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Novel Minkowski Inequalities for Fractional Integrals Involving the Extended Bessel–Maitland Function

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Abstract. This paper establishes Minkowski-type and related integral inequalities for fractional integral operators involving the extended Bessel–Maitland function. New inequalities are obtained for the Bessel–Maitland function with ten, eight, and six parameters. With suitable parameters, the corresponding inequalities involving the Mittag–Leffler function are recovered as particular cases. The results extend existing Minkowski-type inequalities in fractional calculus and provide a unified framework for generalized fractional integral inequalities.

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1. Introduction

Minkowski's and related inequalities got considerable attention due to their wide applications in the diverse fields which include Physics, Engineering, Signal processing and Image processing. In [4], Bougoffa constructed the Minkowski and Hardy integral inequalities and in 2010, Dahamani proved Minkowski and Hermite-Hadamard integral inequalities via fractional integral(see [5]). In 2019, Khan et al. provide the Minkowskis

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inequality for the AB-fractional integral operator [10] and in [13] Mubeen et al. use the generalized k-fractional conformable integral to prove the Minkowski inequalities. The reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function is given in [15]. Recent related work on Minkowski and integral inequalities for fractional operators can be found in [7], [19], [20]. On the other hand, Bessel-Maitland function (BMF) is the generalization of Mittag-Leffler function. The generalized fractional integrals of the generalized Mittag-Leffler function is given in [1] (see also [12]) and in 2022 Ali et al. prove the generalized Bessel-Maitland function to give the generalized fractional integral operator (see [2]). Moreover, some fractional operators with generalized Bessel-Maitland function can be found in [3]. The rich literature related to the Bessel–Maitland function can be found in [8], [9], [17]. In mathematical modeling and analysis the extended version of Bessel-Maitland function (EVBMF) is versatile tool to solve the different problems and is recognized due to its importance in diverse applications in fractional calculus.

Let us describe the basic material related to BMF and its extensions which we use frequently for the construction of our results.

Definition 1.1. [14] *The integral representation of Gamma function is defined, for* $\Re(u) > 0$*, as follows:*

$$\Gamma(\varsigma) = \int_0^\infty \varkappa^{\varsigma-1} e^{-\varkappa} d\varkappa$$

Definition 1.2. [14] *Pochhammer's symbol is defined as:*

$$(\varsigma)_n = \begin{cases} \varsigma(\varsigma+1)(\varsigma+2)\cdots(\varsigma+n-1), & \text{for } n \geq 1 \\ 1, & \text{for } n = 0, \varsigma \neq 0. \end{cases}$$

where $\varsigma \in \mathcal{C}$ *and* $n \in \mathbf{N}$. *Using gamma function, Pochhammer's symbol can be written as*

$$(\varsigma)_n = \frac{\Gamma(\varsigma+n)}{\Gamma(\varsigma)}.$$

Definition 1.3. [16] *The Gauss hyper-geometric function for all* $a, b, c, d \in \mathcal{C}, a \neq 0, -1, -2, -3, \cdots, |s| < 1$ *is defined as;*

$$F_1(b, -d; a; s) = \sum_{n=0}^{\infty} \frac{(b)_n (-d)_n s^n}{(a)_n n!},$$

where $(a)_n, (b)_n$ *and* $(-d)_n$ *are Pochhammer's symbols.*

Definition 1.4. [11] *The BMF is defined as:*

$$\digamma_{\chi}^{\varrho}(s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!\Gamma(\varrho n + \chi + 1)},$$

where

$$\Re(\chi) > -1, \Re(\alpha) \geq 0, \varrho, \chi \in \mathcal{C}.$$

Definition 1.5. [18]*(see* [2]*) The Generalized Bessel-Maitland function (GBMF) is defined as:*

$$\digamma^{\varrho,\beta}_{\chi,\kappa}(s)=\sum_{n=0}^{\infty}\frac{(\beta)_{\kappa n}(-s)^n}{n!\Gamma(\varrho n+\chi+1)},$$

where

$$\Re(\chi)>-1,\Re(\varrho)\geq 0,\Re(\beta)\geq 0,\ \ \varrho,\chi,\beta\in\mathcal{C},\kappa\in(0,1)\cup N,$$

and

$$(\beta)_0=1,(\beta)_{\kappa n}=\frac{\Gamma(\beta+\kappa n)}{\Gamma(\beta)},$$

Definition 1.6. [6] *The Extended Bessel-Maitland function (EBMF) is given as:*

$$\digamma^{\varrho,\beta,\zeta}_{\chi,\kappa,\gamma}(s)=\sum_{n=0}^{\infty}\frac{(\beta)_{\kappa n}(-s)^n}{\Gamma(\varrho n+\chi+1)(\gamma)_{\zeta n}},$$

where

$$\Re(\chi)>-1,\Re(\varrho)\geq 0,\Re(\beta)>0,\Re(\gamma)>0,\varrho,\chi,\beta,\gamma\in\mathcal{C},\zeta,\kappa\geq 0,\kappa<\Re(\varrho)+\zeta.$$

