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Axial Flow Analysis of Burger Fluid with Fractional derivative inside a Right Circular Cylinder

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Abstract. This research examines the unsteady flow characteristics of a fractional Burgers fluid in a right circular cylinder subjected to an axially translating motion at a time varying velocity. First, a generalized mathematical model of the problem was formulated by introducing time-fractional Caputo derivatives into the constitutive equation of the Burgers fluid to extend the classical integer-order model to include the memory and hereditary properties of complex viscoelastic materials. Solutions for the fluid velocity and the related shear stress were found analytically by employing both the Laplace transform and the finite Hankel transforms. The effect of the fractional order of the Caputo derivative, i.e., the order of the time-fractional derivative, on the velocity field was studied using graphical representations produced from numerical simulation performed in MATLAB. The effect of fractional ordering on the flow dynamics is compared to that of classical (integer-order) Burgers fluid. Finally, the obtained solutions are shown to contain other common fluid models such as Maxwell, Oldroyd-B and Newtonian fluids as limiting cases. Thus, the results of this work represent new contributions to the understanding of time-fractional viscoelastic fluid flows in cylindrical geometries.

AMS (MOS) Subject Classification Code: 76A05
Key Words: Burgers Fluid; Fractional Derivative; Analytical Solutions, Laplace and Hankel Transforms.

1. Introduction

The study of non-Newtonian fluids is of significant importance in various industrial and biological applications. The applications range from the flow of drilling muds, extrusion

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of polymer sheets, to the flow of blood in large arteries. Non-Newtonian fluids exhibit a complex nonlinear relationship between stress and shear rate, which leads to some exciting phenomena like stress relaxation, creep, and memory effects. To capture these fascinating traits, various constitutive equations have been developed. Burgers model stands out [5] as it helps us understand viscoelastic characteristics of materials such as bitumen [17, 18, 19, 26], polymers, and some types of biological tissues. Similar viscoelastic creep behavior has also been documented in cement-stabilized soils [32] and, at geological scales, in the Earth's mantle [22, 35].

The Burgers fluid model, an example inside the foundation of rate-type fluids, combines multiple time scales to capture a more intricate range of entertainment. When linked with the upper-convected derivative, it guarantees frame-invariance, making it suitable for interpreting flows with high deformity rates. A lot of research has taken place analyzing these types of fluids in canonical geometries like pipes (cylinders) [24, 30, 2, 28, 29, 16, 25, 13]. A specifically appealing arrangement is the flow inside a porous medium, which introduces a Darcy-type resistance, which is related to enhanced oil recovery, filtration, and groundwater hydrology [1, 3, 11].

The problem of electrically conducting fluid starting impulsively has been extensively studied for fluids like Newtonian, Maxwell, and Oldroyd-B. However, an analysis of Burgers fluid is a more general analysis which reduces to the analysis of before mentioned fluids in special cases. The analysis, however present significant mathematical challenges due to the appearance of higher order temporal derivatives in the constitutive equations. Traditional methods often lead to highly complicated transcendental expressions, which are difficult to deal with analytically. While existing literature [23, 8, 4] has addressed this problem using classical calculus, the intrinsic memory and hereditary properties of viscoelastic fluids suggest that fractional calculus, the branch of mathematics dealing with derivatives of non-integer order, is a more natural and powerful tool for their description.

To study materials with memory, fractional derivatives with a non-local kernel are extremely well-suited, as the current state of stress depends on the history of deformation.

The use of fractional derivatives in constitutive equations leads to more realistic physical models that accurately capture the purely elastic and purely viscous responses.

Recently, several studies have examined Burgers fluid flow over stretching cylinders and surfaces under the influence of magnetic fields, heat and mass transport, modified heat flux and nano-fluid effects using methods like Similarity transforms and Homotopy Analysis method [14, 15, 12, 31]. For fractional Burgers equations, the generalized fractional integral transform technqiue has also been developed [34]. These works focus on transform techniques for PDEs in one-dimensional or sheet/plate configurations.

In this manuscript, we introduce a novel approach to the flow problem of an upper-convected, incompressible Burgers fluid (UCIBF). set in motion by the axial sliding of an infinite circular cylinder. The model is studied using a fractional derivative in the constitutive equations. This approach not only generalizes the classical problem but also provides a more robust framework for understanding the deep-seated memory effects in the fluid's response. The goal is to derive an analytical solution for the shear stress fields and velocity, signifying the influence of key parameters such as the material constants of the Burgers model, and, above all, the fractional order of the time-derivative. We anticipate that this fractional calculus-based analysis will offer greater insights and computational advantages over the classical approach.

2. Problem Statement

We will assume that an upper-convected incompressible Burgers fluid (UCIBF) in an infinite circular cylinder is stationary . At the moment $t = 0^+$, the cylinder starts to slide along its symmetry axis with the velocity $Vf(t)$, where V is fixed and represented in the piecewise function $f(\cdot)$. As the fluid begins to move, due to the shear, we observe the corresponding velocity vector u, which is given by the relation [4]

$$\begin{aligned} u &= u(r,t) \\ u(r,t) &= u(r,t)e_z \end{aligned} \tag{2. 1}$$

in a system of cylindrical coordinates z, θ and r in which e_z is the unit vector along the z-axis.
The following are the relations that represent the constitutive equations of incompressible Burgers fluids [23, 10]

$$T = -pI + S, \quad \left(1 + a_1 \frac{\delta}{\delta t} + a_2 \frac{\delta^2}{\delta t^2}\right) S = 2\mu \left(1 + a_3 \frac{\delta}{\delta t}\right) D, \tag{2. 2}$$

