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Innovative Insights and Fractal-View of Phi-Four Equations with Time Fractal Derivative

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Abstract. In this work, a new analytical technique recognized as the homotopy perturbation method is utilized to identify the general solutions of Phi-4 nonlinear partial differential equations of fractional order with respect to time. The first order general solution is determined by HPM and then compared with the exact result. It can be seen that the result obtained using HPM shows a large convergence rate for Phi-4 partial differential equations with time fractal derivative. The exact graphic solutions have been shown, and their respective errors have also been represented using tables.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Homotopy perturbation method, Fractional order calculus, Fractal Phi-four equation with respect to time, Estimated solution, Caputo operator, Riemann-Liouville fractional integral operator.

1. Introduction

The nonlinear evolution equations have great importance due to their large applications. Because nonlinear inequalities play a key role in different fields of studies, especially in modern technological developments. Moreover, it is an integral part of various fields of learning, such as plasma, streaming, and movement of particles, biomedical, chemical kinetics, plasticity, optical physics, microchip technology, environmental sciences, and others. These are based on fractal systems and consequently on nonlinear inequalities. These nonlinear transformations are helpful in describing the behavior of physical systems, like the behavior of waves. Therefore, these nonlinear evolution equations are frequently used to explain oscillations and ripples. Moreover, there are different types of modeling appearing in real life problems which generate FPDEs: flexibility, flow of fluid, traffic flow, digitization signals and many other fields. Therefore, (Non-Linear Evolution Equations)

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NLEEs are very important and frequently used in various technological fields as discussed above. As we know that NLEEs are helpful to describe the behavior of wavy disturbances and oscillations in different models, that is why the trend of studying these wavy disturbances and oscillations is increasing slowly. Therefore, researchers have developed the latest techniques to explain the travelling waves by Tariq and Akram [25], Akter and Akber [1], Razazadeh et al. [22]. A nonlinear partial differential equation known as the Phi-4 equation is defined as

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + a^2 u + bu^3 = 0. \tag{1.1}$$

Where a and b represent real numbers and x, t are space and time variables. Recently, partial differential Phi-4 equations are helpful to describe the fundamentals of high-energy and nuclear physics.

A Phi-4 partial differential equation of fractional order with respect to time is described in the following way:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + a^2 u + bu^3 = 0, \tag{1.2}$$

$$x \in [0,1], \qquad t > 0, \qquad a > 0, \qquad b > 0 \qquad 1 < \alpha \leqslant 2.$$

Where $\frac{\partial^\alpha}{\partial t^\alpha}$ represents the fractional order derivative operator known as Caputo fractional order derivative operator, and $u(x,t)$ shows the normalized propagation of distance and retracted time t.

Basically, defining the real number powers of the differentiation and integration operators enhanced calculus and developed a calculus for such operators which generalized the classical one known as fractional calculus. L'Hospital was the first who developed the concept of the one-half order derivative. Recently, fractional calculus has made great progress due to its appealing applications in the universe. Oldham and Spanier [18] in 1974 studied its theoretical explanation briefly and elaborately. Afterward, Ross and Miller [16] in 1993 and Podlubny [19] in 1999 played their role in expanding fractional calculus. In the last decades, several mathematicians and researchers have noticed that fractional calculus has a key role in describing the characteristics of natural and physical phenomena. Moreover, relative study of classical and fractional representations has been reached to the conclusion that fractional models are more authentic than classical representations. Presently, many mathematicians have developed new techniques to determine accurate solutions of NLEEs like two-dimensional linear voltra integral equations of the first kind of HPM in Eslami and Mirzazadeh [6] in 2014, exp function method [11] by Khan and Akbar in 2014, and extended trial equation method [4] by Demiray and Bulut in 2015, modified extended tanh function method [29] by Zahran and Khater in 2016, the generalized Kudryashov method [5] by Tuluce Demiray and Bulut in 2017. Furthermore, a new method named as optimal homotopy asymptotic method was also introduced by Marinca and Herisanu [13]-[15]. Many researchers like Iqbal et al. [7]-[9], Alkhalaf [2], Sawer and Rashidi [23], Sawer et al. [24] have proved the accuracy of this method.

Here, the procedure of the latest technique called the Homotopy perturbation method is discussed. The concept of homotopy and perturbation method has been united to find the solution of nonlinear equations. Liao (1992) used the homotopy analysis method [12] to his foundation work. He was the first, who gave the idea of the homotopy perturbation method in (1998). This technique also plays a pivotal role in many branches of science and engineering.

Various researchers [26],[3],[10],[30],[17],[27],[21],[20],[28] have delved deeper into this theory, as well as its applications. Now, the solution of linear and nonlinear fractional order partial differential equations has been obtained using HPM.

The arrangement of this article contains the following sections. Some basic definitions of fractional calculus are described in section 2, whereas section 3 is specified for stability convergence. Moreover, model problems are exhibited in section 4, whereas the residual and flow rate of Phi-4 Equations with time fractal derivative are described in section 5. Furthermore, the consequences and analysis with physical understanding are summarized in section 6, and lastly, section 7 is specified for conclusion.

2. Fundamental Definitions of Fractal Derivatives and Integral

A real valued function $f(s)$, where $s > 0$ belong to a real space C_μ, where $\mu \in \Re$, if there is at least one real number $p > \mu$ to such an extent that $f(s) = s^p h_1(s)$, but then $f_1(s) \in C(0, \infty)$, and it lies in real space C_μ^m if and only if $f^m \in C_\mu$, $m \in N$.

2.1. **Definition.** The Riemann-Liouville fractional integral operator of a function $g(t) \in C_\lambda$ (function space), $\lambda \in \mathcal{R}$ of fractional order $\alpha > 0$ is as follow:

$$I_{a,x}^{-\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\mu)^{\alpha-1}\, g(\mu)\, d\mu, \qquad x > \alpha.$$

$$k-1 < \alpha < K, \qquad K \in Z^+$$

Where Γ denotes gamma function.