Definition 1.7. [3] *The generalization of Generalized Bessel-Maitland function (GGBMF) is given as:*

$$\digamma^{\varrho,\beta,\zeta,\mu}_{\chi,\kappa,\gamma,\ell}(s)=\sum_{n=0}^{\infty}\frac{(\mu)_{\ell n}(\beta)_{\kappa n}(-s)^n}{\Gamma(\varrho n+\chi+1)(\gamma)_{\zeta n}},$$

where

$$\Re(\chi)>-1,\Re(\varrho)\geq 0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\ell)>0,$$

$$\varrho,\chi,\beta,\gamma,\mu\in\mathcal{C},\ and\ \zeta,\kappa,\ell\geq 0.$$

Definition 1.8. [9] *The BMF of ten parameters is given as:*

$$\digamma^{\varrho,\beta,\zeta,\mu,\nu}_{\chi,\kappa,\gamma,\ell,\jmath}(s)=\sum_{n=0}^{\infty}\frac{(\mu)_{\ell n}(\beta)_{\kappa n}(-s)^n}{\Gamma(\varrho n+\chi+1)(\nu)_{\jmath n}(\gamma)_{\zeta n}},$$

where

$$\Re(\chi)>-1,\Re(\varrho)\geq 0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\mu)>0,$$

$$\varrho,\chi,\beta,\gamma,\mu,\nu\in\mathcal{C},\zeta,\kappa,\ell,\jmath\geq 0.$$

Definition 1.9. [9] *The EVBMF is defined as follows:*

$$\digamma^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(s)=\sum_{n=0}^{\infty}\frac{(\beta)_{\kappa n}(\zeta)_{\gamma n}(\mu)_{\ell n}(-s)^n}{\Gamma(\varrho n+\chi+1)(\nu)_{\jmath n}(\lambda)_{\flat n}},\tag{1. 1}$$

where

$$\Re(\chi)>-1,\Re(\varrho)\geq 0,\Re(\beta)>0,\Re(\zeta)>0,\Re(\mu)>0,\Re(\nu)>0,\Re(\lambda)>0,$$

and

$$\beta,\zeta,\mu,\varrho,\chi,\nu,\lambda\in\mathcal{C},\beta,\zeta,\mu<\nu+\lambda+\Re(\varrho),\kappa,\gamma,\ell,\jmath,\flat\geq 0.$$

Definition 1.10. *The fractional integral operator with EVBMF as its kernel* [3] *is defined as:*

$$(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}\daleth)(s)=\int\limits_a^s(s-\varkappa)^{\chi}\digamma^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})\daleth(\varkappa)d\varkappa, \tag{1. 2}$$

where

$$\Re(\chi)>-1,\Re(\varrho)\geq 0,\Re(\beta)>0,\Re(\zeta)>0,\Re(\mu)>0,\Re(\nu)>0,\Re(\lambda)>0,$$

$$\beta,\zeta,\mu,\chi,\nu,\lambda\in\mathcal{C},\beta,\zeta,\mu<\nu+\chi+\Re(\varrho),\kappa,\gamma,\ell,\jmath,\flat\geq 0,$$

Remark 1.11. *The operator $\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}$ in (1. 2) has the following basic analytical properties whenever the integral therein exists for the functions under consideration.*

(1) ***Linearity.*** *For scalars λ, μ and admissible functions f, g,*

$$\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(\lambda f+\mu g)=\lambda\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}f+\mu\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}g.$$

(2) ***Continuity.*** *Fix $T>a$. If $f\in C[a,T]$ and the kernel $(s-\varkappa)^{\chi}\digamma^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})$ is continuous on $\{(\varkappa,s):a\leq\varkappa\leq s\leq T\}$, then the mapping $s\mapsto(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}f)(s)$ is continuous on $[a,T]$. This follows from the Volterra form of (1. 2) together with $\Re(\chi)>-1$.*

(3) ***Boundedness.*** *On $[a,T]$, if $|f(\varkappa)|\leq M$ for all $\varkappa\in[a,T]$ and the kernel is bounded on the triangle $a\leq\varkappa\leq s\leq T$, then*

$$|(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}f)(s)|\leq M\int_a^s|(s-\varkappa)^{\chi}\digamma^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})|\,d\varkappa,$$

so the operator is bounded on $C[a,T]$ with respect to the sup norm.

