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Integrated Entropy-SWARA-ERUNS Algorithm for Optimization and Recycling of Plastic Waste Material using Bipolar Fuzzy Z-Numbers

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Abstract. The concept of bipolar fuzzy Z-numbers (BFZNs) is a new hybrid approach to address vagueness and bipolarity in the real-life circumstances. The idea of BFZNs is a new fuzzy model that integrates bipolar fuzzy numbers and Z-numbers that offer reliability analysis of bipolar fuzzy information. This study introduces a comprehensive multi-criteria decision-making (MCDM) framework that combines the entropy method for objective weighting, the stepwise weight assessment ratio analysis (SWARA) method for expert input, and the evaluation based on relative utility and nonlinear standardization (ERUNS) for ranking alternatives. The approach introduces a novel ”Entropy-SWARA-ERUNS” algorithm based on bipolar fuzzy Z-numbers (BFZNs) to better handle uncertainty and vagueness. The framework enables a balanced evaluation of the recycling plastic waste material, with sensitivity and comparative analyzes that validate its robustness and superiority over conventional methods.

AMS (MOS) Subject Classification Codes: 03B52; 03E72 ; 90B50
Key Words: Bipolar fuzzy Z-numbers; Subjective and objective weights; Entropy; SWARA; ERUNS; Recycling plastic waste material.

1. Introduction

Plastic waste management is a pressing global concern due to its environmental impact and the need for sustainable resource use. Selecting the optimal plastic recycling technology is complex, and existing decision-making methods often overlook the integration of objective data with expert judgment and fail to account for nonlinear relationships among criteria. The problem of plastic pollution constitutes a major environmental issue facing the global community. The extensive distribution of plastic debris across aquatic and terrestrial ecosystems has resulted in significant harm to wildlife, disruption of natural habitats, and the introduction of toxic substances into food chains, thereby posing a threat to human health. The issue is exacerbated by the substantial rise in global plastic production and

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consumption, which persists at alarming rates each year. Numerous regions exhibit insufficient waste management facilities for the processing and recycling of plastic materials, resulting in significant accumulation of plastic waste in landfills, illegal dumping sites, and natural environments. Unmanaged waste streams result in soil and water contamination, greenhouse gas emissions, and prolonged ecological degradation [28, 32].

The development of plastics techniques for recycling are crucial for mitigating the environmental and social consequences associated with plastic pollution. Recycling reduces the volume of waste sent to landfills and ecosystems, while also conserving finite natural resources by recovering and reusing valuable materials. Recycling enhances the economy of circularity by extending material usage, thus decreasing the demand for virgin substances and limiting environmental impacts [11]. Identifying and selecting the most suitable recycling technology presents a complex decision-making challenge. The process entails the integration of various, frequently opposing, criteria, including economic costs, environmental advantages, technical feasibility, operational scalability, energy consumption, emissions, and social acceptance by communities and stakeholders. The variety of plastic types, levels of contamination, and regional policy frameworks complicate the selection process [23]. Consequently, decision-makers need systematic tools that integrate various quantitative data and qualitative expert judgments to identify optimal recycling solutions suited to specific economic and environmental constraints.

Conventional decision-making methods frequently fail to adequately address this complexity, as they may neglect the interaction of various criteria and stakeholder preferences. Therefore, there is a significant necessity for effective decision support tools that can systematically integrate both objective data and subjective expert judgments to assess and prioritize recycling alternatives. The rapid increase in plastic consumption in industrial, commercial, and domestic sectors has led to a worldwide environmental crisis. Plastic waste, characterized by its non-biodegradable properties and intricate chemical structure, presents a considerable risk to ecosystems, public health, and urban infrastructure. Effective management of plastic waste has emerged as a crucial element in achieving sustainability objectives and facilitating the transition to a circular economy [37]. Conventional recycling techniques, although commonly employed, frequently exhibit deficiencies in technological and economic efficiency necessary for large-scale plastic waste management. Recent advancements have introduced innovative and sustainable recycling technologies, including chemical recycling, dissolution–precipitation methods, and design-for-recycling frameworks, which offer enhanced environmental outcomes and economic viability [15, 7]. identifying the appropriate recycling technology presents a complex decision-making challenge, shaped by various conflicting criteria such as cost, efficiency, environmental impact, scalability, and technological maturity [3, 26]. Lim et al. [30] highlighted the necessity of integrating economic feasibility into the development of recycling strategies to enhance overall recycling efficiency. Torkayesh and Simic [45] proposed a stratified hybrid decision model for selecting recycling facilities for healthcare plastic waste, emphasizing the necessity of considering spatial, technical, and social factors during the process of making decisions.

Numerous studies support the incorporation of MCDM techniques to tackle the complexities associated with selecting recycling technologies. Suvitha et al. [44] developed an enhanced MCDM approach to model an ideal plastic waste management system, demonstrating the capacity of structured decision models to incorporate diverse and uncertain data. Tripura et al. [46] utilized Picture Fuzzy SWARA–MARCOS techniques to evaluate and rank plastic recycling technologies, highlighting the significance of fuzzy logic in addressing ambiguity in expert assessments. Novel recycling processes, such as dissolution-precipitation, are emerging as environmentally friendly alternatives due to their capacity to preserve material properties and decrease energy consumption [7]. Jiang and Bateer [26] emphasize the necessity of rigorously assessing environmental trade-offs and economic evaluations to ensure sustainability. Despite the increasing volume of research, a significant need persists for a comprehensive and flexible decision framework that integrates sustainability, technological feasibility, and economic implications in the selection of optimal plastic waste recycling technologies. This study aims to fill this gap by developing a decision support framework that employs advanced MCDM techniques. This study presents an integrated multi-criteria decision-making model that incorporates the Entropy method for objective weighting, SWARA for subjective expert weighting, and the ERUNS method for ranking plastic waste recycling technologies, considering economic and environmental constraints. The framework seeks to enhance decision reliability by integrating quantitative data variability with expert insights. The model's applicability and robustness are demonstrated via sensitivity and comparative analyses. The findings provide essential insights for policymakers, industry stakeholders, and environmental managers in pursuit of sustainable and cost-effective recycling solutions.

1.1. **Literature review.** A foundational technique in decision science, MCDM is intended to get the best answers by taking a variety of factors into account. A group of specialists must evaluate each option according to various criteria and subsequently articulate their conclusions clearly. Conversely, uncertainty affects nearly all decision-making processes in today's complex and demanding environment. Thus, during the interpretation process, it is crucial to consider the possibility of disparities. Data managers often encounter difficulties in delivering precise responses when dealing with imprecise, erroneous, or poorly defined data. Their work clearly reflects these challenges. Such issues can emerge in diverse contexts and at multiple stages in real-world scenarios related to decision-making, organization, and classification of items, and supplier evaluations. In 1965, Zadeh developed fuzzy set (FS) theory and the concept of membership degree (MSD) [53]. First, demonstrating the idea of intuitionistic fuzzy sets (IFSs) initiated by Atanassov [9] with the requirement that the sum of the MSD and non-membership degree (N-MSD) does not exceed unity.

Emerging as a major development from conventional fuzzy logic, bipolar fuzzy set theory offers a more complex picture of human thinking by means of the division of positive and negative preferences. This enhancement has become prominent in MCDM for its capacity to model intricate, real-world decision problems that involve both favorable and unfavorable elements. Alghamdi et al. [2] established the foundational groundwork by investigating various MCDM methods within a bipolar fuzzy context. Their research provided a theoretical framework for the integration of bipolar fuzzy sets in decision-making contexts, highlighting the necessity of addressing both supportive and opposing information in the evaluation of alternatives. In order to explicitly address medical diagnosis concerns, Akram et al. [1] developed bipolar fuzzy extensions of the TOPSIS and ELECTRE-I algorithms. Their approach emphasized the practicality of bipolar fuzzy MCDM in addressing conflicting symptoms and diagnostic criteria. The proposed methods enhanced the decision model's accuracy by integrating both supportive positive evidence for a diagnosis and opposing negative evidence.

Garai and Garg [17] proposed an MCDM framework that uses an interpreter classification index to assess COVID-19 treatments within a fuzzy bipolar context, relevant to pandemic management. Their technique highlighted the necessity of considering dual-sided impacts, such as effectiveness and side effects, of treatment options. The research presented a systematic and comprehensible approach to assist decision-makers in the healthcare sector. Bipolar fuzzy aggregation operators developed by various researchers [39, 47, 51]. Alsolame and Alshehri [6] expanded the VIKOR method to the bipolar fuzzy context, focusing on decision-making scenarios that require compromise solutions. Their approach successfully reconciled opposing criteria, demonstrating the applicability of bipolar fuzzy logic in situations characterized by inherent trade-offs. The study contributes by modifying VIKOR to more accurately represent the bipolar characteristics of real-world data. Garai [18] presented a bipolar expected value-based MCDM approach specifically designed for wastewater management within a trapezoidal bipolar fuzzy context. This study demonstrated the application of bipolar fuzzy numbers in environmental engineering, specifically in the prioritization of sustainable solutions by considering both benefits and risks. Fuzzy credibility numbers (FCNs), which offer a dual-layered depiction of data by matching a fuzzy membership grade with its associated degree of credibility, were first explained by Ye et al. [52] in 2021. Recently, Riaz et al. [38] proposed the idea of the bipolar fuzzy credibility sets (BFCSs) and developed fundamental operations of BFCSs. They developed an MCDM framework to utilize a combined MEREC-EDAS method, capturing both the contradictory elements of human judgment and the accuracy of the information supplied. In parallel with these developments, Zadeh proposed Z-numbers in 2011 [54] to express both uncertain information and its related stability. A restriction, which describes uncertain information, and a reliability measure, which shows certainty in that information, make up a Z-number.

A comprehensive evaluation of the existing uses of Z-numbers in fuzzy environment was carried out by Alam et al. [4]. The study summarized various methodologies that incorporate Z-numbers into fuzzy MCDM models, emphasizing enhancements in modeling uncertainty within complex decision environments. For the purpose of predicting the bearing capacity of circular foundations, Hosseini et al. [21] developed a fuzzy inference technique that is based on the Z-number. This application illustrated the efficacy of Z-numbers in engineering contexts characterized by uncertain and imprecise data, where the reliability of information is paramount. Liu et al. [31] expanded the scope by introducing linguistic q-rung orthopair fuzzy Z-numbers and applying them to MCDM. The flexibility and expressiveness of decision-making processes under uncertainty are enhanced by the integration of Z-number theory with q-rung orthopair fuzzy sets, especially in linguistic assessments. In contexts of group decision-making, Aliyeva [5] utilized fuzzy dynamic group MCDM methods grounded in Z-numbers. This method tackled the difficulties of consolidating expert opinions when both the assessments and their reliabilities fluctuate over time, thereby enhancing consensus and decision-making precision. Hosseini et al. [22]

expanded the use of Z-numbers for assessing the risks associated with vibration impacts from blasting on mining infrastructures. Their methodology integrated causality-weighted fuzzy failure effect analysis, highlighting the significant function of Z-numbers in depicting causal relationships and the dependability of risk factors in engineering risk evaluations.

1.2. **A review of MCDM techniques.** MCDM techniques are acknowledged as effective methods for addressing decision-making issues that involve multiple conflicting criteria. MCDM methods have developed over time by integrating subjective and objective weighting mechanisms alongside fuzzy logic to manage uncertainty and enhance the reliability of decision outcomes.For the purpose of prioritizing advertising organizations, Kumar and Gandotra [29] developed a novel Pythagorean fuzzy entropy measurement to be used with the TOPSIS approach. The model improved the transparency of decision-making in marketing situations by demonstrating how fuzzy sets may express entropy and uncertainty in the generation of objective weights. Wang et al. [49] introduced a MCDM entropy-based objective weighting method for the assessment of significant third-party logistics providers. The study highlighted that the incorporation of sustainability-related criteria and objective weighting improves the consistency of MCDM outcomes, thereby facilitating more sustainable decision-making. Van Dua et al. [48] developed a framework that combines multiple objective weighting methods with various MCDM techniques to enhance the application of MCDM in material selection. This method enhances the alignment between the significance of criteria and the resulting rankings, especially in industrial contexts where trade-offs are critical.

Zolfani [19] conducted a comparative study that critically evaluates four prominent objective weighting techniques: CRITIC, ITARA, MEREC, and others, detailing their mathematical structures, advantages, and limitations. This analysis offers insights into the selection of appropriate weighting methods based on the characteristics of decision data and the context of application. Ghorabaee et al. [43] developed the MEREC method that determines criterion weights by evaluating the impact of removing each criterion on the overall performance of alternatives. The integration with other MCDM tools illustrates the adaptability of MEREC across various decision contexts, particularly in situations where expert opinions differ.

Hybrid approaches have demonstrated enhanced decision accuracy and consistency in engineering applications. An improved LOPCOW-SAW approach was presented by Nurdin et al. [35] for supply chain management's ideal supplier selection. The enhanced LOPCOW technique facilitates data-driven weight determination, while the SAW method offers a straightforward yet robust ranking mechanism, demonstrating efficacy in supplier evaluation. Dündar [16] employed the LOPCOW-CRADIS approach to assess the project performance of Turkish universities. Using fuzzy SWARA approach and a thorough literature analysis, Nezhad et al. [34] developed a readiness assessment model for IoT deployment in the banking industry. Their model illustrated the enhancement of expert-driven weighting through the application of fuzzy logic to address imprecise judgments, particularly in the context of emerging technology assessments.

Derse [14] concentrated on the identification and prioritization of barriers in green reverse logistics through the application of an integrated fuzzy DEMATEL–FUCOM–SWARA model. DEMATEL elucidated the interrelationships among criteria, FUCOM produced

reliable weights, and SWARA enhanced expert opinions. In the framework of fuzzy-based MCDM, Arslan and Cebi [8] proposed expanding the WASPAS approach by using decomposed fuzzy sets. Their methodology more precisely tackled uncertainty in performance data and demonstrated efficacy across multiple industrial optimization challenges. Entropy remains a fundamental method for objective weighting. Biswas et al. [10] presented the ERUNS method for evaluating firm performance within the energy sector. The ERUNS method presents a novel utility-based approach for performance standardization, thereby improving the discriminative capability of MCDM models. For the evaluation of food supply chain performance, Jiang et al. [25] developed a thorough fuzzy decision support system that incorporates ERUNS. The application of fuzzy logic facilitated a more refined approach to uncertainty management in food logistics, thereby enhancing decision-making quality in this essential sector. Mohammadi *et al*. [33] proposed a new MCDM approach based on FMEA-SWARA and MABAC for analyzing risk assessment in complex construction environments. Many researchers developed innovative MCDM techniques for diverse fields such as green sustainability in transportation [12], performance evaluation in South-Eastern European countries [13], Turkish wind turbines evaluation. Wang et al. [50] proposed rough Z-number-based SWARA-TODIM method for supply chain innovation. Many researchers developed Z-numbers based MCDM methods such as z-number linear programming problems [36], IMF D-SWARA—fuzzy ARAS-Z model [27], sustainable transportation challenges [40], integrated SWARA and GRA methods [24]. Zararsiz [55] proposed bipolar fuzzy Aczel–Alsina AOs for MADM. Sengonul and Zararsiz, [42] suggested the idea of fuzzy convergent and fuzzy bounded sequence spaces. Saqlain et al. [41] developed a bibliometric analysis for fuzzy modeling studies in Pakistan and highlighted the past and present developments in fuzzy logic. Hayat et al. [20] proposed the novel concepts of ”intuitionistic fuzzy credibility implicative ideals in BCK-algebra”.
The aforementioned studies illustrate the increasing interest in hybrid, fuzzy, and objective weighting-enhanced MCDM models. Methods including entropy, CRITIC, MEREC, ITARA, SWARA, and LOPCOW have played a crucial role in generating data-driven weights that minimize subjectivity. Simultaneously, advanced methodologies such as ERUNS and decomposed fuzzy models are being developed, providing high-resolution decision support in intricate environments. Notwithstanding these advancements, specific challenges persist. Limited research examines expert consensus modeling or the effects of correlated criteria, which may skew final rankings. Moreover, although fuzzy and hybrid techniques are gaining traction, their computational complexity may impede real-time applications unless optimization is applied.

1.3. **Research gap.**

- Many current models depend exclusively on either expert judgment or statistical data, neglecting to integrate both viewpoints for more balanced and credible assessments.
- Conventional MCDM methods frequently fail to adequately address the uncertainty, vagueness, and hesitancy inherent in expert evaluations, resulting in diminished reliability of outcomes.