2.2. **Definition.** The fractional order $\alpha > 0$, Caputo fractional derivative operator of $f(x)$ is as follows:

$$D_{a,s}^{\alpha} f(x) = \frac{1}{\Gamma(k-\alpha)} \int_a^s (x-\mu)^{\,m-\alpha-1} f^{(m)}(\mu) d\mu.$$

$$x > \alpha, \qquad k-1 < \alpha < k, \qquad m \in \mathbb{Z}^+, \qquad s > 0.$$

2.3. **Definition.** The Gamma function, which is basically the generalized form of the factorial for all real numbers, is defined as:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x \in R^+.$$

Some important formulae are obtained by using recursion relations, which are some unique properties of the Gamma function.

$$\Gamma(x+1) = x\,\Gamma(x), \qquad x \in R^+$$

$$\Gamma(x+1) = x!, \qquad x \in \mathbb{N}.$$

For non-integer values:

$$\Gamma(1/2) = \sqrt{\pi}$$

3. Stability Analysis

Considering the stability analysis of the fractional nonlinear partial differential equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + a^2 u + b u^3 = 0, \tag{3.3}$$

$$x \in [0,1], \qquad t > 0, \qquad a > 0, \qquad b > 0 \qquad 1 < \alpha \leqslant 2.$$

with:

- $x \in [0,1]$,
- $t > 0$,
- Parameters: $a > 0, \qquad b > 0$,
- Time-fractional derivative of order $1 < \alpha \leq 2$.

This is a nonlinear, time-fractional reaction-diffusion equation. We'll carry out a linear stability analysis about the trivial steady state $u(x,t) = 0$.

Here's the step-by-step outline:

Step 1: Identify the Steady States

Steady states satisfy:

$$-\frac{\partial^2 u(x,t)}{\partial x^2} + a^2 u + b u^3 = 0, \tag{3.4}$$

Let's consider small perturbations around the zero steady state, $u = 0$.

Clearly, $u = 0$ is a solution.

Step 2: Linearization Around u=0

Neglect the nonlinear term bu^3. The equation becomes:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - a^2 u(x,t). \tag{3.5}$$

This is a linear fractional PDE.

Step 3: Modal (Eigen Function) Expansion

Use separation of variables and assume homogeneous boundary conditions e.g., Dirichlet: $u(0,t) = u(1,t) = 0$. Try:

$$u(x,t) = \sum_{n=1}^{\infty} U_n(t)\, \sin(n\pi x), \tag{3.6}$$

Plug into the equation. For each mode:

$$\frac{\partial^\alpha u_n(t)}{\partial t^\alpha} = -\left[(n\pi)^2 + a^2\right] u_n(t). \tag{3.7}$$

This is a fractional-order ordinary differential equation (FODE) for each mode.

Step 4: Solution to the FODE

The solution to:

$$\frac{\partial^\alpha y(t)}{\partial t^\alpha} = -\lambda y(t), \qquad y(0) = y_0, \qquad y'(0) = 0, \tag{3.8}$$

with $1 < \alpha \leq 2$, is given by the Mittag-Leffler function $E_\alpha(z)$.

So, the solution becomes:

$$u_n(t) = u_n(0) E_\alpha\left[-\left\{(n\pi)^2 + a^2\right\} t^\alpha\right]. \tag{3.9}$$

Step 5: Stability Criterion

Properties of the Mittag-Leffler function:

- $E_\alpha(-\lambda t^\alpha) \to 0$ as $t \to \infty$, and $\alpha \in (1, 2]$
- This implies asymptotic stability of the linearized system.

Since $(n\pi)^2 + a^2 > 0$, the argument of the Mittag-Leffler function is strictly negative, so all modes decay.

Step 6: Nonlinear Stability (Energy Method Sketch)

We now consider the nonlinear term bu^3, and study Lyapunov-type stability.

Let's define an energy-like functional:

$$E(t) = \frac{1}{2}\int_0^1 u^2(x,t)\,dx. \tag{3.10}$$

Multiply the original equation by u and integrate over $[0, 1]$:

$$\int_0^1 u\frac{\partial^\alpha u}{\partial t^\alpha}dx - \int_0^1 u\frac{\partial^2 u}{\partial x^2}dx + a^2\int_0^1 u^2dx + b\int_0^1 u^4dx = 0, \tag{3.11}$$

Use:

- $\displaystyle\int_0^1 u\,u_{xx}\,dx = -\int_0^1 (u_x)^2\,dx,$
- The first term is trickier due to the fractional derivative, but for Caputo derivative, there's a result:

$$\int_0^1 u(x,t)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}dx = \frac{1}{2}\frac{d^\alpha}{dt^\alpha}\|u(x,t)\|^2 \tag{3.12}$$

So we get:

$$\frac{1}{2}\frac{d^\alpha}{dt^\alpha}\|u\|^2 + \|u_x\|^2 + a^2\|u\|^2 + b\int u^4dx = 0. \tag{3.13}$$

All terms are non-negative, implying:

$$\frac{d^\alpha}{dt^\alpha}\|u\|^2 \le 0. \tag{3.14}$$

This implies energy decay, so $u = 0$ is nonlinearly stable as well. We conclude that the zero steady state $u = 0$ is linearly asymptotically stable for $1 < \alpha \le 2, b > 0$. Moreover, energy method suggests nonlinear stability as well and faster decay as $\alpha \to 2$, slower as $\alpha \to 1^+$.

4. Applications of Phi-4 Partial Differential Equation of Fractional Order

The following section of presentation will demonstrate the application of HPM to solve two specific models. This approach helps to showcase the validity, accuracy, and suitability of the computational scheme in providing solutions to real-world problems and further reinforces its trustworthiness as a computational method.