(4) ***Semigroup property.*** *In general, the operator does not satisfy a semigroup law, since the kernel involves the full EVBMF and is not a one-parameter power kernel. When the parameters are specialized to the Riemann–Liouville case ($\varpi=0$ with χ replaced by $\chi-1$), the classical semigroup property of the Riemann–Liouville fractional integral is recovered; see the corresponding special cases in Section 2.*

For related fractional operators with Bessel–Maitland kernels, we refer to [3].

The aim of this paper is to establish Minkowski-type and related integral inequalities for the fractional operator in (1. 2) with EVBMF kernel. Section 2 develops these bounds under ratio or absolute conditions on the functions v_1 and v_2, records their operator specializations, and includes a numerical illustration of the base Minkowski inequality.

After presenting the existing body of knowledge in the introduction, In Section 2, we shall construct the Minkowski's and related inequalities for the extended version of Bessel-Maitland function.

2. Minkowski inequalities for fractional integrals involving the extended Bessel–Maitland function

This section establishes Minkowski-type and related inequalities for the operator $\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}$. We first prove a base Minkowski bound under a ratio condition on v_1/v_2;

the later results refine or extend this estimate to product, Hölder, Young, sandwich, shifted, and majorization forms.

We begin with the fundamental Minkowski-type inequality for the EVBMF fractional integral operator when $0 < w_1 \leq v_1/v_2 \leq w_2$; this result is used repeatedly in the proofs that follow.

Definition 2.1. *Fix $s > \alpha$ and $p \geq 1$. We write*

$$\mathcal{U}_p(\alpha, s) := \left\{ v : [\alpha, s) \to (0, \infty) : \ \beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v^p(s) < \infty \right\}.$$

For conjugate exponents $p, q \geq 1$ with $1/p + 1/q = 1$, set

$$\mathcal{U}_{p,q}(\alpha, s) := \Bigg\{ v : [\alpha, s) \to (0, \infty) : \beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v^{\frac{1}{p}}(s) < \infty$$
$$\textit{and} \quad \beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v^{\frac{1}{q}}(s) < \infty \Bigg\}.$$

Theorem 2.2. *For $p \geq 1$, let $v_1, v_2 \in \mathcal{U}_p(\alpha, s)$. If $0 < w_1 \leq \frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$, $w_1, w_2 \in \mathbb{R}^+$, for all $\varkappa \in [\alpha, s]$, then the following inequality holds.*

$$\begin{aligned} &\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s)\right)^{\frac{1}{p}} + \left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s)\right)^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1) + (w_2+1)}{(w_1+1)(w_2+1)}\right) \left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} (v_1^p(s) + v_2^p(s))^p\right)^{\frac{1}{p}}. \end{aligned} \tag{2. 3}$$

Proof. Using the assumption $\frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$, we obtain

$$(w_2+1)^p v_1^p(\varkappa) \leq w_2^p (v_1(\varkappa) + v_2(\varkappa))^p. \tag{2. 4}$$

Multiplying both sides of (2. 4) by $(s-\varkappa)^{\chi} \digamma^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})$ and integrating with respect to $\varkappa$ on the interval $[\alpha, s]$, we obtain

$$\begin{aligned} &\int_{\alpha}^{s} (s-\varkappa)^{\chi} \digamma^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})(w_2+1)^p v_1^p(\varkappa) d\varkappa \\ \leq \ &\int_{\alpha}^{s} (s-\varkappa)^{\chi} \digamma^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho}) w_2^p (v_1(\varkappa) + v_2(\varkappa))^p d\varkappa. \end{aligned} \tag{2. 5}$$

We can write (2. 5) using the Definition 1.10 as

$$(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} \leq \frac{w_2}{w_2+1} (\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} (v_1(s) + v_2(s))^p)^{\frac{1}{p}}. \tag{2. 6}$$

On the other hand, the assumption $w_1 \leq \frac{v_1(\varkappa)}{v_2(\varkappa)}$ results that

$$(w_1+1)^p \beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s) \leq \beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} (v_1(s) + v_2(s))^p,$$

which can be written as

$$(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \leq (\frac{1}{w_1+1})(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} (v_1(s) + v_2(s))^p)^{\frac{1}{p}}. \tag{2. 7}$$

Addition of (2. 6) and (2. 7) results the required inequality (2. 3).

