

Weak Forms of $\Psi_\Gamma - C$ Sets

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Abstract. In this study, we introduce $\Gamma - \Psi_\Gamma$ -sets and $\text{pre-}\Gamma - \Psi_\Gamma$ -sets in ideal topological spaces. We investigate various properties of these sets and we obtain new results. Moreover, we analyze the relationships between these sets and some special sets such as $\Psi_\Gamma - C$ set and σ -open set in the literature. Additionally, we observe that the families of $\Gamma - \Psi_\Gamma$ -sets and $\text{pre-}\Gamma - \Psi_\Gamma$ -sets form a supratopology on W in the case $cl(v) \cap \mathcal{I} = \{\emptyset\}$.

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1. INTRODUCTION

The notion of ideal was studied by many authors since it was introduced by Vaidyanathaswamy in [24] and Kuratowski in [7]. Especially, after Natkaniec presented the set operator Ψ [13], a lot of new sets were defined by using local function and the operator Ψ . Ψ -set [2], $\Psi - C$ set [11], $*^\Psi$ -set [5] and Ψ^* -set [12] are examples of these sets.

In 2013, the local closure function and the operator Ψ_Γ were presented in [17]. Later, Pavlović researched that under which cases local function and local closure function are identical in [20]. Tunç and Özen Yıldırım introduced an $\mathcal{I}_\Gamma$ -perfect set [22], a Γ -dense-in-itself set [22] and a $\Psi_\Gamma - C$ set [23] via local closure function and the operator Ψ_Γ . Yalaz and Keskin Kaymakçı also defined a $\Gamma - \mathcal{I}$ -open set and a $\text{pre-}\Gamma - \mathcal{I}$ -open set in [26].

In topological spaces, the concepts of θ -openness [25] and θ -closure of a set [25] were introduced by Velicko. Caldas et al. studied the further properties of θ -closure and θ -interior in [3]. In [17], Al-Omari and Noiri studied these concepts by comparing with the local closure function in ideal topological spaces. On the other side, new types of semi open sets were studied by means of θ -interior and θ -closure in topological spaces in [1]. Besides, there exist studies such as [15, 8, 9] involving new types of functions via semi open and θ -open sets and their relations with each other. In addition to these studies, Tunç and Özen Yıldırım searched the relations between special sets in ideal topological spaces

with θ -open and θ -closed sets in [22, 23]. Especially, they proved that $\Psi_\Gamma - C$ sets are the generalization of θ -open sets in ideal topological spaces and the family of $\Psi_\Gamma - C$ sets may not to be form a topology in [23]. Similarly, Yumak and Keskin Kaymakçı presented some special sets in soft ideal topological spaces and they researched relationships between them in [27]. Parimala et al. defined $nI\alpha g$ -closed sets in nano ideal topological spaces and they also studied normality via these sets in [19]. Apart from these studies, Njamcul and Pavlović studied the topology expansions using ideals in [14]. Al-Omari and Noiri introduced Ψ -operation on an m -space and an ideal m -space in [18].

In this study, we presented the concepts of $\Gamma - \Psi_\Gamma$ -set and pre- $\Gamma - \Psi_\Gamma$ -set in ideal topological space. We investigated the relationships of these new sets with previously defined special sets in general and special cases. Furthermore, we searched whether the families of $\Gamma - \Psi_\Gamma$ -sets and pre- $\Gamma - \Psi_\Gamma$ -sets form topologies. Then, we obtained that they may not form topologies in general. However, we have shown that these families are supra-topologies under the condition $cl(v) \cap \mathfrak{I} = \{\emptyset\}$. Finally, we found that if the ideal is $\{\emptyset\}$, then the family of pre- $\Gamma - \Psi_\Gamma$ -sets forms a topology, but the family of $\Gamma - \Psi_\Gamma$ -sets does not have to.

2. NOTATIONS AND PRELIMINARIES

In this paper, (W, v) represents a topological space and the interior and the closure of a subset H of W are denoted by $int(H)$ and $cl(H)$ in a topological space (W, v) , respectively. $P(W)$ represents the family of all subsets of W .

For (W, v) and $H \subseteq W$, $cl_\theta(H) = \{x \in W : cl(E) \cap H \neq \emptyset \text{ for each } E \in v(x)\}$, where $v(x) = \{E \in v \mid x \in E\}$, is called the θ -closure of H [25]. The θ -interior of H [6], denoted $int_\theta(H)$, consists of those points x of H such that $E \subseteq cl(E) \subseteq H$ for some open set E including x . Furthermore, $W \setminus int_\theta(H) = cl_\theta(W \setminus H)$ [3]. H is called θ -closed [25] if $H = cl_\theta(H)$. If $W \setminus H$ is θ -closed, then H is called θ -open [25] and $v_\theta = \{H \subseteq W \mid H \text{ is } \theta\text{-open}\}$. Besides, v_θ is a topology on W such that $v_\theta \subseteq v$. Additionally, the union of all θ -open subsets of H is $int_\theta(H)$ and so $int_\theta(H)$ is θ -open [25]. H is called regular θ -closed [1] if $H = cl_\theta(int_\theta(H))$. H of W is called semi θ -open [1] if $H \subseteq cl_\theta(int_\theta(H))$. H is called θ -semiopen [4] if there exists a θ -open subset U of W such that $U \subseteq H \subseteq cl(U)$.

An ideal $\mathfrak{I}$ [24, 7] on W is a nonempty collection of subsets of W satisfying the following conditions:

- (i) If $H \in \mathfrak{I}$ and $K \subseteq H$, then $K \in \mathfrak{I}$,
- (ii) If $H \in \mathfrak{I}$ and $K \in \mathfrak{I}$, then $H \cup K \in \mathfrak{I}$.