4.1. **Application -1.** Consider the fractal Phi-four partial differential equation [27]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u + u^3 = 0, \tag{4.15}$$

$$x \in [0, 1.28], \qquad t > 0, \qquad a > 0, \qquad b > 0 \qquad 1 < \alpha \leqslant 2.$$

Initial conditions along boundary values of fractal Phi-four partial differential equation are:

$$u(x,0) = A\left[1 + \frac{\cos(2\pi x)}{1.28}\right], \qquad u_t(x,0) = 0. \tag{4.16}$$

Solution:

Solve this equation using HPM

$$(1-p)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + p\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u + u^3\right] = 0, \tag{4.17}$$

Now comparing the terms with similar powers of p, we obtain the following system of equations

$$p^0: \qquad \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0, \tag{4.18}$$

$$p^1: \qquad \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} - \frac{\partial^2 u_0(x,t)}{\partial x^2} + u_0 + u_0^3 = 0, \tag{4.19}$$

$\vdots$

Approximate Solution Series of 0^{th} order and 1^{st} order

$$u_0(x,t) = A\left[1 + \frac{\cos(2\pi x)}{1.28}\right] \tag{4.20}$$

$$u_1(x,t) = \frac{-t^\alpha A\left[\left(\frac{2\pi}{1.28}\right)^2 \cos\left(\frac{2\pi x}{1.28}\right) + \left(1 + \cos\left(\frac{2\pi x}{1.28}\right)\right) + A^2\left(1 + \cos\left(\frac{2\pi x}{1.28}\right)\right)^3\right]}{\Gamma(1+\alpha)} \tag{4.21}$$

For

$$M = \frac{2\pi}{1.28}, \qquad N = \frac{2\pi x}{1.28},$$

$$P = \cos\left(\frac{2\pi x}{1.28}\right), \qquad Q = 1 + \cos\left(\frac{2\pi x}{1.28}\right)$$

$$u_2(x,t) = \frac{t^{2\alpha} A\left[M^2(1+M)^2 P + 3A^2M^2Q^2(3P-2) + \left(1+M^2\right)Q + A^2Q^3\right]}{\Gamma(1+2\alpha) + A^2\left[(1+M^2)Q + 3A^2Q^3\Gamma(1+4\alpha)\right]} \tag{4.22}$$

Therefore

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) \tag{4.23}$$

The exact solution of the problem-1 in a closed form is:

$$u(x,t) = A\left\{1 + \cos\left(\frac{2\pi x}{1.28}\right)\right\} e^{\left(-\frac{t}{2}\right)}\left\{\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\right\} \tag{4.24}$$

4.2. **Application-2.** Consider the fractal Phi-four partial differential equation [27]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} - u + u^3 = 0, \tag{4.25}$$

$$x \in [0,1], \qquad t > 0, \qquad 1 < \alpha \leqslant 2.$$

Initial conditions along boundary conditions of Phi-4 partial differential equation of fractional order are:

$$u(x,0) = 0, \qquad u_t(x,0) = x.$$

Solution:
Solve this equation using HPM

$$(1-p)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + p\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} - u + u^3\right] = 0 \tag{4.26}$$

Now comparing the terms with like powers of p, we have the following system of equations

$$p^0: \quad \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \tag{4.27}$$

$$p^1: \quad \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} - \frac{\partial^2 u_0(x,t)}{\partial x^2} - u_0 + u_0^3 = 0, \tag{4.28}$$

$$p^2: \quad \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} - \frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1 + u_1^3 = 0. \tag{4.29}$$

$\vdots$

Approximate Solution Series of 0^{th} order and 1^{st} order

$$u_0(x,t) = tx, \tag{4.30}$$

$$u_1(x,t) = \frac{1}{\Gamma(4+\alpha)}\, t^{\alpha+1} x\left[(2+\alpha)(3+\alpha) - 6t^2x^2\right], \tag{4.31}$$

$$u(x,t) = u_0(x,t) + u_1(x,t), \tag{4.32}$$

$$u(x,t) = tx + \frac{1}{\Gamma(4+\alpha)}\, t^{\alpha+1} x\left[(2+\alpha)(3+\alpha) - 6t^2x^2\right]. \tag{4.33}$$

5. The Residual

5.1. **The Residual of Phi-4 Equations with Time Fractal Derivative.** In the context of the governing equations, the residual error computes how the solution successfully regulates these equations. A minor residual error symbolized an appropriate match. To these equations, showing an effective and powerful analytical technique in solving the problem.

$$R_1 = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u + u^3, \tag{5.34}$$

$$R_2 = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} - u + u^3. \tag{5.35}$$

5.2. **Flow Rate of Phi-4 Equations with Time Fractal Derivative.**

$$Q_1 = \int_0^1 u_1(x,t)\,dx, \quad \text{Let } u_1 = u \text{ for Application 1.} \tag{5.36}$$

$$Q_2 = \int_0^1 u_2(x,t)\,dx, \quad \text{Let } u_2 = u \text{ for Application 2.} \tag{5.37}$$

$$Q_1 = 0.800196 + \frac{252.917}{\Gamma(2+2\alpha)} + \frac{2.30784}{\Gamma(2+\alpha)} + \frac{1592.81\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)^3\Gamma(2+4\alpha)} \tag{5.38}$$

$$Q_2 = \frac{1}{4} + \frac{9+\alpha(7+\alpha)}{2\Gamma(5+\alpha)} \tag{5.39}$$

The fractional Phi-4's average values $\overline{u_1}$ and $\overline{v_1}$ are determined by

$$\overline{u_1} = Q_1 \qquad \text{and} \qquad \overline{v_1} = Q_2$$

The graphical representation of the above results

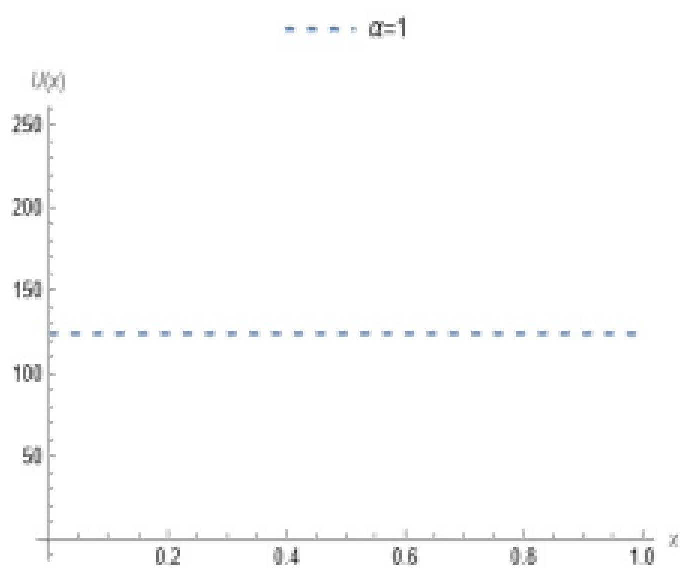

FIGURE 1. Fig. A of application 1.