- Few studies integrates Bipolar Z-numbers to depict positive and negative beliefs with corresponding reliability, despite their capacity to improve decision-making accuracy in uncertain contexts.
- Numerous decision models presuppose linear relationships among criteria, thereby oversimplifying the intricate trade-offs inherent in real-world issues such as the selection of recycling technologies.
- Despite the increasing significance of environmental considerations, limited MCDM studies specifically address the evaluation of plastic recycling technologies within a comprehensive and uncertainty-aware framework.
- Previous models usually overlook sensitivity and comparison analysis, which are critical for evaluating the dependability and efficacy of the method of decision-making.

1.4. **Motivation and contribution.** The impetus for creating the proposed decision-making model arises from the pressing necessity to enhance the selection process for plastic waste recycling technologies due to increasing environmental degradation and resource scarcity. Conventional decision-making methods frequently inadequately address the intricacies of real-world issues, particularly when multiple conflicting criteria, including cost, emissions, efficiency, and public acceptance, need to be evaluated concurrently. Furthermore, these models generally depend exclusively on quantitative data or subjective expert opinions, lacking a balanced and systematic integration of both perspectives. The presence of uncertainty and imprecision in expert judgments and data evaluations is seldom addressed adequately. This study presents a novel integrated MCDM framework that synthesises the strengths of objective (Entropy) and subjective (SWARA) weighting techniques alongside the robust ERUNS method for ranking, addressing existing limitations. The integration of Bipolar Z-numbers improves the model by addressing uncertainty in expert assessments and facilitating more dependable results. The model utilizes complementary techniques to enhance transparency, accuracy, and comprehensiveness in evaluation. This tool is designed to aid policymakers and industry stakeholders in making informed, sustainable decisions that align with circular economy principles and long-term environmental objectives. The highlights of this study are listed blow.

1.5. **Highlights of this study.**

- The aim is to establish a decision support framework that combines objective and subjective weighting methods to evaluate recyclable plastic waste technologies.
- Use the SWARA approach to assign expert-driven subjective weights and the Entropy method to determine objective weights.
- Robust decision-making is made accessible by the thorough rating of alternatives using the ERUNS approach.
- To confirm the stability and dependability of the suggested framework, do sensitivity and comparison assessments.
- The objective is to deliver actionable insights to policymakers and stakeholders aimed at enhancing sustainable plastic waste management practices.

Key contribution of this study are given below.

- This approach integrates Entropy, an objective method, with SWARA, a subjective method, to achieve a balanced evaluation of data-driven insights alongside expert judgment in the determination of criteria weights.
- Employs the ERUNS method to identify complex, nonlinear relationships among criteria, thereby offering a more accurate alternative ranking.
- This method improves the robustness and dependability of decision-making in difficult situations by integrating bipolar Z-numbers to describe uncertainty and reluctance in expert judgments.
- A case study on recycling plastic garbage, a crucial component of environmental sustainability and the circular economy, demonstrates the model's usefulness.
- A comprehensive sensitivity analysis is performed to assess the effects of varying criteria weights on the final rankings, thereby confirming the model's stability and robustness.
- This study compares the proposed model with traditional MCDM techniques, demonstrating its superior performance regarding adaptability, accuracy, and decision reliability.
- Provides practical insights for policymakers and industry leaders to facilitate the adoption of sustainable technologies and enhance waste management strategies.

This paper is organized into different sections. Section 2 outlines the foundational concepts of Bipolar fuzzy Z-numbers, detailing their mathematical formulation, semantic interpretation, and applicability in modeling uncertainty and reliability within complex decision-making contexts. Section 3 outlines the methodology employed in this study. The proposed decision support framework is thoroughly explained, incorporating the Entropy method for objective weight derivation, the SWARA method for eliciting expert-driven subjective weights, and the ERUNS technique for ranking alternatives while considering nonlinear relationships among criteria. Section 4 provides the results and discussion derived from a case study focused on the selection of technologies for recycling plastic waste. The study is concluded in Section 5 with a summary of the main conclusions and their applications, a description of the shortcomings of the suggested model, and recommendations for further research.

2. Preliminaries

In this section, the fundamental concepts of fuzzy modeling such as FS, BFS, and FCS, BFZN are studied.

Definition 1. [53]
Let $\mathcal{Q}$ be the universe. A FS ($\tilde{\mathbb{V}}$) in $\mathcal{Q}$ is described as

$$\tilde{\mathbb{V}} = \left\{ \langle \eth^{Q}, \aleph(\eth^{Q}) \rangle : \eth^{Q} \in \mathcal{Q} \right\} \tag{2.1}$$

where $\aleph : \mathcal{Q} \to [0,1]$ is the function of membership and $\aleph(\eth^{Q})$ is called MSD of an element $\eth^{Q} \in \mathcal{Q}$.

Definition 2. [56] *A BFS ($\tilde{\mathfrak{D}}$) extends the FS concept to deal with bipolarity. It is defined as follows*

$$\tilde{\mathfrak{D}} = \left\{ \left(\eth^{Q}, \aleph^{+}(\eth^{Q}), \aleph^{-}(\eth^{Q}) \right) \mid \eth^{Q} \in \mathcal{Q} \right\} \tag{2.2}$$

where $\aleph^{+}(\eth^{Q}) : \mathcal{Q} \rightarrow [0,1]$ is the positive membership degree (PMD) denoting the degree of satisfaction of some element $\eth^{Q} \in \mathcal{Q}$ corresponding to $\tilde{\mathfrak{D}}$ while $\aleph^{-}(\eth^{Q}) : \mathcal{Q} \rightarrow [-1,0]$ represents negative membership degree (NMD) which denotes degree of satisfaction of counter property of some element $\eth^{Q}$ corresponding to $\tilde{\mathfrak{D}}$, for all $\eth^{Q} \in \mathcal{Q}$.

Definition 3. [52]
A fuzzy credibility set (FCS) ($\mathcal{U}$) in $\mathcal{Q}$ is described as follows:

$$\mathcal{U} = \{(\varpi, \breve{\mu}_{\mathcal{U}}(\varpi), \P_{\mathcal{U}}(\varpi)) \,|\varpi \in \mathcal{Q}\} \tag{2.3}$$

where the functions $\breve{\mu}_{\mathcal{U}}$: $\mathcal{Q}\rightarrow$[0, 1] and $\mathcal{R}_{\mathcal{U}}$: $\mathcal{Q}\rightarrow$[0, 1] give the MSD of an elements in $\mathcal{Q}$ and its credibility degree, respectively. Then the number $(\breve{\mu}_{\mathcal{U}}(\varpi), \mathcal{R}_{\mathcal{U}}(\varpi))$ is called FCN, where $\breve{\mu}_{\mathcal{U}}(\varpi)$ is MSD and $\mathcal{R}_{\mathcal{U}}(\varpi))$ is credibility of MSD, respectively.

Definition 4. [54]
A Z-number is an ordered pair $Z = (\aleph_{\tilde{\mathcal{U}}}, \mathcal{R}_{\mathcal{U}})$, where $\aleph_{\tilde{\mathcal{U}}}$ is a fuzzy number representing MSD, and $\mathcal{R}_{\mathcal{U}}$ is the confidence level (degree of reliability) of A. Thus $\aleph_{\tilde{\mathcal{U}}}$ is the MSD to deal with uncertainty, and $\mathcal{R}_{\mathcal{U}}$ is a fuzzy reliability measure that expresses the degree of confidence or certainty in the statement $\aleph_{\tilde{\mathcal{U}}}$.

2.1. **Bipolar fuzzy Z-number.** A bipolar fuzzy Z-number (BFZN) further extends the concept of Z-number by incorporating positive and negative aspects that address bipolarity and fuzziness in real-life problems. Thus a BFZN assigns a PMD and its reliability as well as NMD and its reliability.

Definition 5. *Let $\mathcal{Q}$ be the universe. Then a BFZN (υ) on universe $\mathcal{Q}$ is defined as:*

$$\upsilon = \left\{ \left(\eth^{Q}, \langle \aleph^{+}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q}) \rangle, \langle \aleph^{-}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q}) \rangle \right) |\eth^{Q} \in \mathcal{Q} \right\} \tag{2.4}$$

where $\aleph^{+}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})$ gives the PMD and its reliability, and $\aleph^{-}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q})$ gives the NMD and its reliability, respectively.
Note that the number $\upsilon = \left(\langle \aleph^{+}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q}) \rangle, \langle \aleph^{-}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q}) \rangle \right)$ is the BFZN.

2.1. **Bipolar fuzzy Z-number.**

Definition 6. *Let $\upsilon = \left(\langle \aleph^{+}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q}) \rangle, \langle \aleph^{-}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q}) \rangle \right)$ and $\Bbbk = \left(\langle \aleph^{+}_{\Bbbk}(\eth^{R}), \mathcal{R}^{+}_{\Bbbk}(\eth^{R}) \rangle, \langle \aleph^{-}_{\Bbbk}(\eth^{R}), \mathcal{R}^{-}_{\Bbbk}(\eth^{R}) \rangle \right)$ be two BFZNs and μ is a scalar. The fundamental operations of BFZNs are listed as follows.*

- *Subset: $\upsilon \subset \Bbbk$ if: $\upsilon \subset \Bbbk$ if*
  $\aleph^{+}_{\mathcal{U}}(\eth^{Q}) \leq \aleph^{+}_{\Bbbk}(\eth^{R})$, $\mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q}) \leq \mathcal{R}^{+}_{\Bbbk}(\eth^{R})$, $\aleph^{-}_{\mathcal{U}}(\eth^{Q}) \geq \aleph^{-}_{\Bbbk}(\eth^{R})$, *and* $\mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q}) \geq \mathcal{R}^{-}_{\Bbbk}(\eth^{R})$.
- *Union:*
$$\begin{aligned}\upsilon \cup \Bbbk = \big(&\big\langle \max\{\aleph^{+}_{\mathcal{U}}(\eth^{Q}), \aleph^{+}_{\Bbbk}(\eth^{Q})\},\ \max\{\mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{+}_{\Bbbk}(\eth^{Q})\}\big\rangle,\\ &\big\langle \min\{\aleph^{-}_{\mathcal{U}}(\eth^{Q}), \aleph^{-}_{\Bbbk}(\eth^{Q})\},\ \min\{\mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{-}_{\Bbbk}(\eth^{Q})\}\big\rangle\big)\end{aligned}$$
- *Intersection:*
$$\begin{aligned}\upsilon \cap \Bbbk = \big(&\big\langle \min\{\aleph^{+}_{\mathcal{U}}(\eth^{Q}), \aleph^{+}_{\Bbbk}(\eth^{Q})\},\ \min\{\mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{+}_{\Bbbk}(\eth^{Q})\}\big\rangle,\\ &\big\langle \max\{\aleph^{-}_{\mathcal{U}}(\eth^{Q}), \aleph^{-}_{\Bbbk}(\eth^{Q})\},\ \max\{\mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{-}_{\Bbbk}(\eth^{Q})\}\big\rangle\big)\end{aligned}$$

- *Complement:*
  $\upsilon^c = \big(\langle 1-\aleph^+_{\mathcal{U}}(\eth^Q),\ 1-\mathcal{R}^+_{\mathcal{U}}(\eth^Q)\rangle,\ \langle |\aleph^-_{\mathcal{U}}(\eth^Q)|-1,\ |\mathcal{R}^-_{\mathcal{U}}(\eth^Q)|-1\rangle\big)$
- *Sum:*

$$\begin{gathered}\upsilon \oplus \Bbbk = \big(\langle \aleph^+_{\mathcal{U}}(\eth^Q)+\aleph^+_{\Bbbk}(\eth^Q)-\aleph^+_{\mathcal{U}}(\eth^Q)\aleph^+_{\Bbbk}(\eth^Q),\\ \mathcal{R}^+_{\mathcal{U}}(\eth^Q)+\mathcal{R}^+_{\Bbbk}(\eth^Q)-\mathcal{R}^+_{\mathcal{U}}(\eth^Q)\mathcal{R}^+_{\Bbbk}(\eth^Q)\rangle,\\ \langle -|\aleph^-_{\mathcal{U}}(\eth^Q)||\aleph^-_{\Bbbk}(\eth^Q)|, -|\mathcal{R}^-_{\mathcal{U}}(\eth^Q)||\mathcal{R}^-_{\Bbbk}(\eth^Q)|\rangle\big)\end{gathered}$$

- *Product:*

$$\upsilon \otimes \Bbbk = \big(\langle \aleph^+_{\mathcal{U}}(\eth^Q)\aleph^+_{\Bbbk}(\eth^Q),\ \mathcal{R}^+_{\mathcal{U}}(\eth^Q)\mathcal{R}^+_{\Bbbk}(\eth^Q)\rangle,$$

$$\langle \aleph^-_{\mathcal{U}}(\eth^Q)\aleph^-_{\Bbbk}(\eth^Q)-|\aleph^-_{\mathcal{U}}(\eth^Q)|-|\aleph^-_{\Bbbk}(\eth^Q)|, \mathcal{R}^-_{\mathcal{U}}(\eth^Q)\mathcal{R}^-_{\Bbbk}(\eth^Q)-|\mathcal{R}^-_{\mathcal{U}}(\eth^Q)|-|\mathcal{R}^-_{\Bbbk}(\eth^Q)|\rangle\big)$$

- *Scalar Multiplication: For a scalar μ,*

$$\mu\upsilon = \big(\langle 1-(1-\aleph^+_{\mathcal{U}}(\eth^Q))^\mu,\ 1-(1-\mathcal{R}^+_{\mathcal{U}}(\eth^Q))^\mu\rangle, \langle -|\aleph^-_{\mathcal{U}}(\eth^Q)|^\mu,\ -|\breve{c}^-_{\mathcal{U}}(\eth^Q)|^\mu\rangle\big)$$

- *Exponent: For a scalar μ,*

$$\upsilon^\mu = \big(\langle (\aleph^+_{\mathcal{U}}(\eth^Q))^\mu,\ (\mathcal{R}^+_{\mathcal{U}}(\eth^Q))^\mu\rangle, \langle -1+|1+\aleph^-_{\mathcal{U}}(\eth^Q)|^\mu,\ -1+|1+\mathcal{R}^-_{\mathcal{U}}(\eth^Q)|^\mu\rangle\big)$$

2.3. Score and accuracy function of BFZNs.

Definition 7. *Let $\upsilon = \big(\langle\aleph^+_{\mathcal{U}},\mathcal{R}^+_{\mathcal{U}}\rangle,\langle\aleph^-_{\mathcal{U}},\mathcal{R}^-_{\mathcal{U}}\rangle\big)$ be a BFZN, then we define score function $\breve{\beta}(\upsilon)$ is defined as*

$$\breve{\beta}(\upsilon) = \frac{2+\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}+\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}}{4},\ \breve{\beta}\in[0,1] \tag{2.5}$$

The second score function $\breve{\beta}_1(\upsilon)$ is defined as

$$\breve{\beta}_1(\upsilon) = \frac{\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}+\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}}{2},\ \breve{\beta}_1\in[-1,1] \tag{2.6}$$

and the accuracy function $\breve{\zeta}(\upsilon)$ is defined as

$$\breve{\zeta}(\upsilon) = \frac{\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}-\aleph^-_{\mathcal{U}}-\mathcal{R}^-_{\mathcal{U}}}{4},\ \breve{\zeta}\in[0,1] \tag{2.7}$$

The second accuracy function $\breve{\zeta}_1(\upsilon)$ is defined as

$$\breve{\zeta}_1(\upsilon) = \frac{\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}-|\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}|}{2},\ \breve{\zeta}_1\in[-1,1] \tag{2.8}$$

Proposition 2.4. *Let $\upsilon = \big(\langle\aleph^+_{\mathcal{U}},\mathcal{R}^+_{\mathcal{U}}\rangle,\langle\aleph^-_{\mathcal{U}},\mathcal{R}^-_{\mathcal{U}}\rangle\big)$ be a BFZN, where $\aleph^+_{\mathcal{U}},\mathcal{R}^+_{\mathcal{U}}\in[0,1]$ and $\aleph^-_{\mathcal{U}},\mathcal{R}^-_{\mathcal{U}}\in[-1,0]$. Then the score function $\tilde{\beta}(\upsilon) = \frac{2+\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}+\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}}{4}$ satisfies $0\le\tilde{\beta}(\upsilon)\le 1$.*

Proof. Since $0\le\aleph^+_{\mathcal{U}}\le 1,\quad 0\le\mathcal{R}^+_{\mathcal{U}}\le 1$, we have $0\le\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}\le 2$. Similarly, because $-1\le\aleph^-_{\mathcal{U}}\le 0,\quad -1\le\mathcal{R}^-_{\mathcal{U}}\le 0$, it follows that $-2\le\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}\le 0$. The above inequalities added together produce $-2\le\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}+\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}\le 2$. Adding 2 throughout gives $0\le 2+\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}+\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}\le 4$. Dividing by 4, we obtain $0\le\frac{2+\aleph^+_{\mathcal{U}}+\mathcal{R}^+_{\mathcal{U}}+\aleph^-_{\mathcal{U}}+\mathcal{R}^-_{\mathcal{U}}}{4}\le 1$. Hence, $\tilde{\beta}(\upsilon)\in[0,1]$. $\square$

Proposition 2.5. *Let* $\upsilon = \big(\langle \aleph^{+}_{\mathcal{U}}, \mathcal{R}^{+}_{\mathcal{U}}\rangle, \langle \aleph^{-}_{\mathcal{U}}, \mathcal{R}^{-}_{\mathcal{U}}\rangle\big)$ *be a BFZN, where* $\aleph^{+}_{\mathcal{U}}, \mathcal{R}^{+}_{\mathcal{U}} \in [0,1]$ *and* $\aleph^{-}_{\mathcal{U}}, \mathcal{R}^{-}_{\mathcal{U}} \in [-1,0]$. *Then the score function* $\tilde{\beta}_1(\upsilon) = \frac{\aleph^{+}_{\mathcal{U}}+\mathcal{R}^{+}_{\mathcal{U}}+\aleph^{-}_{\mathcal{U}}+\mathcal{R}^{-}_{\mathcal{U}}}{2}$ *satisfies* $-1 \leq \tilde{\beta}_1(\upsilon) \leq 1$.