□

Remark 2.3. *When parameters in (1. 2) are specialized, the corresponding fractional integral operators are defined as follows.*

(1) ***Reduced*** $\sqsupset$. *If parameters in (1. 2) are set to zero or fixed values, we write* $\sqsupset^{\varrho,\beta,\ldots}_{\chi,\kappa,\ldots;\alpha^+}$ *with the remaining parameter list for the operator given by (1. 2) with the specialized EVBMF kernel (see* [3], [9]*).*
(2) ***Operator*** $\daleth$. *For* $\gamma = 0$ *with* χ *replaced by* $\chi - 1$*, we denote the resulting operator by* $\daleth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}$*, defined by the same integral formula as (1. 2) under this substitution.*
(3) ***Riemann–Liouville operator*** F. *When* $\varpi = 0$ *and* χ *is replaced by* $\chi - 1$*, we obtain the Riemann–Liouville-type operator*

$$(F^{\beta}_{u^+} f)(s) = \frac{1}{\Gamma(\beta)} \int_u^s (s - \varkappa)^{\beta-1} f(\varkappa)\, d\varkappa, \qquad s > u.$$

Remark 2.4. *The following items are special cases of inequality (2. 3) using the operators in Remark 2.3. The related kernels and operators are already studied in the literature. We refer to Khan et al.* [9] *for the ten-parameter BMF, Ali et al.* [3] *for the eight-parameter case, and Ghayasuddin and Khan* [6] *for the six-parameter case. Further BMF and Mittag-Leffler type limits are connected with* [18], [8], [17]. *For the Riemann–Liouville case see Dahamani* [5]. *In each item below we specialize inequality (2. 3) to the corresponding operator.*

(1) *Substituting* $\gamma = 0$ *in inequality (2. 3), we get*

$$\begin{aligned}&(\sqsupset^{\varrho,\beta,\mu,\nu,\lambda}_{\chi,\kappa,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\sqsupset^{\varrho,\beta,\mu,\nu,\lambda}_{\chi,\kappa,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}}\\ &\le \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\sqsupset^{\varrho,\beta,\mu,\nu,\lambda}_{\chi,\kappa,\ell,\jmath,\flat,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(2) *If we replace* $\gamma = \jmath = 0$ *and* $\nu = 1$ *in inequality (2. 3), then we obtain*

$$\begin{aligned}&(\sqsupset^{\varrho,\beta,\mu,\lambda}_{\chi,\kappa,\ell,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\sqsupset^{\varrho,\beta,\mu,\lambda}_{\chi,\kappa,\ell,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}}\\ &\le \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\sqsupset^{\varrho,\beta,\mu,\lambda}_{\chi,\kappa,\ell,\flat,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(3) *Substituting* $\gamma = \jmath = \ell = 0$ *in inequality (2. 3), we get*

$$\begin{aligned}&(\sqsupset^{\varrho,\beta,\lambda}_{\chi,\kappa,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\sqsupset^{\varrho,\beta,\lambda}_{\chi,\kappa,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}}\\ &\le \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\sqsupset^{\varrho,\beta,\lambda}_{\chi,\kappa,\flat,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(4) *Substituting* $\gamma = \jmath = \ell = 0$ *and* $\flat = \lambda = 1$ *in inequality (2. 3), we have*

$$\begin{aligned}&(\sqsupset^{\varrho,\beta}_{\chi,\kappa,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\sqsupset^{\varrho,\beta}_{\chi,\kappa,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}}\\ &\le \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\sqsupset^{\varrho,\beta}_{\chi,\kappa,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(5) *Substituting $\gamma = \jmath = \ell = \kappa = 0$ and $\flat = \beta = \lambda = 1$ in inequality (2. 3), then we get*

$$\begin{aligned}&(\beth^{\varrho}_{\chi,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\beth^{\varrho}_{\chi,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\beth^{\varrho}_{\chi,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(6) *Substituting $\gamma = 0$ and replacing χ by $\chi - 1$ in inequality (2. 3), then we have*