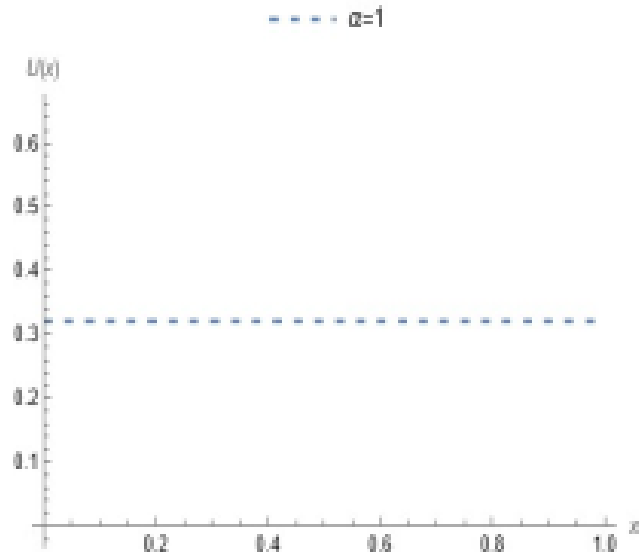

FIGURE 2. Fig. B of application 2.

6. CONSEQUENCES AND ANALYSIS WITH PHYSICAL UNDERSTANDING

Physical phenomena are exhibited by different types of traveling wave solutions. The solution of the Phi-4 partial differential equation of fractional order with respect to time exhibits complexity in nature. If the physical parameters vary, the fact $| u(x,t) |$ characterizes the wave solution becomes explicit when physical parameters take on special values. The important factor whose presence affects the Phi-4 equation solution is the dispersion term. The dispersion term wastes the solitary effect of waves if it is linear, while the formulation of a solitary wave is averted by a nonlinear form of the dispersion term due to transferring pulse energy to a higher level, which is used in the above equation. Interestingly, the propagating term u_{xx} and nonlinear term u^3 are both present in the Phi-4 time fractional order equation, which makes the problem effective and attractive when they are associated with other solutions. Since there are many solutions to model Phi-4 equations which are discussed in the last section. We determine the solution of our model problems using HPM. Section 4 explains the formulation in the model examples, which gives the exact solution without spatial discretization for the problem. When we use HPM, we do not need to find higher order solutions because the solutions of nonlinear Phi-4 partial differential equations of fractional order with respect to time are unique and beyond the bounds of this article.

Distinct values of α, $\alpha = 1.50$, $\alpha = 1.75$, and $\alpha = 2$ are shown in the tables table 1 and 2 of application (1) and (2), which show the order of solution whereas estimated results, exact results and difference between estimated and true solution of application (1) for distinct substitutions of α at fixed value of A=1, time t=0.001 and errors of application (1) have shown in table 1. The result obtained by solving the fractal Phi-four equation of application (2) using the latest HPM are shown is tables 3, 4 and 5 with three other methods

(Adomian Decomposition Technique, Homotopy Analysis Technique, Optimal Homotopy Asymptotic Approach) (Ehsaani and Ehsani 2013) side by side, which also shows exactness authenticity of the method.

Accurate results, 3D and 2D representations of model-1 for a fixed value of $\alpha = 2$ are shown by figures 1 and 2, whereas figures 3, 4 show the 3D and 2D representations of the estimated solution at $\alpha = 2$. In addition, 3D representations of estimated results for distinct substitutions of α, $\alpha = 1.50$, $\alpha = 1.75$, and $\alpha = 2$ are shown by figure 5.Moreover, figure 6 shows 2D graphical representation of results for distinct substitutions of α, $\alpha = 1.50$, $\alpha = 1.75$, and $\alpha = 2$ with corresponding values of A, $A = 1.3$, $A = 1.5$, $A = 1.8$ respectively. Figures 7, 9, 11 show a 3D representation of application-2 for different values of α, $\alpha = 1.50$, $\alpha = 1.75$, and $\alpha = 2$ respectively, and 3D combine results for different values of α, $\alpha = 1.50$, $\alpha = 1.75$, and $\alpha = 2$ is shown by figure 13. Moreover, figures 8, 10, 12 show 2D representation of application (2) for distinct values of α, $\alpha = 1.50$, $\alpha = 1.75$, and $\alpha = 2$ respectively, whereas the 2D combine representation of application-2 for different values of α, $\alpha = 1.50$, $\alpha = 1.75$, and $\alpha = 2$ is shown by figure 14. It is cleared from all the figures that the results obtained by HPM are more accurate, valid and consistent. Moreover, the solutions maintain its accuracy and relevance across the entire domain of interest. In addition, table 2 shows how the approximate solutions and exact solutions are closed to each other and the remaining tables represent the remaining tables represent the validity and consistency of HPM.

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Graphical and Tabular Representation of Application 1

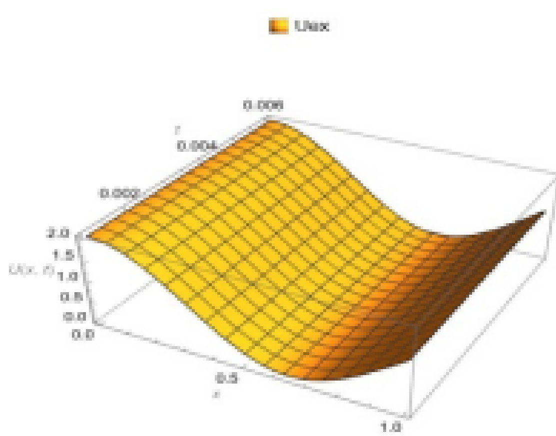

FIGURE 3. 3D graphical behavior of exact solution corresponding to the value at $\alpha = 2$.