An ideal topological space $(W, v, \mathfrak{I})$ is a topological space (W, v) with an ideal $\mathfrak{I}$ on W . By this way, $(W, v, \mathfrak{I})$ represents an ideal topological space in this paper. For $H \subseteq W$, $\Gamma(H)(\mathfrak{I}, v) = \{x \in W \mid H \cap cl(E) \notin \mathfrak{I} \text{ for every } E \in v(x)\}$ is said to be the local closure function [17] of H with respect to $\mathfrak{I}$ and v in $(W, v, \mathfrak{I})$. Briefly, it is denoted by $\Gamma(H)$ instead of $\Gamma(H)(\mathfrak{I}, v)$. In [17], an operator $\Psi_\Gamma : P(W) \mapsto v$ is given as $\Psi_\Gamma(H) = W \setminus \Gamma(W \setminus H)$ and the topologies are defined on W as follows: $\sigma = \{H \subseteq W : H \subseteq \Psi_\Gamma(H)\}$, $\sigma_0 = \{H \subseteq W : H \subseteq int(cl(\Psi_\Gamma(H)))\}$ such that $v_\theta \subseteq \sigma \subseteq \sigma_0$. H is called σ -open [17] (resp. σ_0 -open [17]), if $H \in \sigma$ (resp. $H \in \sigma_0$). Furthermore, Noorie and Goyal defined a $\theta^\mathfrak{I}$ -closed set [16] such that a subset H of W is said to be $\theta^\mathfrak{I}$ -closed if $\Gamma(H) \subseteq H$.

Definition 2.1. [22] A set H is called $\mathfrak{I}_\Gamma$ -perfect (resp. L_Γ -perfect, R_Γ -perfect, $\mathfrak{I}_\Gamma$ -dense, Γ -dense-in-itself) if $H = \Gamma(H)$ (resp. $H \setminus \Gamma(H) \in \mathfrak{I}$, $\Gamma(H) \setminus H \in \mathfrak{I}$, $\Gamma(H) = W$, $H \subseteq \Gamma(H)$) for $H \subseteq W$ in $(W, v, \mathfrak{I})$. Moreover, H is called C_Γ -perfect if H is both L_Γ -perfect and R_Γ -perfect.

Definition 2.2. [23] A set H is said to be a $\Psi_\Gamma - C$ set if $H \subseteq cl(\Psi_\Gamma(H))$ for $H \subseteq W$ in $(W, v, \mathfrak{I})$. The family of all $\Psi_\Gamma - C$ sets in $(W, v, \mathfrak{I})$ is denoted by $\Psi_\Gamma(W, v, \mathfrak{I})$.

Lemma 2.3. [17] In $(W, v, \mathfrak{I})$ for $H \subseteq W$,

(i) if $H \in v$, then $cl(H) = cl_\theta(H)$.

(ii) if $U \in v_\theta$, then $U \cap \Gamma(H) = U \cap \Gamma(U \cap H) \subseteq \Gamma(U \cap H)$.

Theorem 2.4. [17] Let H and K be subsets of W in $(W, v, \mathfrak{I})$. Then, the following properties hold:

(i) If $H \subseteq K$, then $\Gamma(H) \subseteq \Gamma(K)$.

(ii) If $H \subseteq K$, then $\Psi_\Gamma(H) \subseteq \Psi_\Gamma(K)$.

(iii) $\Gamma(H) = cl(\Gamma(H)) \subseteq cl_\theta(H)$ and $\Gamma(H)$ is closed.

(iv) $\Psi_\Gamma(H \cap K) = \Psi_\Gamma(H) \cap \Psi_\Gamma(K)$.

(v) If $H \in \mathfrak{I}$, then $\Gamma(H) = \emptyset$.

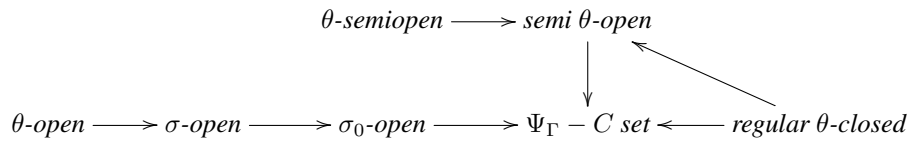
(vi) $\Gamma(\emptyset) = \emptyset$.

Theorem 2.5. [17] In $(W, v, \mathfrak{I})$, where $cl(v) = \{cl(U) : U \in v\}$, $cl(v) \cap \mathfrak{I} = \{\emptyset\}$ if and only if $W = \Gamma(W)$.

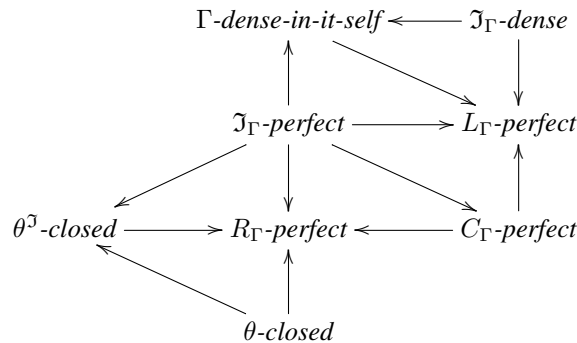
Theorem 2.6. [21] In $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$, $\Psi_\Gamma(H) \subseteq \Gamma(H)$ for each subset H of W .

Corollary 2.7. [17] $U \subseteq \Psi_\Gamma(U)$ for each $U \in v_\theta$ in $(W, v, \mathfrak{I})$.

Corollary 2.8. [17, 1, 23] The following diagram is valid in an ideal topological space.



Remark 2.9. [22] In $(W, v, \mathfrak{I})$, the following diagram holds for $H \subseteq W$:



Theorem 2.10. [23] In $(W, v, \mathfrak{I})$, $\text{int}_\theta(H) \subseteq \Psi_\Gamma(H)$, for each $H \subseteq W$.

Definition 2.11. [26] In $(W, v, \mathfrak{I})$, H is called a $\Gamma - \mathfrak{I}$ -open set if $H \subseteq \text{int}(\Gamma(H))$ for $H \subseteq W$.

Definition 2.12. [10] Let W be a nonempty set and v' be a collection of subsets of W . If $W \in v'$ and v' is closed under arbitrary union, then v' is called a supratopology on W . (W, v') is called a supratopological space (or supraspace).

3. $\Gamma - \Psi_\Gamma$ -SETS

Definition 3.1. A subset H of W is called a $\Gamma - \Psi_\Gamma$ -set if $H \subseteq \Gamma(\Psi_\Gamma(H))$ in $(W, v, \mathfrak{I})$. The family of all $\Gamma - \Psi_\Gamma$ -sets in $(W, v, \mathfrak{I})$ is denoted by $\Gamma - \Psi_\Gamma(W, v, \mathfrak{I})$.

Example 3.2. In the ideal topological space $(\mathbb{R}, \tau_D, \mathfrak{I}_f)$, where $\mathfrak{I}_f$ is the ideal of finite subsets of $\mathbb{R}$ (the set of all real numbers) and τ_D is the usual topology on $\mathbb{R}$, a set $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is not a $\Gamma - \Psi_\Gamma$ -set, but $B = (0, 1]$ is a $\Gamma - \Psi_\Gamma$ -set.