$$\begin{aligned}&(\gimel^{\varrho,\beta,\mu,\nu,\lambda}_{\chi,\kappa,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\gimel^{\varrho,\beta,\mu,\nu,\lambda}_{\chi,\kappa,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\gimel^{\varrho,\beta,\mu,\nu,\lambda}_{\chi,\kappa,\ell,\jmath,\flat,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(7) *Substituting $\gamma = 0, \nu = \jmath = 1$ and replacing χ by $\chi - 1$ in inequality (2. 3), we get*

$$\begin{aligned}&(\gimel^{\varrho,\beta,\mu,\lambda}_{\chi,\kappa,\ell,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\gimel^{\varrho,\beta,\mu,\lambda}_{\chi,\kappa,\ell,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\gimel^{\varrho,\beta,\mu,\lambda}_{\chi,\kappa,\ell,\flat,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(8) *Substituting $\gamma = 0, \zeta = \nu = \jmath = \mu = 1$ and replacing χ by $\chi - 1$ in inequality (2. 3), we have*

$$\begin{aligned}&(\gimel^{\varrho,\beta,\lambda}_{\chi,\kappa,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\gimel^{\varrho,\beta,\lambda}_{\chi,\kappa,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\gimel^{\varrho,\beta,\lambda}_{\chi,\kappa,\flat,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(9) *Substituting $\gamma = 0, \ell = \nu = \jmath = \flat = 1$ and replacing χ by $\chi - 1$ in inequality (2. 3), we have*

$$\begin{aligned}&(\gimel^{\varrho,\beta}_{\chi,\kappa,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\gimel^{\varrho,\beta}_{\chi,\kappa,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\gimel^{\varrho,\beta}_{\chi,\kappa,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(10) *Substituting $\gamma = 0, \ell = \nu = \jmath = \beta = \flat = \lambda = 1$ and replacing χ by $\chi - 1$ in inequality (2. 3), then we obtain*

$$\begin{aligned}&(\gimel^{\varrho}_{\chi,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\gimel^{\varrho}_{\chi,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\gimel^{\varrho}_{\chi,\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(11) *Substituting $\chi = \gamma = 0, \ell = \nu = \jmath = \beta = \flat = \lambda = 1$ in inequality (2. 3), then we get*

$$\begin{aligned}&(\gimel^{\varrho}_{\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\gimel^{\varrho}_{\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \\ &\leq \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\gimel^{\varrho}_{\omega;\alpha^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

(12) *Consider $\varpi = 0$ and replacing χ by $\chi - 1$ in inequality (2. 3), we get*

$$\begin{aligned}&(\digamma^{\beta}_{u^+} v_1^p(s))^{\frac{1}{p}} + (\digamma^{\beta}_{u^+} v_2^p(s))^{\frac{1}{p}}\\ &\le \left(\frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)}\right)(\digamma^{\beta}_{u^+}(v_1^p(s)+v_2^p(s))^p)^{\frac{1}{p}}.\end{aligned}$$

Remark 2.5. *Inequality (2. 3) supports the following uses in mathematical analysis whenever $0 < w_1 \le v_1/v_2 \le w_2$ on $[\alpha, s]$:*

(1) ***Superposition.*** *If a model involves two nonnegative inputs f, g with $f/g \in [w_1, w_2]$, then (2. 3) bounds the sum of the p-th root $\beth$-terms in f^p and g^p by a single term in $(f+g)^p$, with explicit constant $C(w_1, w_2, p)$ from Theorem 2.2.*

(2) ***Comparison estimates.*** *The same bound applies to ratio-controlled pairs arising in barrier, sub-/super-solution, or a priori estimates for Volterra-type fractional integral expressions.*

(3) ***Verifiable specialization.*** *The Riemann–Liouville case in the preceding remark and Example 2.6 provides a concrete setting with checkable constants (Table 1).*

Example 2.6. *To illustrate Theorem 2.2 numerically, we work in the Riemann–Liouville specialization obtained by taking $\varpi = 0$ and replacing χ by $\chi - 1$ (see the last item of the remark after Theorem 2.2). On $[\alpha, s] = [0, s]$ with fractional order $\beta = 0.8$ and exponent $p = 2$, the operator reduces to*

$$(\digamma^{\beta}_{0^+} f)(s) = \frac{1}{\Gamma(\beta)} \int_0^s (s-\varkappa)^{\beta-1} f(\varkappa)\, d\varkappa.$$

We choose

$$v_1(\varkappa) = 1+2\varkappa, \qquad v_2(\varkappa) = 1+\varkappa, \qquad 0 < w_1 = 1 \le \frac{v_1(\varkappa)}{v_2(\varkappa)} \le w_2 = 1.6, \quad \varkappa \in [0, s].$$