Proof. Since $0 \leq \aleph^{+}_{\mathcal{U}} \leq 1$ and $0 \leq \mathcal{R}^{+}_{\mathcal{U}} \leq 1$, it follows that $0 \leq \aleph^{+}_{\mathcal{U}} + \mathcal{R}^{+}_{\mathcal{U}} \leq 2$. Similarly, $-1 \leq \aleph^{-}_{\mathcal{U}} \leq 0$ and $-1 \leq \mathcal{R}^{-}_{\mathcal{U}} \leq 0$, which implies $-2 \leq \aleph^{-}_{\mathcal{U}} + \mathcal{R}^{-}_{\mathcal{U}} \leq 0$. Adding these inequalities yields $-2 \leq \aleph^{+}_{\mathcal{U}} + \mathcal{R}^{+}_{\mathcal{U}} + \aleph^{-}_{\mathcal{U}} + \mathcal{R}^{-}_{\mathcal{U}} \leq 2$. Dividing throughout by 2, we obtain $-1 \leq \frac{\aleph^{+}_{\mathcal{U}}+\mathcal{R}^{+}_{\mathcal{U}}+\aleph^{-}_{\mathcal{U}}+\mathcal{R}^{-}_{\mathcal{U}}}{2} \leq 1$. Hence, $\tilde{\beta}_1(\upsilon) \in [-1,1]$. □

Definition 8. *Let* $\upsilon = \big(\langle \aleph^{+}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})\rangle, \langle \aleph^{-}_{\mathcal{U}}(\eth^{Q}), \mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q})\rangle\big)$ *and* $\Bbbk = \big(\langle \aleph^{+}_{\Bbbk}(\eth^{Q}), \mathcal{R}^{+}_{\Bbbk}(\eth^{Q})\rangle, \langle \aleph^{-}_{\Bbbk}(\eth^{Q}), \mathcal{R}^{-}_{\Bbbk}(\eth^{Q})\rangle\big)$ *be two BFZNs,*

- *If* $\breve{\beta}(\upsilon) < \breve{\beta}(\Bbbk)$, *then* $\upsilon < \Bbbk$.
- *If* $\breve{\beta}(\upsilon) > \breve{\beta}(\Bbbk)$, *then* $\upsilon > \Bbbk$.
- *If* $\breve{\beta}(\upsilon) = \breve{\beta}(\Bbbk)$, *then:*
  - *If* $\breve{\zeta}(\upsilon) < \breve{\zeta}(\Bbbk)$, *then* $\upsilon < \Bbbk$.
  - *If* $\breve{\zeta}(\upsilon) > \breve{\zeta}(\Bbbk)$, *then* $\upsilon > \Bbbk$.
  - *If* $\breve{\zeta}(\upsilon) = \breve{\zeta}(\Bbbk)$, *then* $\upsilon = \Bbbk$.

Example 1. *Consider two BFZNs* $\upsilon = (\langle 0.76, 0.21\rangle, \langle -0.48, -0.34\rangle)$ *and* $\Bbbk = (\langle 0.56, 0.89\rangle, \langle -0.42, -0.57\rangle)$

Then their $\tilde{\beta}$*, using Equation 2.5, is obtained as*

$$\tilde{\beta}(\upsilon) = \frac{2 + 0.76 + 0.21 + (-0.48) + (-0.34)}{4} = 0.5375$$

and

$$\tilde{\beta}(\Bbbk) = \frac{2 + 0.56 + 0.89 + (-0.42) + (-0.57)}{4} = 0.615.$$

As $\tilde{\beta}(\Bbbk) > \tilde{\beta}(\upsilon)$*, so,* $\Bbbk > \upsilon$.

The list of notions, abbreviations and symbols is provided in Table 1.

TABLE 1. List of abbreviations and symbols used in the manuscript.

Notions	Abbreviations	Symbols
Positive membership degree	PMD	$\aleph^{+}(\eth^{Q}) \in [0,1]$
Negative membership grade	NMD	$\aleph^{-}(\eth^{Q}) \in [-1,0]$
Positive reliability degree	PRD	$\mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q}) \in [0,1]$
Negative reliability degree	NRD	$\mathcal{R}^{-}_{\mathcal{U}}(\eth^{Q}) \in [-1,0]$
Score function	SF	$\breve{\beta}(\upsilon) \in [0,1]$
Accuract function	AF	$\breve{\zeta}(\upsilon) \in [0,1]$

Definition 9. *Let* $\upsilon_{\jmath} = \Big(\langle \aleph^{+}_{\mathcal{U}_{\jmath}}, \mathcal{R}^{+}_{\mathcal{U}_{\jmath}}\rangle, \langle \aleph^{-}_{\mathcal{U}_{\jmath}}, \mathcal{R}^{-}_{\mathcal{U}_{\jmath}}\rangle\Big)$ *be the collection of BFZNs and* $\Omega^{\tau}_{\jmath} = (\Omega^{\tau}_1, \Omega^{\tau}_2, ..., \Omega^{\tau}_{\upsilon})^T$ *be the softmax weights of* $\upsilon_{\jmath}(\jmath = 1, 2, ..., r)$*, where* ${\Omega_{\jmath}}^{\tau} > o$*, and*

$\sum_{\jmath=1}^{r} \Omega_{\jmath}{}^{\tau} = 1$. *Then bipolar fuzzy Z-number softmax hamacher weighted averaging (BFZNSHWA) aggregation operator is a mapping BFZNSHWA:* $\mathcal{F}^r \to \mathcal{F}$ *, if*

$$\mathit{BFZNSHWA}(\upsilon_1, \upsilon_2, \dots \upsilon_r) = \sum_{\jmath=1}^{r} \breve{\Omega}_{\jmath} \upsilon_{\jmath} = \breve{\Omega}_1 * \upsilon_1 \tilde{\oplus} \breve{\Omega}_2 * \upsilon_2 \tilde{\oplus} \dots, \tilde{\oplus} \breve{\Omega}_r * \upsilon_r$$

Theorem 1. *Let* $\upsilon_{\jmath} = \left(\langle \aleph^{+}_{\mathcal{U}_{\jmath}}, \mathcal{R}^{+}_{\mathcal{U}_{\jmath}} \rangle, \langle \aleph^{-}_{\mathcal{U}_{\jmath}}, \mathcal{R}^{-}_{\mathcal{U}_{\jmath}} \rangle \right)$ *be the collection of BFZNs, then*

$$\mathit{BFZNSHWA}(\upsilon_1, \upsilon_2, \dots \upsilon_r)$$

$$= \left(\begin{array}{c} \left\langle \dfrac{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\aleph^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}} - \prod_{\jmath=1}^{v}(1-\aleph^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\aleph^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}(1-\aleph^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}}}, \right. \\ \left. \dfrac{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\mathcal{R}^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}} - \prod_{\jmath=1}^{v}(1-\mathcal{R}^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\mathcal{R}^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}(1-\mathcal{R}^{-}_{\mathcal{U}_{\jmath}})^{\Omega^{\tau}}} \right\rangle, \\ \left\langle -\dfrac{\pi^{\varphi}\prod_{\jmath=1}^{v}|\aleph^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_{\jmath}}|))^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}|\aleph^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}}}, \right. \\ \left. -\dfrac{\pi^{\varphi}\prod_{\jmath=1}^{v}|\mathcal{R}^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_{\jmath}}|))^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}|\mathcal{R}^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}}} \right\rangle \end{array} \right) \tag{2.9}$$

where $\Omega_{\jmath}^{\tau} = (\Omega_1^{\tau}, \Omega_2^{\tau}, ..., \Omega_v^{\tau})^T$ *be the softmax weights of* $\upsilon_{\jmath}(\jmath = 1, 2, ..., r)$*, and* $\Omega_{\jmath}{}^{\tau} > 0$*,* $\sum_{\jmath=1}^{r} \Omega_{\jmath}{}^{\tau} = 1$.

Proof. By using mathematical induction on r, we prove Equation (2.9).

• For $r = 2$, we have

$$\Omega_1^{\tau}\upsilon_1 = \left(\begin{array}{c} \left\langle \frac{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}} - \left(1-\aleph^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}}}{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}} + (\pi^{\varphi}-1)\left(1-\aleph^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}}}, \right. \\ \left. \frac{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}} - \left(1-\mathcal{R}^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}}}{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}} + (\pi^{\varphi}-1)\left(1-\mathcal{R}^{+}_{\mathcal{U},1}\right)^{\Omega_1^{\tau}}} \right\rangle, \\ \left\langle -\frac{\pi^{\varphi}\left|\aleph^{-}_{\mathcal{U},1}\right|^{\Omega_1^{\tau}}}{\left(1+(\pi^{\varphi}-1)\left(1-\left|\aleph^{-}_{\mathcal{U},1}\right|\right)\right)^{\Omega_1^{\tau}} + (\pi^{\varphi}-1)\left|\aleph^{-}_{\mathcal{U},1}\right|^{\Omega_1^{\tau}}}, \right. \\ \left. -\frac{\pi^{\varphi}\left|\mathcal{R}^{-}_{\mathcal{U},1}\right|^{\Omega_1^{\tau}}}{\left(1+(\pi^{\varphi}-1)\left(1-\left|\mathcal{R}^{-}_{\mathcal{U},1}\right|\right)\right)^{\Omega_1^{\tau}} + (\pi^{\varphi}-1)\left|\mathcal{R}^{-}_{\mathcal{U},1}\right|^{\Omega_1^{\tau}}} \right\rangle \end{array} \right).$$

$$\Omega_2^{\tau}\upsilon_2 = \left(\begin{array}{c} \left\langle \frac{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}} - \left(1-\aleph^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}}}{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}} + (\pi^{\varphi}-1)\left(1-\aleph^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}}}, \right. \\ \left. \frac{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}} - \left(1-\mathcal{R}^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}}}{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}} + (\pi^{\varphi}-1)\left(1-\mathcal{R}^{+}_{\mathcal{U},2}\right)^{\Omega_2^{\tau}}} \right\rangle, \\ \left\langle -\frac{\pi^{\varphi}\left|\aleph^{-}_{\mathcal{U},2}\right|^{\Omega_2^{\tau}}}{\left(1+(\pi^{\varphi}-1)\left(1-\left|\aleph^{-}_{\mathcal{U},2}\right|\right)\right)^{\Omega_2^{\tau}} + (\pi^{\varphi}-1)\left|\aleph^{-}_{\mathcal{U},2}\right|^{\Omega_2^{\tau}}}, \right. \\ \left. -\frac{\pi^{\varphi}\left|\mathcal{R}^{-}_{\mathcal{U},2}\right|^{\Omega_2^{\tau}}}{\left(1+(\pi^{\varphi}-1)\left(1-\left|\mathcal{R}^{-}_{\mathcal{U},2}\right|\right)\right)^{\Omega_2^{\tau}} + (\pi^{\varphi}-1)\left|\mathcal{R}^{-}_{\mathcal{U},2}\right|^{\Omega_2^{\tau}}} \right\rangle \end{array} \right).$$

Then,

$$BFZNSHWA(\upsilon_1,\upsilon_2) = \Omega^{\tau}{}_1\upsilon_1 \bigoplus \Omega^{\tau}{}_2\upsilon_2$$

$$= \left(\begin{array}{c} \left\langle \frac{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1} - \left(1-\aleph^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1}}{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1} + (\pi^{\varphi}-1)\left(1-\aleph^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1}}, \right. \\ \left. \frac{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1} - \left(1-\mathcal{R}^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1}}{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1} + (\pi^{\varphi}-1)\left(1-\mathcal{R}^{+}_{\mathcal{U}1}\right)^{\Omega^{\tau}1}} \right\rangle, \\ \left\langle \frac{-\pi^{\varphi}|\aleph^{-}_{\mathcal{U}_1}|^{\Omega^{\tau}1}}{\left(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_1}|)\right)^{\Omega^{\tau}1} + (\pi^{\varphi}-1)|\aleph^{-}_{\mathcal{U}_1}|^{\Omega^{\tau}1}}, \right. \\ \left. \frac{-\pi^{\varphi}|\mathcal{R}^{-}_{\mathcal{U}_1}|^{\Omega^{\tau}1}}{\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_1}|)\right)^{\Omega^{\tau}1} + (\pi^{\varphi}-1)|\mathcal{R}^{-}_{\mathcal{U}_1}|^{\Omega^{\tau}1}} \right\rangle \end{array} \right)$$

$$\bigoplus$$

$$\left(\begin{array}{c}\left\langle\frac{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}-\left(1-\aleph^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}}{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}+(\pi^{\varphi}-1)\left(1-\aleph^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}},\right.\\ \left.\frac{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}-\left(1-\mathcal{R}^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}}{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}+(\pi^{\varphi}-1)\left(1-\mathcal{R}^{+}_{\mathcal{U}2}\right)^{\Omega^{\tau}2}}\right\rangle,\\ \left\langle\frac{-\pi^{\varphi}|\aleph^{-}_{\mathcal{U}_2}|^{\Omega^{\tau}2}}{\left(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_2}|)\right)^{\Omega^{\tau}2}+(\pi^{\varphi}-1)|\aleph^{-}_{\mathcal{U}_2}|^{\Omega^{\tau}2}},\right.\\ \left.\frac{-\pi^{\varphi}|\mathcal{R}^{-}_{\mathcal{U}_2}|^{\Omega^{\tau}2}}{\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_2}|)\right)^{\Omega^{\tau}2}+(\pi^{\varphi}-1)|\mathcal{R}^{-}_{\mathcal{U}_2}|^{\Omega^{\tau}2}}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle\frac{\prod_{\jmath=1}^{2}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}-\prod_{\jmath=1}^{2}\left(1-\aleph^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{2}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}+\left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{2}\left(1-\aleph^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}},\right.\\ \left.\frac{\prod_{\jmath=1}^{2}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}-\prod_{\jmath=1}^{2}\left(1-\mathcal{R}^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{2}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}+\left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{2}\left(1-\mathcal{R}^{+}_{\mathcal{U}\jmath}\right)^{\Omega^{\tau}\jmath}}\right\rangle,\\ \left\langle\frac{-\pi^{\varphi}\prod_{\jmath=1}^{2}|\aleph^{-}_{\mathcal{U}_\jmath}|^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{2}\left(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_\jmath}|)\right)^{\Omega^{\tau}\jmath}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{2}|\aleph^{-}_{\mathcal{U}_\jmath}|^{\Omega^{\tau}\jmath}},\right.\\ \left.\frac{-\pi^{\varphi}\prod_{\jmath=1}^{2}|\mathcal{R}^{-}_{\mathcal{U}_\jmath}|^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{2}\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_\jmath}|)\right)^{\Omega^{\tau}\jmath}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{2}|\mathcal{R}^{-}_{\mathcal{U}_\jmath}|^{\Omega^{\tau}\jmath}}\right\rangle\end{array}\right)$$