Proposition 3.3. In $(W, v, \mathfrak{I})$;

- (i) if $\mathfrak{I} = \{\emptyset\}$, then " H is a $\Gamma - \Psi_\Gamma$ -set iff H is semi θ -open" for $H \subseteq W$.
- (ii) if $\mathfrak{I} = P(W)$, then $\Gamma - \Psi_\Gamma(W, v, \mathfrak{I}) = \{\emptyset\}$.

Proof. (i) Let $\mathfrak{I} = \{\emptyset\}$ and $H \subseteq W$. The set H is a $\Gamma - \Psi_\Gamma$ -set

$$\Leftrightarrow H \subseteq \Gamma(\Psi_\Gamma(H)) = \text{cl}_\theta(\Psi_\Gamma(H)) = \text{cl}_\theta(W \setminus \Gamma(W \setminus H)) = \text{cl}_\theta(W \setminus \text{cl}_\theta(W \setminus H)) = \text{cl}_\theta(\text{int}_\theta(H)).$$

$\Leftrightarrow H$ is semi θ -open.

(ii) The proof is obvious. □

Theorem 3.4. H is a $\Gamma - \Psi_\Gamma$ -set iff $\Psi_\Gamma(\Gamma(W \setminus H)) \subseteq W \setminus H$, for $H \subseteq W$ in $(W, v, \mathfrak{I})$.

Proof. Let $H \subseteq W$ in $(W, v, \mathfrak{I})$.

$$H \text{ is a } \Gamma - \Psi_\Gamma\text{-set} \Leftrightarrow H \subseteq \Gamma(\Psi_\Gamma(H)) \Leftrightarrow W \setminus \Gamma(\Psi_\Gamma(H)) = W \setminus \Gamma(W \setminus \Gamma(W \setminus H)) = \Psi_\Gamma(\Gamma(W \setminus H)) \subseteq W \setminus H. \quad \square$$

Theorem 3.5. In $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$, if $H \in \mathfrak{I}$, then $W \setminus H$ is a $\Gamma - \Psi_\Gamma$ -set.

Proof. Let $H \in \mathfrak{I}$ in $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$. Then $\Gamma(\Psi_\Gamma(W \setminus H)) = \Gamma(W \setminus \Gamma(H)) = \Gamma(W) = W$, from the Theorem 2.4 (v) and Theorem 2.5. It implies that $W \setminus H \subseteq \Gamma(\Psi_\Gamma(W \setminus H))$ and so $W \setminus H$ is a $\Gamma - \Psi_\Gamma$ -set. □

Example 3.6. In the ideal topological space $(\mathbb{R}, \tau_F, \mathfrak{I}_f)$, where τ_F is the co-finite topology on $\mathbb{R}$, $\text{cl}(\tau_F) \cap \mathfrak{I}_f = \{\emptyset\}$. As a consequence of the Theorem 3.5, $\mathbb{R} \setminus A$ is a $\Gamma - \Psi_\Gamma$ -set, for every finite subset A of $\mathbb{R}$ in $(\mathbb{R}, \tau_F, \mathfrak{I}_f)$. Therefore, every open set is also a $\Gamma - \Psi_\Gamma$ -set in $(\mathbb{R}, \tau_F, \mathfrak{I}_f)$.

Remark 3.7. In $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$, when $W \setminus H$ is a $\Gamma - \Psi_\Gamma$ -set, H may not be in $\mathfrak{I}$, for $H \subseteq W$. In general, the complement of an element of ideal may not be a $\Gamma - \Psi_\Gamma$ -set.

Example 3.8. Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}, \{r\}, \{n, r\}\}$, $\ell = \{\emptyset, \{n\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$, $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$ and $\{n\} \in \mathfrak{I}$, but the complement of $\{n\}$ is not a $\Gamma - \Psi_\Gamma$ -set. Furthermore, the set $\{r\}$ is a $\Gamma - \Psi_\Gamma$ -set, but $\{m, n, p\} \notin \ell$ in (W, v, ℓ) where $\text{cl}(v) \cap \ell = \{\emptyset\}$.

Theorem 3.9. In $(W, v, \mathfrak{J})$, a $\Gamma - \Psi_\Gamma$ -set is a $\Psi_\Gamma - C$ set.

Proof. Let H be a $\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{J})$. Then, $H \subseteq \Gamma(\Psi_\Gamma(H)) \subseteq cl_\theta(\Psi_\Gamma(H)) = cl(\Psi_\Gamma(H))$, from the Theorem 2.4 (iii) and Lemma 2.3 (i). As a result, H is a $\Psi_\Gamma - C$ set. $\square$

Remark 3.10. In an ideal topological space, a $\Psi_\Gamma - C$ set may not be a $\Gamma - \Psi_\Gamma$ -set.

Example 3.11. In $(\mathbb{R}, P(\mathbb{R}), \mathfrak{J}_f)$, $\mathbb{R}$ is a $\Psi_\Gamma - C$ set, but it is not a $\Gamma - \Psi_\Gamma$ -set.

Proposition 3.12. In an ideal topological space $(W, P(W), \mathfrak{J}_f)$, $\Gamma(H) = \emptyset$, for each subset H of W . As a result, $\Gamma - \Psi_\Gamma(W, P(W), \mathfrak{J}_f) = \{\emptyset\}$ and $\Psi_\Gamma(W, P(W), \mathfrak{J}_f) = P(W)$.

Proof. The proof is obvious. $\square$

Theorem 3.13. Let a subset H of W be a $\Psi_\Gamma - C$ set in $(W, v, \mathfrak{J})$ where $cl(v) \cap \mathfrak{J} = \{\emptyset\}$. If $\Psi_\Gamma(H)$ is closed, then H is a $\Gamma - \Psi_\Gamma$ -set.