here $S = S(r,t)$ denotes the extra-stress tensor, T denotes the stress tensor, D represents the symmetric part of the velocity gradient, I symbolizes the identity tensor, μ represents fluid viscosity, p denotes hydrostatic pressure, and a_1, a_2, a_3 are material constants. If $a_2 = 0$, one finds the constitutive equations of Oldroyd-B fluids [21] and the constants a_1, a_3 are called relaxation and retardation times, respectively, while $\frac{\delta}{\delta t}$ denotes the upper-convected time derivative. Under these suppositions, one can prove that the non-trivial shear stress $S_{rz}(r,t) = \tau(r,t)$ satisfies the partial differential equation (PDE)

$$\left(1 + a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial^2}{\partial t^2}\right) \tau(r,t) = \mu \left(a_3 \frac{\partial}{\partial t} + 1\right) \frac{\partial u(r,t)}{\partial r}; \;\; 0 < r < R_0, \;\; t > 0. \tag{2. 3}$$

Here R_0 represents the cylinder radius.
The following differential equation signifies the reduction of the balance of linear momentum [23]

$$\frac{1}{r}\tau(r,t) + \frac{\partial \tau(r,t)}{\partial r} = \rho \frac{\partial u(r,t)}{\partial t}, \;\; t > 0, \;\; 0 < r < R_0, \tag{2. 4}$$

ρ signifies the fluid's density in the above relation. The next boundary and initial conditions must be satisfied by the unknown functions $\tau(r,t)$ and $u(r,t)$

$$\tau(r,0) = 0, \quad \left.\frac{\partial \tau(r,t)}{\partial t}\right|_{t=0} = 0, \quad u(r,0) = 0, \quad 0 < r < R_0, \tag{2. 5}$$

$$u(R_0,t) = Vf(t); \;\; t \geq 0. \tag{2. 6}$$

We now introduce the next variables and non-dimensional functions

$$\bar{f}(\bar{t}) = f\left(\frac{R_0^2}{\nu}\bar{t}\right), \quad \bar{r} = \frac{1}{R_0} r, \;\; \bar{t} = \frac{\nu}{R_0^2} t, \;\; \bar{u} = \frac{1}{V} u, \;\; \bar{\tau} = \frac{R_0}{\mu V} \tau, \tag{2. 7}$$

We observe that $\nu = \mu/\rho$ denotes kinematic viscosity, and if we drop the bar notation, we see the following dimensionless forms

$$\left(1+\omega\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\tau(r,t)=\frac{\partial u(r,t)}{\partial r}\left(1+\gamma\frac{\partial}{\partial t}\right);\quad 0<r<1,\quad t>0, \tag{2. 8}$$

$$\frac{\partial u(r,t)}{\partial t}=\frac{1}{r}\tau(r,t)+\frac{\partial \tau(r,t)}{\partial r};\ \ 0<r<1,\ \ t>0, \tag{2. 9}$$

In above relations, the constants ω, β, γ can be expressed as

$$\omega=\frac{\nu}{R_0^2}a_1,\ \ \beta=\frac{\nu^2}{R_0^4}a_2,\ \ \gamma=\frac{\nu}{R_0^2}a_3. \tag{2. 10}$$

In the existing literature, the parameter ω is also called dimensionless relaxation time or "Weissenberg number". The initial and boundary conditions (6) and (7) become

$$\tau(r,0)=\left.\frac{\partial \tau(r,t)}{\partial t}\right|_{t=0}=0,\ \ u(r,0)=0,\ \ 0<r<1, \tag{2. 11}$$

$$u(1,t)=f(t);\ \ t\geq 0. \tag{2. 12}$$

2.1. **Fractional model.** In this section, we develop a mathematical model based on the following fractional constitutive equation:

$$\left(1+\omega\frac{\partial^\alpha}{\partial t^\alpha}+\beta\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\tau(r,t)=\left(1+\gamma\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial u(r,t)}{\partial r};\ \ 0<r<1,\ \ t>0, 0<\alpha<1. \tag{2. 13}$$

Here, Caputo time-fractional derivative is characterized by the differential operator $\frac{\partial^\alpha \tau(r,t)}{\partial t^\alpha}$ [7]

$$\frac{\partial^\alpha \tau(r,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-\sigma)^{-\alpha}\frac{\partial \tau(r,\sigma)}{\partial \sigma}\,d\sigma, & 0<\alpha<1,\\[2ex] \dfrac{\partial \tau(r,t)}{\partial t}, & \alpha=1.\end{cases} \tag{2. 14}$$

3. Exact Solutions

Now, we will solve the system of PDEs (9) and (13) using finite Hankel and Laplace transforms, and the boundary and initial conditions (12) and (11).

3.1. **Determination of the Velocity Field** $u(r,t)$**.** If the Laplace transform is applied to Eq. (13) while keeping in mind the initial conditions (11), one will come to the conclusion that the Laplace transform $\overline{\tau}(r,s)$ of $\tau(r,t)$ satisfies the relation

$$\overline{\tau}(r,s)=\frac{(1+\gamma s^\alpha)\frac{\partial}{\partial r}\overline{u}(r,s)}{1+\omega s^\alpha+\beta s^{2\alpha}}. \tag{3. 15}$$

Similarly, the Laplace Transform for velocity using Eq. (9) and $\overline{\tau}(r,s)$ from Eq. (15) gives

$$\frac{\left(1+\omega s^\alpha+\beta s^{2\alpha}\right)}{(1+\gamma s^\alpha)}s\overline{u}(r,s)=\frac{1}{r}\frac{\partial}{\partial r}\overline{u}(r,s)+\frac{\partial^2}{\partial r^2}\overline{u}(r,s),$$

which can be rewritten as

$$a(s)\overline{u}(r,s) = \frac{\partial^2 \overline{u}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u}(r,s)}{\partial r}. \tag{3. 16}$$

