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Numerical Investigation with Stability Assessment of Semi-Analytical Scheme for Time-Fractional Order Heat Type Emden-Fowler Equations

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Abstract. In the present paper, time-fractional order linear and nonlinear heat type Emden-Fowler equations are reformulated from existing classical equations by applying Caputo-Fabrizio time-fractional derivative. Then, a semi-analytical scheme, that is an amalgamation of Laplace transformation and Picard's iterative technique, is exploited to simulate singular initial value problems for corresponding time-fractional order heat type Emden-Fowler equations. Further, the stability of developed scheme is also assessed by exploiting R-stable mapping and Banach contraction principle. Numerical results, error estimation, and comparison of obtained results with exact solutions are presented through graphs and tables to exhibit the efficiency of time-fractional order derivative and implemented semi-analytical scheme.

AMS (MOS) Subject Classification Codes: 26A33; 65M12; 46N20; 35Q99
Key Words: Heat type Emden-Fowler equation; Caputo-Fabrizio time-fractional derivative; Picard's iterative scheme; Stability assessment; Error estimation.

1. Introduction

Fractional partial differential equations emanate as the result of various real-life events in the fields of applied mathematics and science in general [12]. Some of these include heat transfer models, diffusion processes, fluid mechanics, damping laws, electrical circuits, soliton theory and chaos theory, etc [1, 2, 21, 25, 27]. These models are more realistic than classical ones and are taken into account because they also illustrate the succeeding states of a dynamical system rather than only present state of the corresponding system. The main benefit of using fractional differential operators rather than classical ones is their non-local property that helps to describe the complicated dynamical systems. Another benefit of implementing fractional differential operators is that fractional order models give

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more realistic results and interpretation of these results gives more realistic information than classical ones [11, 20, 26].

Considerable fractional order derivatives and their associated integrals are defined in the fractional calculus [30]. Most implemented of them are: Caputo [6], Caputo-Fabrizio [13], Atangana-Baleanu [25], and Riemann-Lioville [27] fractional order derivatives. In this paper, time-fractional order models are reformulated from corresponding existing classical ones by exploiting Caputo-Fabrizio time-fractional derivative (CFTFD) that is direct consequence of Caputo time-fractional derivative (CTFD). Definition of CTFD involves singular kernel which results in simulation complexities while definition of CFTFD includes non-singular kernel which results in reduction of simulation complexities of the corresponding fractional order models [7]. Due to this important feature, CFTFD is implemented here.

Consider the most general singular initial value problem (IVP) for heat type Emden-Fowler equation:

$$\begin{aligned} &\partial_t \Theta(x, t) = \partial_{xx} \Theta(x, t) + \frac{a}{x}\, \partial_x \Theta(x, t) + b\, \mathcal{F}(x, t)\, \mathcal{G}\big(\Theta(x, t)\big) + \mathcal{H}(x, t), \\ &\Theta(x, 0) \;=\; \rho(x)\,; \quad t > 0\,, \quad 0 < x < l\,, \end{aligned} \tag{1. 1}$$

where $\Theta(x, t)$ represents temperature at time t and position x, $\rho(x)$ is the initial condition, and $\mathcal{F}(x, t)\, \mathcal{G}\big(\Theta(x, t)\big) + \mathcal{H}(x, t)$ is the heat source. This equation results in the mathematical modeling of various problems in the field of diffusion of heat orthogonal to the surfaces of parallel planes.

If $\mathcal{H}(x, t) = 0$ and temperature $\Theta(x, t)$ is steady-state, then equation (1. 1) becomes

$$\partial_{xx} \Theta(x, t) + \frac{a}{x}\, \partial_x \Theta(x, t) + b\, \mathcal{F}(x)\, \mathcal{G}\big(\Theta(x, t)\big) = 0\,; \quad 0 < x < l\,, \tag{1. 2}$$

which is recognized as Lane Emden-Fowler equation. For $\mathcal{G}\big(\Theta(x, t)\big) = \Theta^n(x, t)$ and $\mathcal{F}(x) = 1$, equation (1. 2) reduces to standard Lane Emden-Fowler equation of the first kind, whereas for $\mathcal{G}\big(\Theta(x, t)\big) = \mathrm{e}^{\Theta(x, t)}$, equation (1. 2) reduces to standard Lane Emden-Fowler equation of the second kind. These equations appear as a result of modeling various real-life events in the fields of astrophysics and mathematical physics such as: the thermal performance of a spherical cloud of gas, thermal explosion in either an infinite cylinder or sphere, stellar structure, idea of thermionic currents and isothermal gas sphere, etc [8, 9, 19, 28].

Simulation of heat type Emden-Fowler equations is very complicated by exploiting analytical techniques. Therefore, to simulate these type of problems, numerical and semi-analytical techniques are preferred. Considerable numerical and semi-analytical schemes have already been implemented to deal with heat type time-dependent Emden-Fowler equations which are listed as: homotopy analysis method [4], variational iteration method [5], B-spline collocation method [10], modified homotopy perturbation method [22], modified decomposition method [23], successive differentiation method [24], and Adomian decomposition method [29], etc. Each technique have some advantages or drawbacks over the other, for example, calculation of Adomian polynomials in Adomian decomposition method is a tedious work, Lagrange's multiplier in variational iteration scheme is difficult to calculate, while determining auxiliary parameters and functions in homotopy-based approaches require intricate calculations. The implemented semi-analytical technique in this paper is straightforward and does not require any extra computations like the above mentioned methods. Moreover, the combination of Laplace transform (LT) with Picard's iterative scheme (PIS) is well known for its consistent convergence characteristics. Working in the frequency domain is made possible via the Laplace transform, which frequently makes the problem simpler and the Picard's scheme offers an analytical foundation for iterative improvement; which makes this combination a useful tool for solving problems when other numerical methods might be unable to offer a reliable solution. This combination can frequently make it easier to solve difficult differential and integral equations, and can also provide important new information about how the system under study behaves [26].

In this paper, time-fractional order heat type Emden-Fowler equations are redeveloped by exploiting CFTFD. Simulation is performed by employing semi-analytical technique that is a combination of Laplace transform [18] and Picard's iterative scheme [14]. Furthermore, stability of the developed semi-analytical scheme is also analyzed by employing R-stable mapping and Banach contraction principle [15]. Error estimation and comparison of results through graphical and tabular representations are also delineated to reveal and interpret the derived numerical results. Moreover, the obtained results are an innovative contribution to the literature and are significant due to their advantageous usages in scientific and engineering experimentations.

2. Preliminaries

Definition 1. CFTFD of $\Theta(t)$ is demarcated as [3, 13]

$$D_t^{\xi}\left[\Theta(t)\right] = \frac{\mathcal{M}(\xi)}{1-\xi} \int_c^t \Theta^{'}(s) \, \exp\left[-\xi \left(\frac{t-s}{1-\xi}\right)\right] ds, \tag{2. 3}$$

where $\Theta(t) \in H^1(c\,,d),\ c \in (-\infty\,,t),\ t > 0,\ c < d,\ \mathcal{M}(\xi) = 1 - \xi + \dfrac{\xi}{\Gamma(\xi)}$ is normalization function with $\mathcal{M}(1) = \mathcal{M}(0) = 1$ and $\xi \in (0\,,1)$.