• Suppose that for $r=m$ holds,

$$BFZNSHWA(\upsilon_1,\upsilon_2,...,\upsilon_r)=\bigoplus_{\jmath=1}^{m}\Omega^{\tau}{}_{\jmath}\upsilon_{\jmath}$$

$$
= \left(\begin{array}{c}
\left\langle \dfrac{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} - \prod_{\jmath=1}^{m}\left(1-\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} + \left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{\jmath}\left(1-\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}}, \right. \\
\left. \dfrac{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} - \prod_{\jmath=1}^{m}\left(1-\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} + \left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{m}\left(1-\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}} \right\rangle, \\
\left\langle \dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{m}\left|\aleph^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_\jmath}|)\right)^{\Omega^{\tau}{}_\jmath} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{m}\left|\aleph^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}}, \right. \\
\left. \dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{m}\left|\mathcal{R}^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_\jmath}|)\right)^{\Omega^{\tau}{}_\jmath} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{m}\left|\mathcal{R}^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}} \right\rangle
\end{array} \right)
$$

• For $r = m+1$, we have

$$
BFZNSHWA(\upsilon_1, \upsilon_2, ..., \upsilon_r, \upsilon_{m+1}) = \Omega^{\tau}{}_1\upsilon_1 \bigoplus \Omega^{\tau}{}_2\upsilon_2 \bigoplus ... \bigoplus \Omega^{\tau}{}_\jmath\upsilon_\jmath \bigoplus \Omega^{\tau}{}_{m+1}\upsilon_{m+1}
$$

$$
= \left(\begin{array}{c}
\left\langle \dfrac{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} - \prod_{\jmath=1}^{m}\left(1-\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} + \left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{m}\left(1-\aleph^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}}, \right. \\
\left. \dfrac{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} - \prod_{\jmath=1}^{m}\left(1-\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath} + \left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{m}\left(1-\mathcal{R}^{+}_{\mathcal{U}_\jmath}\right)^{\Omega^{\tau}{}_\jmath}} \right\rangle, \\
\left\langle \dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{m}\left|\aleph^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_\jmath}|)\right)^{\Omega^{\tau}{}_\jmath} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{m}\left|\aleph^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}}, \right. \\
\left. \dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{m}\left|\mathcal{R}^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}}{\prod_{\jmath=1}^{m}\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_\jmath}|)\right)^{\Omega^{\tau}{}_\jmath} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{m}\left|\mathcal{R}^{-}_{\mathcal{U}_\jmath}\right|^{\Omega^{\tau}{}_\jmath}} \right\rangle
\end{array} \right)
$$

$$
\bigotimes
$$

$$\left(\begin{array}{c}\frac{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}-\left(1-\aleph^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}}{\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}+\left(\pi^{\varphi}-1\right)\left(1-\aleph^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}},\\ \frac{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}-\left(1-\mathcal{R}^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}}{\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}+\left(\pi^{\varphi}-1\right)\left(1-\mathcal{R}^{+}_{\mathcal{U}\,m+1}\right)^{\Omega^{\tau}m+1}},\\ \frac{-\pi^{\varphi}|\aleph^{-}_{\mathcal{U}_{m+1}}|^{\Omega^{\tau}m+1}}{\left(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_{m+1}}|)\right)^{\Omega^{\tau}m+1}+\left(\pi^{\varphi}-1\right)|\aleph^{-}_{\mathcal{U}_{m+1}}|^{\Omega^{\tau}m+1}},\\ \frac{-\pi^{\varphi}|\mathcal{R}^{-}_{\mathcal{U}_{m+1}}|^{\Omega^{\tau}m+1}}{\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_{m+1}}|)\right)^{\Omega^{\tau}m+1}+\left(\pi^{\varphi}-1\right)|\mathcal{R}^{-}_{\mathcal{U}_{m+1}}|^{\Omega^{\tau}m+1}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle\frac{\prod_{\jmath=1}^{m+1}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}-\prod_{\jmath=1}^{m+1}\left(1-\aleph^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{m+1}\left(1+(\pi^{\varphi}-1)\aleph^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}+\left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{m+1}\left(1-\aleph^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}},\right.\\ \left.\frac{\prod_{\jmath=1}^{m+1}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}-\prod_{\jmath=1}^{m+1}\left(1-\mathcal{R}^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{m+1}\left(1+(\pi^{\varphi}-1)\mathcal{R}^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}+\left(\pi^{\varphi}-1\right)\prod_{\jmath=1}^{m+1}\left(1-\mathcal{R}^{+}_{\mathcal{U}_{\jmath}}\right)^{\Omega^{\tau}\jmath}}\right\rangle,\\ \left\langle\frac{-\pi^{\varphi}\prod_{\jmath=1}^{m+1}|\aleph^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{m+1}\left(1+(\pi^{\varphi}-1)(1-|\aleph^{-}_{\mathcal{U}_{\jmath}}|)\right)^{\Omega^{\tau}\jmath}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{m+1}|\aleph^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}\jmath}},\right.\\ \left.\frac{-\pi^{\varphi}\prod_{\jmath=1}^{m+1}|\mathcal{R}^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}\jmath}}{\prod_{\jmath=1}^{m+1}\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}^{-}_{\mathcal{U}_{\jmath}}|)\right)^{\Omega^{\tau}\jmath}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{m+1}|\mathcal{R}^{-}_{\mathcal{U}_{\jmath}}|^{\Omega^{\tau}\jmath}}\right\rangle\end{array}\right)$$

consequently,according to principle of mathematical induction this holds for any positive integer n. □

Theorem 2. ***(Idempotency)*** *Let* $\upsilon_{\jmath}=\left((\aleph^{+}_{\mathcal{U}_{\jmath}},\mathcal{R}^{+}_{\mathcal{U}_{\jmath}}),(\aleph^{-}_{\mathcal{U}_{\jmath}},\mathcal{R}^{-}_{\mathcal{U}_{\jmath}})\right)$ $(\jmath=1,2,...,r)$ *be a collection of BFZNs such that,* $\upsilon_{\jmath}=\upsilon\ \ \forall\,\jmath=1,2,...,r$*, where* $\upsilon=\left(\aleph^{+}_{\mathcal{U}},\mathcal{R}^{+}_{\mathcal{U}};\aleph^{-}_{\mathcal{U}},\mathcal{R}^{-}_{\mathcal{U}}\right)$. *Then*

$$BFZNSHWA(\upsilon_1,\upsilon_2,...,\upsilon_n)=\upsilon$$

Proof. Since $\sum_{\jmath=1}^{r}\Omega^{\tau_{\jmath}}=1$, therefore

$$\left(\begin{array}{c}\left\langle\dfrac{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}-\prod_{\jmath=1}^{r}\left(1-\aleph_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}\left(1-\aleph_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}},\right.\\ \left.\dfrac{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}-\prod_{\jmath=1}^{r}\left(1-\mathcal{R}_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}\left(1-\mathcal{R}_{\mathcal{U}_\jmath}^{+}\right)^{\Omega^{\tau_\jmath}}}\right\rangle,\\ \left\langle\dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{r}|\aleph_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)(1-|\aleph_{\mathcal{U}_\jmath}^{-}|)\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}|\aleph_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau_\jmath}}},\right.\\ \left.\dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{r}|\mathcal{R}_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}_{\mathcal{U}_\jmath}^{-}|)\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}|\mathcal{R}_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau_\jmath}}}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle\dfrac{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}-\prod_{\jmath=1}^{r}\left(1-\aleph_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}\left(1-\aleph_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}},\right.\\ \left.\dfrac{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}-\prod_{\jmath=1}^{r}\left(1-\mathcal{R}_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}\left(1-\mathcal{R}_{\mathcal{U}}^{+}\right)^{\Omega^{\tau_\jmath}}}\right\rangle,\\ \left\langle\dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{r}|\aleph_{\mathcal{U}}^{-}|^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)(1-|\aleph_{\mathcal{U}}^{-}|)\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}|\aleph_{\mathcal{U}}^{-}|^{\Omega^{\tau_\jmath}}},\right.\\ \left.\dfrac{-\pi^{\varphi}\prod_{\jmath=1}^{r}|\mathcal{R}_{\mathcal{U}}^{-}|^{\Omega^{\tau_\jmath}}}{\prod_{\jmath=1}^{r}\left(1+(\pi^{\varphi}-1)(1-|\mathcal{R}_{\mathcal{U}}^{-}|)\right)^{\Omega^{\tau_\jmath}}+(\pi^{\varphi}-1)\prod_{\jmath=1}^{r}|\mathcal{R}_{\mathcal{U}}^{-}|^{\Omega^{\tau_\jmath}}}\right\rangle\end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \dfrac{(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}} - (1-\aleph_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}}{(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}} + (\pi^{\varphi}-1)(1-\aleph_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}}, \right. \\ \left. \dfrac{(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}} - (1-\mathcal{R}_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}}{(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}} + (\pi^{\varphi}-1)(1-\mathcal{R}_{\mathcal{U}}^{+})^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}} \right\rangle, \\ \left\langle \dfrac{-\pi^{\varphi}|\aleph_{\mathcal{U}}^{-}|^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}}{(1+(\pi^{\varphi}-1)(1-|\aleph_{\mathcal{U}}^{-}|))^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}} + (\pi^{\varphi}-1)|\aleph_{\mathcal{U}}^{-}|^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}}, \right. \\ \left. \dfrac{-\pi^{\varphi}|\mathcal{R}_{\mathcal{U}}^{-}|^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}}{(1+(\pi^{\varphi}-1)(1-|\mathcal{R}_{\mathcal{U}}^{-}|))^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}} + (\pi^{\varphi}-1)|\mathcal{R}_{\mathcal{U}}^{-}|^{\sum_{\jmath=1}^{r}\Omega^{\tau_\jmath}}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \dfrac{(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}}^{+}) - (1-\aleph_{\mathcal{U}}^{+})}{(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}}^{+}) + (\pi^{\varphi}-1)(1-\aleph_{\mathcal{U}}^{+})}, \right. \\ \left. \dfrac{(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}}^{+}) - (1-\mathcal{R}_{\mathcal{U}}^{+})}{(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}}^{+}) + (\pi^{\varphi}-1)(1-\mathcal{R}_{\mathcal{U}}^{+})} \right\rangle, \\ \left\langle \dfrac{-\pi^{\varphi}|\aleph_{\mathcal{U}}^{-}|}{(1+(\pi^{\varphi}-1)(1-|\aleph_{\mathcal{U}}^{-}|)) + (\pi^{\varphi}-1)|\aleph_{\mathcal{U}}^{-}|}, \right. \\ \left. \dfrac{-\pi^{\varphi}|\mathcal{R}_{\mathcal{U}}^{-}|}{(1+(\pi^{\varphi}-1)(1-|\mathcal{R}_{\mathcal{U}}^{-}|)) + (\pi^{\varphi}-1)|\mathcal{R}_{\mathcal{U}}^{-}|} \right\rangle \end{array} \right)$$

$$= \big(\ (\aleph_{\mathcal{U}}^{+}, \mathcal{R}_{\mathcal{U}}^{+}), (\aleph_{\mathcal{U}}^{-}, \mathcal{R}_{\mathcal{U}}^{-}) \ \big) = (\aleph_{\mathcal{U}}^{+}, \mathcal{R}_{\mathcal{U}}^{+}; \aleph_{\mathcal{U}}^{-}, \mathcal{R}_{\mathcal{U}}^{-}) \qquad \square$$

Theorem 3. ***(Monotonicity)*** *Let* $\upsilon_\jmath = \big((\aleph_{\mathcal{U}_{\upsilon_\jmath}}^{+}, \mathcal{R}_{\mathcal{U}_{\upsilon_\jmath}}^{+}), (\aleph_{\mathcal{U}_{\upsilon_\jmath}}^{-}, \mathcal{R}_{\mathcal{U}_{\upsilon_\jmath}}^{-})\big)$ *and* $\Bbbk_\jmath = \big((\aleph_{\mathcal{U}_{\Bbbk_\jmath}}^{+}, \mathcal{R}_{\mathcal{U}_{\Bbbk_\jmath}}^{+}), (\aleph_{\mathcal{U}_{\Bbbk_\jmath}}^{-}, \mathcal{R}_{\mathcal{U}_{\Bbbk_\jmath}}^{-})\big)$ $(\jmath = 1, 2, ..., r)$ *be collection of BFZNs, if* $\upsilon_\jmath \leq \Bbbk_\jmath, \forall\, \jmath = 1, 2, ..., r$. *Then*

$$BFZNSHWA(\upsilon_1, \upsilon_2, ..., \upsilon_r) \leq BFZNSHWA(\Bbbk_1, \Bbbk_2, ..., \Bbbk_r)$$

Theorem 4. ***(Boundedness)*** *Suppose that* $\upsilon_\jmath = \big((\aleph_{\mathcal{U}_\jmath}^{+}, \mathcal{R}_{\mathcal{U}_\jmath}^{+}), (\aleph_{\mathcal{U}_\jmath}^{-}, \mathcal{R}_{\mathcal{U}_\jmath}^{-})\big)$ $(\jmath = 1.2, ..., r)$ *is a collection of BFZNs and* $\upsilon^{-} = \big(\min_\jmath (\aleph_{\mathcal{U}_\jmath}^{+}), \min_\jmath(\mathcal{R}_{\mathcal{U}_\jmath}^{+}); \max_\jmath(\aleph_{\mathcal{U}_\jmath}^{-}) \max_\jmath(\mathcal{R}_{\mathcal{U}_\jmath}^{-})\big)$ *and* $\upsilon^{+} = \big(\max_\jmath (\aleph_{\mathcal{U}_\jmath}^{+}), \max_\jmath(\mathcal{R}_{\mathcal{U}_\jmath}^{+}); \min_\jmath(\aleph_{\mathcal{U}_\jmath}^{-}), \min_\jmath(\mathcal{R}_{\mathcal{U}_\jmath}^{-})\big)$. *Then, we have*

$$\upsilon^{-} \leq BFZNSHWA(\upsilon_1, \upsilon_2, ..., \upsilon_r) \leq \upsilon^{+}$$

3. PROPOSED MCDM: ENTROPY-SWARA-ERUNS TECHNIQUE

The proposed approach introduces a detailed MCDM framework aimed at assessing n alternative plastic waste recycling technologies, represented by ${\iota^{\vartheta}}_{\nu} = \{{\iota^{\vartheta}}_{\nu 1}, {\iota^{\vartheta}}_{\nu 2}, \ldots, {\iota^{\vartheta}}_{\nu n}\}$, in relation to m decision criteria, denoted as $\iota_{\beta}^{\varepsilon} = \{\iota_{\beta_1}^{\varepsilon}, \iota_{\beta_2}^{\varepsilon}, \ldots, \iota_{\beta_m}^{\varepsilon}\}$. This evaluation is informed by a panel of k decision experts who offer structured insights to improve the quality and reliability. The BFZ-Entropy method first establishes objective weights, after which the BFZ-SWARA-ERUNS approach is employed to evaluate these weights, integrating expert input with nonlinear normalisation techniques. The decision-making process includes a multidisciplinary panel of experts, with linguistic assessments modelled using BFZNs, as illustrated in Table 3. The roles and qualifications are outlined in Table 2, facilitating a comprehensive assessment of environmental, economic, and regulatory aspects.

TABLE 2. Decision makers involved in the case study

Profession	**Role and Responsibility**	**Experience and Qualification**
Environmental Engineer	Evaluates environmental criteria such as emissions and energy use.	10+ years in waste management, M.Sc. Environmental Engineering.
Financial Analyst	Assesses capital cost and operating cost.	8 years in project finance, MBA in Finance.
Policy Advisor	Reviews regulatory fit and social acceptance.	12 years in public policy, MPA degree.