Proof. Assume that $H \in \Psi_\Gamma(W, v, \mathfrak{J})$ and $\Psi_\Gamma(H)$ is closed in $(W, v, \mathfrak{J})$ where $cl(v) \cap \mathfrak{J} = \{\emptyset\}$. From the hypothesis, $H \subseteq cl(\Psi_\Gamma(H)) = \Psi_\Gamma(H)$. Then, by the Theorem 2.6, $H \subseteq \Gamma(H)$ and by the Theorem 2.4 (i), $\Gamma(H) \subseteq \Gamma(\Psi_\Gamma(H))$. As a result, H is a $\Gamma - \Psi_\Gamma$ -set. $\square$

Theorem 3.14. In an ideal topological space $(W, v, \{\emptyset\})$, for $H \subseteq W$, H is a $\Gamma - \Psi_\Gamma$ -set iff H is a $\Psi_\Gamma - C$ set.

Proof. The proof is obvious by the Theorem 3.9 and the Lemma 2.3 (i). $\square$

Corollary 3.15. In an ideal topological space $(W, v, \{\emptyset\})$, for $H \subseteq W$, the following diagram is valid.

$$\Gamma - \Psi_\Gamma - \text{set} \longleftrightarrow \text{semi } \theta\text{-open} \longleftrightarrow \Psi_\Gamma - C \text{ set}$$

Proof. The proof is clear by the Proposition 3.3 (i) and the Theorem 3.14. $\square$

Theorem 3.16. Let $cl(v) \cap \mathfrak{J} = \{\emptyset\}$ and $H \subseteq W$ in $(W, v, \mathfrak{J})$.

- (i) If H is a σ -open set,
 - (ii) If H is a θ -open set,
 - (iii) If H is a semi θ -open set,
 - (iv) If H is a regular θ -closed set,
 - (v) If H is a θ -semiopen set,
 - (vi) If $W \setminus H$ is an $\mathfrak{J}_\Gamma$ -perfect set,
- then H is a $\Gamma - \Psi_\Gamma$ -set.

Proof. Let $cl(v) \cap \mathfrak{J} = \{\emptyset\}$ and $H \subseteq W$ in $(W, v, \mathfrak{J})$.

(i) Let H be a σ -open set. By the Theorem 2.6, $H \subseteq \Psi_\Gamma(H) \subseteq \Gamma(H)$. Then, $H \subseteq \Gamma(H) \subseteq \Gamma(\Psi_\Gamma(H))$, from the Theorem 2.4 (i). Finally, H is a $\Gamma - \Psi_\Gamma$ -set.

(ii) It is obvious from the Corollary 2.8 and (i).

(iii) Let H be a semi θ -open set. As $int_\theta(H)$ is θ -open, $int_\theta(H) \subseteq \Psi_\Gamma(int_\theta(H)) \subseteq \Gamma(int_\theta(H))$, from the Corollary 2.7 and the Theorem 2.6. On the other hand, $int_\theta(H) \subseteq \Psi_\Gamma(H)$ and hence by the Theorem 2.4 (i), $\Gamma(int_\theta(H)) \subseteq \Gamma(\Psi_\Gamma(H))$. If we combine these

results, $\text{int}_\theta(H) \subseteq \Gamma(\Psi_\Gamma(H))$. Since $\text{int}_\theta(H)$ is open, $\text{cl}_\theta(\text{int}_\theta(H)) = \text{cl}(\text{int}_\theta(H)) \subseteq \text{cl}(\Gamma(\Psi_\Gamma(H)))$, by the Lemma 2.3 (i). From the hypothesis and the Theorem 2.4 (iii), $H \subseteq \text{cl}_\theta(\text{int}_\theta(H)) \subseteq \Gamma(\Psi_\Gamma(H))$.

(iv)-(v) They are obvious from the Corollary 2.8 and (iii).

(vi) If $W \setminus H$ is an $\mathfrak{I}_\Gamma$ -perfect set, then $H = \Psi_\Gamma(H)$. Thus, H is σ -open and hence from (i), H is a $\Gamma - \Psi_\Gamma$ -set. $\square$

Remark 3.17. In $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$, the above requirements may not be true. Furthermore, even if $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$, the inverses of the above requirements may not be true.

Example 3.18. Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}, \{r\}, \{n, r\}\}$, $\ell = \{\emptyset, \{m\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$, $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$ and W is θ -open (and so σ -open and σ_0 -open), but it is not a $\Gamma - \Psi_\Gamma$ -set. In (W, v, ℓ) where $\text{cl}(v) \cap \ell = \{\emptyset\}$, the set $H = \{n, p\}$ is a $\Gamma - \Psi_\Gamma$ -set, but it is neither a σ -open set nor θ -open set.

Example 3.19. Let $W = \{m, n, p, r, q, s\}$, $\mathfrak{I} = P(\{n, p, r, q, s\})$ and $v = \{\emptyset, \{m\}, \{s\}, \{m, s\}, \{n, s\}, \{m, n, s\}, W\}$. In $(W, v, \mathfrak{I})$, $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$ and W is θ -semiopen (resp. semi θ -open, regular θ -closed), but it is not a $\Gamma - \Psi_\Gamma$ -set. Furthermore, the complement of W is an $\mathfrak{I}_\Gamma$ -perfect set, but W is not a $\Gamma - \Psi_\Gamma$ -set.

Example 3.20. Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{p\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$, $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$ and a set $H = \{m, n, r\}$ is a $\Gamma - \Psi_\Gamma$ -set, but it is not a semi θ -open set (resp. θ -semiopen, regular θ -closed). Moreover, $W \setminus H$ is not an $\mathfrak{I}_\Gamma$ -perfect set.

Remark 3.21. In $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$, a $\Gamma - \Psi_\Gamma$ -set may not be a semi θ -open set. Consequently, $\Gamma - \Psi_\Gamma$ -sets and semi θ -open sets may not be coincide in general.

Example 3.22. Let $W = \{m, n, p, r, q, s\}$, $\mathfrak{I} = P(\{n, p, r, q, s\})$ and $v = \{\emptyset, \{m\}, \{s\}, \{m, s\}, \{n, s\}, \{m, n, s\}, W\}$. In $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$, the set $H = \{m\}$ is a $\Gamma - \Psi_\Gamma$ -set, but it is not a semi θ -open set.

Theorem 3.23. In $(W, v, \mathfrak{I})$, if a subset H of W is both σ -open and $\mathfrak{I}_\Gamma$ -perfect, then it is a $\Gamma - \Psi_\Gamma$ -set.

Proof. Assume that H is a σ -open and $\mathfrak{I}_\Gamma$ -perfect set in $(W, v, \mathfrak{I})$. By the Theorem 2.4 (i), $H = \Gamma(H) \subseteq \Gamma(\Psi_\Gamma(H))$. $\square$

Corollary 3.24. In $(W, v, \mathfrak{I})$, if a subset H of W is both θ -open and $\mathfrak{I}_\Gamma$ -perfect, then it is a $\Gamma - \Psi_\Gamma$ -set.