Definition 2. Caputo-Fabrizio time-fractional integral (CFTFI) of $\Theta(t)$ linked to CFTFD given in above equation (2. 3), is defined by [13]

$$I_t^{\xi}\left[\Theta(t)\right] = \frac{1-\xi}{\mathcal{M}(\xi)}\,\Theta(t) + \frac{\xi}{\mathcal{M}(\xi)} \int_0^t \Theta(s)\,ds, \qquad t \geq 0. \tag{2. 4}$$

Remark. In accordance with Definition 2, CFTFI of a function of order $0 < \xi < 1$ is attained by taking average of function $\Theta(t)$ and its integral of order 1 [13]. Thus, by taking

$$\frac{1-\xi}{\mathcal{M}(\xi)} + \frac{\xi}{\mathcal{M}(\xi)} = 1, \tag{2. 5}$$

we acquire

$$\mathcal{M}(\xi) = 1. \tag{2. 6}$$

Then, by using equation (2. 6), CFTFD described in equation (2. 3) is redefined as [6]

$$D_t^{\xi}\ [\Theta(t)] \;=\; \frac{1}{1-\xi}\int_c^t \Theta^{'}(s)\ \exp\left[-\xi\left(\frac{t-s}{1-\xi}\right)\right] ds. \tag{2. 7}$$

Another definition of CFTFI linked to time-fractional order derivative described in above equation (2. 7) is given by [6,21]

$$I_t^{\xi}\ [\Theta(t)] \;=\; \frac{1}{\xi}\int_0^t \Theta(s)\exp\left[-\frac{1-\xi}{\xi}(t-s)\right] ds, \qquad t\geq 0. \tag{2. 8}$$

Definition 3. Suppose that R is a self-mapping defined on Banach space $(M\,,\,\|\cdot\|)$. Then, the following recursive relation [21]

$$\Theta_{m+1} = R(\Theta_m)\,, \qquad m\,\in\,\mathbb{N}, \tag{2. 9}$$

is R-stable if for

$$\varepsilon_m = \|\Theta_{m+1} - R(\Theta_m)\|\,, \tag{2. 10}$$

we have

$$\lim_{m\to\infty}\varepsilon_m = 0, \quad \text{or} \quad \lim_{m\to\infty}\Theta_m \;=\; r\,, \tag{2. 11}$$

where $\{\Theta_m\}\,\subset\, M$ is a bounded convergent sequence with convergence point $r\in H(R)$ (well-defined collection of distinct fixed points of R).

Definition 4. ($L^1(\Omega)-$ Space) Assume that $(\Omega\,,\,\mu_1)$ is Lebesgue measurable space, such that $\Omega\,=\,(c\,,\,d)$. Then, $L^1(\Omega)$ is demarcated as the space of Lebesgue integrable measurable functions, which is [17]

$$L^1(\Omega) = \left\{\Theta(x):\quad \Theta(x)\in\mu_1(\Omega)\quad\wedge\quad \int_\Omega |\Theta(x)|\,d\mu_1 < \infty\right\}. \tag{2. 12}$$

Introducing equivalence relation on $L^1(\Omega)$ to make it a normed space as:

$$\begin{aligned}\Theta_1\sim\Theta_2 \qquad &\Longleftrightarrow \qquad \Theta_1=\Theta_2 \quad \text{almost everywhere } (a.e),\\ &\Longleftrightarrow \qquad \mu_1\big\{y\in\Omega:\ \Theta_1(y)\neq\Theta_2(y)\big\}=0\,.\end{aligned} \tag{2. 13}$$

Since, the equivalence relation defined above partitioned $L^1(\Omega)$ into equivalence classes, therefore it is the space of equivalence classes. Hence, equivalence class of $\Theta\in L^1(\Omega)$ is defined as [17]

$$[\Theta] = \big\{\Theta\in L^1(\Omega):\quad \Theta\sim\Theta\big\}. \tag{2. 14}$$

Thus, norm on $L^1(\Omega)$ is demarcated by [17]

$$\|\Theta\|_1 = \int_\Omega |\Theta|\,d\,\mu_1\,,\quad \forall\,\Theta\in L^1(\Omega)\,. \tag{2. 15}$$

Theorem 1. Suppose that R is a self-mapping defined on Banach space $(M\,,\,\|\cdot\|)$ such that

$$\|Rx_1 - Rx_2\|\,\leq\,\alpha\,\|x_1 - Rx_1\| + \beta\,\|x_1 - x_2\|\,,\quad \forall\,x_1\,,\,x_2\,\in\,M, \tag{2. 16}$$

where $\alpha\geq 0\,,\ 0\leq\beta<1$ and $Rx_1,\ Rx_2$ are images of $x_1,\ x_2$, respectively. Furthermore, if R has fixed-point, then at that point self-mapping R is Picard's R-stable [16].

Theorem 2. The Laplace transform (LT) of CFTFD of $\Theta(t)$ is given by [7]

$$\mathcal{L}\big\{D_t^{\xi}\,\Theta(t)\big\} \;=\; \frac{q\,\Theta(q)-\Theta(0)}{q+\xi\,(1-q)}\;, \tag{2. 17}$$

where $\Theta(q) \;=\mathcal{L}\,\{\Theta(t)\}$, while q is parameter of the transformation.

Theorem 3. (Banach Contraction Principle) Consider a metric space $(M\,,\,d)$ such that $d(x_{1}\,,\,x_{2}) = \|x_{1}-x_{2}\|\,,\;\forall\,x_{1}\,,\,x_{2}\in M\,,$ and d is a metric [26]. Then, self-mapping R has a unique fixed-point in M if

(1) $(M\,,\,d)$ *is a Banach space*,
(2) R *is contraction mapping*.

Theorem 4. Consider Lebesgue measurable space $(\Omega\,,\,\mu_{1})$ such that $\Omega=(c,\,d)$. Then, $L^1(\Omega)$ forms Banach space with the norm delineated as [17]

$$\|\Theta\|_1 = \int_{\Omega}|\Theta|\;d\mu_{1}\,,\quad \forall\,\Theta\in L^1(\Omega). \tag{2. 18}$$

3. Formation of Time-Fractional Order Emden-Fowler Equation

Consider the most general singular IVP for equation (1. 1) as

$$\begin{aligned}&\partial_t\,\Theta(x\,,\,t)=\partial_{xx}\,\Theta(x\,,\,t)+\frac{a}{x}\;\partial_x\,\Theta(x\,,\,t)+b\,\mathcal{F}(x\,,\,t)\,\mathcal{G}\big(\Theta(x\,,\,t)\big)+\mathcal{H}(x,\,t),\\&\Theta(x\,,\,0)\;=\;\rho(x)\,;\quad t>0\,,\quad 0<x<l\,.\end{aligned} \tag{3. 19}$$

