,c_0,d_0,u_0) = (-\frac{1}{375}, -\frac{3}{250}, -\frac{29}{750}, -\frac{4}{125}, \frac{7}{125})$ on different refinement levels for the sketch of the doll can be viewed in Figure 9. Our finding reveal that for a lower level of rendering the model looks best which reduces the computational cost.

Example 7.3. In this example, the behavior of the CQ_5-*scheme* for a suitable selection of parameters with C^3 convergence is considered to generate resulting curves. The resulting curves are formed for the parameter values: $a_0 = \frac{1}{948}$, $b_0 = -\frac{3913}{562875}$, $c_0 = -\frac{14}{2375}$, $d_0 = \frac{1}{950}$, $u_0 = \frac{71}{2375}$ (blue), $a_0 = \frac{1}{500}$, $b_0 = -\frac{3}{875}$, $c_0 = -\frac{2}{875}$, $d_0 = \frac{1}{350}$, $u_0 = \frac{23}{875}$ (red), $a_0 = \frac{2}{625}$, $b_0 = \frac{3}{1250}$, $c_0 = \frac{2}{625}$, $d_0 = \frac{7}{1250}$, $u_0 = \frac{13}{625}$ (green) and $a_0 = 0$, $b_0 = \frac{11}{175}$, $c_0 = \frac{1}{27}$, $d_0 = -\frac{1}{625}$, $u_0 = -\frac{2}{35}$ (purple) on the sketch of fish in the Figure 10(a) with their amplifying element shown in the Figure 10(b).

Example 7.4. In this example, we visualize the combined behavior of the CQ_5-*scheme* for suitable selection of parameters. The behavior of the CQ_5-*scheme* at $a_0 = -\frac{1}{375}$, $b_0 = -\frac{29}{750}$, $c_0 = -\frac{4}{125}$, $d_0 = -\frac{3}{250}$, $u_0 = \frac{7}{125}$ (red), $a_0 = \frac{4}{875}$, $b_0 = \frac{3}{250}$, $c_0 = \frac{2}{175}$, $d_0 = \frac{17}{1750}$, $u_0 = \frac{11}{875}$ (blue), $a_0 = \frac{3}{2375}$, $b_0 = -\frac{53}{4750}$, $c_0 = -\frac{4}{475}$, $d_0 = -\frac{1}{4750}$, $u_0 = \frac{77}{2375}$

(a) (b)

FIGURE 8. *Local control after second subdivision level.*

(a) First refinement level (b) Second refinement level (c) Third refinement level

FIGURE 9. *Tree sketch on different refinement levels. The dotted lines and the solid boxed represent the control polygon and control points respectively.*

(a) Comparison (b) Amplifying element

FIGURE 10. *Efficiency of the* CQ_5-scheme *for different values of parameters: (a)* $(a_0, b_0, c_0, d_0, u_0)=(\frac{1}{948}, -\frac{3913}{562875}, -\frac{14}{2375}, \frac{1}{950}, \frac{71}{2375})$ *(red),* $(a_0, b_0, c_0, d_0, u_0)=(\frac{1}{500}, -\frac{3}{875}, -\frac{2}{875}, \frac{1}{350}, \frac{23}{875})$ *(blue),* $(a_0, b_0, c_0, d_0, u_0)=(\frac{2}{625}, \frac{3}{1250}, \frac{2}{625}, \frac{7}{1250}, \frac{13}{625})$ *(green), and* $(a_0, b_0, c_0, d_0, u_0)=(0, \frac{11}{175}, \frac{1}{27}, -\frac{1}{625}, -\frac{2}{35})$ *(purple) on second iteration level with their amplifying factor (b).*

(a) Comparison (b) Amplifying element

FIGURE 11. *Efficiency of the* CQ_5-scheme *for different values of parameters: (a)* $(a_0, b_0, c_0, d_0, u_0)=(-\frac{1}{375}, -\frac{29}{750}, -\frac{4}{125}, -\frac{3}{250}, \frac{7}{125})$ *(red),* $(a_0, b_0, c_0, d_0, u_0)=(\frac{4}{875}, \frac{3}{250}, \frac{2}{175}, \frac{17}{1750}, \frac{11}{875})$ *(blue),* $(a_0, b_0, c_0, d_0, u_0)=(\frac{3}{2375}, -\frac{53}{4750}, -\frac{4}{475}, -\frac{1}{4750}, \frac{77}{2375})$ *(navy), and* $(a_0, b_0, c_0, d_0, u_0)=(0, \frac{11}{175}, \frac{1}{27}, -\frac{1}{625}, -\frac{2}{35})$ *(purple) on second iteration level with their amplifying factor (b).*

(navy) and $a_0 = 0$, $b_0 = \frac{11}{175}$, $c_0 = \frac{1}{27}$, $d_0 = -\frac{1}{625}$, $u_0 = -\frac{2}{35}$ (purple) is shown in the sketch of the chicken in Figure 11(a) with their amplifying element which is shown in Figure 11(b).