Proof. By the Theorem 3.23 and the Corollary 2.8, it is obvious. $\square$

Remark 3.25. In an ideal topological space, an $\mathfrak{I}_\Gamma$ -perfect set may not be a $\Gamma - \Psi_\Gamma$ -set. As a similar, a $\Gamma - \Psi_\Gamma$ -set may not be an $\mathfrak{I}_\Gamma$ -perfect set.

Example 3.26. Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$, the set $H = \{m, p\}$ is a $\Gamma - \Psi_\Gamma$ -set, but it is not an $\mathfrak{I}_\Gamma$ -perfect set.

Theorem 3.27. *In $(W, v, \mathfrak{I})$, if a subset H of W is both σ -open and $\Gamma - \mathfrak{I}$ -open, then H is a $\Gamma - \Psi_\Gamma$ -set.*

Proof. Assume that H is a σ -open and $\Gamma - \mathfrak{I}$ -open set in $(W, v, \mathfrak{I})$. Then, $\Gamma(H) \subseteq \Gamma(\Psi_\Gamma(H))$, from the Theorem 2.4 (i). Since H is $\Gamma - \mathfrak{I}$ -open, $H \subseteq \text{int}(\Gamma(H)) \subseteq \Gamma(\Psi_\Gamma(H))$. $\square$

Theorem 3.28. *In $(W, v, \mathfrak{I})$, if H is both a Γ -dense-in-itself set and θ -open set, then H is a $\Gamma - \Psi_\Gamma$ -set.*

Proof. Let H be a Γ -dense-in-itself set and θ -open set in $(W, v, \mathfrak{I})$. Then, $H \subseteq \Psi_\Gamma(H)$, by the Corollary 2.7. It implies that $H \subseteq \Gamma(H) \subseteq \Gamma(\Psi_\Gamma(H))$, from the Theorem 2.4 (i). $\square$

Theorem 3.29. *In $(W, v, \mathfrak{I})$, if a subset H of W is a Γ -dense-in-itself set and $W \setminus H$ is an $\mathfrak{I}_\Gamma$ -perfect set, then H is a $\Gamma - \Psi_\Gamma$ -set.*

Proof. For $H \subseteq W$, let H be a Γ -dense-in-itself set and $W \setminus H$ be an $\mathfrak{I}_\Gamma$ -perfect set in $(W, v, \mathfrak{I})$. Then, $\Psi_\Gamma(H) = H$ and so $H \subseteq \Gamma(\Psi_\Gamma(H))$. $\square$

Remark 3.30. *In an ideal topological space, a Γ -dense-in-itself set may not be a $\Gamma - \Psi_\Gamma$ -set.*

Example 3.31. *Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}, \{r\}, \{n, r\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$, the set $H = \{m\}$ is a Γ -dense-in-itself set, but it is not a $\Gamma - \Psi_\Gamma$ -set.*

Theorem 3.32. *In $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$, if a subset H of W is a $\Gamma - \Psi_\Gamma$ -set, then H is a Γ -dense-in-itself set.*

Proof. Assume that H is a $\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$. Then, $H \subseteq \Gamma(\Psi_\Gamma(H)) \subseteq cl_\theta(\Psi_\Gamma(H)) = cl(\Psi_\Gamma(H))$ from the Lemma 2.3 (i). By the Theorem 2.6, $H \subseteq cl(\Gamma(H)) = \Gamma(H)$ and so H is a Γ -dense-in-itself set. $\square$

Remark 3.33. *In $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$, a Γ -dense-in-itself set may not be a $\Gamma - \Psi_\Gamma$ -set.*

Example 3.34. *Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$, the set $H = \{m\}$ is a Γ -dense-in-itself set, but it is not a $\Gamma - \Psi_\Gamma$ -set.*

Theorem 3.35. *Let $H \subseteq W$ in $(W, v, \mathfrak{I})$. If H is a $\Gamma - \Psi_\Gamma$ -set and $\Psi_\Gamma(H)$ is closed, then H is σ -open and so σ_0 -open.*

Proof. Let H be a $\Gamma - \Psi_\Gamma$ -set and $\Psi_\Gamma(H)$ be closed in $(W, v, \mathfrak{I})$. Then, by the Theorem 2.4 (iii), $H \subseteq \Gamma(\Psi_\Gamma(H)) \subseteq cl_\theta(\Psi_\Gamma(H))$. It implies that $H \subseteq cl_\theta(\Psi_\Gamma(H)) = cl(\Psi_\Gamma(H)) = \Psi_\Gamma(H)$, from the Lemma 2.3 (i). Finally, H is σ -open and so σ_0 -open. $\square$

Remark 3.36. *In $(W, v, \mathfrak{I})$, although $\Psi_\Gamma(H)$ is closed and H is σ -open (or σ_0 -open), a subset H of W may not be a $\Gamma - \Psi_\Gamma$ -set.*

Example 3.37. *Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}, \{r\}, \{n, r\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$, the set $H = \{m, p, r\}$ is σ -open (and so σ_0 -open) and $\Psi_\Gamma(H)$ is closed, but it is not a $\Gamma - \Psi_\Gamma$ -set.*

4. $\text{pre-}\Gamma - \Psi_\Gamma\text{-SETS}$

Definition 4.1. A subset H of W is called a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$ if $H \subseteq \text{int}(\Gamma(\Psi_\Gamma(H)))$ in $(W, v, \mathfrak{I})$. The family of all $\text{pre-}\Gamma - \Psi_\Gamma\text{-sets}$ in $(W, v, \mathfrak{I})$ is denoted by $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I})$.

Example 4.2. In the ideal topological space $(\mathbb{R}, \tau_D, \mathfrak{I}_c)$, where $\mathfrak{I}_c$ is the ideal of countable subsets of $\mathbb{R}$, a set $\mathbb{Q}$ (the set of rational numbers) is not a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$, but $\mathbb{R} \setminus \mathbb{Q}$ is a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$.

Proposition 4.3. In $(W, v, \mathfrak{I})$;

- (i) if $\mathfrak{I} = \{\emptyset\}$, then $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I}) = \sigma_0$.
- (ii) if $\mathfrak{I} = P(W)$, then $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I}) = \{\emptyset\}$.

