

**An Algorithm for Identifying a Precise Location of Digital Receiver Based on Complex Fuzzy Distance Measures**

**Received**  
01 June, 2025

**Revised**  
16 April, 2026

**Accepted**  
21 April, 2026

**Published Online**  
11 June 2026

M. I. Elashiry  
Department of Mathematics,  
College of Science, Northern Border University, Arar 91431, Saudi Arabia  
Department of Mathematics,  
Faculty of Science, Fayoum University, El-Fayoum, Egypt

Muhammad Zeeshan  
Department of Mathematics,  
The University of Agriculture, Dera Ismail Khan, Pakistan

\*Zeeshan Ali  
Department of Information Management,  
National Yunlin University of Science and Technology, Douliou, Taiwan,  
Republic of China

**Abstract.** Distance measures (DMs) within the complex fuzzy set (CFS) environments play a significant role in signal processing systems and complex decision-making (D-making) problems. Various multi-criteria decision-making (MCDM) problems have been effectively resolved based on complex fuzzy (CF) DMs (CFDMs). In the literature, several DMs have been established using CFSs, including Euclidean, normalized Euclidean, Hamming, normalized Hamming, and Zhang DMs. These existing DMs have contributed significantly in MCDM problems; however, they restrict certain limitations, such as violations of metric axioms, imprecise outcomes, and excessive separation effects. To handle these limitations, the proposed study establishes an extended form of the existing DMs aimed at capturing the maximum separation between two CFSs. Several innovative operations, their illustrative examples and fundamental theoretical results are demonstrated to reflect the validity of the newly defined approach. Moreover, an application of the newly defined DMs is presented in a complex D-making problem. Using the proposed DMs, an algorithm is designed to identify a precise location of a digital receiver. Finally, the proposed DMs are compared with existing CFDMs to validate its effectiveness and superiority.

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\*Corresponding Author: zeeshanalinsr@gmail.com

**AMS (MOS) Subject Classification Codes:** 03E72, 28E10, 90B50, 94D05.

**Key Words:** Complex fuzzy sets, distance measures, complex fuzzy operations, digital receiver, signal processing.

## 1. INTRODUCTION

In several domains, such as data science, artificial intelligence, operations research, and MCDM, DMs are essential for solving complex D-making problems. In fuzzy set (FS) theory, DMs reflect a quantitative way to compare entities, assess dissimilarity or similarity, and facilitate the evaluation of alternatives. The applications of DMs facilitate more objective comparisons, more lucid insights, and better decisions in complex, multi-dimensional problems. Common examples of fuzzy DMs include Euclidean distance, Hamming distance, normalized Hamming distance, Minkowski distance, and cosine similarity.

The idea of an FS was given by Zadeh [38] in 1965. the FS model is the extension of crisp sets, that is, its membership function is extended from  $\{0, 1\}$  to  $[0, 1]$ . In applications of real-world problems, the FS approach is crucial. The FS approach has many applications in image processing, engineering design, D-making, economics, operations research, medicine, quantum physics, psychology, mathematical chemistry, and non-equilibrium thermodynamics. He et al. [13] proposed an innovative technique for ranking fuzzy numbers using novel fuzzy distances. Similarly, Nayagam et al. [23] introduced a novel approach for ranking incomplete trapezoidal information, offering a robust mathematical framework to evaluate and order fuzzy numbers when data points are missing. Wang et al. [33] discussed D-making problems using enhancing similarity and DMs of Fermatean FSs. Nanavati et al. [22] constructed distance functions based on fuzzy negations and monotonic fuzzy logic connectives. They presented their real-life applications. Hussain et al. [16] developed the concept of an innovative multiview fuzzy c-means clustering algorithm. Ghosh and Konar proposed an effective fuzzy-pruned high-dimensional clustering approach using minimal fuzzy DMs [11].

The framework of the interval-valued fuzzy set (IVFS) [10] was proposed, which extend the approach of FSs by integrating the grade values as intervals. IVFSs are very useful for capturing higher levels of uncertainty inherent in real-world applications. IVFSs have been successfully employed in many MCDM problems. Singh et al. [27] presented innovative approach of accuracy measures for IVFSs and applied them to address pattern recognition problems. Vo and Nguyen developed a new ranking technique for solving real-world problems within the environments of IVFSs [32]. Cao and Zhong discussed an innovative assessment model for virtual experiment teaching using the framework of IVFSs [5]. Pekala et al. [24] introduced inclusion and similarity measures for IVFSs. Hu et al. [14] introduced a set of multiple approaches for hesitant IVFSs and discussed their applications in D-making problems.

The idea of intuitionistic fuzzy sets (IFSs) [2] was given by Atanassov, which extends the notion of FSs by associating the degree of non-membership function. The IFS model

provides a more robust and reliable structure for modeling uncertainty of real-world applications. IFSs have been widely utilized in many fields, such as medical diagnosis, D-making, engineering, computer science, economics, and management science. Ejegwa et al. [9] presented innovative DMs within the environments of IFSs and applied them in applications to the real-life problems. Dutta and Banik discussed D-making problems based on newly defined intuitionistic fuzzy DMs [7]. Wu et al. demonstrated an innovative algorithm for solving D-making problems based on novel DMs of IFSs [34]. Lakshmi and Dhivya used intuitionistic fuzzy DMs in a D-making algorithm to address medical diagnosis and pattern recognition problems [19]. Xiao et al. [35] defined improved DMs for IFSs and discussed their practical utility in pattern recognition problems. A new intuitionistic fuzzy DM based on the symmetric distance formula was presented by Gohain et al. in [12]. They utilized it in addressing pattern recognition, D-making, and clustering problems. Tripathi et al. [30] proposed a new D-making method for the selections of food waste treatment technology within the IFS environments.

The structure of Pythagorean fuzzy sets (PFSs) [36] was established, which extends the framework of IFSs by integrating Pythagorean flexibility to membership and non-membership degrees. PFSs offer a more reliable structure for addressing MCDM problems. They are widely employed in many real-world D-Making problems. Thakur et al. [29] demonstrated practical utility of the newly defined Pythagorean fuzzy distance metrics in D-making problems. Kumar et al. [18] established  $K - L$  divergence-based DMs for PFSs and discussed their various real-life applications. Bozdog and Kadaifci proposed a distance-based technique to fuzzy cognitive maps within the environments of PFSs [4]. Yin et al. [37] developed a new Pythagorean fuzzy DM for addressing real-world applications. They constructed multiple algorithms for solving pattern recognition, D-making and medical diagnosis problems. Singh and Bisht introduced enhanced dissimilarity measurement for PFSs and used them in real-world problems [28]. Ejegwa et al. [8] demonstrated new tri-parametric Pythagorean fuzzy-based DMs and Pythagorean fuzzy-based operators. They employed the newly defined approaches in medical diagnostic analysis and pattern classification.

The FS, IVFS, IFS, and PFS models are robust and capable of resolving a number of uncertain problems. However, they cannot address problems that are two-dimensional. To handle this limitation, Ramot et al. [25] proposed the concept of CFSs. On a universe of discourse  $U$ , a CFS  $\mathcal{A}$  is defined by a membership function  $\varsigma_{\mathcal{A}}(\mathfrak{x}) = \kappa_{\mathcal{A}}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}}(\mathfrak{x})}$ . Any element  $\mathfrak{x} \in U$  is given a complex-valued grade of membership in  $\mathcal{A}$  by the membership function  $\varsigma_{\mathcal{A}}(\mathfrak{x})$ . The amplitude term in the CFS structure is denoted by  $\kappa_{\mathcal{A}}(\mathfrak{x})$ , whereas the phase term is denoted by  $\dot{C}_{\mathcal{A}}(\mathfrak{x})$ . Both of these functions are real-valued, and the values of  $\kappa_{\mathcal{A}}(\mathfrak{x})$  lies in the interval  $[0, 1]$ . CFSs are useful in many applications of image processing, engineering, computer science, D-making, economics, non-equilibrium thermodynamics, etc. Ullah et al. [31] presented some innovative aggregation operators using CFSs and applied them for solving MCDM problems. Khan et al. [17] established some generalized CFDMs and investigated their practical utility in D-making scenarios. Zeeshan et al. [39] proposed an innovative method for the identification of reference signals based on newly defined CFDMs. Zhang et al. [40] presented an application of signals detection based on  $\delta$ -equality and DMs of CFSs. Hu et al. [15] discussed the applications of newly defined DMs to continuity challenges within the environments of CFSs. Liu et al.

[20] introduced innovative entropy measures using CFSs. They employed them to address MCDM problems. Dai presented several new CF ordered weighted DMs (CFOWDMs) in [6]. He employed the proposed DMs for solving real-life problems. Sarwar et al. [26] investigated several DMs and  $\delta$ -approximations utilizing rough CFSs. They presented their practical applications in classifying unknown building materials. New methods for quantifying the distance and similarity based on CFSs have been examined by McCulloch [21]. He employed them to address complex D-making problems. Bi et al. [3] introduced two different types of entropy measures within the framework of CFSs. They investigated their key features and applications. Zhang et al. [41] proposed a DM based on CF relations. Furthermore, they applied the newly defined approaches for addressing real-world applications.

**1.1. Motivation of the Proposed WORK.** The theory of CFSs has contributed significantly in applications of signal processing due to its capability to handle uncertainty involving both phase and amplitude information. Zhang et al. [40] presented foundational concepts within the environments of CFSs. They demonstrated fundamental operational properties and  $\delta$ -equalities based on innovative CFDMs, offering a comprehensive theoretical basis for further developments. Zeeshan et al. [39] established new DMs for CFSs and discussed their applications signal-related problems, illustrating the significance of DMs in practical domains. Furthermore, Alkouri and Salleh proposed multiple DMs within the environment of CFSs, further highlighting the significance of DMs in capturing uncertainty. Despite the above essential contributions, existing DMs may not effectively capture the maximal deviation between CFSs, which is critical in precise signal identification tasks. Therefore, the proposed work is highly motivated by the need to establish such DMs for CF environments and to apply it in signal processing applications, especially for the resemblance of individual signals. The proposed DMs are significantly effective in measuring the highly discrepancy between two CFSs, provides a favorable outcome. This new approach aims to improve discrimination ability and offer a more reliable structure for analyzing CF data in real-world scenarios.