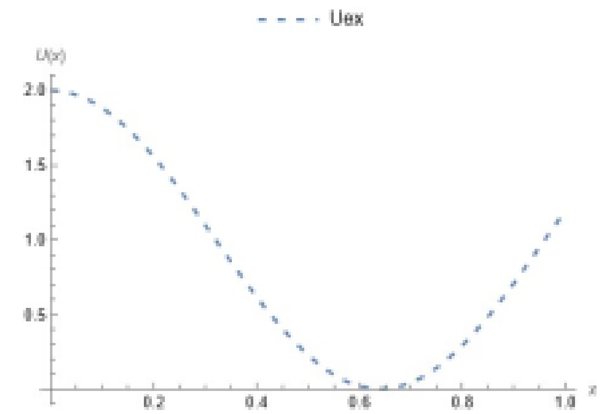

FIGURE 4. 2D graphical behavior of exact solution corresponding to the value at $\alpha = 2$.

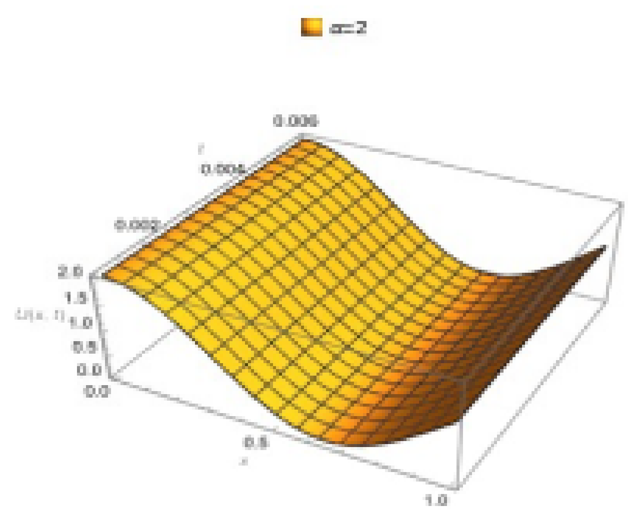

FIGURE 5. 3D graphical behavior of estimated solution corresponding to the value at $\alpha = 2$.

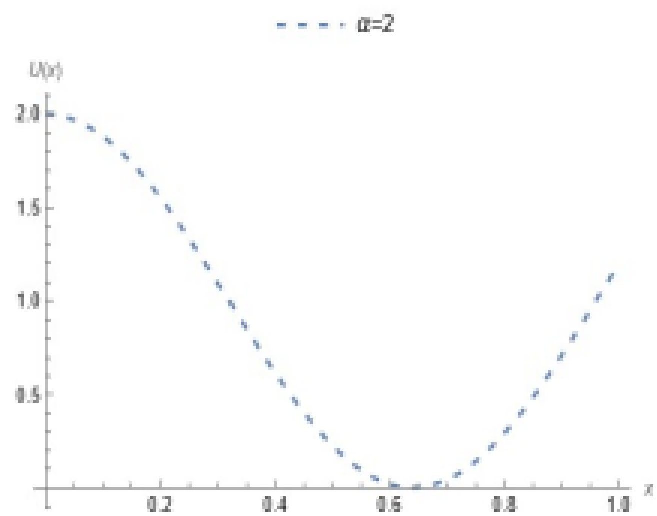

FIGURE 6. 2D graphical behavior of estimated solution corresponding to the value at $\alpha = 2$.

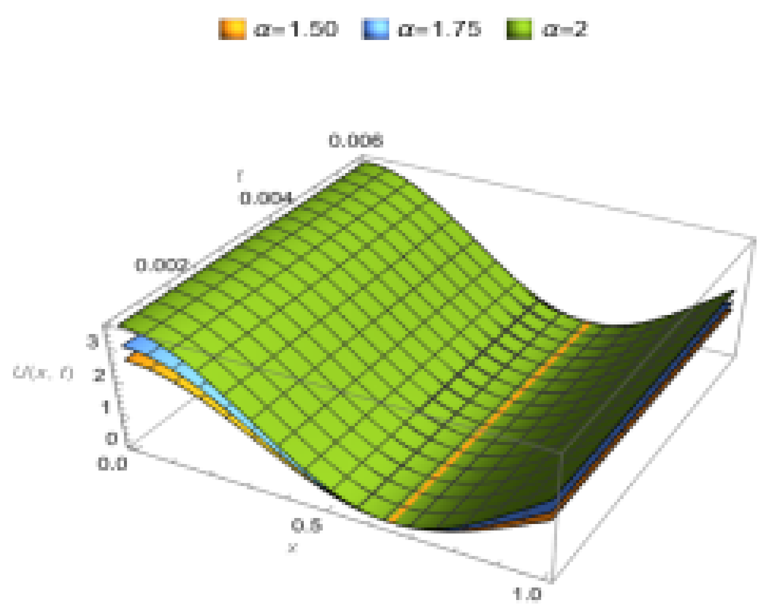

FIGURE 7. 3D graph of estimated solution at distinct values of $\alpha = 1.5, 1.75, 2$.