Proof. In $(W, v, \mathfrak{I})$,

(i) suppose that $\mathfrak{I} = \{\emptyset\}$ and $H \subseteq W$. $\text{int}(cl(\Psi_\Gamma(H))) = \text{int}(cl_\theta(\Psi_\Gamma(H))) = \text{int}(\Gamma(\Psi_\Gamma(H)))$ from the Lemma 2.3 (i). Therefore, H is a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set} \Leftrightarrow H$ is a σ_0 -open set.

(ii) the proof is obvious. □

Theorem 4.4. In $(W, v, \mathfrak{I})$, $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I}) \subseteq \Gamma - \Psi_\Gamma(W, v, \mathfrak{I})$.

Proof. The proof is obvious. □

Corollary 4.5. In $(W, v, \mathfrak{I})$, $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I}) \subseteq \Psi_\Gamma(W, v, \mathfrak{I})$.

Proof. From the Theorem 4.4 and the Theorem 3.9, it is obvious. □

Theorem 4.6. In $(W, v, \mathfrak{I})$, $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I}) \subseteq \sigma_0$.

Proof. Let $H \in \dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I})$ in $(W, v, \mathfrak{I})$. Then, $A \subseteq \text{int}(\Gamma(\Psi_\Gamma(H))) \subseteq \text{int}(cl_\theta(\Psi_\Gamma(H)))$, from the Theorem 2.4 (iii). By the Lemma 2.3 (i), $H \subseteq \text{int}(cl_\theta(\Psi_\Gamma(H))) = \text{int}(cl(\Psi_\Gamma(H)))$ and so $H \in \sigma_0$. □

Remark 4.7. In an ideal topological space, neither a σ_0 -open set nor a $\Gamma - \Psi_\Gamma\text{-set}$ may be a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$. Moreover, neither a $\Psi_\Gamma - C$ set nor a σ -open set may be a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$.

Example 4.8. Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}, \{r\}, \{n, r\}\}$, $\ell = \{\emptyset, \{p\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$, the set $A = \{r\}$ is both σ -open and σ_0 -open, but it is not a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$. Furthermore, in $(W, v, \mathfrak{I})$, the set $B = \{m, n, p\}$ is a $\Gamma - \Psi_\Gamma\text{-set}$, but it is not a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$. In (W, v, ℓ) , the set $C = \{m, n\}$ is a $\Psi_\Gamma - C$ set, but it is not a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$.

Theorem 4.9. For $H \subseteq W$, in $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$;

- (i) if H is a σ -open set,
 - (ii) if H is a θ -open set,
 - (iii) if $W \setminus H$ is an $\mathfrak{I}_\Gamma$ -perfect set,
- then H is a $\text{pre-}\Gamma - \Psi_\Gamma\text{-set}$.

Proof. Let $H \subseteq W$ in $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$.

(i) Let H be a σ -open set. Then, $H \subseteq \Psi_\Gamma(H) \subseteq \Gamma(H)$ from the Theorem 2.6. Thus, $\Psi_\Gamma(H) \subseteq \Gamma(H) \subseteq \Gamma(\Psi_\Gamma(H))$ and so $H \subseteq \text{int}(\Psi_\Gamma(H)) = \Psi_\Gamma(H) \subseteq \text{int}(\Gamma(\Psi_\Gamma(H)))$.

(ii) The proof is obvious from (i) and the Corollary 2.8.

(iii) Let $W \setminus H$ be an $\mathfrak{I}_\Gamma$ -perfect set. Then, $H = \Psi_\Gamma(H)$ and so H is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set from (i). $\square$

Remark 4.10. In $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$, a $\text{pre-}\Gamma - \Psi_\Gamma$ -set may not be a θ -open set.

Example 4.11. Let $W = \{m, n, p, r\}, \mathfrak{I} = \{\emptyset, \{p\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$, the set $H = \{m, n, r\}$ is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set, but it is not a θ -open set.

Theorem 4.12. In $(W, v, \mathfrak{I})$, if a subset H of W is both σ -open and $\Gamma - \mathfrak{I}$ -open, then it is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Proof. Suppose that H is a σ -open and $\Gamma - \mathfrak{I}$ -open set in $(W, v, \mathfrak{I})$. Then, $\Gamma(H) \subseteq \Gamma(\Psi_\Gamma(H))$ and hence $H \subseteq \text{int}(\Gamma(H)) \subseteq \text{int}(\Gamma(\Psi_\Gamma(H)))$. $\square$

Theorem 4.13. In an ideal topological space $(W, v, \{\emptyset\})$, for $H \subseteq W$, if H is both an open set and a $\Psi_\Gamma - C$ set, then it is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Proof. Suppose that $H \in v$ and H is a $\Psi_\Gamma - C$ set in $(W, v, \{\emptyset\})$. Then, $H \subseteq cl(\Psi_\Gamma(H)) = cl_\theta(\Psi_\Gamma(H)) = \Gamma(\Psi_\Gamma(H))$. Therefore, $H \subseteq \text{int}(\Gamma(\Psi_\Gamma(H)))$. $\square$

Corollary 4.14. In an ideal topological space $(W, v, \{\emptyset\})$, for $H \subseteq W$, if H is both an open set and a regular θ -closed set, then H is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Proof. By the Corollary 2.8 and the Theorem 4.13, it is obvious. $\square$

Theorem 4.15. In $(W, v, \mathfrak{I})$, for $H \subseteq W$, if H is both an open set and a $\Gamma - \Psi_\Gamma$ -set, then it is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Proof. The proof is obvious. $\square$

Corollary 4.16. In $(W, v, \mathfrak{I})$, for $H \subseteq W$, if H is both a θ -open set and a $\Gamma - \Psi_\Gamma$ -set, then it is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Proof. Since $v_\theta \subseteq v$ and by the Theorem 4.15, it is obvious. $\square$

Corollary 4.17. In $(W, v, \mathfrak{I})$, if a subset H of W is both θ -open and $\mathfrak{I}_\Gamma$ -perfect, then it is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Proof. It is clear by the Corollary 3.24 and the Corollary 4.16. $\square$

Corollary 4.18. Let $H \subseteq W$ in $(W, v, \mathfrak{I})$. If H is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set and $\Psi_\Gamma(H)$ is closed, then H is σ -open and hence σ_0 -open.