TABLE 3. Linguistic terms and corresponding BFZNs for evaluating plastic recycling technologies

Linguistic Term	Abbreviation	BFZNs
Transformational Impact	TI	$(\langle 0.96, 0.28\rangle, \langle -0.08, -0.24\rangle)$
Outstanding Sustainability	OS	$(\langle 0.88, 0.43\rangle, \langle -0.18, -0.57\rangle)$
Advanced Performance	AP	$(\langle 0.80, 0.39\rangle, \langle -0.32, -0.45\rangle)$
Enhanced Feasibility	EF	$(\langle 0.70, 0.27\rangle, \langle -0.42, -0.33\rangle)$
Moderate Optimization	MO	$(\langle 0.59, 0.79\rangle, \langle -0.48, -0.72\rangle)$
Basic Compliance	BC	$(\langle 0.47, 0.55\rangle, \langle -0.59, -0.36\rangle)$
Emerging Potential	EP	$(\langle 0.38, 0.50\rangle, \langle -0.68, -0.29\rangle)$
Marginal Adoption	MA	$(\langle 0.29, 0.74\rangle, \langle -0.79, -0.47\rangle)$
Unsuitable Option	UO	$(\langle 0.17, 0.60\rangle, \langle -0.91, -0.22\rangle)$

Algorithm: BFZNs based entropy and SWARA-ERUNS technique

Step 1: A panel of specialists evaluates each alternative plastic recycling technology based on a number of criteria as the first stage in the decision-making process. Experts examine each alternative's performance in relation to all criteria, methodically recording their findings. The options are represented by rows in this matrix-formatted assessment, while the criteria are represented by columns.

$$\begin{array}{cc} & \begin{array}{cccc} \iota^{\varepsilon}_{\beta_1} & \iota^{\varepsilon}_{\beta_2} & \cdots & \iota^{\varepsilon}_{\beta_n} \end{array} \\ \begin{array}{c} \iota^{\vartheta}{}_{\nu 1} \\ \iota^{\vartheta}{}_{\nu 2} \\ \vdots \\ \iota^{\vartheta}{}_{\nu m} \end{array} & \begin{bmatrix} \langle \aleph^{+}_{\mathcal{U}}(\eth^{Q})_{11}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{11}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{11}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{11}\rangle & \langle \gamma^{\mu}_{12}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{12}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{12}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{12}\rangle & \cdots & \langle \gamma^{\mu}_{1n}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{1n}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{1n}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{1n}\rangle \\ \langle \gamma^{\mu}_{21}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{21}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{21}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{21}\rangle & \langle \gamma^{\mu}_{22}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{22}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{22}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{22}\rangle & \cdots & \langle \gamma^{\mu}_{2n}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{2n}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{2n}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \gamma^{\mu}_{m1}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{m1}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{m1}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{m1}\rangle & \langle \gamma^{\mu}_{m2}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{m2}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{m2}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{m2}\rangle & \cdots & \langle \gamma^{\mu}_{mn}, \mathcal{R}^{+}_{\mathcal{U}}(\eth^{Q})_{mn}\rangle, \langle |\aleph^{-}_{\mathcal{U}}|(\eth^{Q})_{mn}, |\mathcal{R}^{-}_{\mathcal{U}}|(\eth^{Q})_{mn}\rangle \end{bmatrix} \end{array}$$

Step 2: Calculate the score matrix ($\zeta_{(S_{\imath\jmath})}$) by using the following score function in Equation 3.10.

$$\breve{\beta}(\upsilon) = \frac{2 + \aleph^{+}_{\mathcal{U}} + \mathcal{R}^{+}_{\mathcal{U}} + \aleph^{-}_{\mathcal{U}} + \mathcal{R}^{-}_{\mathcal{U}}}{4}, \ \breve{\beta} \in [0, 1] \tag{3.10}$$

Step 3: Equation 3.11 may normalise the softmax function $\nu_{\imath\jmath}^{\tau,\ell}$. The function can be normalised using Equation 3.12.

$$\nu_{\imath\jmath}^{\tau,\,\ell} = \frac{\exp\left(\frac{\subset_{\imath\jmath}{}^{k}}{\tau}\right)}{\sum_{\jmath=1}^{v} \exp\left(\frac{\subset_{\imath\jmath}{}^{k}}{\tau}\right)} \quad \imath = 1, 2, \ldots, n;\ \jmath = 1, 2, \ldots, m; \tag{3.11}$$

$$\Omega_{\imath\jmath}^{\tau,\,\ell} = \frac{\nu_{\imath\jmath}^{\tau,\,\ell}}{\sum_{\jmath=1}^{v} \nu_{\imath\jmath}^{\tau,\,\ell}} \quad \imath = 1, 2, \ldots, n;\ \jmath = 1, 2, \ldots, m; \tag{3.12}$$

Step 4: Utilize the bipolar fuzzy Z-number softmax hamacher weighted average (BPFZNSHWA) operator on the evaluation data of each decision maker to obtain the aggregated values. The Equation (3.13) is employed for this purpose.

$$= \left(\begin{array}{c} \left\langle \dfrac{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}} - \prod_{\jmath=1}^{v}(1-\aleph_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\aleph_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}(1-\aleph_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}}}, \right. \\ \left. \dfrac{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}} - \prod_{\jmath=1}^{v}(1-\mathcal{R}_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)\mathcal{R}_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}(1-\mathcal{R}_{\mathcal{U}_\jmath}^{+})^{\Omega^{\tau}}} \right\rangle, \\ \left\langle -\dfrac{\pi^{\varphi}\prod_{\jmath=1}^{v}|\aleph_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)(1-|\aleph_{\mathcal{U}_\jmath}^{-}|))^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}|\aleph_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau}}}, \right. \\ \left. -\dfrac{\pi^{\varphi}\prod_{\jmath=1}^{v}|\mathcal{R}_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau}}}{\prod_{\jmath=1}^{v}(1+(\pi^{\varphi}-1)(1-|\mathcal{R}_{\mathcal{U}_\jmath}^{-}|))^{\Omega^{\tau}} + (\pi^{\varphi}-1)\prod_{\jmath=1}^{v}|\mathcal{R}_{\mathcal{U}_\jmath}^{-}|^{\Omega^{\tau}}} \right\rangle \end{array} \right) \tag{3.13}$$

Step 5
Find the score matrix $(\zeta(S_{\imath\jmath}))_{m\times n}$ for each BFZN using Equation (3.14):

$$\breve{\beta} = \frac{2 + \aleph_{\mathcal{U}}^{+} + \mathcal{R}_{\mathcal{U}}^{+} + \aleph_{\mathcal{U}}^{-} + \mathcal{R}_{\mathcal{U}}^{-}}{4}. \tag{3.14}$$

3.1. **Entropy method. Step 6:** Normalisation entails modifying data inputs in a decision-making matrix to ensure they are confined within a range of 0 to 1. This method is crucial for quantifying the criteria, thereby streamlining the assessment process. The entropy-based method utilized Equations (3.15) and (3.16) during the data formalisation for benefit-type and cost-type criteria.

$$\Psi\eta_{\imath\jmath} = \frac{\zeta_{S_{\imath\jmath}} - \underset{i}{min}(\zeta_{S_{\imath\jmath}})}{\underset{i}{max}(\zeta_{S_{\imath\jmath}}) - \underset{i}{min}(\zeta_{S_{\imath\jmath}})} \quad \imath = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m; \tag{3.15}$$

and

$$\Psi\eta_{\imath\jmath} = \frac{\underset{i}{max}(\zeta_{S_{\imath\jmath}}) - \zeta_{S_{\imath\jmath}}}{\underset{i}{max}(\zeta_{S_{\imath\jmath}}) - \underset{i}{min}(\zeta_{S_{\imath\jmath}})} \quad \imath = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{3.16}$$

Step 7: The standardized value, represented as $\vartheta_{\imath\jmath}$, is calculated using Equation (3.17) in this phase.

$$\vartheta_{\imath\jmath} = \frac{\Psi\eta_{\imath\jmath}}{\sum_{i=1}^{m}(\Psi\eta_{\imath\jmath})} \quad \imath = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m; \tag{3.17}$$

The weight exhibits a significant increase when $\vartheta_{\imath\jmath} = 0$. In these cases, $\xi_\jmath$ is determined by utilizing the condition $\vartheta_{\imath\jmath}\ln(\vartheta_{\imath\jmath}) = 0$. This phenomenon occurs because of the reduction in entropy values for these measurements [57], which can be linked to the significant presence of zero values in the dataset $\vartheta_{\imath\jmath}$. The standardization process described in Equation (3.17) is adjusted to eliminate zero values in the normalised dataset, as shown in Equation (3.18).

$$\vartheta_{\imath\jmath} = \frac{\Psi\eta_{\imath\jmath} + k}{\sum_{i=1}^{m}(\Psi\eta_{\imath\jmath} + k)} \quad \imath = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m \tag{3.18}$$

Where k is a constant satisfying the condition,

$$\Psi\eta_{\imath\jmath} + k > 0.$$

Step 8: The application of the standardized value in the computation of entropy for the $\jmath$th criteria is illustrated by the Equation (3.19).

$$\xi_\jmath = A\sum_{i=1}^{m}(\vartheta_{\imath\jmath} ln(\vartheta_{\imath\jmath})) \quad j = 1, 2, \ldots, m; \tag{3.19}$$

Where,

$$A = \frac{1}{ln(p)}$$

Step 9: Calculating the degree of divergence, represented as $\varphi_\jmath$, for each criteria is the first step in establishing the weights for the $\jmath$th criterion.

$$\varphi_\jmath = 1 - \xi_\jmath \quad \jmath = 1, 2, \ldots, m;$$

Based on divergence measure weights are calculated as,

$$\chi(V)_\jmath = \frac{\varphi_\jmath}{\sum_{j=1}^{n}(\varphi_\jmath)} \quad \jmath = 1, 2, \ldots, m; \tag{3.20}$$

such that, $\chi(V)_\jmath \in [0, 1]$ and $\sum_{i=1}^{n}\chi(V)_\jmath = 1$.

SWARA. Step 10: The attribute coefficient for each decision-maker is calculated using Equation 3.21.

$$\delta_{\jmath} = \begin{cases} 1 & \text{if } j = 1 \\ \chi(V) + 1 & \text{if } j > 1 \end{cases} \quad \jmath = 1, \ldots, m \tag{3.21}$$

Step 11: The initial weight of criteria for each alternative is calculated at this stage using Equation 3.22.

$$\eth^{Q}{}_{\jmath} = \begin{cases} 1 & \text{if } j = 1 \\ \frac{\eth^{Q}{}_{\jmath}}{\delta_{\jmath}} & \text{if } j > 1 \end{cases} \quad \jmath = 1, \ldots, m \tag{3.22}$$

Step 12 The relative weight of a criterion for each alternative is determined using Equation 3.23.

$$W_{\jmath} = \frac{\eth^{Q}{}_{\jmath}}{\sum_{j=1}^{n} \eth^{Q}{}_{\jmath}} \tag{3.23}$$

3.2. **ERUNS method. Step 13:** To normalize the elements of the matrix $\Xi = \left[\zeta_{S_{\imath\jmath}}\right]_{r\times o}$, the Equation (3.24) is used that maps the criterion intervals into an arbitrary interval $[\zeta, \beta]$, where ζ indicates the right limit and β signifies the left limit of the interval.

$$\Phi_{\imath\jmath} = \left(\frac{\zeta_{S_{\jmath}}^{\min}}{\zeta_{S_{\imath\jmath}}} \right)^{3} \beta + \frac{\zeta}{\zeta_{S_{\jmath}}^{\min}} \quad \imath = 1, 2, \ldots, n; j = 1, 2, \ldots, m \tag{3.24}$$

Here, $\zeta_{S_{\jmath}}^{\min} = \min\limits_{1 \le i \le r} (\zeta_{S_{\imath\jmath}})$. To translate the elements of the matrix Ξ to the interval $[1, 9]$, for example, use the value of $\zeta = 1$ and $\beta = 9$. The values of the interval $[\zeta, \beta]$ are determined by the decisions made by the decision-maker and the specifics of the problem-solving process.

Step 14: $\varepsilon_{\imath\jmath}$ represents values meet the maximum type requirement.To edit, use Equation (3.25).

$$\varepsilon_{\imath\jmath} = \begin{cases} -\Phi_{\imath\jmath} + \max\limits_{1 \le i \le m} (\Phi_{\imath\jmath}) + \min\limits_{1 \le i \le m} (\Phi_{\imath\jmath}), & j \in \omega^{\nu}{}_{b} \\ \Phi_{\imath\jmath}, & j \in \omega^{\nu}{}_{c} \end{cases} \tag{3.25}$$

Step 15: Apply equation (3.26) to obtain the weighted standardized decision matrix.

$$\varpi_{\imath\jmath} = \frac{\exp\left(\sqsupset(\varepsilon_{\imath\jmath})/\zeta\right) W_{\jmath}}{\sum_{j=1}^{n} \exp\left(\sqsupset(\varepsilon_{\imath\jmath})/\zeta\right) W_{\jmath}} \quad \imath = 1, 2, \ldots, n; j = 1, 2, \ldots, m \tag{3.26}$$

Where $\sqsupset(\varepsilon_{\imath\jmath}) = \frac{\varepsilon_{\imath\jmath}}{\sum_{j=1}^{n} \varepsilon_{\imath\jmath}}, j \in \{1, 2, 3 \ldots n\}$.

Step 16: Assess the utility degrees of both the ideal and anti-ideal solutions. The utility values of the i^{th} option for the ideal and anti-ideal solutions are presented as follows:

$$\ell_{\imath}^{+} = \frac{\prod\limits_{j=1}^{n} (\varepsilon_{\imath\jmath})^{\varpi_{\imath\jmath}}}{\sum\limits_{j=1}^{n} \varpi_{\jmath}^{+}}, \quad \imath = 1, 2, \ldots, n \tag{3.27}$$

$$
\begin{aligned}
\ell_\imath^- &= -\frac{\sum_{\jmath=1}^n \varpi_\jmath^-}{\prod_{\jmath=1}^n \left(\varepsilon_{\imath\jmath}\right)^{\varpi_{\imath\jmath}}} + \max_{1\le i\le m}\left(\frac{\sum_{\jmath=1}^n \varpi_\jmath^-}{\prod_{\jmath=1}^n \left(\varepsilon_{\imath\jmath}\right)^{\varpi_{\imath\jmath}}}\right) + \min_{1\le i\le m}\left(\frac{\sum_{\jmath=1}^n \varpi_\jmath^-}{\prod_{\jmath=1}^n \left(\varepsilon_{\imath\jmath}\right)^{\varpi_{\imath\jmath}}}\right), \\
\varpi_\jmath^+ &= \max_{1\le i\le m}\left(\varepsilon_{\imath\jmath}\cdot W_\jmath\right), \\
\varpi_\jmath^- &= \min_{1\le i\le m}\left(\varepsilon_{\imath\jmath}\cdot W_\jmath\right),
\end{aligned}
\tag{3.28}
$$

Step 17: The Equation (3.29) can be utilized to ascertain the values of the utility function in relation to the overall levels of utility.

$$
\begin{aligned}
\beth\left(\ell_\imath^+\right) &= \frac{\ell_\imath^+}{\ell_\imath^+ + \ell_\imath^-} \\
\beth\left(\ell_\imath^-\right) &= \frac{\ell_\imath^-}{\ell_\imath^+ + \ell_\imath^-} \quad \imath = 1, 2, \ldots, n
\end{aligned}
\tag{3.29}
$$

Step 18: The evaluation scores of the options are determined using Equation (3.30) and the parameter $\rho \in [0, 1]$.

$$
F_\imath = \left(\ell_\imath^+ + \ell_\imath^-\right)\frac{\left(1+\beth\left(\ell_\imath^+\right)\right)^\rho\left(1+\beth\left(\ell_\imath^-\right)\right)^{1-\rho} - \left(1-\beth\left(\ell_\imath^+\right)\right)^\rho\left(1-\beth\left(\ell_\imath^-\right)\right)^{1-\rho}}{\left(1+\beth\left(\ell_\imath^+\right)\right)^\rho\left(1+\beth\left(\ell_\imath^-\right)\right)^{1-\rho} + \left(1-\beth\left(\ell_\imath^+\right)\right)^\rho\left(1-\beth\left(\ell_\imath^-\right)\right)^{1-\rho}} \quad \imath = 1, 2, \ldots, n \tag{3.30}
$$

Figure 1 presents the complete algorithm of the proposed model.