For each $s \in (0, 1]$, the left-hand side (LHS) and right-hand side (RHS) of inequality (2. 3) are computed by adaptive Simpson quadrature on $[0, s]$; the coefficient in (2. 3) equals

$$C(w_1, w_2, p) = \frac{w_2(w_1+1)+(w_2+1)}{(w_1+1)(w_2+1)} = \frac{29}{26} \approx 1.1154.$$

Table 1 reports the computed values. In every tested case we have $\mathrm{LHS} \le \mathrm{RHS}$, with ratio $\mathrm{LHS}/\mathrm{RHS} \approx 0.897$. Figure 1 displays the same comparison for $s \in [0.1, 1]$.

TABLE 1. Numerical check of inequality (2. 3) in the Riemann–Liouville case of Example 2.6.

s	*LHS*	*RHS*	RHS − LHS	LHS/RHS
0.15	*1.093670*	*1.219674*	*0.126005*	*0.896690*
0.27	*1.516144*	*1.690485*	*0.174342*	*0.896869*
0.39	*1.910417*	*2.129714*	*0.219297*	*0.897030*
0.51	*2.299612*	*2.563228*	*0.263616*	*0.897155*
0.64	*2.691815*	*3.000093*	*0.308279*	*0.897244*
0.76	*3.090535*	*3.444253*	*0.353718*	*0.897302*
0.88	*3.497444*	*3.897585*	*0.400141*	*0.897336*
1.00	*3.913355*	*4.361002*	*0.447647*	*0.897352*

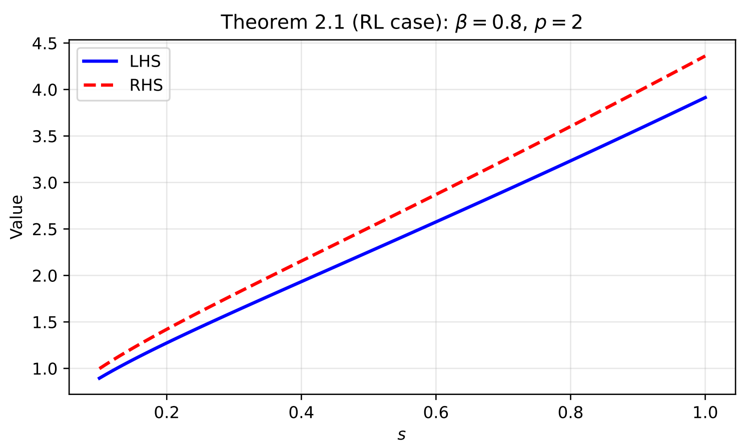

FIGURE 1. Plot of the LHS and RHS of inequality (2. 3) for the data in Example 2.6.

The computed values show that the fractional integral operator produces finite, increasing outputs on the chosen inputs and that the Minkowski bound is non-trivial: $\mathrm{LHS} < \mathrm{RHS}$ *at every tested* s*, with gap* $\mathrm{RHS} - \mathrm{LHS}$ *increasing from about* 0.13 *to* 0.45 *on* $[0.15, 1]$. *This illustrates the practical significance of the operator specialization in (1. 2) on explicit data.*

Multiplying the two one-sided bounds obtained in the proof of Theorem 2.2 yields the following product-type refinement, which compares the product of the fractional integral terms with the sum of their squares.

Theorem 2.7. *Let the assumption of the Theorem 2.2 be satisfied. Then the following inequality holds:*

$$\left(\frac{(w_1+1)(w_2+1)}{w_2}-2\right)\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s)\right)^{\frac{1}{p}}\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s)\right)^{\frac{1}{p}}$$
$$\leq\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s)\right)^{\frac{2}{p}}+\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s)\right)^{\frac{2}{p}}. \tag{2. 8}$$

Proof. Taking the product of (2. 6) and (2. 7) we have

$$\begin{aligned}&\frac{(w_1+1)(w_2+1)}{w_2}\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s)\right)^{\frac{1}{p}}\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s)\right)^{\frac{1}{p}}\\ &\leq\left(\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)+v_2(s))^p\right)^{\frac{1}{p}}\right)^2. \qquad (2.\ 9)\end{aligned}$$