Proof. By the Theorem 3.35 and the Theorem 4.4, the proof is clear. $\square$

Remark 4.19. In $(W, v, \mathfrak{I})$, although $\Psi_\Gamma(H)$ is closed and H is σ -open, a subset H of W may not be a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Example 4.20. In $(W, v, \mathfrak{I})$ in the Example 4.8, the set $B = \{m, n, p\}$ is σ -open and $\Psi_\Gamma(A)$ is closed, but it is not a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Theorem 4.21. In $(W, v, \mathfrak{I})$, for $H \subseteq W$, if H is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set and $\Psi_\Gamma(H)$ is a $\theta^\mathfrak{I}$ -closed set, then H is a σ -open set.

Proof. Let H be a $\text{pre-}\Gamma - \Psi_\Gamma$ -set and $\Psi_\Gamma(H)$ be a $\theta^\mathfrak{I}$ -closed set in $(W, v, \mathfrak{I})$. Then, $H \subseteq \text{int}(\Gamma(\Psi_\Gamma(H))) \subseteq \Gamma(\Psi_\Gamma(H)) \subseteq \Psi_\Gamma(H)$. As a result, H is a σ -open set. $\square$

Corollary 4.22. In $(W, v, \mathfrak{I})$, for $H \subseteq W$, if H is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set and $\Psi_\Gamma(H)$ is θ -closed, then $H \in \sigma$.

Proof. The proof is obvious from the Remark 2.9 and the Theorem 4.21. $\square$

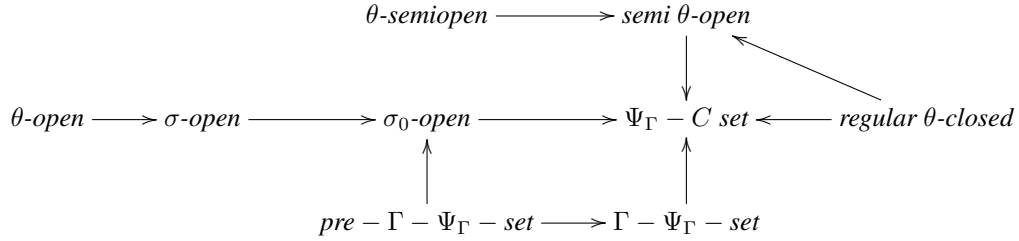
Theorem 4.23. Let $H \subseteq W$ in $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$. If H is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set, then it is Γ -dense-in-itself.

Proof. Let H be a $\text{pre-}\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$. Then, $H \subseteq \text{int}(\Gamma(\Psi_\Gamma(H))) \subseteq \Gamma(\Psi_\Gamma(H)) \subseteq cl_\theta(\Psi_\Gamma(H)) = cl(\Psi_\Gamma(H)) \subseteq cl(\Gamma(H)) = \Gamma(H)$. $\square$

Remark 4.24. In $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$, a Γ -dense-in-itself set may not be a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

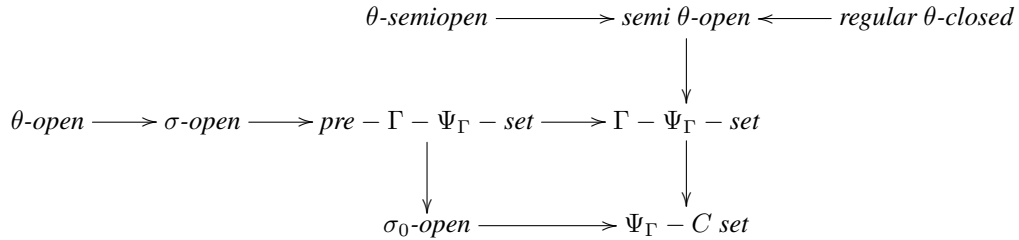
Example 4.25. In $(W, v, \mathfrak{I})$ in the Example 4.11, the set $C = \{m, p\}$ is Γ -dense-in-itself, but it is not a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Corollary 4.26. The following diagram is valid in an ideal topological space.



Proof. It is clear from the Corollary 2.8, the Theorem 3.9, the Theorem 4.6, and the Theorem 4.4. $\square$

Corollary 4.27. The following diagram is valid in $(W, v, \mathfrak{I})$ where $cl(v) \cap \mathfrak{I} = \{\emptyset\}$.



Proof. By the Corollary 4.26, the Theorem 4.9 and the Theorem 3.16, it is clear. $\square$

5. FURTHER PROPERTIES OF $\Gamma - \Psi_\Gamma$ -SETS AND $\text{pre-}\Gamma - \Psi_\Gamma$ -SETS

Theorem 5.1. *The set W is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$.*

Proof. Let $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$ in $(W, v, \mathfrak{I})$. Then, $\text{int}(\Gamma(\Psi_\Gamma(W))) = \text{int}(\Gamma(W)) = W$ by the Theorem 2.5. $\square$

Corollary 5.2. *The set W is a $\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$.*

Proof. By the Theorem 5.1 and the Theorem 4.4, the proof is clear. $\square$

Remark 5.3. *In $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$, the set W may not be a $\Gamma - \Psi_\Gamma$ -set and a $\text{pre-}\Gamma - \Psi_\Gamma$ -set. In the Example 3.18, W is neither a $\Gamma - \Psi_\Gamma$ -set nor a $\text{pre-}\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} \neq \{\emptyset\}$.*

Theorem 5.4. *If $\{H_\alpha : \alpha \in \Delta\}$ is a collection of nonempty $\Gamma - \Psi_\Gamma$ -sets, then $\bigcup_{\alpha \in \Delta} H_\alpha \in \Gamma - \Psi_\Gamma(W, v, \mathfrak{I})$ in $(W, v, \mathfrak{I})$.*

Proof. Let $\{H_\alpha : \alpha \in \Delta\}$ be a collection of nonempty $\Gamma - \Psi_\Gamma$ -sets in $(W, v, \mathfrak{I})$. Then, $\Psi_\Gamma(H_\alpha) \subseteq \Psi_\Gamma(\bigcup_{\alpha \in \Delta} H_\alpha)$ and so $\Gamma(\Psi_\Gamma(H_\alpha)) \subseteq \Gamma(\Psi_\Gamma(\bigcup_{\alpha \in \Delta} H_\alpha))$, for each $\alpha \in \Delta$. Since H_α is a $\Gamma - \Psi_\Gamma$ -set, $H_\alpha \subseteq \Gamma(\Psi_\Gamma(H_\alpha)) \subseteq \Gamma(\Psi_\Gamma(\bigcup_{\alpha \in \Delta} H_\alpha))$, for each $\alpha \in \Delta$. It implies that $\bigcup_{\alpha \in \Delta} H_\alpha \subseteq \Gamma(\Psi_\Gamma(\bigcup_{\alpha \in \Delta} H_\alpha))$. $\square$