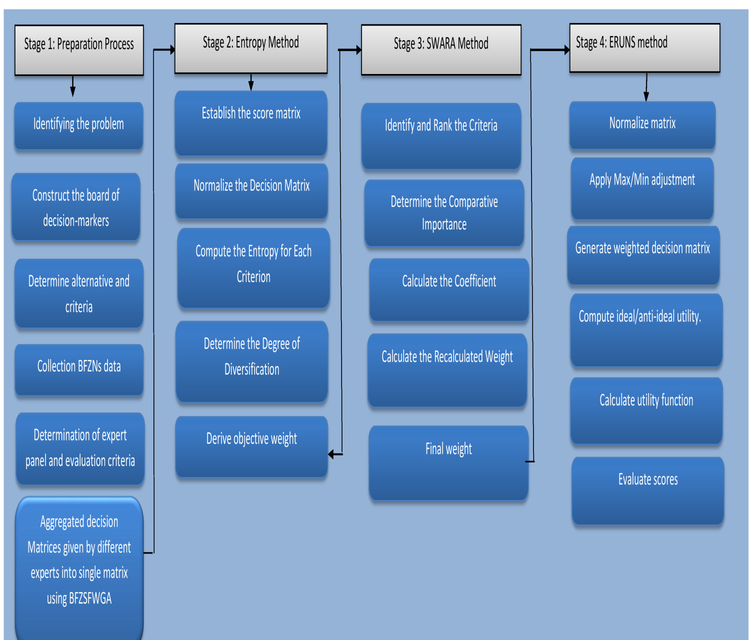

FIGURE 1. Representation of the proposed MCDM method

4. Case study

Plastic waste represents a significant environmental challenge globally, contributing notably to land, water, and air pollution, while also exacerbating climate change through greenhouse gas emissions. Recycling presents a practical approach to diminishing plastic waste and preserving natural resources; however, identifying the most efficient technology for plastic waste recycling is a complex and multifaceted challenge. The complexity stems from the necessity to balance various and frequently conflicting criteria, such as economic feasibility, environmental impact, energy consumption, technological readiness, regulatory compliance, and social acceptance. Existing methods for selecting recycling technologies exhibit several significant limitations. Many individuals depend significantly on subjective expert opinions while failing to sufficiently integrate objective data, potentially introducing bias and diminishing the reliability of decisions.In contrast, certain methods emphasize quantitative metrics exclusively, neglecting the significant experiential and contextual insights offered by experts. Secondly, conventional decision-making models typically presume straightforward linear relationships between criteria and alternatives, neglecting the complex nonlinear interactions and trade-offs present in real-world situations. This simplification may result in erroneous rankings and suboptimal technology selections. Third, the decision-making process frequently lacks transparency and robustness, which undermines stakeholders' ability to trust and confidently implement the outcomes.

The deficiencies in current methodologies introduce considerable uncertainty for policymakers, industry leaders, and environmental managers responsible for choosing recycling technologies that optimise economic returns while minimising environmental impact and achieving public acceptance. A comprehensive and systematic decision support framework is essential, integrating objective data-driven weighting with subjective expert insights and accounting for nonlinearities; otherwise, decisions may prove ineffective or detrimental to sustainable development goals. This study proposes an integrated MCDM framework to address these challenges, incorporating the Entropy method for objective criteria weighting, the SWARA technique for expert judgment capture, and the ERUNS method for robust alternative ranking. This integrated method seeks to harmonise quantitative data with qualitative insights, adeptly address nonlinear relationships among criteria, and establish a transparent and reliable foundation for assessing plastic waste recycling technologies. The proposed model aims to assist policymakers and industry stakeholders in making informed, sustainable, and practical decisions that enhance efficient plastic waste management and promote the advancement of a circular economy.

4.1. **Definition of Alternatives.**

(1) ${\iota^{\vartheta}}_{\nu 1}$**Solvent-based purification**
Solvent-based purification, or dissolution recycling, employs selective solvents to dissolve particular polymers from plastic waste mixtures, facilitating the recovery of purified materials while preserving polymer chain integrity. The dissolved plastic undergoes separation from additives and contaminants, followed by re-precipitation, filtration, and drying to obtain clean polymer for reuse. This method accommodates multi-layer packaging, coloured plastics, and blends that present challenges for mechanical recycling. This method maintains the integrity of the

polymer more effectively than mechanical recycling and produces high-quality recyclate. Careful selection of solvents, implementation of recovery systems, and compliance with environmental regulations are essential due to the use of chemicals.

(2) $\iota^{\vartheta}{}_{\nu 2}$: **Chemical recycling**
In order to produce virgin-quality plastic, chemical recycling breaks down polymers into monomers or basic chemicals using techniques like depolymerisation, glycolysis, or pyrolysis. Chemical recycling differs from mechanical recycling by modifying the molecular structure of plastic waste, thereby enabling the recycling of complex, contaminated, or mixed plastics. The outputs, including monomers, fuels, and chemical feedstocks, are reusable within the petrochemical industry for the synthesis of new high-quality polymers. Chemical recycling, while more costly and energy-demanding, presents significant potential for a circular plastic economy and aligns with existing petrochemical infrastructures.

(3) $\iota^{\vartheta}{}_{\nu 3}$: **Mechanical recycling**
Mechanical recycling is a physical process encompassing the collection, sorting, cleaning, shredding, melting, and re-extruding of plastic waste into new products, while maintaining the integrity of the polymer's chemical structure. This method is optimal for homogeneous, uncontaminated plastic waste streams, specifically PET and HDPE. Following sorting, the plastic undergoes washing to eliminate impurities, is shredded into flakes, melted, and subsequently reformed into pellets for the production of new products. It is economically efficient and commonly utilized; however, it diminishes the quality of the plastic after several recycling cycles. Mechanical recycling faces constraints in processing mixed or multi-layered plastics, as well as contaminated materials.

(4) $\iota^{\vartheta}{}_{\nu 4}$: **Gasification**
In a regulated setting with limited access to oxygen or steam, gasification is a high-temperature process that turns waste plastic into synthesis gas (syngas), which is mostly composed of hydrogen and carbon monoxide. The generated syngas can be used to heat buildings, generate power, or transform it into chemicals and fuels. For materials that are not suitable for conventional recycling techniques, gasification offers a workable alternative for the efficient processing of polluted and heterogeneous plastic waste. The process involves significant technological complexity and capital investment, necessitating stringent emission controls and sophisticated gas cleaning systems. This technology serves mainly as a waste-to-energy solution.

(5) $\iota^{\vartheta}{}_{\nu 5}$: **Pyrolysis**
By heating waste plastic in an oxygen-free atmosphere, a thermochemical process known as pyrolysis breaks down the material into liquid oil, gas and solid char. Pyrolysis works best with trash that contains a lot of polyolefins and operates at temperatures between 350 and 700°C. The pyrolysis oil produced can be refined into diesel or utilized as a chemical feedstock. This process offers a method for energy recovery from non-recyclable plastics. Nonetheless, it incurs significant capital and energy expenses and generates emissions unless adequately managed. This

method is appropriate for plastic waste streams that cannot be processed through mechanical or chemical recycling.

(6) ${\iota^{\vartheta}}_{\nu 6}$: **Biological recycling**
Biological recycling utilizes microorganisms or enzymes to decompose specific plastics into biodegradable by-products within regulated environmental parameters. This method remains in the research and development phase and is primarily relevant to biodegradable or bio-based plastics. Certain microbial strains and enzymes exhibit the capacity to degrade PET, PU, and various synthetic plastics. Biological recycling, despite being environmentally friendly and low-energy, faces limitations such as extended processing times, scalability challenges, and low degradation efficiency for conventional petrochemical plastics. This approach is regarded as a viable solution for future biotechnological recycling frameworks.

4.2. **Definition of Criteria.**

(1) **Economic criteria**
- $\iota^{\varepsilon}_{\beta_1}$: **Capital cost**
The initial capital required for establishing the recycling facility encompasses equipment, installation, and infrastructure costs. Increased capital costs elevate financial burdens and risks, particularly for small and medium enterprises, thereby rendering technologies with reduced capital costs more appealing for early adoption.
- $\iota^{\varepsilon}_{\beta_2}$: **Operating cost**
The continuous costs associated with the operation of recycling technology per tonne of plastic processed, including labour, energy, and maintenance. High operating costs diminish profit margins and threaten long-term sustainability; therefore, technologies that offer lower operating costs are favoured to enhance efficiency.
- $\iota^{\varepsilon}_{\beta_3}$: **Return on investment**
The anticipated financial return in relation to the capital invested over a defined time frame. An elevated ROI signifies enhanced financial viability and appeal to investors, thereby facilitating expedited deployment and scalability of the technology.

(2) **Environmental criteria**
- $\iota^{\varepsilon}_{\beta_4}$: **Emission reduction**
The reduction in greenhouse gas emissions associated with recycling technology relative to conventional disposal methods. Increased emission reduction complies with environmental regulations and sustainability objectives, rendering these technologies more advantageous for stakeholders concerned with carbon impact.
- $\iota^{\varepsilon}_{\beta_5}$: **Energy consumption**
The energy required to process one tonne of plastic waste using this technology. Reduced energy consumption leads to lower operational costs and a smaller environmental footprint; therefore, energy-efficient technologies are favoured for sustainable operations.

- $\iota^{\varepsilon}_{\beta_6}$**: Landfill diversion**
  The percentage of plastic waste redirected from landfills via recycling or recovery methods. Increased landfill diversion mitigates environmental pollution and preserves landfill capacity, thereby promoting circular economy principles and improving social acceptance.

(3) **Technical and social criteria**

- $\iota^{\varepsilon}_{\beta_7}$**: Technology maturity**
  The degree of technological advancement, dependability, and industrial preparedness of the recycling process. Mature technologies demonstrate proven performance and reduced risk, facilitating large-scale implementation, while emerging technologies present greater innovation potential accompanied by increased uncertainty.
- $\iota^{\varepsilon}_{\beta_8}$**: Regulatory & social acceptance**
  The degree of support or approval for recycling technology by governments, regulators, and local communities. High acceptance enhances the efficiency of permitting, funding, and community collaboration, whereas low acceptance may result in project delays or heightened operational risks.

4.3. **Experts Evaluation. Step 1:** The decision makers utilized the terminology presented in Table 3 to articulate their ideas, with the responses documented in Table 4.

TABLE 4. Linguistic evaluation matrix by decision makers

DM	Alt.	$\iota^{\varepsilon}_{\beta_1}$	$\iota^{\varepsilon}_{\beta_2}$	$\iota^{\varepsilon}_{\beta_3}$	$\iota^{\varepsilon}_{\beta_4}$	$\iota^{\varepsilon}_{\beta_5}$	$\iota^{\varepsilon}_{\beta_6}$	$\iota^{\varepsilon}_{\beta_7}$	$\iota^{\varepsilon}_{\beta_8}$
	${\iota^{\vartheta}}_{\nu 1}$	EF	AP	MO	EP	OS	MA	TI	BC
	${\iota^{\vartheta}}_{\nu 2}$	MO	MO	EF	T1	EF	AP	MO	EP
DM_1	${\iota^{\vartheta}}_{\nu 3}$	OS	EP	T1	EF	MA	M0	BC	AP
	${\iota^{\vartheta}}_{\nu 4}$	AP	EF	MO	MO	EP	TI	UO	MA
	${\iota^{\vartheta}}_{\nu 5}$	MA	EP	OS	EF	BC	MO	MO	TI
	${\iota^{\vartheta}}_{\nu 6}$	TI	BC	EF	MO	EP	UO	EP	AP
	${\iota^{\vartheta}}_{\nu 1}$	MO	EP	TI	AP	UO	EF	MA	BC
	${\iota^{\vartheta}}_{\nu 2}$	BC	MO	MO	EP	TI	OS	EF	UO
DM_2	${\iota^{\vartheta}}_{\nu 3}$	TI	MA	BC	MO	MO	EP	AP	EF
	${\iota^{\vartheta}}_{\nu 4}$	EP	UO	EF	BC	AP	TI	MA	OS
	${\iota^{\vartheta}}_{\nu 5}$	EF	OS	EP	MO	MA	UO	TI	BC
	${\iota^{\vartheta}}_{\nu 6}$	AP	TI	UO	MA	EF	BC	OS	EP
	${\iota^{\vartheta}}_{\nu 1}$	AP	MA	BC	EF	TI	BC	EP	MO
	${\iota^{\vartheta}}_{\nu 2}$	TI	BC	OS	MA	AP	EF	MO	EP
DM_3	${\iota^{\vartheta}}_{\nu 3}$	MO	AP	EP	UO	OS	TI	BC	MA
	${\iota^{\vartheta}}_{\nu 4}$	EF	OS	TI	BC	MA	MO	OS	TI
	${\iota^{\vartheta}}_{\nu 5}$	BC	EP	MA	UO	MO	AP	OS	TI
	${\iota^{\vartheta}}_{\nu 6}$	UO	EF	AP	MO	OS	BC	TI	MA

Step 2: The score values and ${\subset_{\imath\jmath}}^k$ matrices for each matrix provided by decision-makers are calculated using Equation (3.10).

$$
{\subset_{\imath\jmath}}^1 = \begin{bmatrix}
0.5550 & 0.6050 & 0.5450 & 0.4775 & 0.6400 & 0.4425 & 0.7300 & 0.5175 \\
0.5450 & 0.5450 & 0.5550 & 0.7300 & 0.5550 & 0.6050 & 0.5450 & 0.4775 \\
0.6400 & 0.4775 & 0.7300 & 0.5550 & 0.4425 & 0.5450 & 0.5175 & 0.6050 \\
0.6050 & 0.5550 & 0.5450 & 0.5450 & 0.4775 & 0.7300 & 0.4100 & 0.4425 \\
0.4425 & 0.4775 & 0.6400 & 0.5550 & 0.5175 & 0.5450 & 0.5450 & 0.7300 \\
0.7300 & 0.5175 & 0.5550 & 0.5450 & 0.4775 & 0.4100 & 0.4775 & 0.6050
\end{bmatrix}
$$

$$
{\subset_{\imath\jmath}}^2 = \begin{bmatrix}
0.5450 & 0.4775 & 0.7300 & 0.6050 & 0.4100 & 0.5550 & 0.4425 & 0.5175 \\
0.5175 & 0.5450 & 0.5450 & 0.4775 & 0.7300 & 0.6400 & 0.5550 & 0.4100 \\
0.7300 & 0.4425 & 0.5175 & 0.5450 & 0.5450 & 0.4775 & 0.6050 & 0.5550 \\
0.4775 & 0.4100 & 0.5550 & 0.5175 & 0.6050 & 0.7300 & 0.4425 & 0.6400 \\
0.5550 & 0.6400 & 0.4775 & 0.5450 & 0.4425 & 0.4100 & 0.7300 & 0.5175 \\
0.6050 & 0.7300 & 0.4100 & 0.4425 & 0.5550 & 0.5175 & 0.6400 & 0.4775
\end{bmatrix}
$$

$$
{\subset_{\imath\jmath}}^3 = \begin{bmatrix}
0.6050 & 0.4425 & 0.5175 & 0.5550 & 0.7300 & 0.4100 & 0.4775 & 0.5450 \\
0.7300 & 0.5175 & 0.6400 & 0.4425 & 0.6050 & 0.5550 & 0.5450 & 0.4775 \\
0.5450 & 0.6050 & 0.4775 & 0.4100 & 0.6400 & 0.7300 & 0.5175 & 0.4425 \\
0.5550 & 0.6400 & 0.7300 & 0.5175 & 0.4425 & 0.5450 & 0.4775 & 0.4100 \\
0.5175 & 0.4775 & 0.4425 & 0.4100 & 0.5450 & 0.6050 & 0.6400 & 0.7300 \\
0.4100 & 0.5550 & 0.6050 & 0.5450 & 0.6400 & 0.5175 & 0.7300 & 0.4425
\end{bmatrix}
$$

Step 3: The softmax function for each matrix provided by DMs is computed using Equation (3.11). The results are normalised using Equation (3.12), with $\tau = 2$ taken into account.

$$
\nu_{\imath\jmath}^{\tau,\,1} = \begin{bmatrix}
0.3312 & 0.3248 & 0.3543 & 0.3370 & 0.3285 & 0.3326 & 0.3313 & 0.3334 \\
0.3387 & 0.3129 & 0.3284 & 0.3284 & 0.3486 & 0.3412 & 0.3404 & 0.3396 \\
0.3308 & 0.3304 & 0.3242 & 0.3503 & 0.3456 & 0.3338 & 0.3317 & 0.3309 \\
0.3263 & 0.3549 & 0.3450 & 0.3256 & 0.3278 & 0.3267 & 0.3283 & 0.3266 \\
0.3408 & 0.3114 & 0.3123 & 0.3394 & 0.3429 & 0.3428 & 0.3425 & 0.3410 \\
0.3173 & 0.3888 & 0.3296 & 0.3332 & 0.3288 & 0.3198 & 0.3168 & 0.3155
\end{bmatrix}
$$

$$
\nu_{\imath\jmath}^{\tau,\,2} = \begin{bmatrix}
0.3352 & 0.3240 & 0.3219 & 0.3444 & 0.3457 & 0.3323 & 0.3359 & 0.3342 \\
0.3450 & 0.3065 & 0.3274 & 0.3293 & 0.3320 & 0.3411 & 0.3437 & 0.3444 \\
0.3260 & 0.3703 & 0.3276 & 0.3188 & 0.3251 & 0.3266 & 0.3241 & 0.3257 \\
0.3499 & 0.3172 & 0.3029 & 0.3140 & 0.3304 & 0.3436 & 0.3471 & 0.3487 \\
0.3226 & 0.3462 & 0.3631 & 0.3375 & 0.3327 & 0.3255 & 0.3222 & 0.3231 \\
0.3268 & 0.3350 & 0.3720 & 0.3295 & 0.3225 & 0.3247 & 0.3266 & 0.3268
\end{bmatrix}
$$

$$
\nu_{\imath\jmath}^{\tau,\,3} = \begin{bmatrix}
0.3336 & 0.3513 & 0.3238 & 0.3185 & 0.3259 & 0.3351 & 0.3328 & 0.3324 \\
0.3163 & 0.3806 & 0.3442 & 0.3423 & 0.3194 & 0.3177 & 0.3159 & 0.3160 \\
0.3432 & 0.2993 & 0.3481 & 0.3309 & 0.3294 & 0.3397 & 0.3442 & 0.3434 \\
0.3238 & 0.3279 & 0.3520 & 0.3604 & 0.3418 & 0.3297 & 0.3246 & 0.3247 \\
0.3366 & 0.3424 & 0.3246 & 0.3231 & 0.3244 & 0.3317 & 0.3353 & 0.3358 \\
0.3560 & 0.2762 & 0.2984 & 0.3373 & 0.3488 & 0.3555 & 0.3567 & 0.3577
\end{bmatrix}
$$

Step 4: Utilize the BPZNSHWA operator on the evaluation data of each decision maker to obtain the aggregated values. The Equation (3.13) is employed for this purpose. The results are presented in Table 5.