To formulate time-fractional order heat type Emden-Fowler equation, the above equation (3. 19) is supposed to be in non-dimensional form, then without loss of generality, the integer-order time derivative in equation (3. 19) can be switched to CFTFD $\big(D_t^{\xi}\big)$. Thus, the corresponding time-fractional order model is

$$\begin{aligned}&D_t^{\xi}\,[\Theta(x\,,\,t)]=\partial_{xx}\,\Theta(x\,,\,t)+\frac{a}{x}\;\partial_x\,\Theta(x\,,\,t)+b\,\mathcal{F}(x\,,\,t)\,\mathcal{G}\big(\Theta(x\,,\,t)\big)+\mathcal{H}(x,\,t),\\&\Theta(x\,,\,0)\;=\;\rho(x)\,;\quad t>0\,,\quad 0<x<l\,.\end{aligned} \tag{3. 20}$$

4. Implementation of Semi-Analytical Scheme

Consider the most general singular IVP for time-fractional order heat type Emden-Fowler equation (3. 20)

$$\begin{aligned}&D_t^{\xi}\,[\Theta(x\,,\,t)]=\partial_{xx}\,\Theta(x\,,\,t)+\frac{a}{x}\,\partial_x\,\Theta(x\,,\,t)+b\,\mathcal{F}(x\,,\,t)\,\mathcal{G}\big(\Theta(x\,,\,t)\big)+\mathcal{H}(x,\,t),\\&\Theta(x\,,\,0)\;=\;\rho(x).\end{aligned}$$

Implementing LT $(\mathcal{L})$ on above equation, we acquire

$$\mathcal{L}\left\{D_t^{\xi}\,[\Theta(x\,,t)]\right\}=\mathcal{L}\left\{\partial_{xx}\,\Theta(x\,,t)+\frac{a}{x}\,\partial_x\,\Theta(x\,,t)+b\,\mathcal{F}(x\,,t)\mathcal{G}\big(\Theta(x\,,t)\big)+\mathcal{H}(x\,,t)\right\}. \tag{4. 21}$$

Employing Theorem 2 on the right hand side of above equation, we obtain

$$\frac{q\Theta(x\,,q)-\Theta(x\,,0)}{q+\xi(1-q)}=\mathcal{L}\left\{\partial_{xx}\Theta(x\,,t)+\frac{a}{x}\,\partial_{x}\Theta(x\,,t)+b\,\mathcal{F}(x\,,t)\mathcal{G}\big(\Theta(x\,,t)\big)+\mathcal{H}(x\,,t)\right\} \tag{4. 22}$$

By simplifying equation (4. 22), we get

$$\begin{aligned}\Theta(x\,,\,q) &= \frac{\Theta(x\,,\,0)}{q}+\frac{q+\xi\,(1-q)}{q}\\ &\quad\times\mathcal{L}\left\{\partial_{xx}\Theta(x\,,\,t)+\frac{a}{x}\,\partial_{x}\Theta(x\,,\,t)+b\,\mathcal{F}(x\,,\,t)\mathcal{G}\big(\Theta(x\,,\,t)\big)\;+\;\mathcal{H}(x\,,\,t)\right\}.\end{aligned}$$

Exploiting inverse Laplace transform $(\mathcal{L}^{-1})$ on above equation, we attain

$$\begin{aligned}\Theta(x\,,\,t) \;&=\; \Theta(x\,,\,0)+\mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right) \qquad (4.\ 23)\\ &\times\;\; \mathcal{L}\left\{\partial_{xx}\,\Theta(x\,,\,t)\;+\;\frac{a}{x}\,\partial_{x}\,\Theta(x\,,\,t)+b\,\mathcal{F}(x\,,\,t)\mathcal{G}\big(\Theta(x\,,\,t)\big)\;+\;\mathcal{H}(x\,,\,t)\right\}\Bigg].\end{aligned}$$

Then, Picard's iterative scheme that provide approximate solution for singular IVP (3. 20) is expressed as:

$$\begin{aligned}&\Theta_0(x\,,\,t)=\Theta(x\,,\,0)=\rho(x),\\ &\Theta_{m+1}(x\,,\,t)\;=\;\Theta_m(x\,,\,t)\;+\;\mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right) \qquad (4.\ 24)\\ &\times\mathcal{L}\left\{\partial_{xx}\,\Theta_m(x\,,\,t)+\frac{a}{x}\,\partial_{x}\,\Theta_m(x\,,\,t)+b\,\mathcal{F}(x\,,\,t)\mathcal{G}\big(\Theta_m(x\,,\,t)\big)+\mathcal{H}(x\,,\,t)\right\}\Bigg],\end{aligned}$$

where $m\,=\,0,1,2,\ldots.$ Thus, approximate solution of singular IVP for time-fractional order heat type Emden-Fowler equation (3. 20) in $L^1(\Omega)$ is obtained by Picard's iterative scheme (4. 24) such as

$$\Theta(x\,,\,t)\;=\lim_{m\,\rightarrow\,\infty}\Theta_m(x,\,t),\qquad m\in\mathbb{N}. \tag{4. 25}$$

5. Solutions of Time-Fractional Order Emden-Fowler Equations

The semi-analytical scheme is implemented on selected test examples to derive the corresponding approximate solutions.

Example 1. Consider equation (3. 20) by taking $a=2,$ and the heat source

$$b\,\mathcal{F}(x\,,\,t)\mathcal{G}\big(\Theta(x\,,\,t)\big)+\mathcal{H}(x\,,\,t)=-\big(6+4\,x^2-\cos t\big)\,\Theta(x\,,\,t),$$

with exact solution [10]

$$\Theta(x\,,\,t)=e^{\,x^2\,+\sin t}.$$

The corresponding time-fractional order linear model is

$$D_t^{\xi}\,[\Theta(x\,,\,t)]=\partial_{xx}\Theta(x\,,\,t)+\frac{2}{x}\,\partial_{x}\Theta(x\,,\,t)-(6+4\,x^2-\cos t)\Theta(x\,,\,t),\;\Theta(x\,,\,0)=e^{\,x^2}. \tag{5. 26}$$

Then, approximate solution of above IVP (5. 26) is retrieved by employing Picard's iterative scheme (4. 24) for $a = 2$ and heat source

$$b\,\mathcal{F}(x\,,\,t)\mathcal{G}\big(\Theta(x\,,\,t)\big) + \mathcal{H}(x\,,\,t) = -\left(6 + 4\,x^2 - \cos t\right)\Theta(x\,,\,t) \quad \text{as}$$

$$\begin{aligned}
&\Theta_0(x\,,\,t) = \Theta(x\,,\,0) = e^{\,x^2},\\
&\Theta_{m+1}(x\,,\,t) = \Theta_m(x\,,\,t) \,+\, \mathcal{L}^{-1}\left[\left(\frac{q + \xi\,(1-q)}{q}\right)\right.\\
&\left.\times\mathcal{L}\left\{\partial_{xx}\Theta_m(x\,,t) + \frac{2}{x}\,\partial_x\Theta_m(x\,,t) - (6 + 4\,x^2 - \cos t)\Theta_m(x\,,t)\right\}\right],
\end{aligned} \tag{5. 27}$$

where $m = 0, 1, 2, \ldots.$ From numerical scheme (5. 27), we obtain the following sequence of functions with the help of computer package Maple:

$$\begin{aligned}
&\Theta_0(x\,,\,t) = \Theta(x\,,\,0) = e^{\,x^2},\\
&\Theta_1(x\,,\,t) = e^{\,x^2} + (\xi t - \xi + 1)\,e^{\,x^2}\cos t\,,\\
&\Theta_2(x\,,\,t) = e^{\,x^2} + (2\,\xi\,t - 2\,\xi + 2)\,e^{\,x^2}\cos t\\
&\qquad + \left(\frac{1}{2}\,\xi^2\,t^2 - 2\,\xi^2 t + 2\,\xi\,t + \xi^2 - 2\,\xi + 1\right)e^{\,x^2}\cos t^2,\\
&\quad\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned} \tag{5. 28}$$