Theorem 5.5. *If $\{H_\alpha : \alpha \in \Delta\}$ is a collection of nonempty $\text{pre-}\Gamma - \Psi_\Gamma$ -sets, then $\bigcup_{\alpha \in \Delta} H_\alpha \in \dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I})$ in $(W, v, \mathfrak{I})$.*

Proof. The proof is obvious. $\square$

Corollary 5.6. *In $(W, v, \mathfrak{I})$ where $\text{cl}(v) \cap \mathfrak{I} = \{\emptyset\}$, both $\Gamma - \Psi_\Gamma(W, v, \mathfrak{I})$ and $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I})$ form a supratopology on W .*

Proof. From the Theorem 5.1, the Corollary 5.2, the Theorem 5.4, and the Theorem 5.5, it is clear. $\square$

Remark 5.7. *The intersection of two $\Gamma - \Psi_\Gamma$ -sets need not be a $\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$. Even though $\mathfrak{I} = \{\emptyset\}$, the intersection of two $\Gamma - \Psi_\Gamma$ -sets need not be a $\Gamma - \Psi_\Gamma$ -set. As a result, the collection $\Gamma - \Psi_\Gamma(W, v, \mathfrak{I})$ may not be a topology on W .*

Example 5.8. *Let $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{p\}\}$, $\ell = \{\emptyset\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$. In both $(W, v, \mathfrak{I})$ and (W, v, ℓ) ; the set $H = \{n, r\}$ and the set $K = \{m, n, p\}$ are $\Gamma - \Psi_\Gamma$ -sets, but $H \cap K$ is not a $\Gamma - \Psi_\Gamma$ -set.*

Remark 5.9. *When $\mathfrak{I} = \{\emptyset\}$, then $\dot{\Gamma} - \Psi_\Gamma(W, v, \mathfrak{I}) = \sigma_0$ is a topology on W from the Proposition 4.3 (i). Therefore, if $\mathfrak{I} = \{\emptyset\}$, the intersection of two $\text{pre-}\Gamma - \Psi_\Gamma$ -sets is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$.*

On the other hand, when $\mathfrak{I} = \{\emptyset\}$, $\Gamma - \Psi_\Gamma(W, v, \mathfrak{I}) = \Psi_\Gamma(W, v, \mathfrak{I}) = \{H \subseteq W \mid H \text{ is semi } \theta\text{-open}\}$ may not be a topology on W by the Corollary 3.15 and the Example 5.8.

Remark 5.10. *In $(W, v, \mathfrak{I})$, a subset of a $\Gamma - \Psi_\Gamma$ -set need not be a $\Gamma - \Psi_\Gamma$ -set. In the Example 3.2, $A \subseteq B$ and B is a $\Gamma - \Psi_\Gamma$ -set, but A is not a $\Gamma - \Psi_\Gamma$ -set. Therefore, $\Gamma - \Psi_\Gamma$ -sets do not have a heredity property.*

Remark 5.11. A subset of a $\text{pre-}\Gamma - \Psi_\Gamma$ -set may not be a $\text{pre-}\Gamma - \Psi_\Gamma$ -set in an ideal topological space. For example, in $(W, v, \mathfrak{I})$ where $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{p\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$, the set $\{m, n, r\}$ is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set, but $\{m, n\}$ is not a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Theorem 5.12. The closure of a $\Gamma - \Psi_\Gamma$ -set is also a $\Gamma - \Psi_\Gamma$ -set in an ideal topological space.

Proof. Let H be a $\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$. We have $H \subseteq \Gamma(\Psi_\Gamma(H)) \subseteq \Gamma(\Psi_\Gamma(\text{cl}(H)))$. Then, by the Theorem 2.4 (iii), $\text{cl}(H) \subseteq \Gamma(\Psi_\Gamma(\text{cl}(H)))$. $\square$

Remark 5.13. Although a set is not a $\Gamma - \Psi_\Gamma$ -set, the closure of a set may be a $\Gamma - \Psi_\Gamma$ -set. Furthermore, the closure of a $\text{pre-}\Gamma - \Psi_\Gamma$ -set may not be a $\text{pre-}\Gamma - \Psi_\Gamma$ -set in an ideal topological space. For example, in $(W, v, \mathfrak{I})$ where $W = \{m, n, p, r\}$, $\mathfrak{I} = \{\emptyset, \{n\}, \{r\}, \{n, r\}\}$ and $v = \{\emptyset, \{r\}, \{m, p\}, \{m, p, r\}, W\}$, the set $\{m\}$ is not a $\Gamma - \Psi_\Gamma$ -set, the closure of $\{m\}$ is a $\Gamma - \Psi_\Gamma$ -set. Moreover, $\{m, p\}$ is a $\text{pre-}\Gamma - \Psi_\Gamma$ -set, but its closure $\{m, n, p\}$ is not a $\text{pre-}\Gamma - \Psi_\Gamma$ -set.

Theorem 5.14. If H is a $\Gamma - \Psi_\Gamma$ -set and U is θ -open, then $U \cap H$ is a $\Gamma - \Psi_\Gamma$ -set in $(W, v, \mathfrak{I})$.

Proof. Let H be a $\Gamma - \Psi_\Gamma$ -set and U be a θ -open set in $(W, v, \mathfrak{I})$. Then, $U \cap H \subseteq U \cap \Gamma(\Psi_\Gamma(H)) \subseteq \Gamma(U \cap \Psi_\Gamma(H))$, from the Lemma 2.3 (ii). By the Corollary 2.7, $U \cap H \subseteq \Gamma(\Psi_\Gamma(U) \cap \Psi_\Gamma(H)) = \Gamma(\Psi_\Gamma(U \cap H))$. $\square$

6. CONCLUSION

This paper is based on new types of sets called $\Gamma - \Psi_\Gamma$ -sets and $\text{pre-}\Gamma - \Psi_\Gamma$ -sets in ideal topological spaces via local closure function. Their relations are investigated with existing some special sets in the literature and the reverse requirements are explained with counter examples. The fundamental properties of these sets are researched.

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9. CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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