With the initial condition $\overline{u}(1,s) = F(s)$

$$a(s) = \left(\frac{\beta s^{2\alpha+1} + \omega s^{\alpha+1} + s}{1+\gamma s^{\alpha}}\right). \tag{3. 17}$$

Now, we apply the Finite Henkel transform to Eq. (16) [27] to obtain

$$\overline{u}_H(r_n,s) = \frac{r_n J_1(r_n) F(s)}{a(s) + {r_n}^2}.$$

It can be rewritten as

$$\overline{u}_H(r_n,s) = \frac{1}{r_n} J_1(r_n) F(s) - \frac{1}{r_n}\frac{a(s)}{a(s)+{r_n}^2} J_1(r_n) F(s). \tag{3. 18}$$

The inverse Hankel transform gives

$$\overline{u}_H(r,s) = -F(s)a(s)\overline{u}_H^{-1}\left(\frac{1}{r_n(a(s)+{r_n}^2)} J_1(r_n)\right) + F(s). \tag{3. 19}$$

Now, using $[\bar{u}_H^{-1}\left(\frac{1}{r_n(a(s)+{r_n}^2)} J_1(r_n)\right) = \sum_{n=1}^{\infty} \frac{1}{{r_n}^2 + a(s)} \frac{J_0(r r_n)}{r_n J_1(r_n)}$, we obtain

$$\overline{u}(r,s) = F(s) - 2F(s)\sum_{n=1}^{\infty} \frac{a(s)}{(a(s)+{r_n}^2)} \frac{J_0(r r_n)}{r_n J_1(r_n)}. \tag{3. 20}$$

In the above relations, $J_0(\cdot)$ and $J_1(\cdot)$ are standard Bessel functions of the first kind of zero and one order, respectively.
Using the value of $a(s)$ from Eq. (17), we get

$$\frac{a(s)}{{r_n}^2 + a(s)} = 1 - {r_n}^2 \frac{(1+\gamma s^{\alpha})}{\beta s^{\alpha+1}\left[\left(s^{\alpha} + \frac{\omega}{\beta}\right) + \frac{1}{\beta}\left(\gamma {r_n}^2 s^{-1} + s^{-\alpha} + {r_n}^2 s^{-\alpha-1}\right)\right]}. \tag{3. 21}$$

Now using

$$\frac{1}{z+a} = \sum_{k=0}^{\infty} \frac{(-z)^k}{a^{k+1}}, \left|\frac{z}{a}\right| < 1,$$

we get

$$\begin{aligned}
\frac{a(s)}{a(s)+{r_n}^2} &= 1-\frac{{r_n}^2}{\beta}\left(\gamma s^{-1}+s^{-\alpha-1}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\left[\frac{1}{\beta}\left(\gamma {r_n}^2 s^{-1}+s^{-\alpha}+{r_n}^2 s^{-\alpha-1}\right)\right]^k}{\left(s^\alpha+\frac{\omega}{\beta}\right)^{k+1}}\\
&= 1-\frac{{r_n}^2}{\beta}\left(\gamma s^{-1}+s^{-\alpha-1}\right)\sum_{k=0}^{\infty}\sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=k}}(-1)^k\frac{k!}{k_1!k_2!k_3!\beta^k\left(s^\alpha+\frac{\omega}{\beta}\right)^{k+1}}\\
&\times\left[\gamma^{k_1}{r_n}^{2(k_1+k_2)}s^{-k_1-\alpha k_2-\alpha k_3-k_3}\right]\\
&= 1-{r_n}^2\sum_{k=0}^{\infty}\sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=k}}(-1)^k\frac{k!}{k_1!k_2!k_3!\beta^{k+1}\left(s^\alpha+\frac{\omega}{\beta}\right)^{k+1}}\\
&\times\left[\gamma^{k_1}{r_n}^{2(k_1+k_2)}\left(\gamma s^{-1-k_1-\alpha k_2-\alpha k_3-k_3}+s^{-\alpha-1-k_1-\alpha k_2-\alpha k_3-k_3}\right)\right].
\end{aligned}$$

Using $G_{a;b;c}(\mathrm{t,d})=\mathcal{L}^{-1}\left\{\frac{s^b}{(s^a-d)^c}\right\}$, if $\mathrm{Re}(ac-b)>0$, where $G_{a;b;c}(t,d)$ is the Lorenzo-Hartley G-function [9, 20], we finally obtain

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{a(s)}{a(s)+{r_n}^2}\right\} &= \delta(t)-{r_n}^2\sum_{k=0}^{\infty}\sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=k}}(-1)^k\frac{k!}{\beta^{k+1}k_2!k_1!k_3!}\\
&\times\left[\gamma^{k_1}{r_n}^{2(k_2+k_1)}\left(\gamma G_{\alpha;-1-k_1-\alpha(k_3+k_2)-k_3;k+1}(t,-\omega/\beta)\right.\right.\\
&\left.\left.+G_{\alpha;-1-k_1-\alpha-\alpha(k_2+k_3)-k_3;k+1}(t,-\omega/\beta)\right)\right]. \qquad (22^*)
\end{aligned}$$

Therefore, the velocity $u(r,t)$ becomes

$$\begin{aligned}
u(r,t) &= f(t)-2f(t)*\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(r_n)}\left[\delta(t)-r_n^2\sum_{k=0}^{\infty}\sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=k}}(-1)^k\frac{k!}{k_1!k_2!k_3!\beta^{k+1}}\right.\\
&\times\left[\gamma^{k_1}r_n^{2(k_1+k_2)}\left(\gamma G_{\alpha;-g(k_1,k_2,k_3);k+1}(t,-\omega/\beta)\right.\right.\\
&\left.\left.\left.+G_{\alpha;-\alpha-g(k_1,k_2,k_3);k+1}(t,-\omega/\beta)\right)\right]\right],
\end{aligned} \qquad (3.\ 22)$$

where $g(k_1,k_2,k_3)=-1-k_1-\alpha(k_2+k_3)-k_3$.