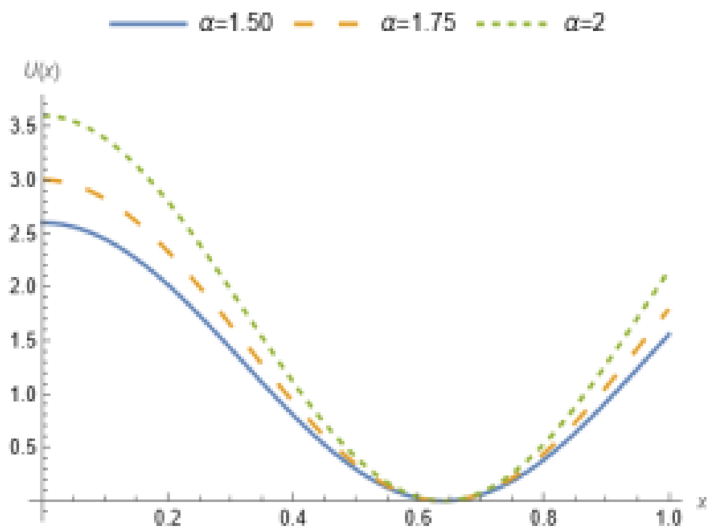

FIGURE 8. 2D graph of estimated solution at distinct values of $\alpha = 1.5, 1.75, 2$.

TABLE 1. Shows the solutions and absolute error of application-1 for various values of α.

x	$\alpha = 1.50$	$\alpha = 1.70$	$\alpha = 2$	Exact	Error
0.0	1.9991894	1.9998808	1.9998929	2	0.0008096
0.1	1.8812127	1.8818171	1.8819064	1.8819203	0.0007077
0.2	1.5555125	1.5555084	1.5555699	1.5555695	0.0000443
0.3	1.0979033	1.0980004	1.0980147	1.0980168	0.0001113
0.4	0.6175157	0.6173458	0.6173208	0.6173163	0.0001994
0.5	0.2274252	0.2270538	0.2270047	0.2269969	0.0003475
0.6	0.0197764	0.0192793	0.0192265	0.0192174	0.0005617
0.7	0.04362071	0.0431401	0.0430712	0.0430596	0.0005475
0.8	0.2932910	0.2929517	0.2929016	0.2928931	0.0003979
0.9	0.7098564	0.7097360	0.7097183	0.7097150	0.0001414
1.0	1.1949094	1.1950637	1.1950865	1.1950897	0.0001803

Graphical and Tabular Representation of Application 2

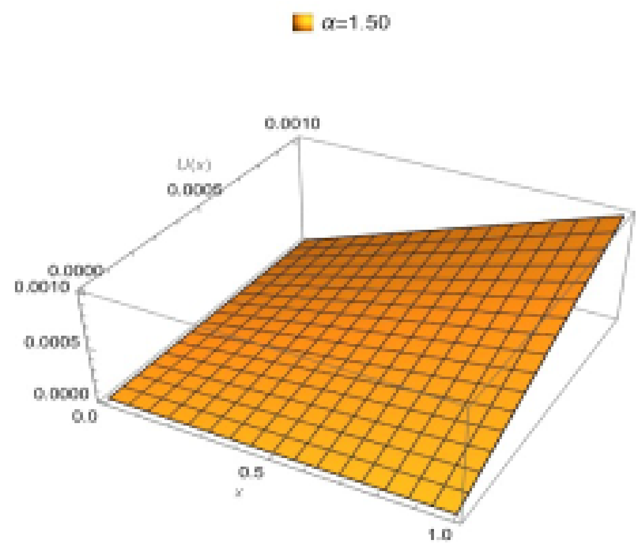

FIGURE 9. 3D graph of estimated solution for the value of $\alpha = 1.50$.

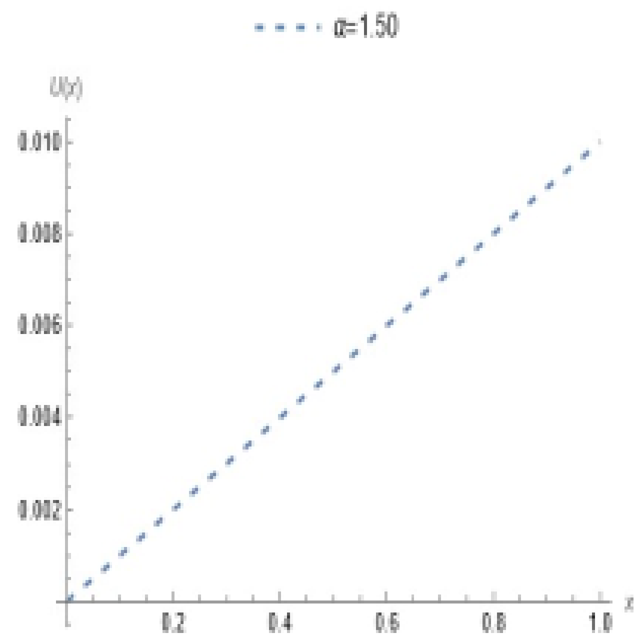

FIGURE 10. 2D graph of estimated solution for the value of $\alpha = 1.50$.

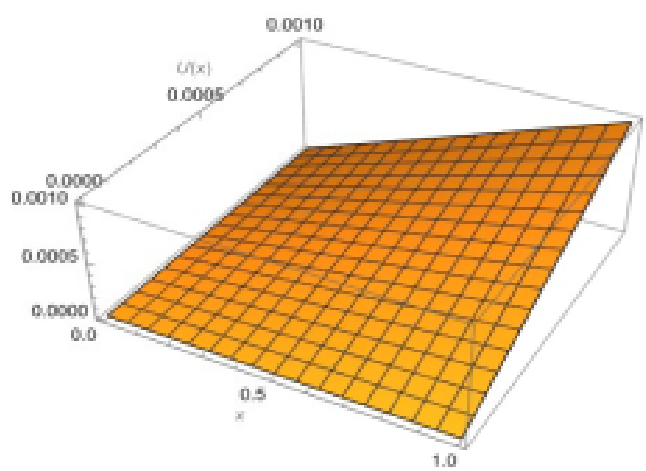

FIGURE 11. 3D graph of estimated solution for the value of $\alpha = 1.75$.