**1.2. Major Contributions.** DMs within the environments of CFSs perform huge contributions in signal processing systems and complex D-making problems. Various MCDM problems have been effectively resolved using CFDMs. In the literature, several DMs have been established using CFSs, including Euclidean, normalized Euclidean, Hamming, normalized Hamming, and Zhang DMs. This paper focuses on developing a generalized CFDM for measuring the similarities and distances between CFSs. To demonstrate the effectiveness of these techniques, a recommendation system is created, utilizing relative comparisons of subjective data. The primary research aims and objectives of this paper include:

- i). To further explore the theory of CFSs.
- ii). To establish some new set-theoretic results based on innovative operations.
- iii). To propose some improved DMs within the environments of CFSs.
- iv). To investigate their key characteristics.
- v). To design a new algorithm for signals identification based on the newly defined DMs.
- vi). To apply the proposed algorithm in signal processing applications.

vii). To show the significance and advantages of the improved DMs by comparing them with the current literature.

**1.3. Summary of the Manuscript.** The overall structure of the paper is as follows. Section 2 presents the fundamental concepts of CFS theory. Some new CF operations and set-theoretic results are given in Section 3. Proposed DMs for CFSs are defined in Section 4. Some basic results of the newly defined DMs are also established in this section. Section 5 is concerned with applications of the proposed CFDMs. Section 6 provides a comparative analysis of the newly defined DMs with existing CFDMs. Section 7 presents the conclusion of the proposed work.

## 2. PRELIMINARIES

This section presents the fundamental frameworks of CFSs. In addition, the basic operational laws and properties associated with the CFSs are systematically demonstrated to facilitate a clear theoretical foundation for subsequent developments.

[25] On a universal set  $U$ , a CFS  $\mathcal{A}$  is defined by a grade value  $\varsigma_{\mathcal{A}}(\mathfrak{x}) = \kappa_{\mathcal{A}}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}}(\mathfrak{x})}$ . Any element  $\mathfrak{x} \in U$  is given a complex-valued grade of membership in  $\mathcal{A}$ , represented by  $\varsigma_{\mathcal{A}}(\mathfrak{x})$ . The expression  $\kappa_{\mathcal{A}}(\mathfrak{x})$  termed amplitude and the expression  $\dot{C}_{\mathcal{A}}(\mathfrak{x})$  called phase term. Both of these functions are real-valued, and the values of  $\kappa_{\mathcal{A}}(\mathfrak{x})$  lies in the closed interval  $[0, 1]$ .

Mathematically, a CFS  $\mathcal{A}$  can be interpreted as follows:

$$\begin{aligned}\mathcal{A} &= \{(\mathfrak{x}; \varsigma_{\mathcal{A}}(\mathfrak{x})) : \mathfrak{x} \in U\}, \\ &= \{(\mathfrak{x}; \kappa_{\mathcal{A}}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}}(\mathfrak{x})}) : \mathfrak{x} \in U\}.\end{aligned}$$

[40] Let  $\mathcal{A}_i$ ,  $i = 1, 2, 3, \dots, n$  be  $n$  CFSs defined on  $U$  and  $\varsigma_{\mathcal{A}_i}(\mathfrak{x}) = \kappa_{\mathcal{A}_i}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_i}(\mathfrak{x})}$  represents the grade value of  $\mathcal{A}_i$ . The CF Cartesian product of these  $n$  CFSs, represented by  $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \dots \times \mathcal{A}_n$  and is defined by a function

$$\begin{aligned}\varsigma_{\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \dots \times \mathcal{A}_n}(\mathfrak{x}) &= \kappa_{\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \dots \times \mathcal{A}_n}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \dots \times \mathcal{A}_n}(\mathfrak{x})}, \\ &= \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}_1), \kappa_{\mathcal{A}_2}(\mathfrak{x}_2), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x}_n)) \\ &\quad e^{i \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_1), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_2), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x}_n))}.\end{aligned}$$

[25] Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs on  $U$ , and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  represent the membership functions of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. The CF intersection of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  is denoted by  $\mathcal{A}_1 \cap \mathcal{A}_2$ , and is defined by a function

$$\begin{aligned}\mathcal{A}_1 \cap \mathcal{A}_2 &= \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})} \cap \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}, \\ &= \min[\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})] e^{i \min[\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})]}.\end{aligned}$$

[25] Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs on  $U$ , and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  represent the membership functions of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. The CF

union of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  is expressed by  $\mathcal{A}_1 \cup \mathcal{A}_2$ , and is defined by a function

$$\begin{aligned} \mathcal{A}_1 \cup \mathcal{A}_2 &= \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})} \cup \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}, \\ &= \max[\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})] e^{i \max[\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})]}. \end{aligned}$$

### 3. SOME NEW OPERATIONS ON COMPLEX FUZZY SETS

In this subsection, we present some innovative operations within the CF environment, together with their particular examples and basic results. Note that, in the proposed concepts, the phase term  $\dot{C}_{\mathcal{A}_i}(\mathfrak{x}) \in [0, 2\pi]$  for all  $\mathfrak{x} \in U$ .

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  represent the membership functions of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. The CF complement union of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , denoted by  $\mathcal{A}_1 \Upsilon \mathcal{A}_2$ , and is defined by a function

$$\begin{aligned} \varsigma_{\mathcal{A}_1 \Upsilon \mathcal{A}_2}(\mathfrak{x}) &= \kappa_{\mathcal{A}_1 \Upsilon \mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1 \Upsilon \mathcal{A}_2}(\mathfrak{x})} \\ &= \max(1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}), 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})) \cdot e^{i \max(2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x}), 2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}))}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{A}_1 &= \frac{0.3e^{i0.5\pi}}{\mathfrak{x}_1} + \frac{0.6e^{i0.7\pi}}{\mathfrak{x}_2} + \frac{0.1e^{i1.\pi}}{\mathfrak{x}_3}, \\ \mathcal{A}_2 &= \frac{0.4e^{i0.2\pi}}{\mathfrak{x}_1} + \frac{0.5e^{i1.\pi}}{\mathfrak{x}_2} + \frac{1e^{i2\pi}}{\mathfrak{x}_3}, \end{aligned}$$

then

$$\mathcal{A}_1 \Upsilon \mathcal{A}_2 = \frac{0.7e^{i1.8\pi}}{\mathfrak{x}_1} + \frac{0.5e^{i1.3\pi}}{\mathfrak{x}_2} + \frac{0.9e^{i1.\pi}}{\mathfrak{x}_3}.$$

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs defined on  $U$ . The following result is satisfied by the CF union and CF complement union.

$$(\mathcal{A}_1 \Upsilon \mathcal{A}_2)^c = \mathcal{A}_1 \cup \mathcal{A}_2.$$

*Proof.* Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  their grade values, respectively. Then, the grade value of the right-hand side of the given identity can be interpreted as follows:

$$\begin{aligned} \varsigma_{\mathcal{A}_1 \cup \mathcal{A}_2}(\mathfrak{x}) &= \kappa_{\mathcal{A}_1 \cup \mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1 \cup \mathcal{A}_2}(\mathfrak{x})}, \\ &= \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})) \cdot e^{i \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}))}. \end{aligned} \tag{1}$$

Also, the grade value of the left-hand side of the given identity can be demonstrated by the following:

$$\begin{aligned} \varsigma_{(\mathcal{A}_1 \Upsilon \mathcal{A}_2)^c}(\mathfrak{x}) &= \kappa_{(\mathcal{A}_1 \Upsilon \mathcal{A}_2)^c}(\mathfrak{x})e^{i\dot{C}_{(\mathcal{A}_1 \Upsilon \mathcal{A}_2)^c}(\mathfrak{x})}, \\ &= \max(1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), 1 - (1 - \kappa_{\mathcal{A}_2}(\mathfrak{x}))) \\ &\quad e^{i \max(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})))} \\ &= \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})) \cdot e^{i \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}))}. \end{aligned} \tag{2}$$

Since the established grade values on both sides are same, therefore the result follows that

$$(\mathcal{A}_1 \curlyvee \mathcal{A}_2)^c = \mathcal{A}_1 \cup \mathcal{A}_2.$$

□

Let  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$  be  $n$  CFSs defined on  $U$ . The following result is satisfied by the CF union, CF intersection and CF complement union.

$$(\mathcal{A}_1 \curlyvee (\bigcap_{i=1}^n \mathcal{A}_i))^c = \bigcap_{i=1}^n (\mathcal{A}_1 \cup \mathcal{A}_i).$$

*Proof.* Let  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$  be  $n$  CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$ ,  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$ , ...,  $\varsigma_{\mathcal{A}_n}(\mathfrak{x}) = \kappa_{\mathcal{A}_n}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_n}(\mathfrak{x})}$  their grade values, respectively. Then, the grade value of the left-hand side of the given identity is illustrated as follows:

$$\begin{aligned} \varsigma_{(\mathcal{A}_1 \curlyvee (\bigcap_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x}) &= \kappa_{(\mathcal{A}_1 \curlyvee (\bigcap_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x})e^{i\dot{C}_{(\mathcal{A}_1 \curlyvee (\bigcap_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x})}, \\ &= \max \left( \begin{array}{l} 1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), \\ 1 - (1 - \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x}))) \end{array} \right) \quad (3) \\ &e^{i \max(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}. \end{aligned}$$

Also, the grade value of the right-hand side of the given identity is given as follows:

$$\begin{aligned} \varsigma_{\bigcap_{i=1}^n (\mathcal{A}_1 \cup \mathcal{A}_i)}(\mathfrak{x}) &= \kappa_{\bigcap_{i=1}^n (\mathcal{A}_1 \cup \mathcal{A}_i)}(\mathfrak{x})e^{i\dot{C}_{\bigcap_{i=1}^n (\mathcal{A}_1 \cup \mathcal{A}_i)}(\mathfrak{x})}, \\ &= \min \left( \begin{array}{l} \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x})), \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})), \\ \dots, \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_n}(\mathfrak{x})) \end{array} \right) \quad (4) \\ &e^{i \min(\max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})), \dots, \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}. \end{aligned}$$

We consider a few cases to verify the indicated result.