3.2. **Another form of the Velocity Field** $u(r,t)$**.** Eq. (16) can be written as

$$\overline{u}(r,s)\left(-\left(r\sqrt{a(s)}\right)^2\right)+r\frac{\partial\overline{u}(r,s)}{\partial r}+r^2\frac{\partial^2\overline{u}(r,s)}{\partial r^2}=0, \qquad (3.\ 23)$$

which is a modified Bessel equation, whose solution is given by

$$\overline{u}(r,s)=A_1I_0\left(r\sqrt{a(s)}\right)+A_2K_0\left(r\sqrt{a(s)}\right). \qquad (3.\ 24)$$

As we want to have a finite solution at $r = 0$, we must have $A_2 = 0$ because $\lim_{r\to 0} K_0(r) = +\infty$. Now we are left with $\overline{u}(r,s) = \left(r\sqrt{a(s)}\right) A_1 I_0$. When the boundary condition is imposed, the Laplace Transform of the velocity field is given by

$$\overline{u}(r,s) = F(s)\frac{I_0\left(\sqrt{a(s)}r\right)}{\left(\sqrt{a(s)}\right) I_0}. \tag{3. 25}$$

Using the Stehfest algorithm, we can determine the inverse Laplace transform of Eq. (25) [33]. According to this algorithm, the inverse Laplace transform of Eq. (25) is given by

$$u(r,t) = \frac{\ln(2)}{t}\sum_{j=1}^{2p}\left\{(-1)^{p+j}\sum_{i=floor\left(\frac{j+1}{2}\right)}^{\frac{1}{2}(j+p-|j-p|)}\frac{i^p(2i)!}{i!(p-i)!(i-1)!(j-i)!(2i-j)!}\overline{u}\left(r,\frac{j\ln(2)}{t}\right)\right\}. \tag{3. 26}$$

3.3. **Determination of the Shear Stress** $\tau(r,t)$**.** Using Eq.(21) in Eq. (15), we have

$$\overline{\tau}(r,s) = 2F(s)\frac{(1+\gamma s^\alpha)}{1+\omega s^\alpha+\beta s^{2\alpha}}\sum_{n=1}^{\infty}\frac{a(s)}{(a(s)+{r_n}^2)}\frac{J_1(rr_n)}{J_1(r_n)}. \tag{3. 27}$$

Now, the inverse Laplace transform of Eq. (27) is given by

$$\tau(r,t) = 2\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(r_n)}\mathcal{L}^{-1}\{F(s)H(s)G_n(s)\}, \tag{3. 28}$$

where

$$H(s) = \frac{1+\gamma s^\alpha}{1+\omega s^\alpha+\beta s^{2\alpha}}, G_n(s) = \frac{a(s)}{a(s)+{r_n}^2}.$$

As before, we can show that

$$\mathcal{L}^{-1}\left\{\frac{1+\gamma s^\alpha}{1+\omega s^\alpha+\beta s^{2\alpha}}\right\} = \mathcal{L}^{-1}\left\{\frac{-1+\gamma m_1}{\beta(m_2-m_1)(s^\alpha-m_1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1+\gamma m_2}{\beta(m_2-m_1)(s^\alpha-m_2)}\right\},$$

with $m_{1,2} = \frac{-\omega\pm\sqrt{\omega^2-4\beta}}{2\beta}$.

By using $\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha-m}\right\} = t^{\alpha-1}E_{\alpha,\alpha}(mt^\alpha)$, we get

$$h(t) = \frac{t^{\alpha-1}}{\beta(m_2-m_1)}\left[(-1+\gamma m_1)E_{\alpha,\alpha}(m_1t^\alpha)+(1+\gamma m_2)E_{\alpha,\alpha}(m_2t^\alpha)\right], \tag{3. 29}$$

where

$$\tau(r,t) = 2\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(r_n)}\left(f(t)*h(t)*g_n(t)\right), \tag{3. 30}$$

with $g_n(t)$ coming from Eq. (22*).

4. Special Cases

4.1. **Case** $\alpha = 1$**.**

4.1.1. *Expression for velocity* $u(r,t)$. In the case $\alpha = 1$, Eq. (21) can be written as

$$\frac{a(s)}{a(s)+{r_n}^2} = 1 - \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(-1)^k k!(\gamma {r_n}^2+1)^m {r_n}^{2(k-m+1)}}{\beta^{k+1} m!(-m+k)!}\left[\frac{\gamma s^{m-2k-1}}{\left(s+\frac{\omega}{\beta}\right)^{k+1}} + \frac{s^{m-2k-2}}{\left(s+\frac{\omega}{\beta}\right)^{k+1}}\right],$$

which has an inverse Laplace transform

$$\frac{a(s)}{a(s)+{r_n}^2} = \delta(t) - \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(-1)^k k!(\gamma {r_n}^2+1)^m {r_n}^{2(-m+k+1)}}{\beta^{k+1} m!(-m+k)!} \times \left[\gamma G_{1;m-2k-1;+1+k}\left(t,\frac{-\omega}{\beta}\right) + G_{1;m-2k-2;+1+k}\left(t,\frac{-\omega}{\beta}\right)\right], \tag{31*}$$

which leads to the expression

$$u(r,t) = f(t) - 2f(t) * \sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n J_1(r_n)}\Bigg[\delta(t) - \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(-1)^k k!(\gamma {r_n}^2+1)^m {r_n}^{2(k-m+1)}}{\beta^{k+1} m!(-m+k)!} \times \left[\gamma G_{1;m-2k-1;k+1}\big(t,-\tfrac{\omega}{\beta}\big) + G_{1;m-2k-2;k+1}\big(t,-\tfrac{\omega}{\beta}\big)\right]\Bigg]. \tag{4. 31}$$