Thus, approximate solution of singular IVP for time-fractional order linear heat type Emden-Fowler equation (5. 26) is retrieved from the sequence of functions given in equation (5. 28) as

$$\Theta(x,\,t) \,=\, \lim_{m\,\to\,\infty} \Theta_m(x,\,t), \qquad m \in \mathbb{N}\,. \tag{5. 29}$$

Example 2. Consider equation (3. 20) by taking $a = 5$ and the heat source

$$b\,\mathcal{F}(x\,,\,t)\mathcal{G}\big(\Theta(x\,,\,t)\big) \,+\, \mathcal{H}(x\,,\,t) = -\big(24\,t + 16\,t^2x^2\big)\,e^{\Theta\,(x\,,\,t)} - 2\,x^2\,e^{\frac{\Theta\,(x\,,\,t)}{2}},$$

with exact solution [10]

$$\Theta(x\,,\,t) = -2\,\ln(1 + t\,x^2).$$

The corresponding time-fractional order nonlinear model is

$$\begin{aligned}
&D_t^{\xi}\,[\Theta(x\,,t)] = \partial_{xx}\Theta(x\,,t) + \frac{5}{x}\,\partial_x\Theta(x\,,t) - \big(24\,t + 16\,t^2x^2\big)\,e^{\Theta\,(x,t)} - 2\,x^2e^{\frac{\Theta\,(x,t)}{2}},\\
&\Theta(x\,,0) = 0.
\end{aligned} \tag{5. 30}$$

Then, approximate solution of above IVP (5. 30) is retrieved by employing Picard's iterative scheme (4. 24) for $a = 5$ and heat source

$$b\,\mathcal{F}(x\,,\,t)\mathcal{G}\big(\Theta(x\,,\,t)\big) \,+\, \mathcal{H}(x\,,\,t) = -\big(24\,t + 16\,t^2x^2\big)\,e^{\Theta\,(x\,,\,t)} - 2\,x^2\,e^{\frac{\Theta\,(x\,,\,t)}{2}} \quad \text{as}$$

$$\Theta_0(x\,,\,t)=\Theta(x,\,0)=0,$$
$$\Theta_{m+1}(x\,,\,t)=\Theta_m(x\,,\,t)\;+\;\mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right) \tag{5. 31}$$
$$\times\mathcal{L}\left\{\partial_{xx}\Theta_m(x\,,t)+\frac{5}{x}\,\partial_x\Theta_m(x\,,t)-\left(24\,t+16\,t^2x^2\right)e^{\Theta_m(x\,,t)}-2\,x^2e^{\frac{\Theta_m(x\,,t)}{2}}\right\}\Bigg],$$

where $m=0,1,2,\ldots$. From numerical scheme (5. 31), we obtain the following sequence of functions with the help of computer package Maple:

$$\Theta_0(x\,,\,t)=\Theta(x,\,0)=0\,,$$
$$\Theta_1(x\,,t)=-\frac{16}{3}\,\xi x^2t^3+\left(16[\xi-1]x^2-12\,\xi\right)t^2+\left(24[\xi-1]-2\,\xi x^2\right)t+2\,[\xi-1]x^2,$$
$$\Theta_2(x\,,t)=-\frac{2048}{81}\,\xi^3x^6t^9+\left(\frac{3584}{9}\big[\xi^3-\xi^2\big]x^6-\frac{512}{3}\,\xi^3x^4\right)t^8-\Bigg(\frac{7168}{3}\big[\xi^2-\xi^3\big]x^4$$
$$+\frac{106048}{63}\xi^3x^6+\frac{34816}{21}\big[\xi-2\,\xi^2\big]x^6+384\,\xi^3x^2\Bigg)t^7+\Bigg(\frac{7040}{3}\,\xi^3x^6+2048[3\,\xi-1]x^6$$
$$-288\xi^3+4736\big[\xi^3-\xi^2\big]x^2-\frac{19328}{3}\xi^2x^6-8816\xi^3x^4-8704\,\xi x^4+\frac{156800}{9}\xi^2x^4\Bigg)t^6$$
$$+\Bigg(\frac{416768}{15}\,\xi\,x^4-9216\,x^4-\frac{433856}{15}\,\xi^2x^4+\frac{15552}{5}\big[\xi^3-\xi^2\big]+\frac{12608}{15}\big[2\,\xi^2-\xi\big]x^6$$
$$+\frac{150848}{5}\,\xi^2x^2-848\,\xi^3x^6+\frac{51776}{5}\,\xi^3x^4-\frac{75264}{5}\,\xi\,x^2-\frac{75876}{5}\,\xi^3x^2\Bigg)t^5+\Bigg(256\,x^4$$
$$+41984\,\xi x^2-13824\,x^2-43064\,\xi^2x^2+\frac{17404}{3}\,\xi^2x^4-3280\,\xi\,x^4+\frac{1904}{3}\,\xi^3x^6$$
$$+8696\,\xi-8640\,\xi^3+14904\,\xi^3x^2-\frac{5360}{3}\,\xi^2x^6-2783\,\xi^3x^4+\;576\,[3\,\xi-1]\,x^6\Bigg)t^4$$
$$+\Bigg(6912\big[\xi^3-1\big]-21088\big[\xi^2-\xi\big]-1728\,x^4-\frac{305}{3}\,\xi^3x^6+1832\,\xi^3x^4-\frac{16024}{3}\,\xi^2x^4$$
$$+\frac{304}{3}\big[2\,\xi^2-\xi\big]x^6-768\big[3\,\xi^3-1\big]x^2+5396\,\xi^2x^2+\frac{15712}{3}\,\xi x^4-\frac{11552}{3}\,\xi x^2\Bigg)t^3$$
$$+\Bigg(385\,\xi^2x^4-264\,\xi x^4+1296\,\xi^3x^2+4016\,\xi x^2-168\,\xi^3x^4+384+372\,\xi^2+48\,x^4$$
$$+50\,\xi^3x^6+48[3\,\xi-1]x^6-1328\big[3\,\xi^2+1\big]x^2-792\,\xi-146\,\xi^2x^6\Bigg)t^2+\Bigg(48\big[\xi^2-1\big]$$
$$+72\big[\xi^3-1\big]x^4+72\big[\xi^2+1\big]x^2-148\,\xi x^2-220\big[\xi^2-\xi\big]x^4-3\big[\xi^3-2\,\xi^2+\xi\big]x^6\Bigg)t$$
$$+\big[\xi^3-3\,\xi^2+3\,\xi-1\big]x^6+2\big[\xi^2-2\,\xi+1\big]x^4+4[\xi-1]x^2-24\big[\xi^2-2\,\xi+1\big],$$
$$\vdots \qquad\qquad \vdots \tag{5. 32}$$

Thus, approximate solution of singular IVP for time-fractional order nonlinear heat type Emden-Fowler equation (5. 30) is retrieved from the sequence of functions given in equation (5. 32) as

$$\Theta(x\,,\,t) \;=\; \lim_{m\,\rightarrow\,\infty} \Theta_m(x\,,\,t), \qquad m \in \mathbb{N}\,. \tag{5. 33}$$

6. Stability Assessment

The stability of numerical schemes (5. 27) and (5. 31) for time-fractional order heat type Emden-Fowler equations (5. 26) and (5. 30) is analyzed by implementing R-stable mapping along with Banach contraction principle(BCP).