TABLE 5. Aggregated values by applying the BFZNSHWA

C_1^F	C_2^F	C_3^F	C_4^F	C_5^F	C_6^F	C_7^F	C_8^F
$\langle(0.7057, 0.5214), (-0.2607, -0.2886)\rangle$	$\langle(0.5267, 0.5593), (-0.3155, -0.2569)\rangle$	$\langle(0.7765, 0.5693), (-0.1954, -0.2546)\rangle$	$\langle(0.6552, 0.3892), (-0.2786, -0.2383)\rangle$	$\langle(0.8094, 0.4448), (-0.1700, -0.2166)\rangle$	$\langle(0.4104, 0.5573), (-0.3499, -0.2248)\rangle$	$\langle(0.6897, 0.5303), (-0.2250, -0.2232)\rangle$	$\langle(0.5118, 0.6443), (-0.3171, -0.2797)\rangle$
$\langle(0.7720, 0.5711), (-0.1977, -0.2551)\rangle$	$\langle(0.5526, 0.7257), (-0.3044, -0.3217)\rangle$	$\langle(0.7478, 0.5333), (-0.2273, -0.3025)\rangle$	$\langle(0.6902, 0.5280), (-0.2249, -0.2226)\rangle$	$\langle(0.8630, 0.3137), (-0.1652, -0.2271)\rangle$	$\langle(0.8059, 0.3650), (-0.2064, -0.2760)\rangle$	$\langle(0.6302, 0.6582), (-0.2842, -0.3147)\rangle$	$\langle(0.3098, 0.5356), (-0.3742, -0.1948)\rangle$
$\langle(0.8696, 0.5318), (-0.1470, -0.2802)\rangle$	$\langle(0.5259, 0.5615), (-0.3158, -0.2575)\rangle$	$\langle(0.7292, 0.4489), (-0.2123, -0.2101)\rangle$	$\langle(0.5089, 0.5870), (-0.3197, -0.2458)\rangle$	$\langle(0.6461, 0.6774), (-0.2579, -0.3255)\rangle$	$\langle(0.7549, 0.5549), (-0.2036, -0.2426)\rangle$	$\langle(0.6067, 0.4987), (-0.2905, -0.2553)\rangle$	$\langle(0.6325, 0.4890), (-0.2866, -0.2647)\rangle$
$\langle(0.6522, 0.3901), (-0.2796, -0.2378)\rangle$	$\langle(0.6515, 0.4417), (-0.2578, -0.2335)\rangle$	$\langle(0.8186, 0.4843), (-0.1834, -0.2501)\rangle$	$\langle(0.5119, 0.6445), (-0.3171, -0.2797)\rangle$	$\langle(0.5304, 0.5586), (-0.3144, -0.2575)\rangle$	$\langle(0.9085, 0.4871), (-0.1172, -0.2317)\rangle$	$\langle(0.2806, 0.6237), (-0.3834, -0.2185)\rangle$	$\langle(0.5325, 0.6020), (-0.2893, -0.2536)\rangle$
$\langle(0.5041, 0.5402), (-0.3248, -0.2523)\rangle$	$\langle(0.6139, 0.4771), (-0.2674, -0.2432)\rangle$	$\langle(0.5857, 0.5708), (-0.2766, -0.2696)\rangle$	$\langle(0.5089, 0.5870), (-0.3197, -0.2458)\rangle$	$\langle(0.4548, 0.7058), (-0.3344, -0.2962)\rangle$	$\langle(0.5574, 0.6167), (-0.3028, -0.2615)\rangle$	$\langle(0.8695, 0.5320), (-0.1471, -0.2802)\rangle$	$\langle(0.8950, 0.3765), (-0.1243, -0.2006)\rangle$
$\langle(0.7789, 0.4300), (-0.1952, -0.2067)\rangle$	$\langle(0.8010, 0.3711), (-0.1897, -0.2161)\rangle$	$\langle(0.5985, 0.4287), (-0.2946, -0.2222)\rangle$	$\langle(0.4969, 0.7744), (-0.3215, -0.3389)\rangle$	$\langle(0.7018, 0.4017), (-0.2437, -0.2498)\rangle$	$\langle(0.3742, 0.5673), (-0.3553, -0.2162)\rangle$	$\langle(0.8412, 0.4061), (-0.1594, -0.2326)\rangle$	$\langle(0.5267, 0.5593), (-0.3155, -0.2569)\rangle$

Step 5: The score matrix $\zeta_{(S_{\imath\jmath})}$ was calculated using Equation (3.14) as follows:

$$\zeta_{(S_{\imath\jmath})} = \begin{bmatrix} 0.9441 & 0.9146 & 0.9489 & 0.8903 & 0.9102 & 0.8856 & 0.9171 & 0.9382 \\ 0.9490 & 0.9761 & 0.9527 & 0.9164 & 0.8922 & 0.9133 & 0.9718 & 0.8536 \\ 0.9571 & 0.9152 & 0.9001 & 0.9154 & 0.9767 & 0.9390 & 0.9128 & 0.9182 \\ 0.8899 & 0.8961 & 0.9341 & 0.9383 & 0.9152 & 0.9361 & 0.8765 & 0.9193 \\ 0.9053 & 0.9004 & 0.9257 & 0.9154 & 0.9478 & 0.9346 & 0.9572 & 0.8991 \\ 0.9027 & 0.8945 & 0.8860 & 0.9829 & 0.8993 & 0.8783 & 0.9098 & 0.9146 \end{bmatrix}$$

4.4. **Entropy method for criteria weights.** **Step 6:** Data normalisation involved converting the input data from the decision-making matrix into a binary scale that ranges from zero to one. Equation (3.15) was employed for benefit-type criteria, while Equation (3.16) was utilized for cost-type criteria.

$$\Psi\eta_{\imath\jmath} = \begin{bmatrix} 0.8059 & 0.2467 & 0.9433 & 0 & 0.2126 & 0.8789 & 0.5749 & 1.0000 \\ 0.8788 & 1.0000 & 1.0000 & 0.2818 & 0 & 0.4231 & 0 & 0 \\ 1.0000 & 0.2535 & 0.2117 & 0.2702 & 1.0000 & 0 & 0.6196 & 0.7633 \\ 0 & 0.0197 & 0.7209 & 0.5179 & 0.2721 & 0.0476 & 1.0000 & 0.7766 \\ 0.2291 & 0.0725 & 0.5946 & 0.2702 & 0.6576 & 0.0722 & 0.1537 & 0.5376 \\ 0.1905 & 0 & 0 & 1.0000 & 0.0832 & 1.0000 & 0.6509 & 0.7209 \end{bmatrix}$$

Step 7: In this phase, the standardized value $\vartheta_{\imath\jmath}$ was calculated using Equation (3.18) with $k = 0.5$.

$$\vartheta_{\imath\jmath} = \begin{bmatrix} 0.2139 & 0.1626 & 0.2231 & 0.0936 & 0.1364 & 0.2543 & 0.1792 & 0.2206 \\ 0.2259 & 0.3266 & 0.2318 & 0.1464 & 0.0957 & 0.1703 & 0.0833 & 0.0735 \\ 0.2457 & 0.1641 & 0.1100 & 0.1442 & 0.2871 & 0.0922 & 0.1866 & 0.1858 \\ 0.0819 & 0.1132 & 0.1887 & 0.1906 & 0.1478 & 0.1010 & 0.2500 & 0.1878 \\ 0.1194 & 0.1247 & 0.1692 & 0.1442 & 0.2215 & 0.1055 & 0.1090 & 0.1526 \\ 0.1131 & 0.1089 & 0.0773 & 0.2809 & 0.1116 & 0.2767 & 0.1918 & 0.1796 \end{bmatrix}$$

Step 8: The entropy value for the $\jmath$th criterion was determined using Equation (3.19).

$$\xi_\jmath = \begin{bmatrix} 0.9578 & 0.9516 & 0.9652 & 0.9679 & 0.9575 & 0.9453 & 0.9674 & 0.9753 \end{bmatrix}$$

Step 9: The weights for the $\jmath$th criterion were determined using Equation (3.20).

$$\varphi_\jmath = \begin{bmatrix} 0.1353 & 0.1552 & 0.1115 & 0.1029 & 0.1361 & 0.1752 & 0.1045 & 0.0793 \end{bmatrix}$$

Step 10: The attribute coefficient (Ξ) for each criterion is calculated using Equation (3.21), with results presented in Table 6.

TABLE 6. Relative importance

$\iota^{\varepsilon}_{\beta_1}$	$\iota^{\varepsilon}_{\beta_2}$	$\iota^{\varepsilon}_{\beta_3}$	$\iota^{\varepsilon}_{\beta_4}$	$\iota^{\varepsilon}_{\beta_5}$	$\iota^{\varepsilon}_{\beta_6}$	$\iota^{\varepsilon}_{\beta_7}$	$\iota^{\varepsilon}_{\beta_8}$
0	1.1552	1.1115	1.1029	1.1361	1.1752	1.1045	1.0793

Step 11: The initial weight of the criteria is determined using Equation (3.22), with results presented in Table 7.

TABLE 7. Initial weight of criteria

$\iota^{\varepsilon}_{\beta_1}$	$\iota^{\varepsilon}_{\beta_2}$	$\iota^{\varepsilon}_{\beta_3}$	$\iota^{\varepsilon}_{\beta_4}$	$\iota^{\varepsilon}_{\beta_5}$	$\iota^{\varepsilon}_{\beta_6}$	$\iota^{\varepsilon}_{\beta_7}$	$\iota^{\varepsilon}_{\beta_8}$
1.0000	0.8656	0.7788	0.7061	0.6215	0.5289	0.4788	0.4437

Step 12 The relative weight of each criterion is determined using Equation (3.23), as presented in Table 8.

TABLE 8. Relative weight

$\iota^{\varepsilon}_{\beta_1}$	$\iota^{\varepsilon}_{\beta_2}$	$\iota^{\varepsilon}_{\beta_3}$	$\iota^{\varepsilon}_{\beta_4}$	$\iota^{\varepsilon}_{\beta_5}$	$\iota^{\varepsilon}_{\beta_6}$	$\iota^{\varepsilon}_{\beta_7}$	$\iota^{\varepsilon}_{\beta_8}$
0.1844	0.1596	0.1436	0.1302	0.1146	0.0975	0.0883	0.0818

The weights obtained by Entropy and SWARA methods were aggregated and shown in Table 9.

TABLE 9. Combined weights

$\iota^{\varepsilon}_{\beta_1}$	$\iota^{\varepsilon}_{\beta_2}$	$\iota^{\varepsilon}_{\beta_3}$	$\iota^{\varepsilon}_{\beta_4}$	$\iota^{\varepsilon}_{\beta_5}$	$\iota^{\varepsilon}_{\beta_6}$	$\iota^{\varepsilon}_{\beta_7}$	$\iota^{\varepsilon}_{\beta_8}$
0.1598	0.1574	0.1275	0.1166	0.1254	0.1363	0.0964	0.0805

TABLE 10. Provided Matrix Data

$\iota^{\varepsilon}_{\beta_1}$	$\iota^{\varepsilon}_{\beta_2}$	$\iota^{\varepsilon}_{\beta_3}$	$\iota^{\varepsilon}_{\beta_4}$	$\iota^{\varepsilon}_{\beta_5}$	$\iota^{\varepsilon}_{\beta_6}$	$\iota^{\varepsilon}_{\beta_7}$	$\iota^{\varepsilon}_{\beta_8}$
0.9441	0.9146	0.9489	0.8903	0.9102	0.8856	0.9171	0.9382
0.9490	0.9761	0.9527	0.9164	0.8922	0.9133	0.9718	0.8536
0.9571	0.9152	0.9001	0.9154	0.9767	0.9390	0.9128	0.9182
0.8899	0.8961	0.9341	0.9383	0.9152	0.9361	0.8765	0.9193
0.9053	0.9004	0.9257	0.9154	0.9478	0.9346	0.9572	0.8991
0.9027	0.8945	0.8860	0.9829	0.8993	0.8783	0.9098	0.9146

4.5. **Proposed Method: ERUNS.** **Step 13:** The Equation (3.24) is employed to normalise the matrix $\Xi = \left[\zeta_{S_{\imath\jmath}}\right]$. The values are assigned to the interval $[1, 100]$ as presented below.

$$\Phi_{\imath\jmath} = \begin{bmatrix} 84.8761 & 94.6586 & 82.5160 & 101.1232 & 95.3168 & 98.6666 & 88.4637 & 76.4762 \\ 83.5855 & 78.0766 & 81.5506 & 92.8227 & 101.1208 & 90.0625 & 74.5128 & 101.1715 \\ 81.4968 & 94.4892 & 96.4932 & 93.1453 & 77.3477 & 82.9594 & 89.6917 & 81.5128 \\ 101.1237 & 100.5803 & 86.4587 & 86.5629 & 93.7717 & 83.7194 & 101.1409 & 81.2195 \\ 96.1063 & 99.1587 & 88.8113 & 93.1453 & 84.5443 & 84.1170 & 77.9337 & 86.7438 \\ 96.9269 & 101.1180 & 101.1287 & 75.4426 & 98.7938 & 101.1386 & 90.5669 & 82.4627 \end{bmatrix}$$

Step 14: Equation (3.25) modifies the values of $\Xi^N = [\Lambda_{\imath\jmath}]_{m\times n}$, after which the elements of the matrix are standardized to the range $[1, 100]$. The maximal type serves as the standard. The resulting matrix is presented below.

$$\varepsilon_{\imath\jmath} = \begin{bmatrix} 97.7444 & 84.5359 & 100.1633 & 75.4426 & 83.1517 & 85.4314 & 87.1900 & 101.1715 \\ 99.0350 & 101.1180 & 101.1287 & 83.7431 & 77.3477 & 94.0356 & 101.1409 & 76.4762 \\ 101.1237 & 84.7053 & 86.1861 & 83.4206 & 101.1208 & 101.1386 & 85.9620 & 96.1350 \\ 81.4968 & 78.6142 & 96.2206 & 90.0029 & 84.6968 & 100.3787 & 74.5128 & 96.4282 \\ 86.5142 & 80.0358 & 93.8680 & 83.4206 & 93.9241 & 99.9811 & 97.7199 & 90.9039 \\ 85.6936 & 78.0766 & 81.5506 & 101.1232 & 79.6746 & 82.9594 & 85.0868 & 95.1851 \end{bmatrix}$$

Step 15: The weighted standardized decision matrix V is presented using Equation (3.26) after its acquisition.

$$\varpi_{\imath\jmath} = \begin{bmatrix} 0.1616 & 0.1563 & 0.1294 & 0.1143 & 0.1243 & 0.1355 & 0.0961 & 0.0818 \\ 0.1611 & 0.1591 & 0.1289 & 0.1151 & 0.1227 & 0.1365 & 0.0975 & 0.0787 \\ 0.1617 & 0.1558 & 0.1264 & 0.1152 & 0.1269 & 0.1379 & 0.0956 & 0.0809 \\ 0.1585 & 0.1555 & 0.1292 & 0.1171 & 0.1250 & 0.1389 & 0.0947 & 0.0816 \\ 0.1591 & 0.1553 & 0.1282 & 0.1156 & 0.1261 & 0.1382 & 0.0975 & 0.0806 \\ 0.1598 & 0.1557 & 0.1267 & 0.1192 & 0.1243 & 0.1358 & 0.0963 & 0.0816 \end{bmatrix}$$

Step 16, 17, and 18: Tables 11 and 12 present the usefulness levels for the ideal, anti-ideal, and utility functions using Equations (3.27) to (3.30). The alternatives appraisal scores are calculated using the parameters $\rho = 0.5$.