4.1.2. *Expression for shear stress* $\tau(r,t)$. The shear stress becomes (From Eq. 28)

$$\tau(r,t) = 2\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(r_n)}\left(f(t) * h(t) * g_n(t)\right), \tag{4. 32}$$

where $h(t) = \frac{1}{\beta(m_2-m_1)}\left[(-1+\gamma m_1)e^{m_1 t} + (1+\gamma m_2)e^{m_2 t}\right]$ and $g_n(t)$ coming from Eq. (31*).

We point out that equations (31) and (32) for the velocity and shear stress corresponding to the flow of ordinary Burgers fluid ($\alpha = 1$), are identical to equations (24), respectively (29) obtained by Fetecau and Vieru [6] , in the case of the magnetic parameter M equal to zero.

4.2. **Oldroyd-B Fluid (**$\beta = 0$**).** In the case of $\beta = 0$, the quantity $\frac{a(s)}{a(s)+{r_n}^2}$ can be simplified to

$$\frac{a(s)}{a(s)+{r_n}^2} = 1 - {r_n}^2\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(-1)^k {r_n}^{2k}}{\omega^{k+1}}\frac{k!\gamma^m}{(k-m)!m!}\left[\frac{s^{\alpha m-(k+1)}}{\left(s^\alpha+\frac{1}{\omega}\right)^{+1+k}} + \frac{\gamma s^{\alpha(m+1)-(k+1)}}{\left(s^\alpha+\frac{1}{\omega}\right)^{k+1}}\right]. \tag{4. 33}$$

Consequently, its inverse Laplace transform in light of [9] becomes

$$\mathcal{L}^{-1}\left\{\frac{a(s)}{a(s)+{r_n}^2}\right\} = \delta(t) - {r_n}^2\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(-1)^k {r_n}^{2k}}{\omega^{k+1}}\frac{k!\gamma^m}{(k-m)!m!} \times \left[G_{\alpha;\alpha m-(k+1);k+1}\big(t,-\tfrac{1}{\omega}\big) + G_{\alpha;\alpha(m+1)-(k+1);k+1}\big(t,-\tfrac{1}{\omega}\big)\right]. \tag{34*}$$

Hence, in the case of $\beta = 0$, the velocity $u(r,t)$ becomes

$$u(r,t) = f(t) - 2f(t) * \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)} \left[\delta(t) - r_n^2 \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k r_n^{2k}}{\omega^{k+1}} \frac{k!\gamma^m}{(k-m)!m!} \right. \\ \left. \times \left[G_{\alpha;\alpha m-(k+1);k+1}\big(t, -\tfrac{1}{\omega}\big) + G_{\alpha;\alpha(m+1)-(k+1);k+1}\big(t, -\tfrac{1}{\omega}\big) \right] \right]. \tag{4. 34}$$

The corresponding shear stress in light of Eq. (28) with $\beta = 0$ becomes

$$\tau(r,t) = 2 \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(r_n)} \left(f(t) * h(t) * g_n(t) \right),$$

where $h(t) = \frac{1}{\omega} \left[\gamma\delta(t) + \left(1 - \frac{\gamma}{\omega}\right) t^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-t^\alpha}{\omega}\right) \right]$ and $g_n(t)$ coming from (34*).

4.3. **Maxwell Fluid ($\beta = \gamma = 0$).** We get simpler expressions in this case, Eq. (21) takes the form

$$\frac{a(s)}{a(s) + {r_n}^2} = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k {r_n}^{2(k+1)}}{\omega^{k+1}} \frac{s^{-(1+k)}}{\left(s^\alpha + \frac{1}{\omega}\right)^{k+1}}. \tag{35*}$$

The velocity transforms to

$$u(r,t) = -2f(t) + f(t) * \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)} \left[\delta(t) - \sum_{k=0}^{\infty} G_{\alpha;-k-1;k+1}\left(t, \frac{-1}{\omega}\right) \right], \tag{4. 35}$$

and the shear stress transforms to

$$\tau(r,t) = 2 \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(r_n)} \left(f(t) * h(t) * g_n(t) \right),$$

where $h(t) = \frac{1}{\omega} t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{t^\alpha}{\omega}\right)$ and $g_n(t)$ coming from Eq. (35*).

4.4. **Newtonian Fluid ($\beta = \gamma = \omega = 0$).** In such a case, $\frac{a(s)}{a(s)+{r_n}^2} = 1 - \frac{{r_n}^2}{s+{r_n}^2}$, which gives

$$u(r,t) = -2f(t) + f(t) * \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(r_n)} \left[\delta(t) - {r_n}^2 e^{-{r_n}^2 t} \right], \tag{4. 36}$$

and the shear stress becomes

$$\tau(r,t) = 2 \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(r_n)} \left(f(t) * \delta(t) * \left(\delta(t) - {r_n}^2 e^{-{r_n}^2 t} \right) \right). \tag{4. 37}$$

5. Graphical Analysis

5.1. **Solution validation.** To validate the analytical solution (20), obtained using the integral transform method, we also determined the solution (25) which is the solution of the modified Bessel equation (23). The velocity profiles given by the two equations are represented in Figure A that clearly highlights the coincidence of the two solutions. To achieve Figure A we considered the cylinder velocity given by the function $f(t) = 1 - \cos(t)e^{-t}, t \geq 0$. The curves corresponding to equations (20) and (25) were plotted for three values of the fractional parameter $\alpha \in \{0.4, 0.8, 1.0\}$ and for two values of time t.