Theorem 5. (Stability Analysis of Example 1) The self-mapping $R_{_1} \,:\, L^1(\Omega) \longrightarrow L^1(\Omega)$ defined by

$$R_{_1}\,(\Theta_m(x\,,\,t)) = \Theta_{m+1}(x\,,\,t) = \Theta_m(x\,,\,t) \,+\, \mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right) \tag{6. 34}$$

$$\times\,\mathcal{L}\left\{\partial_{xx}\Theta_m(x\,,\,t) + \frac{2}{x}\,\partial_x\Theta_m(x\,,\,t) - \big(6+4\,x^2-\cos t\big)\,\Theta_m(x\,,\,t)\right\}\Bigg],$$

is $R_{_1}$-stable in Banach space $L^1(\Omega)$ if

$$1 \,+\, \alpha_1\, f_1(\varsigma) \,+ \alpha_2\,\alpha_3\, g_1(\varsigma) \,+\, \alpha_4\, h_1(\varsigma) \;<\; 1\,, \tag{6. 35}$$

where f_1, g_1 and h_1 came from $\mathcal{L}^{-1}\left[\left(\dfrac{q+\xi\,(1-q)}{q}\right)\cdot\mathcal{L}\{\bullet\}\right]$.

Proof. First, we prove that $R_{_1}$ acquires a fixed point in $L^1(\Omega)$ by applying Banach contraction principle. For this, we consider the recursive relation $\forall\ (m\,,n\,)\ \in\ \mathbb{N}\times\mathbb{N}$ as follows:

$$R_{_1}\big(\Theta_m(x\,,t)\big) - R_{_1}\big(\Theta_n(x\,,t)\big) = [\Theta_m(x\,,t) - \Theta_n(x\,,t)] + \mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right)$$

$$\times\mathcal{L}\Bigg\{\partial_{xx}\,[\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)] + \frac{2}{x}\,\partial_x\,[\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)]$$

$$-(6+4\,x^2-\cos t)\,[\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)]\Bigg\}\Bigg]. \tag{6. 36}$$

Exploiting the norm and triangular inequality on equation (6. 36), we acquire

$$\big\|R_{_1}\big(\Theta_m(x\,,t)\big) - R_{_1}\big(\Theta_n(x\,,t)\big)\big\| \leq \|\Theta_m(x\,,t) - \Theta_n(x\,,t)\| + \mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right)$$

$$\times\,\mathcal{L}\Bigg\{\partial_{xx}\,\|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\| + \left\|\frac{2}{x}\right\|\partial_x\,\|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\|$$

$$+\,\big\|6+4\,x^2-\cos t\big\|\,\|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\|\Bigg\}\Bigg]. \tag{6. 37}$$

Since, $0 < x < l$ and $\| \cos t\| \leq 1$, therefore

$$\begin{aligned} \left\| \frac{2}{x} \right\| &= \frac{2}{x} < \alpha_2 , \\ \left\| 6 + 4\,x^2 - \cos t \right\| &\leq 6 + 4\,x^2 + \| \cos t \| \leq 6 + 4\,x^2 + 1 < 7 + 4\,l^2 = \alpha_4 , \end{aligned} \tag{6. 38}$$

where α_2 and α_4 are positive constants. Then, equation (6. 37) becomes

$$\begin{aligned} \left\| R_1\big(\Theta_m(x,\,t)\big) - R_1\big(\Theta_n(x,\,t)\big) \right\| &< [1 + \alpha_1\, f_1(\varsigma) + \alpha_2\, \alpha_3\, g_1(\varsigma) + \alpha_4\, h_1(\varsigma)] \\ &\times \|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\| , \end{aligned} \tag{6. 39}$$

where $\alpha_1 = \partial_{xx}$, $\alpha_3 = \partial_x$, while f_1 , g_1 and h_1 came from

$$\mathcal{L}^{-1}\left[\left(\frac{q + \xi\,(1 - q)}{q} \right) \cdot \mathcal{L}\{\bullet\} \right].$$

The self-mapping R_1 is contraction mapping on $L^1(\Omega)$ if

$$1 + \alpha_1\, f_1(\varsigma) + \alpha_2\, \alpha_3\, g_1(\varsigma) + \alpha_4\, h_1(\varsigma) < 1. \tag{6. 40}$$

By exploiting Banach contraction principle, linear self-mapping R_1 acquires a fixed point. Furthermore, R_1 accomplishes Theorem 1 with

$$\alpha = 0\,, \quad \text{and} \quad \beta = 1 + \alpha_1\, f_1(\varsigma) + \alpha_2\, \alpha_3\, g_1(\varsigma) + \alpha_4\, h_1(\varsigma). \tag{6. 41}$$

Thus, the Picard's iterative scheme (5. 27) is R_1-stable in Banach space $L^1(\Omega)$.

Theorem 6. (Stability Analysis of Example 2) The self-mapping $R_2 : L^1(\Omega) \longrightarrow L^1(\Omega)$ defined by

$$\begin{aligned} R_2\big(\Theta_m(x,\,t)\big) &= \Theta_{m+1}(x\,,\,t) = \Theta_m(x\,,\,t) + \mathcal{L}^{-1}\Bigg[\left(\frac{q + \xi\,(1 - q)}{q} \right) \\ &\times \mathcal{L}\left\{ \partial_{xx}\Theta_m(x,\,t) + \frac{5}{x}\,\partial_x\Theta_m(x,\,t) - \big(24\,t + 16\,t^2x^2\big)\,\mathrm{e}^{\Theta_m(x,\,t)} - 2\,x^2\,\mathrm{e}^{\frac{\Theta_m(x\,,t)}{2}} \right\} \Bigg], \end{aligned} \tag{6. 42}$$

is R_2-stable in Banach space $L^1(\Omega)$ if

$$1 + \alpha_1\, f_2(\varsigma) + \alpha_3\, \gamma_1\, g_2(\varsigma) + \gamma_2\, h_2(\varsigma) + \gamma_3\, (A + B) k_2(\varsigma) + \gamma_4\, l_2(\varsigma) < 1\,, \tag{6. 43}$$

where f_2, g_2, h_2, k_2 and l_2 are obtained from $\mathcal{L}^{-1}\left[\left(\frac{q + \xi\,(1 - q)}{q} \right) \times \mathcal{L}\{\bullet\} \right]$.