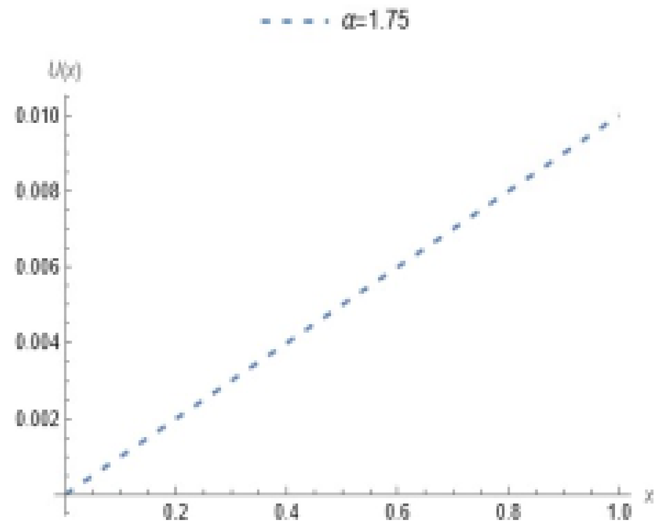

FIGURE 12. 2D graph of estimated solution for the value of $\alpha = 1.75$.

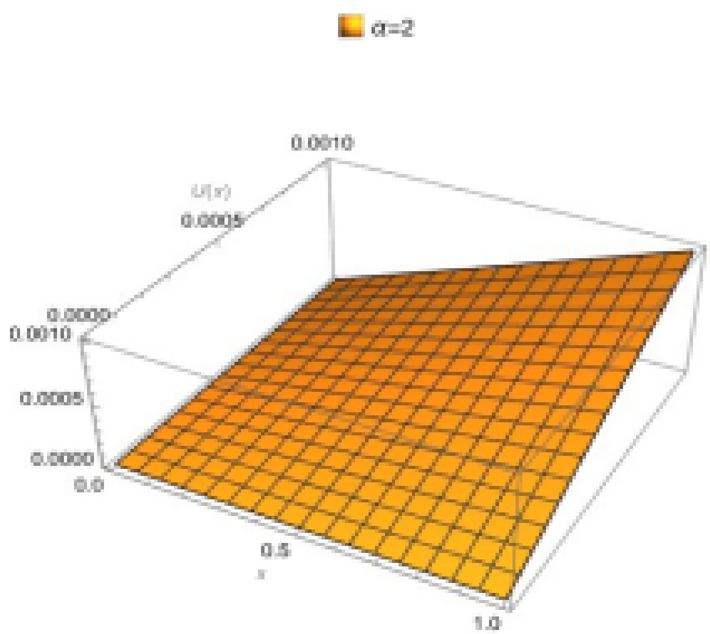

FIGURE 13. 3D graph of estimated solution for the value of $\alpha = 2$.

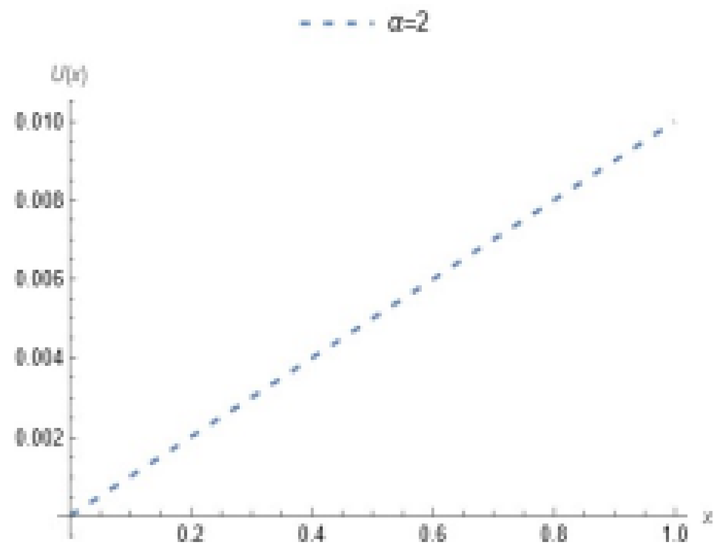

FIGURE 14. 2D graph of estimated solution for the value of $\alpha = 2$.

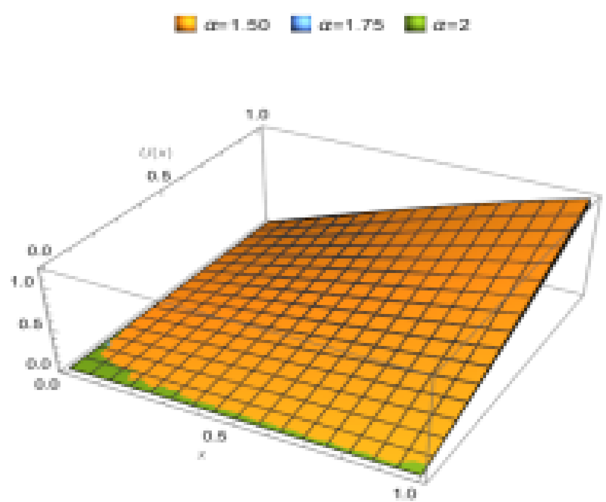

FIGURE 15. 3D graph of estimated solution for various values of α.

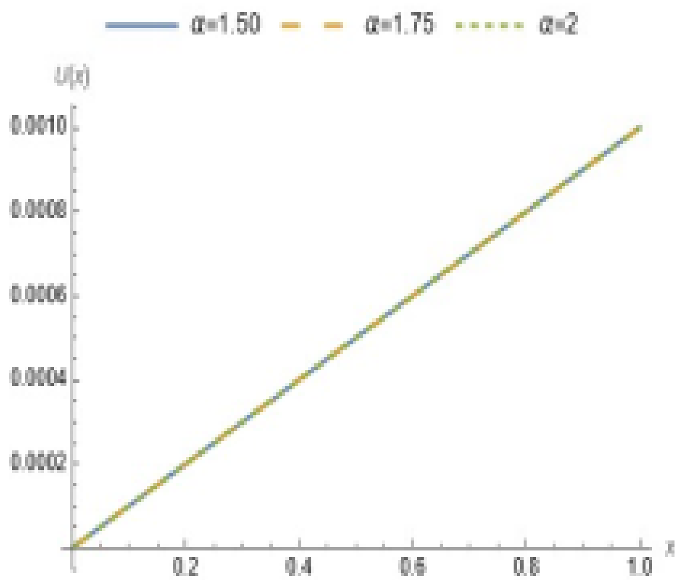

FIGURE 16. 2D graph of estimated solution for various values of α.