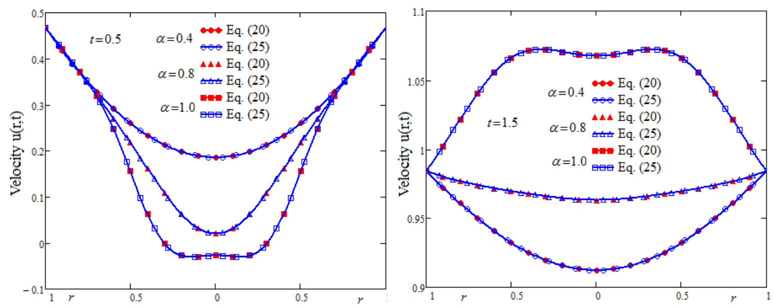

FIGURE A. The coincidence of the solutions given by equations (20) and (25) for material parameters $\omega = 0.4, \beta = 0.2, \gamma = 0.3$.

5.2. **Velocity profile with** $f(t) = 1$**.** The graphs in Figures 1-3 signify the fractional parameter α's influence on the fluid velocity, in the case where the cylinder is translated along its axis of symmetry with constant velocity $f(t) = 1$.

Figure 1 shows the velocity profiles versus time t, at position $r = 0.5$ of the cylinder and for four values of the fractional parameter α. It is observed in this figure that the velocity of the ordinary Burgers fluid ($\alpha = 1$) shows significant variations in relation to the velocity corresponding to the fractional model $0 < \alpha < 1$. These variations are greater for values of time t less than 2.5. When observed, it is seen that for large values of time t, values of velocity become uniform and move towards the constant value of 1.

The significant differences between the fractional Burgers fluid and the velocity of the ordinary Burgers fluid are also highlighted by the graphs in Figures 2 and 3. The velocity of the fractional Burgers fluid has significantly lower values than that of the ordinary Burgers fluid. This is due to the fractional derivative kernel, which has the effect of damping the velocity gradient, leading to smaller values of the fluid velocity.

5.3. **Velocity profile with** $f(t) = 1 - e^{-t}$**.** If we talk about the case where the cylinder is translated with a time-dependent velocity $f(t) = 1 - e^{-t}$, we discuss their fluid velocity profiles in this particular section. In this case, the cylinder is not suddenly moved from its rest position but moves with a slowly increasing velocity. As a result, in this case, the differences between the velocity of the ordinary and fractional fluid are smaller than in the case presented in the previous paragraph.

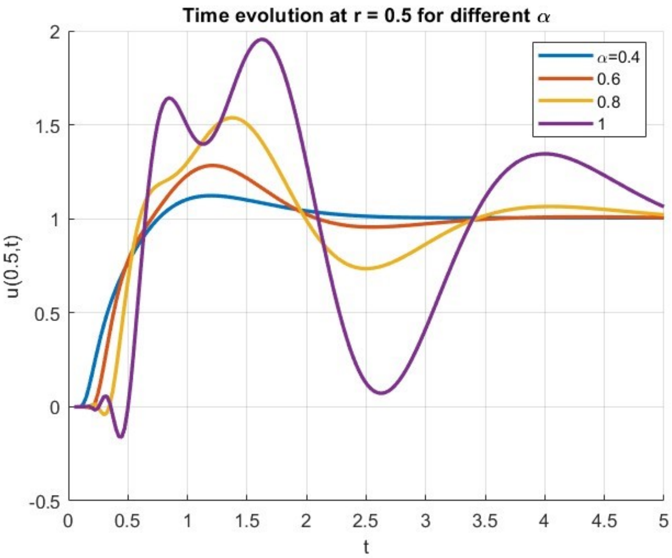

FIGURE 1. Velocity profiles versus time for $r = 0.5$ and different values of the fractional parameter α.

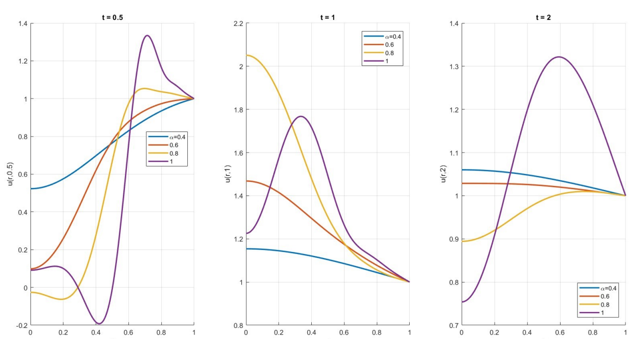

FIGURE 2. Velocity profiles versus radial coordinate r for different values of time t and fractional parameter α.