**Case 1.** If  $\kappa_{\mathcal{A}_1}(\mathfrak{x}) \leq \kappa_{\mathcal{A}_2}(\mathfrak{x}) \leq \dots \leq \kappa_{\mathcal{A}_n}(\mathfrak{x})$  and  $\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \leq \dot{C}_{\mathcal{A}_2}(\mathfrak{x}) \leq \dots \leq \dot{C}_{\mathcal{A}_n}(\mathfrak{x})$  then (3) and (4) implies that

$$\begin{aligned} \varsigma_{(\mathcal{A}_1 \curlyvee (\bigcap_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x}) &= \max \left( \begin{array}{l} 1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), \\ 1 - (1 - \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x}))) \end{array} \right) \\ &e^{i \max(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\ &= \max(1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), 1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}))). \\ &e^{i \max(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - 2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), \\ &= \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), (\kappa_{\mathcal{A}_1}(\mathfrak{x})) \cdot e^{i \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x}))}, \\ &= \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \quad (5) \end{aligned}$$

$$\begin{aligned}
 \varsigma_{\bigcap_{i=1}^n (\mathcal{A}_1 \cup \mathcal{A}_i)}^n(\mathfrak{x}) &= \min \left( \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x})), \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})), \dots, \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_n}(\mathfrak{x})) \right) \\
 &\quad e^{i \min(\max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})), \dots, \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\
 &= \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x})) \\
 &\quad e^{i \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x}))}, \\
 &= \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \tag{6}
 \end{aligned}$$

On the based of equations (5) and (6), the identity is verified, that is,

$$(\mathcal{A}_1 \Upsilon (\bigcap_{i=1}^n \mathcal{A}_i))^c = \bigcap_{i=1}^n (\mathcal{A}_1 \cup \mathcal{A}_i).$$

**Case 2.** If  $\kappa_{\mathcal{A}_1}(\mathfrak{x}) \geq \kappa_{\mathcal{A}_2}(\mathfrak{x}) \geq \dots \geq \kappa_{\mathcal{A}_n}(\mathfrak{x})$  and  $\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \geq \dot{C}_{\mathcal{A}_2}(\mathfrak{x}) \geq \dots \geq \dot{C}_{\mathcal{A}_n}(\mathfrak{x})$  then (3) and (4) implies that

$$\begin{aligned}
 \varsigma_{(\mathcal{A}_1 \Upsilon (\bigcap_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x}) &= \max \left( \begin{aligned} &1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), \\ &1 - (1 - \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x}))) \end{aligned} \right) \\
 &\quad e^{i \max(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\
 &= \max(1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), 1 - (1 - (\kappa_{\mathcal{A}_n}(\mathfrak{x}))). \\
 &\quad e^{i \max(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\
 &= \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), (\kappa_{\mathcal{A}_n}(\mathfrak{x})) . e^{i \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x}))}, \\
 &= \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \tag{7}
 \end{aligned}$$

$$\begin{aligned}
 \varsigma_{\bigcap_{i=1}^n (\mathcal{A}_1 \cup \mathcal{A}_i)}^n(\mathfrak{x}) &= \min \left( \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x})), \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})), \dots, \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_n}(\mathfrak{x})) \right) \\
 &\quad e^{i \min(\max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})), \dots, \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\
 &= \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_1}(\mathfrak{x})) \\
 &\quad e^{i \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_1}(\mathfrak{x}))}, \\
 &= \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \tag{8}
 \end{aligned}$$

Similarly, on the based of equations (7) and (8), the identity is proved.

A similar method can be utilized for the remaining cases. □

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  represent the grade values of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. The CF complement intersection of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , denoted by  $\mathcal{A}_1 \wedge \mathcal{A}_2$ , is defined by a function

$$\begin{aligned}
 \varsigma_{\mathcal{A}_1 \wedge \mathcal{A}_2}(\mathfrak{x}) &= \kappa_{\mathcal{A}_1 \wedge \mathcal{A}_2}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1 \wedge \mathcal{A}_2}(\mathfrak{x})} \\
 &= \min(1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}), 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})) . e^{i \min(2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x}), 2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}))}.
 \end{aligned}$$

Consider CFSs  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are given in Example 1. Then, the CF complement intersection is

$$\mathcal{A}_1 \wedge \mathcal{A}_2 = \frac{0.6e^{i1.5\pi}}{\mathfrak{x}_1} + \frac{0.4e^{i1.\pi}}{\mathfrak{x}_2} + \frac{0.e^{i0.\pi}}{\mathfrak{x}_3}.$$

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs defined on U. The following identity is satisfied by the CF intersection and CF complement intersection

$$(\mathcal{A}_1 \wedge \mathcal{A}_2)^c = \mathcal{A}_1 \cap \mathcal{A}_2.$$

*Proof.* Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  their grade values, respectively. Then, the grade value of the right-hand side of the given identity is given as follows:

$$\begin{aligned} \varsigma_{\mathcal{A}_1 \cap \mathcal{A}_2}(\mathfrak{x}) &= \kappa_{\mathcal{A}_1 \cap \mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1 \cap \mathcal{A}_2}(\mathfrak{x})}, \\ &= \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})) \cdot e^{i \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}))}. \end{aligned} \quad (8)$$

Also, the grade value of the right-hand side of the given identity can be interpreted as follows:

$$\begin{aligned} \varsigma_{(\mathcal{A}_1 \wedge \mathcal{A}_2)^c}(\mathfrak{x}) &= \kappa_{(\mathcal{A}_1 \wedge \mathcal{A}_2)^c}(\mathfrak{x})e^{i\dot{C}_{(\mathcal{A}_1 \wedge \mathcal{A}_2)^c}(\mathfrak{x})}, \\ &= \min(1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), 1 - (1 - \kappa_{\mathcal{A}_2}(\mathfrak{x}))) \\ &\quad e^{i \min(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})))} \\ &= \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})) \cdot e^{i \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}))}. \end{aligned} \quad (9)$$

On the based of equations (8) and (9), the identity is verified, that is,

$$(\mathcal{A}_1 \wedge \mathcal{A}_2)^c = \mathcal{A}_1 \cap \mathcal{A}_2.$$

□

Let  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$  be  $n$  CFSs defined on U. The following identity is satisfied by the CF union, CF intersection and CF complement intersection

$$(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c = \bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i). \quad (10)$$

*Proof.* Let  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$  be  $n$  CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$ ,  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$ , ...,  $\varsigma_{\mathcal{A}_n}(\mathfrak{x}) = \kappa_{\mathcal{A}_n}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_n}(\mathfrak{x})}$  their grade values, respectively. Then, the grade value of the left-hand side of the given identity is given as follows:

$$\begin{aligned} \varsigma_{(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x}) &= \kappa_{(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x})e^{i\dot{C}_{(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x})}, \\ &= \min \left( \begin{array}{c} 1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), \\ 1 - (1 - \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x}))) \end{array} \right) \\ &\quad e^{i \min(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}. \end{aligned} \quad (11)$$

Also, the grade value of the right-hand side of the given identity is demonstrated as follows:

$$\begin{aligned} \varsigma_{\bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i)}^n(\mathfrak{x}) &= \kappa_{\bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i)}^n(\mathfrak{x}) e^{i \dot{C}_{\bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i)}(\mathfrak{x})}, \\ &= \max \left( \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x})), \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})), \dots, \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_n}(\mathfrak{x})) \right) \\ &e^{i \max(\min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})), \dots, \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}. \end{aligned} \tag{12}$$

We consider a few cases to verify the indicated result.