TABLE 11. Usefullness levels for ideal and anti-ideal solutions

	$\iota^{\varepsilon}_{\beta_1}$	$\iota^{\varepsilon}_{\beta_2}$	$\iota^{\varepsilon}_{\beta_3}$	$\iota^{\varepsilon}_{\beta_4}$	$\iota^{\varepsilon}_{\beta_5}$	$\iota^{\varepsilon}_{\beta_6}$	$\iota^{\varepsilon}_{\beta_7}$	$\iota^{\varepsilon}_{\beta_8}$
$\ell-$	16.1596	15.9160	12.8939	11.7910	12.6805	13.7852	9.7500	8.1443
$\ell+$	13.0232	12.2892	10.3977	8.7966	9.6994	11.3074	7.1830	6.1563

TABLE 12. Utility functions and Ranking

Alternatives	$\sqsupset(\ell+)$	$\sqsupset(\ell-)$	$F_\imath$	Rank
${\iota^{\vartheta}}_{\nu 1}$	0.4962	0.5038	0.8824	4
${\iota^{\vartheta}}_{\nu 2}$	0.4963	0.5037	0.9188	2
${\iota^{\vartheta}}_{\nu 3}$	0.4963	0.5037	0.9224	1
${\iota^{\vartheta}}_{\nu 4}$	0.4960	0.5040	0.8694	5
${\iota^{\vartheta}}_{\nu 5}$	0.4960	0.5040	0.9006	3
${\iota^{\vartheta}}_{\nu 6}$	0.4963	0.5037	0.8455	6

4.6. **Sensitive analysis.** Table 13 demonstrates the sensitivity of the ranking outcomes of alternatives to changes in the parameters ρ and ζ, which influence the aggregation mechanism within the decision-making model. The values $\left({\iota^{\vartheta}}_{\nu 1}\right)$ to $\left({\iota^{\vartheta}}_{\nu 6}\right)$ represent the performance scores of six alternatives under various configurations of these parameters. Across all combinations of ρ and ζ, the ranking order remains consistent: ${\iota^{\vartheta}}_{\nu 3} \succ {\iota^{\vartheta}}_{\nu 2} \succ {\iota^{\vartheta}}_{\nu 5} \succ {\iota^{\vartheta}}_{\nu 1} \succ {\iota^{\vartheta}}_{\nu 4} \succ {\iota^{\vartheta}}_{\nu 6}$. The observed consistency demonstrates the robustness of the decision-making model, indicating that variations in parameter values do not substantially affect the relative superiority of the alternatives. It is clear that the performance scores for all alternatives increase progressively with rising values of ρ and ζ. This indicates the model's sensitivity to these parameters regarding intensity, but not in terms of ranking order. Significant increases are evident at elevated values of $\rho = 0.9$ and $\zeta = 1000$, especially for ${\iota^{\vartheta}}_{\nu 3}$, which consistently attains the highest performance score, thereby affirming its status as the leading alternative. The findings indicate that although the absolute evaluation scores are influenced by parameter settings, the decision outcome, specifically the ranking, remains consistent, thereby ensuring decision reliability across different conditions. Figure 2 presents the sensitivity analysis results through a line graph, while Figure 3 displays the same results in a bar graph format. The visualization illustrate the impact of differing ρ values on alternative performance at various fixed ζ levels.

TABLE 13. The effect of parameters ζ and ρ on the outcome of the alternatives

ρ	ζ	$\left(\iota^{\vartheta}{}_{\nu 1}\right)$	$\left(\iota^{\vartheta}{}_{\nu 2}\right)$	$\left(\iota^{\vartheta}{}_{\nu 3}\right)$	$\left(\iota^{\vartheta}{}_{\nu 4}\right)$	$\left(\iota^{\vartheta}{}_{\nu 5}\right)$	$\left(\iota^{\vartheta}{}_{\nu 6}\right)$	Ranking
	$\zeta=5$	1.0643	1.1077	1.1087	1.0432	1.0800	1.0197	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=10$	1.0685	1.1102	1.1121	1.0465	1.0827	1.0221	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
$\rho=0.1$	$\zeta=100$	1.0728	1.1128	1.1156	1.0499	1.0853	1.0245	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=500$	1.0771	1.1153	1.1190	1.0533	1.0880	1.0269	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=1000$	1.0814	1.1178	1.1225	1.0567	1.0907	1.0293	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=5$	1.0857	1.1203	1.1260	1.0601	1.0933	1.0317	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=10$	1.0900	1.1228	1.1294	1.0635	1.0960	1.0341	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
$\rho=0.3$	$\zeta=100$	1.0943	1.1253	1.1329	1.0669	1.0987	1.0365	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=500$	1.0986	1.1278	1.1364	1.0703	1.1013	1.0389	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=1000$	1.1588	1.1628	1.1849	1.1179	1.1387	1.0725	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=5$	1.1072	1.1328	1.1433	1.0771	1.1067	1.0437	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=10$	1.1115	1.1353	1.1468	1.0805	1.1093	1.0461	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
$\rho=0.6$	$\zeta=100$	1.1158	1.1378	1.1502	1.0839	1.1120	1.0485	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=500$	1.1201	1.1403	1.1537	1.0873	1.1147	1.0509	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=1000$	1.1244	1.1428	1.1572	1.0907	1.1173	1.0533	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=5$	1.1287	1.1453	1.1606	1.0941	1.1200	1.0557	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=10$	1.1330	1.1478	1.1641	1.0975	1.1227	1.0581	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
$\rho=0.9$	$\zeta=100$	1.1373	1.1503	1.1676	1.1009	1.1253	1.0605	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=500$	1.1416	1.1528	1.1710	1.1043	1.1280	1.0629	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$
	$\zeta=1000$	1.1717	1.1703	1.1953	1.1281	1.1467	1.0797	$\iota^{\vartheta}{}_{\nu 3} \succ \iota^{\vartheta}{}_{\nu 2} \succ \iota^{\vartheta}{}_{\nu 5} \succ \iota^{\vartheta}{}_{\nu 1} \succ \iota^{\vartheta}{}_{\nu 4} \succ \iota^{\vartheta}{}_{\nu 6}$

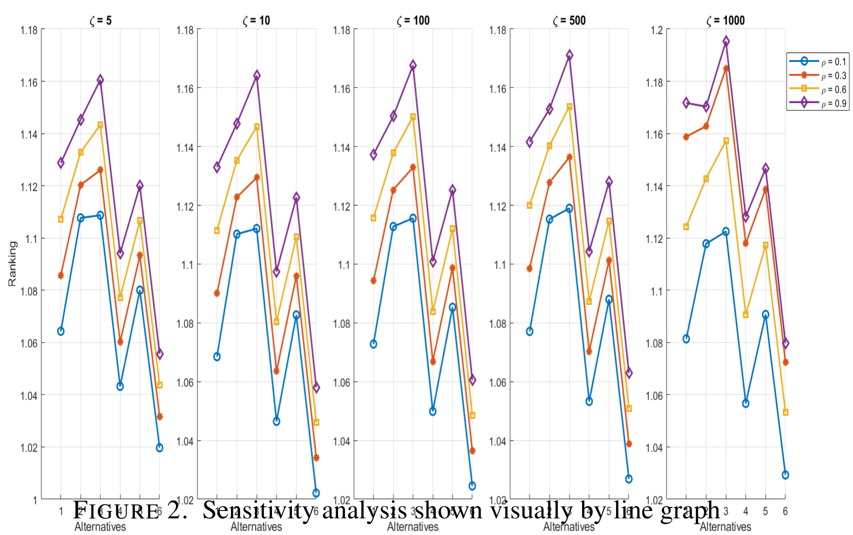

FIGURE 2. Sensitivity analysis shown visually by line graph

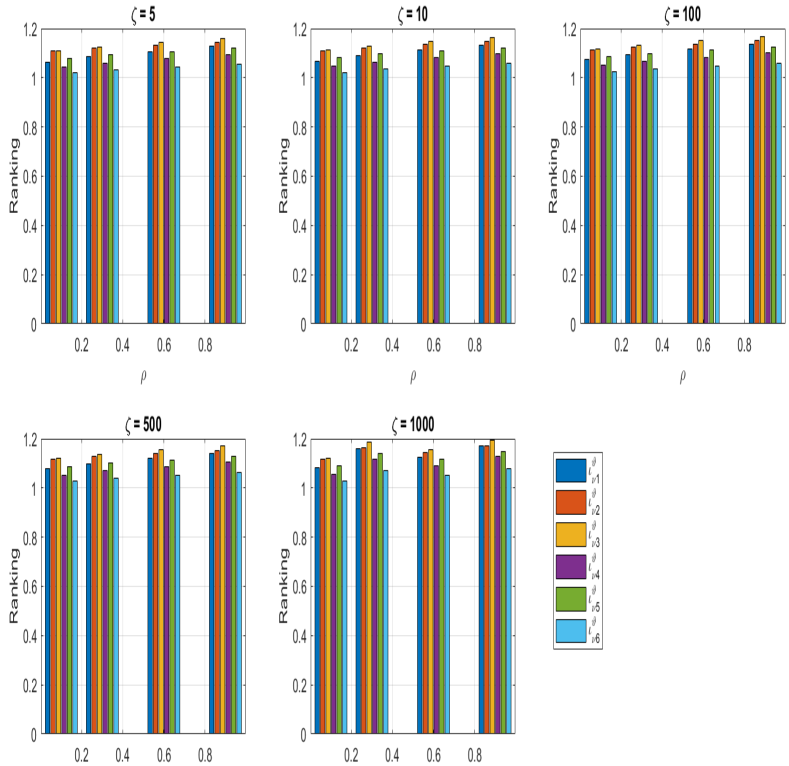

FIGURE 3. Sensitivity analysis shown visually by bar graph

4.7. **Limitations of the model.** While the proposed integrated MCDM framework offers several advantages, it is important to recognise its limitations. The model depends on the availability and quality of data, including both objective criteria values and expert judgments; inaccuracies or incompleteness in the data may compromise the robustness of the outcomes. The model incorporates both subjective and objective weighting methods; however, it remains reliant on expert opinions, which may introduce bias, particularly if the expert panel lacks diversity or representation. The model's complexity may present challenges for stakeholders who are not well-versed in advanced decision-making techniques, which could hinder its practical adoption in the absence of sufficient training or user-friendly software support. The model presently fails to explicitly incorporate dynamic temporal changes, including shifts in regulatory frameworks, market conditions, or technological progress, which may influence the pertinence of the criteria or alternatives. The framework posits independence among criteria; however, it does not explicitly model interdependencies or correlations, potentially oversimplifying certain real-world interactions that influence the decision-making process. Future research should aim to address these limitations by integrating dynamic weighting mechanisms, implementing bias reduction strategies via diverse expert panels, and developing interactive user interfaces to facilitate broader adoption. Furthermore, the incorporation of machine learning algorithms alongside probabilistic or fuzzy techniques may enhance adaptability and the management of uncertainty. Explicit modeling of inter-criteria dependencies would improve the framework's capacity to represent complex decision-making environments.

4.8. **Discussion of Ranking.** Mechanical recycling is the most widely adopted method among plastic recycling techniques, noted for its cost-effectiveness and simplicity of process. This technique is established and economically viable for recycling clean, single-type plastics such as PET and HDPE. The process does not modify the polymer's chemical structure and contributes to the circular economy in plastic production. However, its primary limitation is the difficulty in processing mixed, contaminated, or multi-layered plastics, as well as the decline in polymer quality with repeated recycling cycles. Chemical recycling ranks second, as it involves the breakdown of plastic polymers into monomers or chemical feedstocks via processes like depolymerisation or glycolysis. This method provides the benefit of generating virgin-quality materials, rendering it appropriate for more intricate and contaminated plastic waste streams that mechanical recycling is unable to handle. Major limitations include elevated operational costs, significant energy demands, and the necessity for advanced infrastructure, which currently restricts widespread industrial application. Pyrolysis is the third most significant technology for waste-to-energy conversion. The process entails heating plastic waste in an oxygen-free environment to produce pyrolysis oil, gas, and char, demonstrating particular efficacy for polyolefin-rich materials such as LDPE and PP. Pyrolysis offers a feasible method for energy recovery from non-recyclable plastics; however, it is associated with significant capital and energy expenditures and may present environmental hazards through emissions if inadequately controlled. It plays a significant role in the management of plastics that are not addressed by conventional recycling methods. Solvent-based purification, referred to as dissolution recycling, ranks fourth. This process facilitates the selective recovery of high-quality polymers from multi-layered, coloured, or additive-laden plastics while preserving the integrity of the polymer chains. This method maintains the structural integrity of the material more effectively than mechanical recycling, resulting in higher quality recyclate. The necessity for meticulous solvent selection, recovery systems, and compliance with environmental and safety regulations renders it a less prevalent commercial option and constrains its scalability. Gasification is positioned as the fifth method, primarily owing to its technological complexity and substantial capital demands. Plastic waste is converted into synthesis gas (syngas), consisting of hydrogen and carbon monoxide, which can be utilized for electricity generation, heating, or as a chemical feedstock. Gasification is notably efficient for contaminated and heterogeneous plastic waste; however, its primary application is energy recovery rather than genuine recycling, which constrains its contribution to circular plastic economies. Biological recycling is positioned sixth, primarily due to its current status in the research and development phase. This method employs microorganisms or enzymes to decompose biodegradable plastics and, in certain instances, synthetic plastics such as PLA and PET. Although it holds potential as an environmentally sustainable and low-energy approach, it faces limitations due to slow degradation rates, low efficiency for conventional plastics, and difficulties in scaling to industrial levels. Biological recycling possesses long-term potential, as advancements in biotechnology may enhance its applicability and effectiveness significantly.

5. Conclusion

The study proposes a comprehensive MCDM framework that integrates Entropy, SWARA, and ERUNS methods to establish a decision support system for the selection of plastic waste recycling technologies, considering economic and environmental constraints. The notion of bipolar fuzzy Z-numbers (BFZNs) is a new hybrid approach that integrates bipolar fuzzy numbers and Z-numbers that offers reliability components of both positive and negative membership grades. The proposed framework integrates objective, data-driven weighting techniques with subjective expert evaluations, effectively addressing the complexity and uncertainty involved in assessing multiple recycling alternatives across various criteria. The ranking results indicated that mechanical recycling is the most appropriate technology, succeeded by chemical recycling and pyrolysis. The results correspond with the priorities of economic viability and environmental sustainability. Sensitivity and comparative analyses confirmed the framework's robustness and reliability, underscoring its practical significance for stakeholders and decision-makers. Despite its limitations, including the assumption of criteria independence and reliance on expert judgment, the model serves as a transparent, comprehensive, and adaptable decision support tool. Future research may enhance the model by integrating dynamic factors, modeling inter-criteria relationships, and incorporating machine learning techniques. The framework significantly contributes to the advancement of sustainable plastic waste management by facilitating informed and balanced technology selection.

5.1. **Future Directions.** To improve the robustness, flexibility, and applicability of the proposed framework, various future research directions warrant exploration. The recommendations highlight the importance of adaptability to changing conditions, the modeling of inter-dependencies, the management of uncertainty, the inclusion of stakeholders, the integration with machine learning, and the application of broader sustainability principles. The future directions are listed as follows.

- Integrate dynamic decision making with reliability components incorporating evolving regulations, sustainable development, and circular economy.
- Model and integrate inter-dependencies and relationship among criteria to more efficiently reflect real-world issues.
- Increase the size and diversity of the expert panel to improve the reliability of subjective weightings. Modeling uncertainty with BFZNs that effectively address data vagueness and bipolarity.
- Prioritizing uncertainty modeling, by fuzzy logic and linguistic variables, to effectively manage data vagueness and ambiguity.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Rana Shehzad Waheed Akhtar: Conceptualization, Data Curation, Writing Original Draft, Investigation, Methodology, Edit-ing, and Software. Muhammad Aslam Malik: Review and Supervision. The final version of the manuscript has been read and approved by all authors.

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