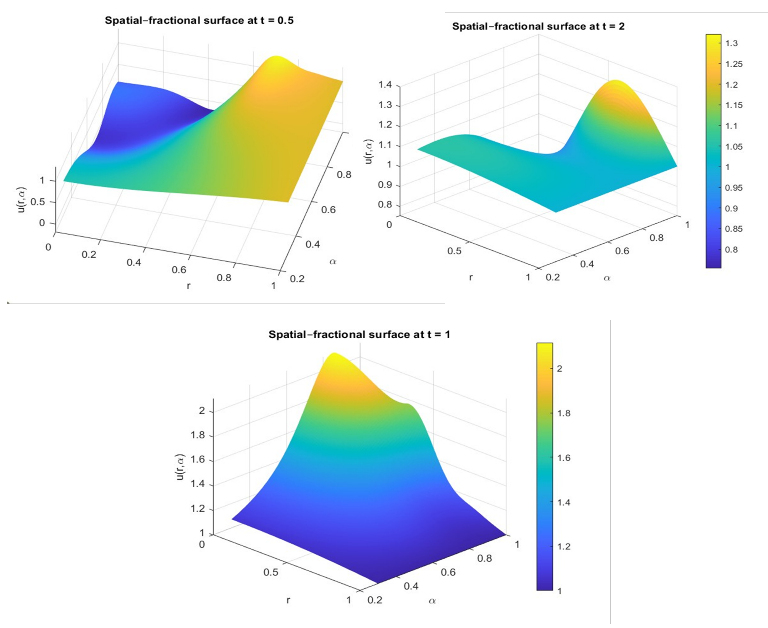

FIGURE 3. Surface plot of the velocity $u(r,t)$ for $(r,\alpha) \in [0,1] \times [0.2,1]$ and $t \in \{0.5, 1, 2\}$.

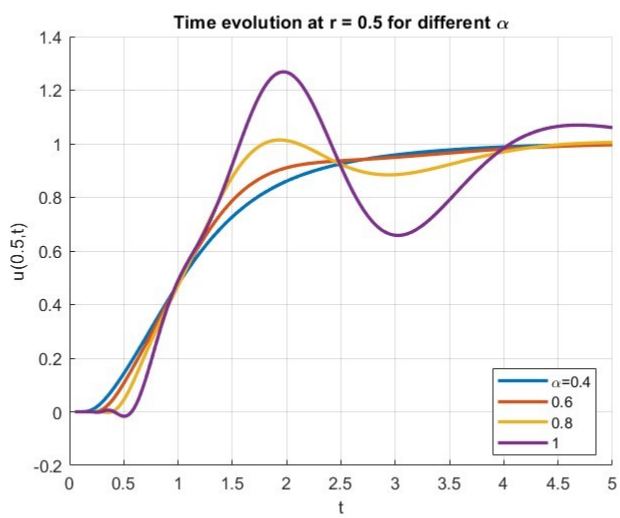

FIGURE 4. Velocity profiles versus time for $r = 0.5$ and different values of the fractional parameter α.

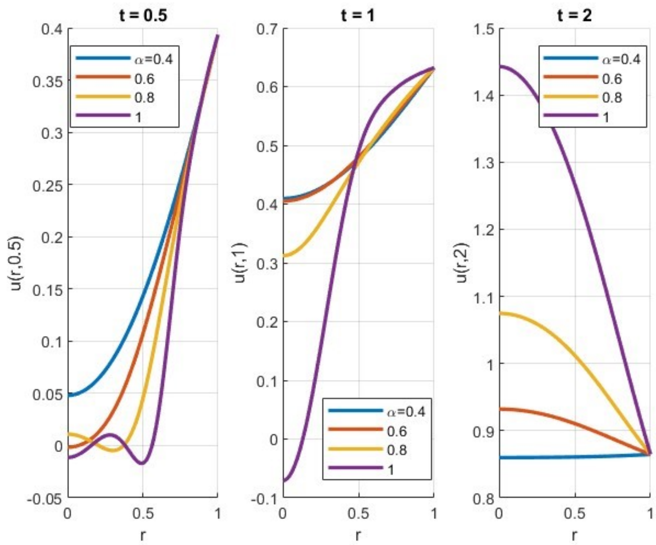

FIGURE 5. Velocity profiles versus radial coordinate r for different values of time t and fractional parameter α.

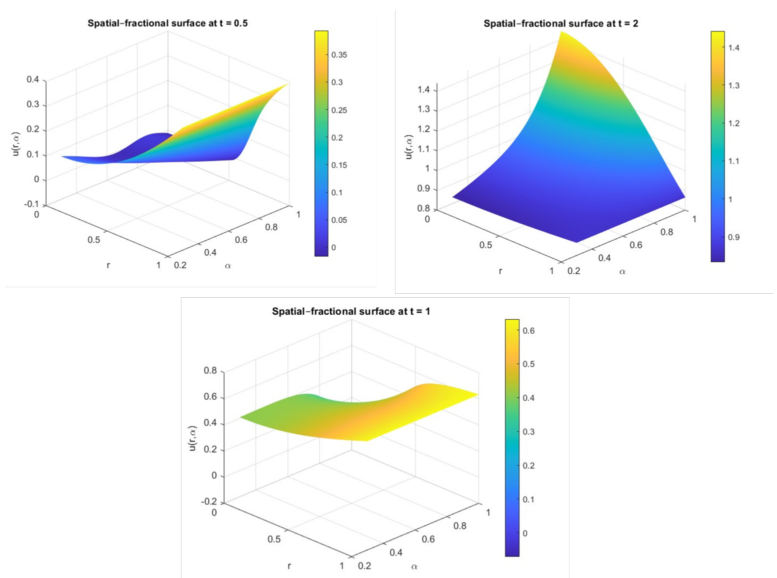

FIGURE 6. Surface plot of the velocity $u(r, t)$ for $(r, \alpha) \in [0, 1] \times [0.2, 1]$ and $t \in \{0.5, 1, 2\}$.

5.4. **Velocity profile with $f(t) = \sin\left(\frac{\pi t}{4}\right)$.** In the case of an oscillatory translation of the cylinder with the velocity $f(t) = \sin\left(\frac{\pi t}{4}\right)$, the fluid velocity profiles are presented in Figures 7-9

As can be seen from these figures, the oscillatory motion of the cylinder leads to the appearance of reverse flow at certain times.