**Case 1.** If  $\kappa_{\mathcal{A}_1}(\mathfrak{x}) \leq \kappa_{\mathcal{A}_2}(\mathfrak{x}) \leq \dots \leq \kappa_{\mathcal{A}_n}(\mathfrak{x})$  and  $\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \leq \dot{C}_{\mathcal{A}_2}(\mathfrak{x}) \leq \dots \leq \dot{C}_{\mathcal{A}_n}(\mathfrak{x})$  then (11) and (12) implies that

$$\begin{aligned} \varsigma_{(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x}) &= \min \left( \begin{aligned} &1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), \\ &1 - (1 - \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x}))) \end{aligned} \right) \\ &e^{i \min(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\ &= \min(1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), 1 - (1 - (\kappa_{\mathcal{A}_n}(\mathfrak{x}))) \\ &e^{i \min(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\ &= \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), (\kappa_{\mathcal{A}_n}(\mathfrak{x})) . e^{i \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x}))}, \\ &= \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \end{aligned} \tag{13}$$

$$\begin{aligned} \varsigma_{\bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i)}^n(\mathfrak{x}) &= \max \left( \begin{aligned} &\min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x})), \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})), \\ &\dots, \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_n}(\mathfrak{x})) \end{aligned} \right) \\ &e^{i \max(\min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})), \dots, \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\ &= \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_1}(\mathfrak{x})) \\ &e^{i \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_1}(\mathfrak{x}))}, \\ &= \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \end{aligned} \tag{14}$$

Equations (13) and (14) imply that

$$(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c = \bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i).$$

**Case 2.** If  $\kappa_{\mathcal{A}_1}(\mathfrak{x}) \geq \kappa_{\mathcal{A}_2}(\mathfrak{x}) \geq \dots \geq \kappa_{\mathcal{A}_n}(\mathfrak{x})$  and  $\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \geq \dot{C}_{\mathcal{A}_2}(\mathfrak{x}) \geq \dots \geq \dot{C}_{\mathcal{A}_n}(\mathfrak{x})$  then (3) and (4) implies that

$$\begin{aligned} \varsigma_{(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c}(\mathfrak{x}) &= \min \left( \begin{array}{l} 1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), \\ 1 - (1 - \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x}))) \end{array} \right) \\ &= e^{i \min(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x})))}, \\ &= \min(1 - (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})), 1 - (1 - (\kappa_{\mathcal{A}_1}(\mathfrak{x}))). \\ &= e^{i \min(2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), 2\pi - (2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x}))),} \\ &= \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), (\kappa_{\mathcal{A}_1}(\mathfrak{x})) \cdot e^{i \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x}))}, \\ &= \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \end{aligned} \quad (15)$$

$$\begin{aligned} \varsigma_{\bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i)}(\mathfrak{x}) &= \max \left( \begin{array}{l} \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_1}(\mathfrak{x})), \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x})), \\ \dots, \min(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_n}(\mathfrak{x})) \end{array} \right) \\ &= e^{i \max(\min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x})), \dots, \min(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_n}(\mathfrak{x}))),} \\ &= \max(\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}), \dots, \kappa_{\mathcal{A}_n}(\mathfrak{x})) \\ &= e^{i \max(\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}), \dots, \dot{C}_{\mathcal{A}_n}(\mathfrak{x})),} \\ &= \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}. \end{aligned} \quad (16)$$

On the based of equations (15) and (16), we have

$$(\mathcal{A}_1 \wedge (\bigcup_{i=1}^n \mathcal{A}_i))^c = \bigcup_{i=1}^n (\mathcal{A}_1 \cap \mathcal{A}_i).$$

Similarly, we can prove the other cases.  $\square$

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  represent their grade values, respectively. The CF complement product of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , denoted by  $\mathcal{A}_1 \circ \mathcal{A}_2$ , is defined by a function

$$\begin{aligned} \varsigma_{\mathcal{A}_1 \circ \mathcal{A}_2}(\mathfrak{x}) &= \kappa_{\mathcal{A}_1 \circ \mathcal{A}_2}(\mathfrak{x}) e^{i \dot{C}_{\mathcal{A}_1 \circ \mathcal{A}_2}(\mathfrak{x})} \\ &= (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})) \circ (1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})) \cdot e^{i 2\pi \left( \frac{2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}{2\pi} \right) \circ \left( \frac{2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}{2\pi} \right)}, \end{aligned}$$

where  $\circ$  denotes the ordinary multiplication.

Consider  $\mathcal{A}_1$  and  $\mathcal{A}_2$  in Example 1. Then, the CF complement product is

$$\mathcal{A}_1 \circ \mathcal{A}_2 = \frac{0.42e^{i1.35\pi}}{\mathfrak{x}_1} + \frac{0.2e^{i0.65\pi}}{\mathfrak{x}_2} + \frac{0.e^{i0.\pi}}{\mathfrak{x}_3}.$$

For any two CFSs  $\mathcal{A}_1$  and  $\mathcal{A}_2$ ,

- i).  $\mathcal{A}_1 \vee \mathcal{A}_2$  is a CFS;
- ii).  $\mathcal{A}_1 \wedge \mathcal{A}_2$  is a CFS;
- iii).  $\mathcal{A}_1 \circ \mathcal{A}_2$  is a CFS.

*Proof.* *i).* Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  their grade values, respectively. Then, by using Definition 5, we have

$$\kappa_{\mathcal{A}_1 \vee \mathcal{A}_2}(\mathfrak{x}) = \max(1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}), 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})),$$

since  $\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}) \in [0, 1]$  then  $0 \leq 1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}) \leq 1, 0 \leq 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x}) \leq 1$  and hence

$$0 \leq \max(1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}), 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})) \leq 1.$$

Also,

$$\dot{C}_{\mathcal{A}_1 \vee \mathcal{A}_2}(\mathfrak{x}) = \max(2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x}), 2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})),$$

since  $2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})$  and  $2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})$  are real-valued functions so their max is also real-valued.

Thus  $\mathcal{A}_1 \vee \mathcal{A}_2$  is a CFS.

*ii).* By using Definition 6, we have

$$\kappa_{\mathcal{A}_1 \wedge \mathcal{A}_2}(\mathfrak{x}) = \min(1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}), 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})),$$

since  $\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}) \in [0, 1]$  then  $0 \leq 1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}) \leq 1$ , and  $0 \leq 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x}) \leq 1$ . Thus,

$$0 \leq \min(1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}), 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})) \leq 1.$$

Also,

$$\dot{C}_{\mathcal{A}_1 \wedge \mathcal{A}_2}(\mathfrak{x}) = \min(2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x}), 2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})),$$

since  $2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})$  and  $2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})$  are real-valued functions so their min is also real-valued.

Thus  $\mathcal{A}_1 \wedge \mathcal{A}_2$  is a CFS.

$$\begin{aligned} \varsigma_{\mathcal{A}_1 \odot \mathcal{A}_2}(\mathfrak{x}) &= \kappa_{\mathcal{A}_1 \odot \mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1 \odot \mathcal{A}_2}(\mathfrak{x})} \\ &= (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})) \circ (1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})) \cdot e^{i2\pi \left( \frac{2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}{2\pi} \right) \circ \left( \frac{2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}{2\pi} \right)}, \end{aligned}$$

*iii).* By using Definition 7, we have

$$\kappa_{\mathcal{A}_1 \odot \mathcal{A}_2}(\mathfrak{x}) = (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})) \circ (1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})),$$

since  $\kappa_{\mathcal{A}_1}(\mathfrak{x}), \kappa_{\mathcal{A}_2}(\mathfrak{x}) \in [0, 1]$  implies that  $0 \leq 1 - \kappa_{\mathcal{A}_1}(\mathfrak{x}) \leq 1$ , and  $0 \leq 1 - \kappa_{\mathcal{A}_2}(\mathfrak{x}) \leq 1$  then,

$$0 \leq (1 - \kappa_{\mathcal{A}_1}(\mathfrak{x})) \circ (1 - \kappa_{\mathcal{A}_2}(\mathfrak{x})) \leq 1.$$

Also,

$$\dot{C}_{\mathcal{A}_1 \odot \mathcal{A}_2}(\mathfrak{x}) = 2\pi \left( \frac{2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}{2\pi} \right) \circ \left( \frac{2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}{2\pi} \right),$$

since  $\dot{C}_{\mathcal{A}_1}(\mathfrak{x}), \dot{C}_{\mathcal{A}_2}(\mathfrak{x}) \in [0, 2\pi]$  implies  $(2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})), (2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})) \in [0, 2\pi]$  then  $\left( \frac{2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}{2\pi} \right), \left( \frac{2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}{2\pi} \right) \in [0, 1]$ , and hence  $2\pi \left( \frac{2\pi - \dot{C}_{\mathcal{A}_1}(\mathfrak{x})}{2\pi} \right) \circ \left( \frac{2\pi - \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}{2\pi} \right) \in [0, 2\pi]$ , which is a real-valued function. Thus,  $\mathcal{A}_1 \odot \mathcal{A}_2$  is a CFS.  $\square$

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  their grade values, respectively. Then, the arithmetic mean operation between them is denoted by  $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ , is defined by a function

$$\begin{aligned}\mathcal{A}_1 \boxtimes \mathcal{A}_2 &= \varsigma_{\mathcal{A}_1 \boxtimes \mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_1 \boxtimes \mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1 \boxtimes \mathcal{A}_2}(\mathfrak{x})}, \\ &= \frac{\kappa_{\mathcal{A}_1}(\mathfrak{x}) + \kappa_{\mathcal{A}_2}(\mathfrak{x})}{2} \cdot e^{i\left(\frac{\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) + \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}{2}\right)}.\end{aligned}$$

Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  their grade values, respectively. Then, the geometric mean operation between them is denoted by  $\mathcal{A}_1 \square \mathcal{A}_2$ , is defined by a function

$$\begin{aligned}\mathcal{A}_1 \square \mathcal{A}_2 &= \varsigma_{\mathcal{A}_1 \square \mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_1 \square \mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1 \square \mathcal{A}_2}(\mathfrak{x})}, \\ &= \sqrt{\kappa_{\mathcal{A}_1}(\mathfrak{x}) \cdot \kappa_{\mathcal{A}_2}(\mathfrak{x})} \cdot e^{i2\pi\left(\sqrt{\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \cdot \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}\right)}.\end{aligned}$$

For any two CFSs  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , we have

- i).  $\mathcal{A}_1 \boxtimes \mathcal{A}_2$  is a CFS;
- ii).  $\mathcal{A}_1 \square \mathcal{A}_2$  is a CFS.