Proof. First, we prove that R_2 acquires a fixed point in $L^1(\Omega)$ by exploiting Banach contraction principle. For this, we consider the recursive relation $\forall\ (m\,,n) \in \mathbb{N}\times\mathbb{N}$ as follows:

$$R_2\big(\Theta_m(x,\,t)\big) \;-\; R_2\big(\Theta_n(x,\,t)\big) = [\Theta_m(x,\,t) - \Theta_n(x,\,t)] \;+\; \mathcal{L}^{-1}\Bigg[\bigg(\frac{q+\xi\,(1-q)}{q}\bigg) \times \mathcal{L}\bigg\{\partial_{xx}\,[\Theta_m(x,\,t) - \Theta_n(x,\,t)] + \frac{5}{x}\,\partial_x\,[\Theta_m(x,\,t) - \Theta_n(x,\,t)] - \big(24\,t + 16\,t^2x^2\big)\,e^{\Theta_m(x,\,t)-\Theta_n(x,\,t)} - 2\,x^2\,e^{\frac{\Theta_m(x,\,t)-\Theta_n(x,\,t)}{2}}\bigg\}\Bigg]. \tag{6. 44}$$

Further, assume that $\Theta(x,\,t)$ is small enough such that

$$e^{\Theta\,(x\,,\,t)} \;\approx\; 1 + \Theta(x\,,\,t) + \frac{1}{2}\,\Theta^2(x\,,\,t)\,. \tag{6. 45}$$

Then,

$$e^{\Theta_m(x\,,\,t)-\Theta_n(x\,,\,t)} \;\approx\; 1 + \big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big] + \frac{1}{2}\,\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big]^2\,, \tag{6. 46}$$

and

$$e^{\frac{\Theta_m(x\,,\,t)-\Theta_n(x\,,\,t)}{2}} \;\approx\; 1 + \frac{1}{2}\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big] + \frac{1}{8}\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big]^2\,. \tag{6. 47}$$

Then, equation (6. 44) becomes

$$R_2\big(\Theta_m(x,\,t)\big) \;-\; R_2\big(\Theta_n(x,\,t)\big) \approx \big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big] \;+\; \mathcal{L}^{-1}\Bigg[\bigg(\frac{q+\xi\,(1-q)}{q}\bigg) \times \mathcal{L}\bigg\{\partial_{xx}\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big] + \frac{5}{x}\,\partial_x\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big] - \big(24\,t + 16\,t^2x^2\big) \times \bigg(1 + \big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big] + \frac{1}{2}\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big]^2\bigg) - 2\,x^2 \times \bigg(1 + \frac{1}{2}\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big] + \frac{1}{8}\big[\Theta_m(x,\,t) - \Theta_n(x,\,t)\big]^2\bigg)\bigg\}\Bigg]. \tag{6. 48}$$

On simplifying equation (6. 48), we have

$$R_2\big(\Theta_m(x\,,\,t)\big) \;-\; R_2\big(\Theta_n(x\,,\,t)\big) \approx \big[\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\big] \;+\; \mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right) \times \mathcal{L}\Bigg\{\partial_{xx}\big[\Theta_m(x\,,t) - \Theta_n(x\,,t)\big] + \frac{5}{x}\,\partial_x\big[\Theta_m(x\,,t) - \Theta_n(x\,,t)\big] - \big(24\,t + 16\,t^2x^2 + x^2\big) \times \big[\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\big] - \left(12\,t + 8\,t^2x^2 + \frac{1}{4}\,x^2\right)\big[\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\big]^2 - \big(24\,t + 16\,t^2x^2 + 2\,x^2\big)\Bigg\}\Bigg]. \tag{6. 49}$$

Exploiting the norm and triangular inequality on above equation (6. 49), we get

$$\big\|R_2\big(\Theta_m(x\,,t)\big) - R_2\big(\Theta_n(x\,,t)\big)\big\| \le \|\Theta_m(x\,,t) - \Theta_n(x\,,t)\| + \mathcal{L}^{-1}\Bigg[\left(\frac{q+\xi\,(1-q)}{q}\right) \times \mathcal{L}\Bigg\{\partial_{xx}\,\|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\| + \left\|\frac{5}{x}\right\|\partial_x\,\|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\| + \big\|24\,t + 16\,t^2x^2 + x^2\big\|\,\|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\| + \left\|12\,t + 8\,t^2x^2 + \frac{1}{4}\,x^2\right\| \times \|\Theta_m(x\,,\,t) - \Theta_n(x\,,\,t)\|^2 + \big\|24\,t + 16\,t^2x^2 + 2\,x^2\big\|\Bigg\}\Bigg]. \tag{6. 50}$$

Since, $0 < x < l$ and $0 < t \le T$, therefore

$$\begin{aligned} \left\|\frac{5}{x}\right\| \;&=\; \frac{5}{x} \;<\; \gamma_1\,, \\ \big\|24\,t + 16\,t^2x^2 + x^2\big\| \;&<\; 24\,T + 16\,T^2l^2 + l^2 = \gamma_2\,, \\ \left\|12\,t + \;8\,t^2x^2 + \frac{x^2}{4}\right\| \;&<\; 12\,T + 8\,T^2l^2 + \frac{l^2}{4} = \gamma_3\,. \end{aligned} \tag{6. 51}$$

where $\gamma_1,\ \gamma_2,\ \gamma_3$ are positive constants and T is the final time. Also, $\{\Theta_m(x\,,\,t)\}$ and $\{\Theta_n(x\,,\,t)\}$ are bounded sequences, then there exist distinct constants $A,\ B$ such that

$$\|\Theta_m(x,t)\| \le A\,, \quad \|\Theta_n(x,t)\| \le B\,, \quad \forall\;\; t > 0\,. \tag{6. 52}$$

Thus, inequality (6. 50) becomes

$$\big\|R_2\big(\Theta_m(x\,,t)\big) - R_2\big(\Theta_n(x\,,t)\big)\big\| < \|\Theta_m(x\,,t) - \Theta_n(x\,,t)\| \cdot \Bigg[1 + \alpha_1 f_2(\varsigma) + \alpha_3\gamma_1 g_2(\varsigma) + \gamma_2\;h_2(\varsigma) + \gamma_3\;\;\|\Theta_m(x\,,t) - \Theta_n(x\,,t)\|\;\;k_2(\varsigma) + \frac{\gamma_2}{\|\Theta_m(x\,,t) - \Theta_n(x\,,t)\|}\;l_2(\varsigma)\Bigg], \tag{6. 53}$$

where f_2, g_2, h_2, k_2 and l_2 are obtained from $\mathcal{L}^{-1}\left[\left(\dfrac{q+\xi\,(1-q)}{q}\right)\times\mathcal{L}\{\bullet\}\right]$.
Further,

$$\left|\|\Theta_m(x,t)\|-\|\Theta_n(x,t)\|\right| \;\le\; \|\Theta_m(x,t)-\Theta_n(x,t)\| \;\le\; \|\Theta_m(x,t)\|+\|\Theta_n(x,t)\|. \tag{6. 54}$$