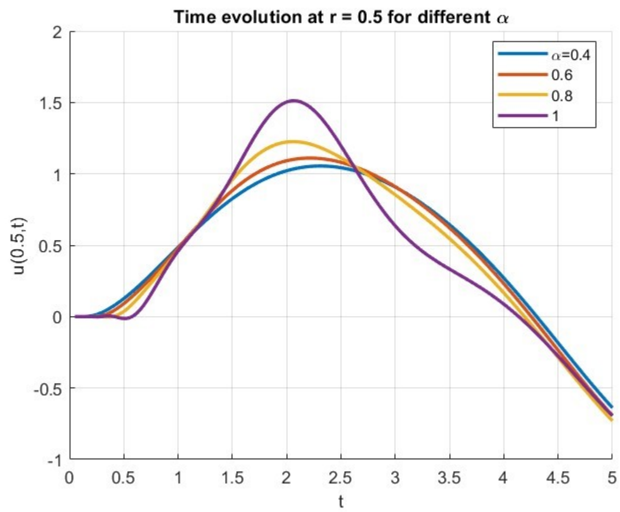

FIGURE 7. Velocity profiles versus time for $r = 0.5$ and different values of the fractional parameter α.

6. CONCLUSION

- A complete analytical investigation of an unsteady fractional Burgers fluid flow inside a right circular cylinder with a time-varying axial motion is presented here.
- A time-fractional Caputo derivative based on a generalized Burgers fluid model is proposed as a new constitutive equation to account for the memory and viscoelastic properties of complex fluids.
- Velocity fields and shear stresses are obtained through the use of Laplace and finite Hankel transforms from the initial boundary value problem formulation.
- The solutions show the ability of the analytical method to provide exact solutions and illustrate its applicability and efficacy.
- Temporal variations of the velocity profiles due to changes in the fractional order significantly affect the fluid dynamics of the system and confirm the significance of fractional modeling for capturing non-local and hereditary effects not accounted for by the classical integer-order models.
- Further validation of the analytical results can be found in the numerical results and graphical representations provided in the paper to better describe the physical behavior of the system.

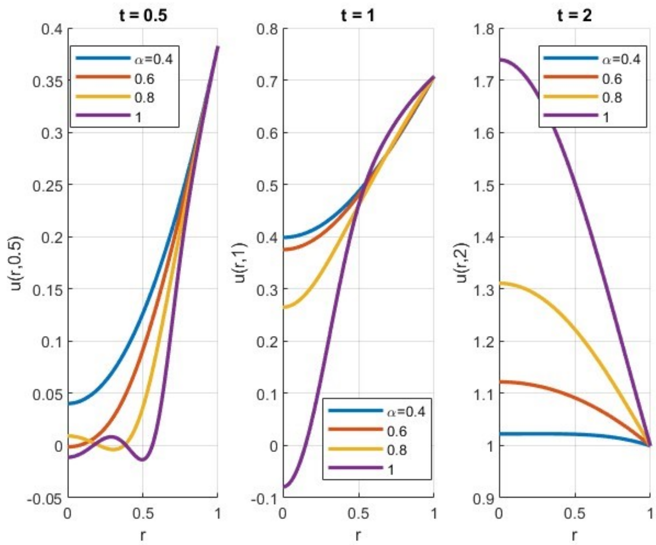

FIGURE 8. Velocity profiles versus radial coordinate r for different values of time t and different values of the fractional parameter α.

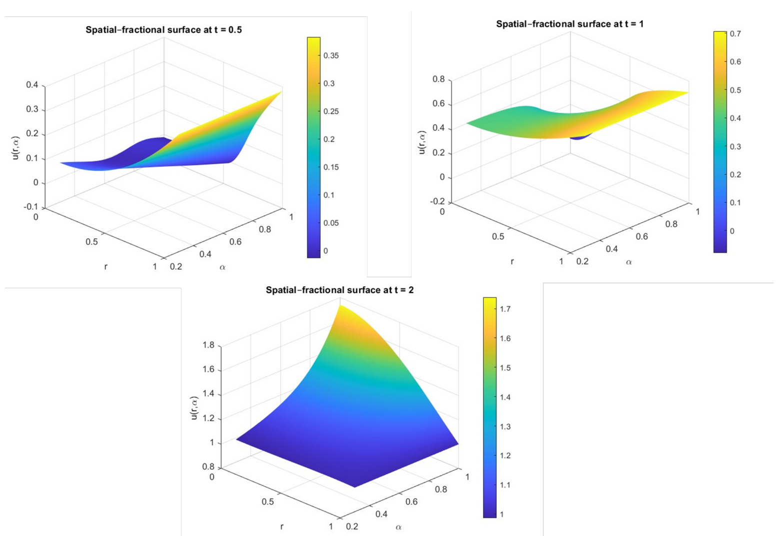

FIGURE 9. Surface plot of the velocity $u(r, t)$ for $(r, \alpha) \in [0, 1] \times [0.2, 1]$ and $t \in \{0.5, 1, 2\}$.

Limitations of the study.

- The model assumes incompressible, isothermal flow.

- Porosity, magnetic effects and thermal transport are not included.
- Numerical inversion of Laplace transforms is required for practical visualization.

Analytical solutions provide benchmark results for validating numerical methods such as bvp4c, HAM and shooting techniques widely used in Burgers fluid literature. Other analytical approaches such as Homotopy Analysis Method (HAM), Adomian Decomposition, and Eigenfunction Expansion could be applied. However, these methods often produce slowly convergent series solutions, whereas the Laplace–Hankel technique yields compact closed-form expressions in transform space and preserves boundary conditions exactly.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

All authors have contributed equally to this work.

DECLARATIONS

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Data Availability: All the data used during this study is accessible within the manuscript.

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