*Proof.* i). Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two CFSs, and  $\varsigma_{\mathcal{A}_1}(\mathfrak{x}) = \kappa_{\mathcal{A}_1}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_1}(\mathfrak{x})}$  and  $\varsigma_{\mathcal{A}_2}(\mathfrak{x}) = \kappa_{\mathcal{A}_2}(\mathfrak{x})e^{i\dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  their grade values, respectively. Then, by using Definition 8, we have

$$\kappa_{\mathcal{A}_1 \boxtimes \mathcal{A}_2}(\mathfrak{x}) = \frac{\kappa_{\mathcal{A}_1}(\mathfrak{x}) + \kappa_{\mathcal{A}_2}(\mathfrak{x})}{2},$$

since  $0 \leq \kappa_{\mathcal{A}_1}(\mathfrak{x}) \leq 1$ ,  $0 \leq \kappa_{\mathcal{A}_2}(\mathfrak{x}) \leq 1$  then  $0 \leq \frac{\kappa_{\mathcal{A}_1}(\mathfrak{x}) + \kappa_{\mathcal{A}_2}(\mathfrak{x})}{2} \leq 1$ .

Also,

$$\dot{C}_{\mathcal{A}_1 \boxtimes \mathcal{A}_2}(\mathfrak{x}) = \frac{\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) + \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}{2},$$

since  $\dot{C}_{\mathcal{A}_1}(\mathfrak{x})$  and  $\dot{C}_{\mathcal{A}_2}(\mathfrak{x})$  are real-valued functions so their mean is also real-valued.

Thus  $\mathcal{A}_1 \boxtimes \mathcal{A}_2$  is a CFS.

ii). By using Definition 9, we have

$$\kappa_{\mathcal{A}_1 \square \mathcal{A}_2}(\mathfrak{x}) = \sqrt{\kappa_{\mathcal{A}_1}(\mathfrak{x}) \cdot \kappa_{\mathcal{A}_2}(\mathfrak{x})},$$

since  $0 \leq \kappa_{\mathcal{A}_1}(\mathfrak{x}) \leq 1$ ,  $0 \leq \kappa_{\mathcal{A}_2}(\mathfrak{x}) \leq 1$  then

$$0 \leq \sqrt{\kappa_{\mathcal{A}_1}(\mathfrak{x})} \leq 1, 0 \leq \sqrt{\kappa_{\mathcal{A}_2}(\mathfrak{x})} \leq 1,$$

and

$$0 \leq \sqrt{\kappa_{\mathcal{A}_1}(\mathfrak{x}) \cdot \kappa_{\mathcal{A}_2}(\mathfrak{x})} \leq 1.$$

Also,

$$\dot{C}_{\mathcal{A}_1 \square \mathcal{A}_2}(\mathfrak{x}) = 2\pi \left( \sqrt{\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \cdot \dot{C}_{\mathcal{A}_2}(\mathfrak{x})} \right),$$

since  $\sqrt{\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \cdot \dot{C}_{\mathcal{A}_2}(\mathfrak{x})}$  is real-valued so  $2\pi \left( \sqrt{\dot{C}_{\mathcal{A}_1}(\mathfrak{x}) \cdot \dot{C}_{\mathcal{A}_2}(\mathfrak{x})} \right)$  is also real-valued.

Thus,  $\mathcal{A}_1 \square \mathcal{A}_2$  is a CFS.  $\square$

4. DISTANCE MEASURES OF COMPLEX FUZZY SETS

We recall some DMs for CFSs such as Zeeshan DM [39], Zhang DM [40], Normalized Hamming DM, and Hamming DM.

i). The Zeeshan distance [39],

$$d(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2n} \sum_{r=1}^n \left[ \frac{|\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)|}{1 + |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)|} + \frac{|\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)|}{2\pi + |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)|} \right]. \tag{17}$$

ii). The Zhang distance [40],

$$d(\mathcal{A}_1, \mathcal{A}_2) = \max \left[ \sup_{\mathfrak{x}_r \in U} |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)|, \frac{1}{2\pi} \sup_{\mathfrak{x}_r \in U} |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)| \right]. \tag{18}$$

iii). The Hamming distance [1],

$$d(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2} \left[ \sum_{r=1}^n |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)| + \frac{1}{2\pi} \sum_{r=1}^n |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)| \right]. \tag{19}$$

iv). The Normalized Hamming distance [1],

$$d(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2n} \left[ \sum_{r=1}^n |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)| + \frac{1}{2\pi} \sum_{r=1}^n |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)| \right]. \tag{20}$$

Now we propose another DM under the environment of CFSs.

Let  $\mathcal{A}(U)$  represent a collections of CFSs. A function  $\hat{H} : \mathcal{A}(U) \times \mathcal{A}(U) \rightarrow [0, 1]$  is said to be a DM of CFSs if the following properties hold.

- i).  $0 \leq \hat{H}(\mathcal{A}_1, \mathcal{A}_2) \leq 1$ .
- ii).  $\hat{H}(\mathcal{A}_1, \mathcal{A}_2) = 0$  if and only if  $\mathcal{A}_1 = \mathcal{A}_2$ .
- iii).  $\hat{H}(\mathcal{A}_1, \mathcal{A}_2) = \hat{H}(\mathcal{A}_2, \mathcal{A}_1)$ .
- iiii).  $\hat{H}(\mathcal{A}_1, \mathcal{A}_3) \leq \hat{H}(\mathcal{A}_1, \mathcal{A}_2) + \hat{H}(\mathcal{A}_2, \mathcal{A}_3)$ .

We introduce the DM  $\hat{H}$  as:

$$\hat{H}(\mathcal{A}_1, \mathcal{A}_2) = \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_1} d(\mathfrak{x}_i, \mathcal{A}_2), \bigvee_{\mathfrak{x}_j \in \mathcal{A}_2} d(\mathfrak{x}_j, \mathcal{A}_1) \right], \tag{21}$$

where  $d$  is defined in equations (17)-(20).

Let

$$\mathcal{A}_1 = \frac{0.3e^{i0.5\pi}}{\mathfrak{x}_1} + \frac{0.6e^{i0.7\pi}}{\mathfrak{x}_2} + \frac{0.1e^{i1.\pi}}{\mathfrak{x}_3},$$

$$\mathcal{A}_2 = \frac{0.4e^{i0.2\pi}}{\mathfrak{x}_1} + \frac{0.5e^{i1.\pi}}{\mathfrak{x}_2} + \frac{1e^{i2\pi}}{\mathfrak{x}_3}.$$

To compute  $\hat{H}(\mathcal{A}_1, \mathcal{A}_2)$  using Zhang DM  $d(\mathcal{A}_1, \mathcal{A}_2)$ . For this we have

$$\begin{aligned}\bigvee_{\mathfrak{x}_i \in \mathcal{A}_1} d(\mathfrak{x}_i, \mathcal{A}_2) &= \max_{\mathfrak{x}_i \in \mathcal{A}_1} [0.7, 0.65, 0.9], \\ &= 0.9. \\ \bigvee_{\mathfrak{x}_j \in \mathcal{A}_2} d(\mathfrak{x}_j, \mathcal{A}_1) &= \max_{\mathfrak{x}_j \in \mathcal{A}_1} [0.4, 0.4, 0.9], \\ &= 0.9.\end{aligned}$$

Now,

$$\begin{aligned}\hat{H}(\mathcal{A}_1, \mathcal{A}_2) &= \max [0.9, 0.9], \\ &= 0.9.\end{aligned}$$

Similarly, we can use the DMs defined in (17), (19), and (20) to compute  $\hat{H}(\mathcal{A}_1, \mathcal{A}_2)$ .

The function  $\hat{H}$  defined by the equation (21) is a distance function of CFSs on  $\mathcal{U}$ .

*Proof.* The condition  $\hat{H}(\mathcal{A}_i, \mathcal{A}_j) \geq 0$  is straightforward. Also, we have

$$\begin{aligned}\hat{H}(\mathcal{A}_i, \mathcal{A}_j) &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_j \in \mathcal{A}_j} d(\mathfrak{x}_j, \mathcal{A}_i) \right], \\ &= \max [1, 1], \\ &= 1.\end{aligned}$$

Therefore  $0 \leq \hat{H}(\mathcal{A}_i, \mathcal{A}_j) \leq 1$ , and

$$\begin{aligned}\hat{H}(\mathcal{A}_i, \mathcal{A}_i) &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_i), \bigvee_{\mathfrak{x}_j \in \mathcal{A}_i} d(\mathfrak{x}_j, \mathcal{A}_i) \right], \\ &= \max [0, 0], \\ &= 0.\end{aligned}$$

The condition (iii) can be easily prove by definition. To prove (iv); we have

$$\begin{aligned}\hat{H}(\mathcal{A}_i, \mathcal{A}_k) &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_k), \bigvee_{\mathfrak{x}_j \in \mathcal{A}_k} d(\mathfrak{x}_j, \mathcal{A}_i) \right], \\ &= \max \left[ \begin{aligned} &\bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) + d(\mathcal{A}_j, \mathcal{A}_k), \\ &\bigvee_{\mathfrak{x}_j \in \mathcal{A}_k} d(\mathfrak{x}_j, \mathcal{A}_j) + d(\mathcal{A}_j, \mathcal{A}_i) \end{aligned} \right], \\ &\leq \max \left[ \begin{aligned} &\bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) + \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_j, \mathcal{A}_k), \\ &\bigvee_{\mathfrak{x}_j \in \mathcal{A}_k} d(\mathfrak{x}_j, \mathcal{A}_j) + \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_j, \mathcal{A}_i) \end{aligned} \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_j, \mathcal{A}_i) \right] + \\ &\max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_j, \mathcal{A}_k), \bigvee_{\mathfrak{x}_j \in \mathcal{A}_k} d(\mathfrak{x}_j, \mathcal{A}_j) \right], \\ &= \hat{H}(\mathcal{A}_i, \mathcal{A}_j) + \hat{H}(\mathcal{A}_j, \mathcal{A}_k).\end{aligned}$$

Thus  $\hat{H}(\mathcal{A}_i, \mathcal{A}_k) \leq \hat{H}(\mathcal{A}_i, \mathcal{A}_j) + \hat{H}(\mathcal{A}_j, \mathcal{A}_k)$ .  $\square$

Let  $\hat{H}$  be a DM of CFSs. Then, the max – min DM of two CFSs  $\mathcal{A}_i$  and  $\mathcal{A}_j$  is denoted by  $\hat{H}(\mathcal{A}_i \cup \mathcal{A}_j, \mathcal{A}_i \cap \mathcal{A}_j)$  and is defined by

$$\hat{H}(\mathcal{A}_i \cup \mathcal{A}_j, \mathcal{A}_i \cap \mathcal{A}_j) = \max \left[ \begin{array}{l} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bigvee \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \\ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bar{\wedge} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \end{array} \right],$$

where the max and min operators of CFSs are shown by the symbols  $\bigvee$  and  $\bar{\wedge}$ , respectively.