Applying Minkowski's inequality to the right side of the above inequality, we obtain

$$\begin{aligned}&\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}((v_1(s)+v_2(s))^p)^{\frac{1}{p}}\right)^2\\ &\leq\left(\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s)\right)^{\frac{1}{p}}+(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s))^{\frac{1}{p}}\right)^2\\ &=\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s)\right)^{\frac{2}{p}}+(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s))^{\frac{2}{p}}\\ &+2\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s)\right)^{\frac{1}{p}}\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s)\right)^{\frac{1}{p}} \qquad (2.\ 10)\end{aligned}$$

Hence from (2. 9) and (2. 10)we obtain

$$\begin{aligned}&\frac{(w_1+1)(w_2+1)}{w_2}(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s))^{\frac{1}{p}}(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s))^{\frac{1}{p}}\\ &\leq(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s))^{\frac{2}{p}}+(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s))^{\frac{2}{p}}\\ &+2((\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s))^{\frac{1}{p}}(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s))^{\frac{1}{p}}.\end{aligned}$$

This results the required inequality (2. 8), which completes the proof. □

Remark 2.8. *Special cases of inequality (2. 8) follow from the parameter substitutions listed in the remark after Theorem 2.2. Representative operator reductions are $\gamma = 0$ (ten-parameter BMF), $\gamma = \jmath = \ell = 0$ (six-parameter BMF), and $\varpi = 0$ with χ replaced by $\chi - 1$ (Riemann–Liouville operator $F^{\beta}_{\alpha^+}$). The literature sources are given there.*

Having established the basic and product Minkowski-type bounds, we now treat Hölder- and Young-type inequalities for the same operator under the ratio condition on v_1/v_2. Theorem 2.9 gives the Hölder-type bound with conjugate exponents p and q.

Theorem 2.9. *For $p, q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, let $v_1, v_2 \in \mathcal{U}_{p,q}(\alpha, s)$. If $0 < w_1 \leq \frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$ for $w_1, w_2 \in \mathbb{R}^+$, for all $\varkappa \in [\alpha, s]$, then the following inequality holds:*

$$\begin{aligned}&\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1(s)\right)^{\frac{1}{p}}(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2(s))^{\frac{1}{q}}\\ \leq\ &(\frac{w_2}{w_1})^{\frac{1}{pq}}(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^{\frac{1}{p}}(s))(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^{\frac{1}{q}}(s)) \qquad (2.\ 11)\end{aligned}$$

Proof. Solving for the assumption $\frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$, we get

$$v_2^{\frac{1}{q}}(\varkappa) \geq w_2^{\frac{-1}{q}}\, v_1^{\frac{1}{q}}(\varkappa)$$

Multiplying both sides by $v_1^{\frac{1}{p}}(\varkappa)$, and making necessary settings, we obtain

$$w_2^{\frac{-1}{q}}\int_{\alpha}^{s}(s-\varkappa)^{\chi}F_{\chi,\kappa,\gamma,\ell,\jmath,\flat}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(\varpi(s-\varkappa)^{\varrho})v_1(\varkappa)d\varkappa$$
$$\leq \int_{\alpha}^{s}(s-\varkappa)^{\chi}F_{\chi,\kappa,\gamma,\ell,\jmath,\flat}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(\varpi(s-\varkappa)^{\varrho})v_1^{\frac{1}{p}}(\varkappa)v_2^{\frac{1}{q}}(\varkappa)d\varkappa.$$

This results

$$w_2^{\frac{-1}{pq}}(\beth_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}v_1(s))^{\frac{1}{p}} \leq (\beth_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(v_1^{\frac{1}{p}}(s)v_2^{\frac{1}{q}}(s)))^{\frac{1}{p}}. \tag{2. 12}$$

Now considering $w_1 \leq \frac{v_1(\varkappa)}{v_2(\varkappa)}$, we get

$$w_1^{\frac{1}{p}}v_2^{\frac{1}{p}}(\varkappa) \leq v_1^{\frac{1}{p}}(\varkappa)$$

Multiplying both sides by $v_2^{\frac{1}{q}}(\varkappa)$, and making necessary settings, we obtain

$$w_1^{\frac{1}{p}}\int_{\alpha}^{s}(s-\varkappa)^{\chi}F_{\chi,\kappa,\gamma,\ell,\jmath,\flat}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(\varpi(s-\varkappa)^{\varrho})v_2(\varkappa)d\varkappa$$
$$\leq \int_{\alpha}^{s}(s-\varkappa)^{\chi}F_{\chi,\kappa,\gamma,\ell,\jmath,\flat}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(\varpi(s-\varkappa)^{\varrho})v_1^{\frac{1}{p}}(\varkappa)v_2^{\frac{1}{q}}(\varkappa)d\varkappa.$$

This results

$$w_1^{\frac{1}{pq}}(\beth_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}v_2(s))^{\frac{1}{q}} \leq (\beth_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(v_1^{\frac{1}{p}}(s)v_2^{\frac{1}{q}}(s)))^{\frac{1}{q}}. \tag{2. 13}$$

Multiplying (2. 12) and (2. 13) results the required inequality (2. 11).