7.5. **Comparison with existing schemes.** This section compares the CQ_5-*scheme* with the existing old ones and brand new schemes. The comparative analysis involves an interpolating and approximating scheme and existing combined schemes. The beauty of the CQ_5-*scheme* is that it offers the highest flexibility in generating smoother shapes according to the requirements with less computational cost than the existing scheme. The resulting curves are observed on second refinement levels.

(a) Limit curve of [7] (b) Limit curve of [19] (d) Limit curve of CQ_5-*scheme*

FIGURE 12. *Comparison with the existing interpolating and approximating schemes.*

(a) Limit curve of [7] (b) Limit curve of [19]

(c) Limit curve of [14] (d) Limit curve of CQ_5-*scheme*

FIGURE 13. *Comparison with the existing combined scheme.*

TABLE 4. Comparison of CQ_5-*scheme* with existing schemes.

Scheme	**Type**	**Continuity**	**Support size**
Ghaffar et al. [4]	Approximating	2	3.5
Siddiqi and Younis [12]	Approximating	2	3.6
Tariq et al. [14]	Combined	3	4.6
CQ_5-*scheme*	Combined	3	5

Example 7.6. In this example, we present a comparative analysis of the CQ_5-*scheme* with the recently existing interpolating and approximating scheme. For the suitable values of parameters $a_0 = \frac{4}{875}$, $b_0 = \frac{3}{250}$, $c_0 = \frac{2}{175}$, $d_0 = \frac{17}{1750}$, $u_0 = \frac{11}{875}$ the CQ_5-*scheme* turns into a new approximating scheme. So, for these suitable parameter choices, the comparison of the CQ_5-*scheme* with the binary approximating scheme [7] for $\omega = 1$ and ternary scheme [19] for $u = \frac{1}{61}$ is presented in Figure 12.

Example 7.7. In this example, we present a comparative analysis of the CQ_5-*scheme* with the recently existing combined scheme. For the suitable values of parameters $a_0 = \frac{4}{875}$, $b_0 = \frac{3}{250}$, $c_0 = \frac{2}{175}$, $d_0 = \frac{17}{1750}$, $u_0 = \frac{11}{875}$ the CQ_5-*scheme* turns into a new approximating scheme. So, for these suitable parameter choices, the comparison of the CQ_5-*scheme* with the combined binary scheme [7] for $\alpha = 0.7531$ and $\beta = 0$, combined ternary scheme [19] for $d = -\frac{1}{28}$ and with the combined quaternary scheme [14] for $\alpha = 0$ and $\beta = 2$ is presented in Figure 13.

Table 4 presents a comparison of the proposed scheme with some existing schemes.

8. CONCLUSIONS

This study presents a 5-point combined quinary refining scheme and conducts a full examination of its features. The scheme attains C^3 convergence while maintaining a compact support $\left[-\frac{5}{2}, \frac{5}{2}\right]$ than the three-point approximating quinary refinement scheme. Fractal curves corresponding to the scheme are conducted and illustrated. The parameter c_0 for fractal curve production has a range of $-\frac{7}{25} < c_0 < -\frac{1}{5}$. Moreover, the application of fractal curves produced by the CQ_5-scheme demonstrates its capacity to generate dense fractal depths, opens new possibilities for artistic expression and scientific visualization. Additionally, numerical examples of the proposed scheme illustrate the fact that the parameters selection significantly impacts the curve shape. Alternative parameters tend to produce inwardly shrinking curves, while positive parameters lead to outward expansion. This flexibility allows designers to fine-tune the appearance of their models according to their requirements by choosing appropriate parameters. These comprehensive analysis collectively contribute to the better understanding of the proposed scheme.

AUTHOR CONTRIBUTIONS

Conceptualization, Pakeeza Ashraf and Robina Bashir; methodology, Pakeeza Ashraf and Kaneez Fatima; software, Kaneez Fatima; validation, Pakeeza Ashraf; formal analysis, Pakeeza Ashraf and Robina Bashir; investigation, Pakeeza Ashraf and Robina Bashir; data curation, Kaneez Fatima; writing-original draft preparation, Pakeeza Ashraf and Kaneez Fatima; writing-review and editing, Robina Bashir and Kaneez Fatima; visualization, Pakeeza Ashraf; funding acquisition, Robina Bashir. All authors have read and agreed to the published version of the manuscript.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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