Let  $\hat{H}$  be a DM of CFSs. Then, the following hold

- i).  $\hat{H}(\mathcal{A}_i \cup \mathcal{A}_j, \mathcal{A}_i \cap \mathcal{A}_j) = \hat{H}(\mathcal{A}_i, \mathcal{A}_j)$ ,
- ii).  $\hat{H}(\mathcal{A}_i, \mathcal{A}_i \cap \mathcal{A}_j) = \hat{H}(\mathcal{A}_j, \mathcal{A}_i \cup \mathcal{A}_j)$ ,
- iii).  $\hat{H}(\mathcal{A}_i, \mathcal{A}_i \cup \mathcal{A}_j) = \hat{H}(\mathcal{A}_j, \mathcal{A}_i \cap \mathcal{A}_j)$ .

*Proof.* To prove (i); there are two cases arise here.

**Case 1.**

$$\bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \geq \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i),$$

$$\begin{aligned} \hat{H}(\mathcal{A}_i \cup \mathcal{A}_j, \mathcal{A}_i \cap \mathcal{A}_j) &= \max \left[ \begin{array}{l} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bigvee \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \\ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bar{\wedge} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \end{array} \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \hat{H}(\mathcal{A}_i, \mathcal{A}_j). \end{aligned}$$

**Case 2.**

$$\bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \leq \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i),$$

$$\begin{aligned} \hat{H}(\mathcal{A}_i \cup \mathcal{A}_j, \mathcal{A}_i \cap \mathcal{A}_j) &= \max \left[ \begin{array}{l} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bigvee \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \\ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bar{\wedge} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \end{array} \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \hat{H}(\mathcal{A}_i, \mathcal{A}_j). \end{aligned}$$

*ii*). To prove (ii); if  $\bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \geq \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i)$  then,

$$\begin{aligned} \hat{H}(\mathcal{A}_i, \mathcal{A}_i \cap \mathcal{A}_j) &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bar{\wedge} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \vee \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \hat{H}(\mathcal{A}_i, \mathcal{A}_i \cup \mathcal{A}_j). \end{aligned}$$

*iii*). To prove (iii); if  $\bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \leq \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i)$  then,

$$\begin{aligned} \hat{H}(\mathcal{A}_i, \mathcal{A}_i \cup \mathcal{A}_j) &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \vee \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \right], \\ &= \max \left[ \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i), \bigvee_{\mathfrak{x}_i \in \mathcal{A}_i} d(\mathfrak{x}_i, \mathcal{A}_j) \bar{\wedge} \bigvee_{\mathfrak{x}_i \in \mathcal{A}_j} d(\mathfrak{x}_i, \mathcal{A}_i) \right], \\ &= \hat{H}(\mathcal{A}_j, \mathcal{A}_i \cap \mathcal{A}_j). \end{aligned}$$

□

## 5. APPLICATIONS OF COMPLEX FUZZY SETS

This section presents a D-making algorithm within the environments of CFSs. The newly defined DMs are employed to propose a new technique which is crucial for indentifying a precise location of digital receiver. This innovative technique is based on the framework of inverse discrete Fourier transform (IDFT), which apply to both the received signals and known signal. This method allows the transformation of frequency-domain expressions into the time domain, processing of the signal characteristics and supporting effective analysis. The concept of IDFT is demonstrated as follows.

**5.1. Inverse Discrete Fourier Transform.** Suppose the DFT  $X : \mathbb{Z} \rightarrow \mathbb{C}$  of the unknown signal is:

$$X(k) = \sum_{\mathfrak{x}=0}^{N-1} \Gamma(\mathfrak{x}) e^{-i2\pi \frac{k\mathfrak{x}}{N}}; \quad k, \mathfrak{x} = 0, 1, 2, \dots, N-1.$$

The IDFT of  $X$  is defined as the signal  $\Gamma : [0, N - 1] \rightarrow \mathbb{C}$  with components  $\Gamma(\mathfrak{x})$  given by the expression

$$\Gamma(\mathfrak{x}) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{i2\pi \frac{k\mathfrak{x}}{N}}; \quad (22)$$

$$k, \mathfrak{x} = 0, 1, 2, \dots, N - 1.$$

Note that in the IDFT,  $X(k)$  has different values.

In the procedure that follows, we consider the specific scenario when  $X(k)$  is limited to a closed interval  $[0, 1]$  because the amplitude term in CFS contains all of the values in the closed interval  $[0, 1]$ .

Thus in equation (22), the term  $X(k) e^{i2\pi \frac{k\mathfrak{x}}{N}}$  becomes a CFS.

For the digital receiver, we propose an algorithm based on the newly defined DMs within the environment of CFSs. Figure 1 reflects the schematic illustration of the proposed algorithm.

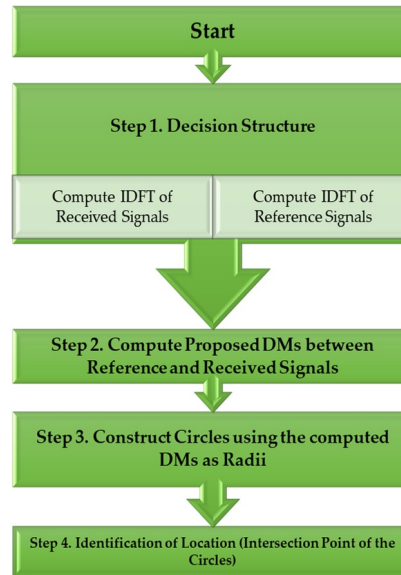
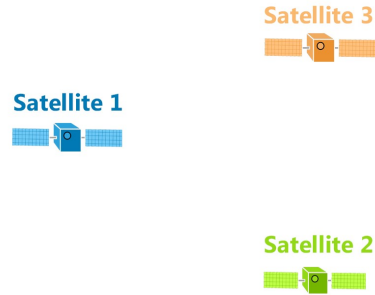


Figure 1. Schematic illustration of the proposed algorithm.

**5.2. Algorithm.** The subsequent steps highlight the method for employing DMs  $\hat{H}$  constructed under CFSs to identify the location of the digital receiver. This process systematically integrates the uncertainty and phase-related assessment captured by the framework of CFSs, allowing a more reliable evaluation in the localization procedure. These steps are demonstrated as follows.

**Step 1.**

Let us consider three satellites, each having a known location, as represented in Figure 2.



**Figure 2.** Location of three different satellites.

The primary function of all satellites is to transmit signals toward the Earth that a digital receiver will be able to receive at a particular time and location. The digital receiver receives a signal called  $\Gamma^{(1)}(\mathfrak{x})$  that is broadcast by the first satellite. The IDFT of the signal  $\Gamma^{(1)}(\mathfrak{x})$  is given by

$$\Gamma^{(1)}(\mathfrak{x}) = \frac{1}{N} \sum_{k=0}^{N-1} X^{(1)}(k) e^{i2\pi \frac{k\mathfrak{x}}{N}},$$

$$k, \mathfrak{x} = 0, 1, 2, \dots, N - 1. \quad (23)$$

Similarly, the second and third satellites broadcast signals  $\Gamma^{(2)}(\mathfrak{x})$  and  $\Gamma^{(3)}(\mathfrak{x})$  that eventually hit the digital receiver. The IDFT of the signals  $\Gamma^{(2)}(\mathfrak{x})$  and  $\Gamma^{(3)}(\mathfrak{x})$  are given by

$$\Gamma^{(2)}(\mathfrak{x}) = \frac{1}{N} \sum_{k=0}^{N-1} X^{(2)}(k) e^{i2\pi \frac{k\mathfrak{x}}{N}},$$

$$k, \mathfrak{x} = 0, 1, 2, \dots, N - 1. \quad (24)$$

$$\Gamma^{(3)}(\mathfrak{x}) = \frac{1}{N} \sum_{k=0}^{N-1} X^{(3)}(k) e^{i2\pi \frac{k\mathfrak{x}}{N}},$$

$$k, \mathfrak{x} = 0, 1, 2, \dots, N - 1. \quad (25)$$

Furthermore, the IDFT of the known signal, or reference signal, is provided by

$$\mathbb{R}(\mathfrak{x}) = \frac{1}{N} \sum_{k=0}^{N-1} R(k)e^{i2\pi \frac{k\mathfrak{x}}{N}},$$

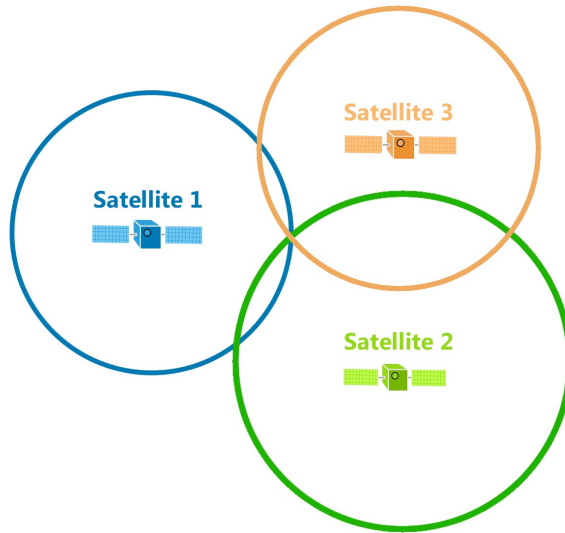
$$k, \mathfrak{x} = 0, 1, 2, \dots, N - 1. \tag{26}$$

**Step 2.**

Compute the newly defined DMs  $\hat{H}_i; i = 1, 2, 3$  between a reference signal and received signals.