Therefore

$$\begin{aligned} \|\Theta_m(x,\,t)-\Theta_n(x,\,t)\| \;&\le\; \|\Theta_m(x,\,t)\|+\|\Theta_n(x,\,t)\|\le A+B\,,\\ \|\Theta_m(x,\,t)-\Theta_n(x,\,t)\| \;&\ge\; \left|\|\Theta_m(x,\,t)\|-\|\Theta_n(x,\,t)\|\right| \quad \text{or}\\ \frac{1}{\|\Theta_m(x,\,t)-\Theta_n(x,\,t)\|} \;&\le\; \frac{1}{\left|\|\Theta_m(x,\,t)\|-\|\Theta_n(x,\,t)\|\right|} \quad \text{and}\\ \frac{\gamma_2}{\|\Theta_m(x,\,t)-\Theta_n(x,\,t)\|} &\le \frac{\gamma_2}{\left|\|\Theta_m(x,\,t)\|-\|\Theta_n(x,\,t)\|\right|}=\gamma_4\,, \end{aligned} \tag{6. 55}$$

where γ_4 is positive constant. Then, inequality (6. 53) becomes

$$\begin{aligned} &\left\|R_2\big(\Theta_m(x,\,t)\big)\;-\;R_2\big(\Theta_n(x,\,t)\big)\right\|<\|\Theta_m(x,\,t)-\Theta_n(x,\,t)\|\\ &\times\Big[1\,+\,\alpha_1\,f_2(\varsigma)\,+\,\alpha_3\,\gamma_1\,g_2(\varsigma)+\gamma_2\,h_2(\varsigma)+\gamma_3\,(A+B)\,k_2(\varsigma)+\gamma_4\,l_2(\varsigma)\Big]. \end{aligned} \tag{6. 56}$$

The self-mapping R_2 is contraction mapping on $L^1(\Omega)$ if

$$1\,+\,\alpha_1\,f_2(\varsigma)\,+\,\alpha_3\,\gamma_1\,g_2(\varsigma)+\gamma_2\,h_2(\varsigma)+\gamma_3\,(A+B)\,k_2(\varsigma)+\gamma_4\,l_2(\varsigma)\;<\;1\,. \tag{6. 57}$$

By using Banach contraction principle, mapping R_2 acquires a fixed-point. Furthermore, R_2 accomplishes Theorem 1 with

$$\alpha=0 \;\text{ and }\; \beta=1+\alpha_1 f_2(\varsigma)+\alpha_3\gamma_1\,g_2(\varsigma)+\gamma_2 h_2(\varsigma)+\gamma_3(A+B)k_2(\varsigma)+\gamma_4 l_2(\varsigma). \tag{6. 58}$$

Thus, the Picard's iterative scheme (5. 31) is R_2-stable in Banach space $L^1(\Omega)$.

7. Numerical Results and Discussion

In this section, temperature distributions retrieved from the solutions of time-fractional order heat type Emden-Fowler equations (5. 26) and (5. 30) are exhibited in Figures 1, 2 and 3, 4 respectively for different values of fractional parameter ξ. Tables 1 and 2 are presenting the numerical values for the solutions of time-fractional order heat type Emden-Fowler equations for diverse values of fractional parameter ξ, time t and position x. Furthermore, these tables contain the comparison of obtained results for $\xi=1$ with exact solutions along with error estimation.

Figures 1 and 2 show that the absolute temperature decreases when the values of ξ approaches 1 and time varies, but in figures 3 and 4 the absolute temperature increases with increase in the values of ξ. Moreover, figures 2 and 4 show that the approximate solutions $\Theta(x,\,t)$ for $\xi=1$ are in excellent agreement with the exact solutions. Tables 1 and 2 reveal that absolute error between the approximate solutions $\Theta(x,\,t)$ for $\xi=1$ and exact solutions is very small provide the evidence of efficiency and effectiveness of the developed semi-analytical scheme.

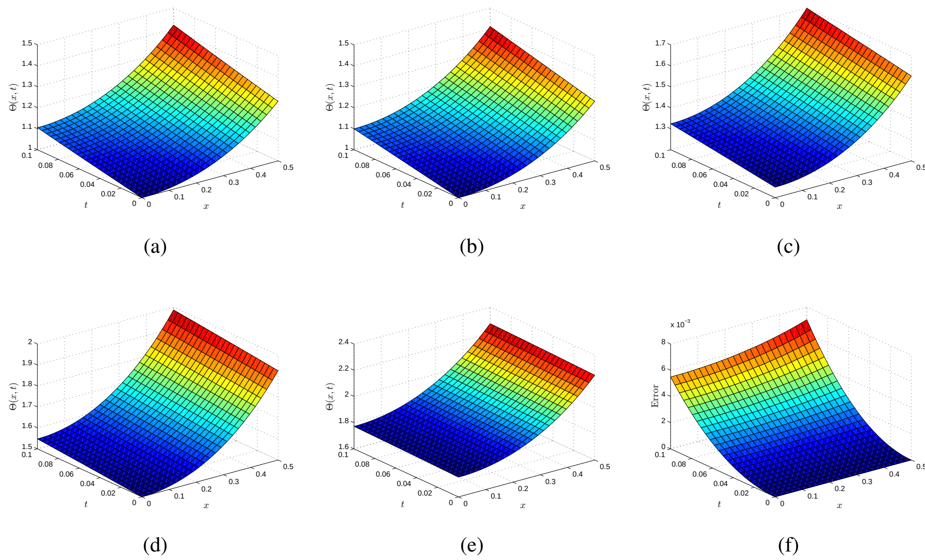

Figure 1: Graphical demonstration of solution $\Theta(x\,,\,t)$ of equation (5. 26) where (a) exact solution (b) $\xi = 1$ (c) $\xi = 0.75$ (d) $\xi = 0.5$ (e) $\xi = 0.25$ (f) absolute error.

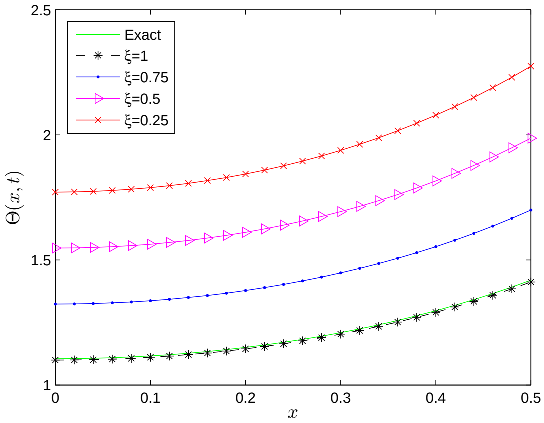

Figure 2: Comparison of the solution $\Theta(x\,,\,t)$ of equation (5. 26) for different values of ξ at $t = 0.1$.