TABLE 2. Shows the solutions of application-2 for various values of α.

x	$\alpha = 1.50$	$\alpha = 1.70$	$\alpha = 2$
0.0	0	0	0
0.1	1.0003×10^{-3}	1.0001×10^{-3}	1.0000×10^{-3}
0.2	2.0006×10^{-3}	2.0001×10^{-3}	2.0000×10^{-3}
0.3	3.0009×10^{-3}	3.0002×10^{-3}	3.0000×10^{-3}
0.4	4.0012×10^{-3}	4.0003×10^{-3}	4.0001×10^{-3}
0.5	5.0015×10^{-3}	5.0004×10^{-3}	5.0001×10^{-3}
0.6	6.0018×10^{-3}	6.0004×10^{-3}	6.0001×10^{-3}
0.7	7.0021×10^{-3}	7.0005×10^{-3}	7.0001×10^{-3}
0.8	8.0024×10^{-3}	8.0006×10^{-3}	8.0001×10^{-3}
0.9	9.0027×10^{-3}	9.0006×10^{-3}	9.0001×10^{-3}
1.0	1.00030×10^{-2}	1.00007×10^{-2}	1.00002×10^{-2}

Comparative Tables for Homotopy Perturbation Method (HPM)

TABLE 3. Comparison of HPM solutions with other methods for $x = 0.3$ at $\alpha = 2$.

t	HPM Solution	ADM Solution	HAM Solution	OHAM Solution
0.006	1.80001×10^{-3}	1.80001×10^{-3}	1.80001×10^{-3}	1.80001×10^{-3}
0.01	3.00005×10^{-3}	3.00005×10^{-3}	3.00004×10^{-3}	3.00005×10^{-3}
0.05	1.50063×10^{-2}	1.50063×10^{-2}	1.50063×10^{-2}	1.50063×10^{-2}
0.1	3.005×10^{-2}	3.005×10^{-2}	3.00409×10^{-2}	3.005×10^{-2}
0.15	4.51688×10^{-2}	3.98512×10^{-2}	4.51378×10^{-2}	4.5169×10^{-2}
0.2	6.04003×10^{-2}	6.04007×10^{-2}	6.03265×10^{-2}	6.0400×10^{-2}
0.25	7.57822×10^{-2}	7.57823×10^{-2}	7.576374×10^{-2}	7.57799×10^{-2}

TABLE 4. Comparison of HPM solutions with other methods for $x = 0.6$ at $\alpha = 2$.

t	HPM Solution	ADM Solution	HAM Solution	OHAM Solution
0.006	3.60002×10^{-3}	3.60002×10^{-3}	3.60002×10^{-3}	3.60002×10^{-3}
0.01	6.0001×10^{-3}	6.0001×10^{-3}	6.0008×10^{-3}	6.0001×10^{-3}
0.05	3.00125×10^{-2}	3.00125×10^{-2}	3.00102×10^{-2}	3.00125×10^{-2}
0.1	6.00999×10^{-2}	6.01×10^{-2}	6.00816×10^{-2}	6.00999×10^{-2}
0.15	9.03371×10^{-2}	9.03374×10^{-2}	9.02752×10^{-2}	9.03370×10^{-2}
0.2	1.20798×10^{-1}	1.20799×10^{-1}	1.20651×10^{-1}	1.20797×10^{-1}
0.25	1.51557×10^{-1}	1.51558×10^{-1}	1.51268×10^{-1}	1.51552×10^{-1}

TABLE 5. Comparison of HPM solutions with other methods for $x = 1.3$ at $\alpha = 2$.

t	ADM	HAM	OHAM	HPM
0.006	7.80005×10^{-3}	7.80005×10^{-3}	7.80004×10^{-3}	7.80005×10^{-3}
0.01	1.30002×10^{-2}	1.30002×10^{-2}	1.30002×10^{-2}	1.30002×10^{-2}
0.05	6.50271×10^{-2}	6.50271×10^{-2}	6.50221×10^{-2}	6.50271×10^{-2}
0.1	1.30216×10^{-1}	1.30216×10^{-1}	1.30176×10^{-1}	1.30216×10^{-1}
0.15	1.95724×10^{-1}	1.95724×10^{-1}	1.95591×10^{-1}	1.95722×10^{-1}
0.2	2.61701×10^{-1}	2.61703×10^{-1}	2.61388×10^{-1}	2.61698×10^{-1}
0.25	3.28288×10^{-1}	3.28290×10^{-1}	3.27679×10^{-1}	3.28278×10^{-1}

7. CONCLUSION

The new progressive wave solution to an important nonlinear chromatic dispersion equation known as Phi-4 partial differential equation of fractal order with respect to time has been determined using a novel technique. The models discussed above have a vital role when analyzing different physical phenomena which are applied to various kinds of nonlinear applications in the natural sciences. Moreover, the tedious algebraic calculations involved in the above models have been handled using *Mathematica*. The consequences of this technique represent the excellent bond between approximate solutions and exact solutions. Since HPM is promptly focused, it demonstrates strong potential and capability for finding solutions to nonlinear real-life problems.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Rashida Qayyum Khan and H. M. Younas: Conceptualization, methodology, writing-original draft preparation **Rashida Qayyum Khan:** software, validation, visualization **Shaukat Ali Tahir and H. M. Younas:** formal analysis, investigation, writing review, and editing.
All authors have read and agreed to the published version of the manuscript.

DECLARATION

Conflict of Interest:The authors declare no conflict of interest.

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