**Step 3.**

Draw three circles equal in all directions of radius  $\hat{H}_i; i = 1, 2, 3$ , as illustrated in Figure 3.



**Figure 3.** Construction of circles using the computed DMs as Radii.

**Step 4.**

The above figure, reveals the true location of the digital receiver where all three circles intersect.

Using three DMs of CFSs, trilateration can pinpoint the precise location of the digital receiver. Each satellite is at the center of a sphere, where they are all intersect, represents the position of the digital receiver.

Each circle's radius, or distance, will vary in accordance with the digital receiver's position.

**5.3. Example.** Let  $\Gamma^{(1)}(\mathfrak{x})$ ,  $\Gamma^{(2)}(\mathfrak{x})$ , and  $\Gamma^{(3)}(\mathfrak{x})$  be three signals from three different satellites detected by a digital receiver. Each of these time domain signals is sampled four times. Let  $\mathbb{R}(\mathfrak{x})$  be the known signal. The IDFT of the  $\Gamma^{(1)}(\mathfrak{x})$ ,  $\Gamma^{(2)}(\mathfrak{x})$ ,  $\Gamma^{(3)}(\mathfrak{x})$  and

$\mathbb{R}(\mathfrak{x})$  are given by

$$\Gamma^{(1)}(\mathfrak{x}) = \frac{1}{4} \sum_{k=0}^3 X^{(1)}(k) e^{i2\pi \frac{k\mathfrak{x}}{4}}; k, \mathfrak{x} = 0, 1, 2, 3, \quad (27)$$

$$\Gamma^{(2)}(\mathfrak{x}) = \frac{1}{4} \sum_{k=0}^3 X^{(2)}(k) e^{i2\pi \frac{k\mathfrak{x}}{4}}; k, \mathfrak{x} = 0, 1, 2, 3, \quad (28)$$

$$\Gamma^{(3)}(\mathfrak{x}) = \frac{1}{4} \sum_{k=0}^3 X^{(3)}(k) e^{i2\pi \frac{k\mathfrak{x}}{4}}; k, \mathfrak{x} = 0, 1, 2, 3, \quad (29)$$

$$\mathbb{R}(\mathfrak{x}) = \frac{1}{4} \sum_{k=0}^3 R(k) e^{i2\pi \frac{k\mathfrak{x}}{4}}; k, \mathfrak{x} = 0, 1, 2, 3, \quad (30)$$

where  $X^{(i)}(k), R(k) \in [0, 1]$ .

**Step1.**

The sample values of  $\Gamma^{(1)}(\mathfrak{x}), \Gamma^{(2)}(\mathfrak{x}), \Gamma^{(3)}(\mathfrak{x})$ , and  $\mathbb{R}(\mathfrak{x})$  can be rearranged using the following table by incorporating the specific values of amplitude terms and phase terms to describe our suggested technique.

Signals	$\Gamma^{(1)}(\mathfrak{x})$	$\Gamma^{(2)}(\mathfrak{x})$	$\Gamma^{(3)}(\mathfrak{x})$	$\mathbb{R}(\mathfrak{x})$
0	$0.4e^{i1\pi}$	$0.5e^{i0.4\pi}$	$0.3e^{i1.4\pi}$	$0.8e^{i2\pi}$
1	$0.6e^{i2\pi}$	$0.8e^{i0.2\pi}$	$1e^{i0.8\pi}$	$0.4e^{i1\pi}$
2	$0.8e^{i1.4\pi}$	$0.2e^{i0.9\pi}$	$0.8e^{i1\pi}$	$0.9e^{i2\pi}$
3	$0.2e^{i1.8\pi}$	$1e^{i2\pi}$	$0.7e^{i2\pi}$	$0.5e^{i1\pi}$

**Table 1.** Sampling values of signals

**Step 2.**

Compute the CFDMs  $\hat{H}_1(\Gamma^{(1)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})), \hat{H}_1(\Gamma^{(2)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})), \hat{H}_1(\Gamma^{(3)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x}))$  between the reference signal and received signals, i.e.,

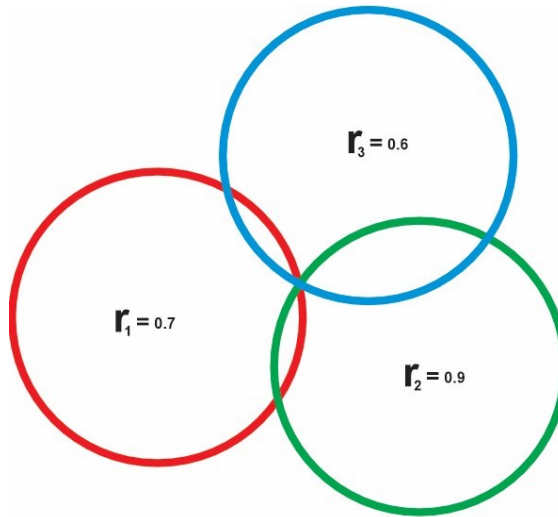
$$\begin{aligned} \hat{H}_1(\Gamma^{(1)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) &= \max \left[ \bigvee_{\mathfrak{x} \in \Gamma^{(1)}(\mathfrak{x})} d(\mathfrak{x}_i, \mathbb{R}(\mathfrak{x})), \bigvee_{\mathfrak{x} \in \mathbb{R}(\mathfrak{x})} d(\mathfrak{x}_j, \Gamma^{(1)}(\mathfrak{x})) \right], \\ &= \max \left[ \begin{array}{l} \max_{\mathfrak{x} \in \Gamma^{(1)}(\mathfrak{x})} \{0.5, 0.5, 0.4, 0.7\}, \\ \max_{\mathfrak{x} \in \mathbb{R}(\mathfrak{x})} \{0.6, 0.5, 0.7, 0.5\} \end{array} \right], \\ &= 0.7. \end{aligned}$$

$$\begin{aligned} \hat{H}_2(\Gamma^{(2)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) &= \max \left[ \bigvee_{\mathfrak{x} \in \Gamma^{(2)}(\mathfrak{x})} d(\mathfrak{x}_i, \mathbb{R}(\mathfrak{x})), \bigvee_{\mathfrak{x} \in \mathbb{R}(\mathfrak{x})} d(\mathfrak{x}_j, \Gamma^{(2)}(\mathfrak{x})) \right], \\ &= \max \left[ \begin{array}{l} \max_{\mathfrak{x} \in \Gamma^{(1)}(\mathfrak{x})} \{0.8, 0.9, 0.7, 0.6\}, \\ \max_{\mathfrak{x} \in \mathbb{R}(\mathfrak{x})} \{0.9, 0.6, 0.9, 0.5\} \end{array} \right], \\ &= 0.9. \end{aligned}$$

$$\begin{aligned} \hat{H}_3(\Gamma^{(3)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) &= \max \left[ \bigvee_{\mathfrak{x} \in \Gamma^{(3)}(\mathfrak{x})} d(\mathfrak{x}_i, \mathbb{R}(\mathfrak{x})), \bigvee_{\mathfrak{x} \in \mathbb{R}(\mathfrak{x})} d(\mathfrak{x}_j, \Gamma^{(3)}(\mathfrak{x})) \right], \\ &= \max \left[ \begin{array}{l} \max_{\mathfrak{x} \in \Gamma^{(1)}(\mathfrak{x})} \{0.6, 0.6, 0.5, 0.5\}, \\ \max_{\mathfrak{x} \in \mathbb{R}(\mathfrak{x})} \{0.6, 0.6, 0.6, 0.5\} \end{array} \right], \\ &= 0.6. \end{aligned}$$

**Step 3.**

Now, we draw three circles equal in all directions of radii  $\hat{H}_1(\Gamma^{(1)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) = r_1 = 0.7$ ,  $\hat{H}_2(\Gamma^{(2)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) = r_2 = 0.9$ ,  $\hat{H}_3(\Gamma^{(3)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) = r_3 = 0.6$ , which is illustrated in Figure 4.



**Figure 4.** Construction of circles using  $r_1 = 0.7$ ,  $r_2 = 0.9$ ,  $r_3 = 0.6$ .

**Step 4.**

Thus the intersecting point of all the three circles of radii 0.6, 0.9, and 0.7 reveals the true location of digital receiver.

## 6. COMPARISON ANALYSIS

In this section, we present the comparison analysis of the newly defined DMs within the environments of CFSs with Zeeshan distance [39], Zhang distance [40], Normalized Hamming distance [1] and Hamming distance [1].

Here, we will discuss that our proposed DM is the extended approach of all the DMs. The comparison of the proposed DMs of CFSs with Zeeshan distance [39], Zhang distance [40], Normalized Hamming distance and Hamming distance is demonstrated in Remarks 1-4. Additionally, we have examined the illustrative numerical data in the context of other DMs.

**Remark 1.** Our proposed DM within the environments of CFSs reduces to Zeeshan DM [39] if we consider the second part of  $\hat{H}$  is zero and define the DM as:

$$d(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2n} \sum_{r=1}^n \left[ \frac{|\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)|}{1 + |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)|} + \frac{|\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)|}{2\pi + |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)|} \right],$$

then our proposed DM reduces to Zeeshan's DM.