Table 1: Table presenting numerical values of solution $\Theta(x\,,\,t)$ of equation (5. 26) for diverse values of ξ along with absolute error between the approximate solution for $\xi = 1$ and exact solution.

t	x	$\xi = 0.25$	$\xi = 0.50$	$\xi = 0.75$	$\xi = 1.00$	Exact Sol.	|Error|
0.00	0.0	1.7500000	1.5000000	1.2500000	1.0000000	1.0000000	0.0000000
	0.3	1.9148050	1.6412614	1.3677179	1.0941743	1.0941743	0.0000000
	0.5	2.2470445	1.9260381	1.6050318	1.2840254	1.2840254	0.0000000
0.05	0.0	1.7615471	1.5243439	1.2871407	1.0499375	1.0512492	0.0013117
	0.3	1.9274395	1.6678979	1.4083563	1.1488146	1.1502498	0.0014352
	0.5	2.2618712	1.9572963	1.6527214	1.3481465	1.3498307	0.0016842
0.10	0.0	1.7711282	1.5472523	1.3233764	1.0995004	1.1049868	0.0054864
	0.3	1.9379230	1.6929637	1.4480044	1.2030451	1.2090482	0.0060031
	0.5	2.2741737	1.9867113	1.6992489	1.4117865	1.4188312	0.0070447

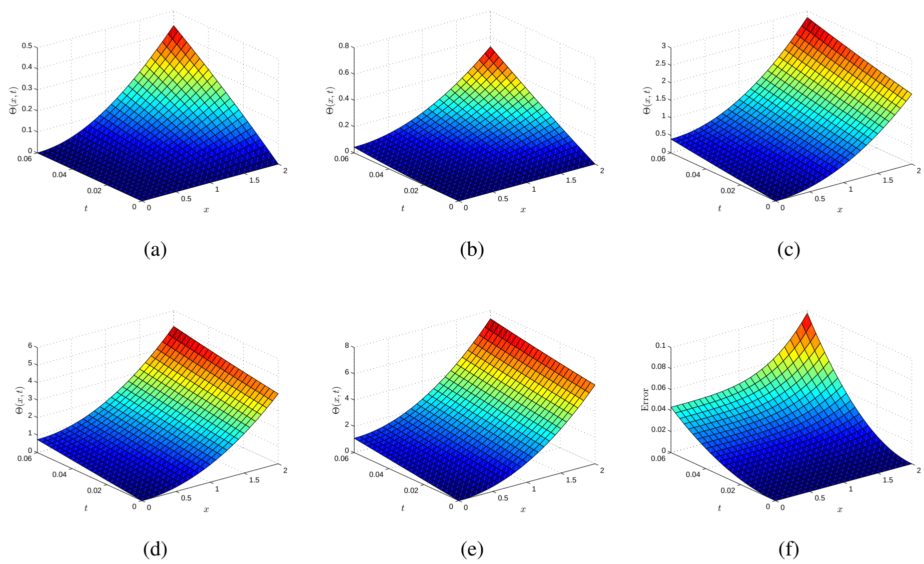

Figure 3: Graphical demonstration of solution $\Theta(x\,,\,t)$ of equation (5. 30) where (a) exact solution (b) $\xi = 1$ (c) $\xi = 0.75$ (d) $\xi = 0.5$ (e) $\xi = 0.25$ (f) absolute error.

8. Conclusion

Comprehensively, a semi-analytical scheme, that is the amalgamation of LT and PIS is exploited as mathematical method to simulate time-fractional order heat type Emden-Fowler

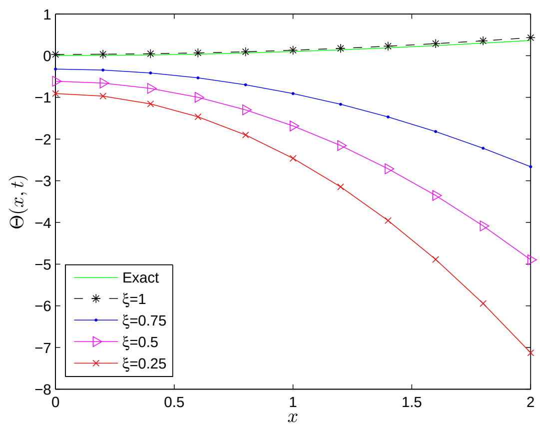

Figure 4: Comparison of the solution $\Theta(x\,,\,t)$ of equation (5. 30) for different values of ξ at $t = 0.05$.

Table 2: Table presenting numerical values of solution $\Theta(x\,,\,t)$ of equation (5. 30) for diverse values of ξ along with absolute error between the approximate solution for $\xi = 1$ and exact solution.

t	x	$\xi = 0.25$	$\xi = 0.50$	$\xi = 0.75$	$\xi = 1.00$	Exact Sol.	|Error|
0.00	0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
	0.5	0.3750000	0.2500000	0.1250000	0.0000000	0.0000000	0.0000000
	1.0	1.5000000	1.0000000	0.5000000	0.0000000	0.0000000	0.0000000
	1.5	3.3750000	2.2500000	1.1250000	0.0000000	0.0000000	0.0000000
	2.0	6.0000000	4.0000000	2.0000000	0.0000000	0.0000000	0.0000000
0.03	0.0	0.5427000	0.3654000	0.1881000	0.0108000	0.0000000	0.0108000
	0.5	0.9241590	0.6247180	0.3252770	0.0258360	0.0149440	0.0108919
	1.0	2.0685360	1.4026720	0.7368080	0.0709440	0.0591176	0.0118264
	1.5	3.9758310	2.6992620	1.4226930	0.1461240	0.1306389	0.0154850
	2.0	6.6460440	4.5144880	2.3829320	0.2513760	0.2266573	0.0247186
0.06	0.0	1.0908000	0.7416000	0.3924000	0.0432000	0.0000000	0.0432000
	0.5	1.4841720	1.0139440	0.5437160	0.0734880	0.0297772	0.0437107
	1.0	2.6642880	1.8309760	0.9976640	0.1643520	0.1165378	0.0478141
	1.5	4.6311480	3.1926960	1.7542440	0.3157920	0.2532653	0.0625267
	2.0	7.3847520	5.0991040	2.8134560	0.5278080	0.4302227	0.0975852

equations, is provided in this attempt. The illustration is made through time-fractional order heat type Emden-Fowler equations with CFTFD. Moreover, stability of the semi-analytical scheme éy éxploiting R-stable mapping and éanach éontraction érinciple és also analyzed. Graphical and tabulated éiscussions éor éistinct values éf along with érror analysis have shown the éfficiency é f t he é xploited s cheme é or t ime-fractional é rder h eat t ype Emden-Fowler équations.

Obviously, the time-fractional order heat type Emden-Fowler models are very challenging. The analysis designated in this paper is a substantial development in the study of heat type Emden-Fowler equations. The present research exploration can be protracted for time-fractional order heat type Emden-Fowler equations consist of diverse forms of fractional order derivatives along with singular convolution kernels taking distinct IBCs (initial and boundary conditions).

Credit Authorship Contribution's Statement

Saif Ullah: Conceptualization, formal analysis, investigation, methodology, project ad-ministration, resources, supervision, validation, data analysis, original draft preparation, review & editing.

Noor Fatima: Data curation, methodology, resources, software, validation, visualization, data analysis, writing original draft, review & editing.

DECLARATIONS

Conflict of Interest. The authors declare that they have no conflict of interest.
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