**Remark 2.** The proposed DM within the environments of CFSs simplifies to Zhang's DM if we assess the second part of  $\hat{H}$  as zero and define the DM as:

$$d(\mathcal{A}_1, \mathcal{A}_2) = \max \left[ \sup_{\mathfrak{x}_r \in \mathcal{U}} |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)|, \frac{1}{2\pi} \sup_{\mathfrak{x}_r \in \mathcal{U}} |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)| \right].$$

Here,  $d$  is Zhang's DM within the environment of CFSs.

**Remark 3.** Our proposed CFDM becomes the Hamming DM if we take the second part of  $\hat{H}$  to be zero and introduce  $d$  as:

$$d(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2} \left[ \sum_{r=1}^n |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)| + \frac{1}{2\pi} \sum_{r=1}^n |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)| \right],$$

which represents the complex fuzzy Hamming DM.

**Remark 4.** The proposed CFDM can be reduced to Normalized Hamming DM if the second part of  $\hat{H}$  becomes zero and defined the DM as:

$$d(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2n} \left[ \sum_{r=1}^n |\kappa_{\mathcal{A}_1}(\mathfrak{x}_r) - \kappa_{\mathcal{A}_2}(\mathfrak{x}_r)| + \frac{1}{2\pi} \sum_{r=1}^n |\dot{C}_{\mathcal{A}_1}(\mathfrak{x}_r) - \dot{C}_{\mathcal{A}_2}(\mathfrak{x}_r)| \right].$$

The distance function  $d$  defined above represents the Normalized Hamming DM.

Note that we only consider Remark 1 for the identification of the location of the GPS receiver.

The specific values of the amplitude and phase terms of the signals  $\Gamma^{(1)}(\mathfrak{x})$ ,  $\Gamma^{(2)}(\mathfrak{x})$ ,  $\Gamma^{(3)}(\mathfrak{x})$ , and  $\mathbb{R}(\mathfrak{x})$  are given in Table 2.

Signals	$\Gamma^{(1)}(\mathfrak{x})$	$\Gamma^{(2)}(\mathfrak{x})$	$\Gamma^{(3)}(\mathfrak{x})$	$\mathbb{R}(\mathfrak{x})$
0	$0.4e^{i1\pi}$	$0.5e^{i0.4\pi}$	$0.3e^{i1.4\pi}$	$0.8e^{i2\pi}$
1	$0.6e^{i2\pi}$	$0.8e^{i0.2\pi}$	$1e^{i0.8\pi}$	$0.4e^{i1\pi}$
2	$0.8e^{i1.4\pi}$	$0.2e^{i0.9\pi}$	$0.8e^{i1\pi}$	$0.9e^{i2\pi}$
3	$0.2e^{i1.8\pi}$	$1e^{i2\pi}$	$0.7e^{i2\pi}$	$0.5e^{i1\pi}$

**Table 2.** Particular values of known and received signal

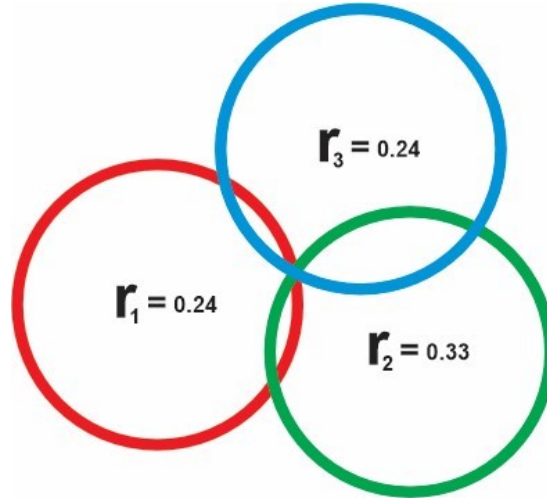
Compute the CFDMs  $d(\Gamma^{(1)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x}))$ ,  $d(\Gamma^{(2)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x}))$ ,  $d(\Gamma^{(3)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x}))$  of the received signals and reference signals, i.e.,

$$\begin{aligned} d(\Gamma^{(1)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) &= \frac{1}{2(4)} \sum_{\mathfrak{x}=1}^3 \left[ \frac{|\kappa_{\Gamma^{(1)}(\mathfrak{x})}(\mathfrak{x}) - \kappa_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|}{1 + |\kappa_{\Gamma^{(1)}(\mathfrak{x})}(\mathfrak{x}) - \kappa_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|} + \frac{|\dot{C}_{\Gamma^{(1)}(\mathfrak{x})}(\mathfrak{x}) - \dot{C}_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|}{2\pi + |\dot{C}_{\Gamma^{(1)}(\mathfrak{x})}(\mathfrak{x}) - \dot{C}_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|} \right], \\ &= \frac{1}{8} [0.29 + 0.33 + 0.17 + 0.33 + 0.09 + 0.23 + 0.23 + 0.27], \\ &= 0.24. \end{aligned}$$

$$\begin{aligned} d(\Gamma^{(2)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) &= \frac{1}{2(4)} \sum_{\mathfrak{x}=1}^3 \left[ \frac{|\kappa_{\Gamma^{(2)}(\mathfrak{x})}(\mathfrak{x}) - \kappa_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|}{1 + |\kappa_{\Gamma^{(2)}(\mathfrak{x})}(\mathfrak{x}) - \kappa_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|} + \frac{|\dot{C}_{\Gamma^{(2)}(\mathfrak{x})}(\mathfrak{x}) - \dot{C}_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|}{2\pi + |\dot{C}_{\Gamma^{(2)}(\mathfrak{x})}(\mathfrak{x}) - \dot{C}_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|} \right], \\ &= \frac{1}{8} [0.23 + 0.44 + 0.29 + 0.29 + 0.41 + 0.35 + 0.33 + 0.33], \\ &= 0.33. \end{aligned}$$

$$\begin{aligned} d(\Gamma^{(3)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) &= \frac{1}{2(4)} \sum_{\mathfrak{x}=1}^3 \left[ \frac{|\kappa_{\Gamma^{(3)}(\mathfrak{x})}(\mathfrak{x}) - \kappa_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|}{1 + |\kappa_{\Gamma^{(3)}(\mathfrak{x})}(\mathfrak{x}) - \kappa_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|} + \frac{|\dot{C}_{\Gamma^{(3)}(\mathfrak{x})}(\mathfrak{x}) - \dot{C}_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|}{2\pi + |\dot{C}_{\Gamma^{(3)}(\mathfrak{x})}(\mathfrak{x}) - \dot{C}_{\mathbb{R}(\mathfrak{x})}(\mathfrak{x})|} \right], \\ &= \frac{1}{8} [0.33 + 0.23 + 0.38 + 0.09 + 0.09 + 0.33 + 0.17 + 0.33], \\ &= 0.24. \end{aligned}$$

Draw three circles equal in all directions of radii  $\hat{H}_1(\Gamma^{(1)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) = 0.24$ ,  $\hat{H}_2(\Gamma^{(2)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) = 0.33$ ,  $\hat{H}_3(\Gamma^{(3)}(\mathfrak{x}), \mathbb{R}(\mathfrak{x})) = 0.24$ , which is provided in Figure 5.



**Figure 5.** Construction of circles using  $r_1 = 0.24$ ,  $r_2 = 0.33$ ,  $r_3 = 0.24$ .

Thus the intersecting point of all three circles of radii 0.24, 0.33, and 0.24 reveals the true location of the digital receiver.

**6.1. Advantages/Significance of Proposed Algorithm.** Our proposed algorithm is based on a generalized CFDMs. Here are some advantages of using the proposed algorithm and proposed CFDMs.

i). The newly defined DMs can be employed to sets of points, curves, shapes, or any type of evaluation data that can be reflected as sets. This versatility allows this new algorithm applicable for a wide range of practical utility.

ii). The proposed DMs illustrates the maximum separation between two CFSs, which is crucial in various real-world applications.

iii). The newly defined DMs can handle local differences between two CFSs. It considers the maximum closest distance from each point in one set to the other, which makes it sensitive to localized differences in shape or point distribution. Thus, the demonstrated technique is applicable for shape matching, object recognition, and image registration tasks.

But the existing CFDMs can not be applied to sets of points, curves, shapes, object recognition, image registration tasks or any type of data. Moreover, they can not represent the maximum separation between two CFSs.

## 7. CONCLUSION

D-making problems within the environments of multi-dimensional are the most challenging circle of exploration. DMs with complexity are very useful in signals and systems and complex D-making problems. To discuss such problems under CF information, DMs play a crucial role. In this paper, a generalized CFDMs is developed which is based on Hamming DM, Normalized Hamming DM, Zhang DM, etc. Some theoretical results and particular examples of the generalized CFDMs were discussed. In addition, the standard CF union, CF intersection, and CF complement were used to define some new operations on CFSs. A real-life application to identify a precise locations of a digital receiver based on the proposed CF DM was discussed. Some comparative studies of the generalized CFDMs with Hamming DM, Normalized Hamming DM, Zhang DM, and Zeeshan DM were carried out to show the superiority of the proposed DM.

**7.1. Future Research.** Future research may explore the newly defined DMs approach to more advanced fuzzy frameworks, such as interval-valued CFSs, complex IFSs, complex PFSs, complex picture fuzzy sets, etc., to capture higher levels of uncertainty. It would also be significant to analyze its integration with optimization techniques and machine learning for enhanced pattern recognition and signal classification. Furthermore, employing the proposed approach to real-world datasets in domains like communication systems, biomedical signal analysis, and image processing could additionally support its effectiveness and broaden its practical significance.

## CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

All the authors contributed equally.

## DECLARATIONS

**Conflict of Interest:** The authors declared that there is no conflict of interest regarding the publication of this research.

**Funding:** This work was supported in part by the National Science and Technology Council, Taiwan, under Grant NSTC 114-2410-H-224-001 and internal number 114-1011.

**Data Availability:** Our manuscript has no associated data.

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