□

Now we will describe some special cases of the Theorem 2.9 in next remark.

Remark 2.10. *Special cases of the Hölder-type inequality (2. 11) are obtained by the same substitutions as in the remark after Theorem 2.2. For instance, $\gamma = 0$ yields the ten-parameter operator, and the Riemann–Liouville choice $\varpi = 0$ with $\chi \to \chi - 1$ gives the corresponding inequality for $F_{\alpha^+}^{\beta}$. Further reductions to eight-, six-, and four-parameter BMF operators are as in Theorem 2.2.*

The next result is a Young-type inequality for $\beth(v_1v_2)$; it follows from Theorem 2.9 by the standard Hölder–Young argument applied to the EVBMF fractional integral operator.

Theorem 2.11. *Let the assumptions of the Theorem 2.9 be satisfied. Then the following inequality holds:*

$$\begin{aligned}\beth_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(v_1(s)v_2(s)) \quad &\leq \tfrac{2^{p-1}w_2^p}{p(w_2+1)^p}(\beth_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(v_1^p(s)+v_2^p(s)))\\ &+\tfrac{2^{q-1}}{q(w_1+1)^q}(\beth_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}(v_1^q(s)+v_2^q(s)))\end{aligned} \tag{2. 14}$$

Proof. The assumption $\frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$, results

$$(w_2+1)^p v_1^p(\varkappa) \leq w_2^p(v_1(\varkappa)+v_2(\varkappa))^p.$$

After making some necessary changes, we have

$$\begin{aligned}&\int_{\alpha}^{s}(s-\varkappa)^{\chi}\mathcal{F}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})(w_2+1)^{p}v_1^{p}(\varkappa)d\varkappa\\ \leq&\int_{\alpha}^{s}(s-\varkappa)^{\chi}\mathcal{F}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})w_2^{p}(v_1(\varkappa)+v_2(\varkappa))^{p}d\varkappa.\end{aligned}$$

This implies that

$$\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^{p}(s)\leq\frac{w_2^{p}}{(w_2+1)^{p}}\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)+v_2(s))^{p}. \tag{2. 15}$$

Similarly, using the assumption $w_1\leq\frac{v_1(\varkappa)}{v_2(\varkappa)}$, we have

$$\begin{aligned}&\int_{\alpha}^{s}(s-\varkappa)^{\chi}\mathcal{F}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})(w_1+1)^{q}v_2^{q}(\varkappa)d\varkappa\\ \leq&\int_{\alpha}^{s}(s-\varkappa)^{\chi}\mathcal{F}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})(v_1(\varkappa)+v_2(\varkappa))^{q}d\varkappa.\end{aligned}$$

This can be written as

$$\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^{q}(s)\leq\frac{1}{(w_1+1)^{q}}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)+v_2(s))^{q}). \tag{2. 16}$$

Now using young's inequality, that is

$$v_1(\varkappa)v_2(\varkappa)\leq\frac{v_1^{p}(\varkappa)}{p}+\frac{v_2^{q}(\varkappa)}{q}. \tag{2. 17}$$

Multiplying (2. 17) by $(s-\varkappa)^{\chi}\mathcal{F}^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})$ on both sides and integrating with respect to $\varkappa$ on $[\alpha,s]$, we get

$$\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)v_2(s))\leq\frac{\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^{p}(s)}{p}+\frac{\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^{q}(s)}{q}. \tag{2. 18}$$

Combining inequalities (2. 15), (2. 16) and (2. 18), we get

$$\begin{aligned}&\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)v_2(s))\\ \leq&\frac{w_2^{p}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)+v_2(s))^{p})}{p(w_2+1)^{p}}+\frac{\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)+v_2(s))^{q}}{q(w_1+1)^{q}}\end{aligned} \tag{2. 19}$$

Using the inequality

$$(\lambda+\kappa)^{j}\leq 2^{j-1}(\lambda^{j}+\kappa^{j}),\lambda,\kappa,j>0,$$

we obtained

$$\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)+v_2(s))^{p}\leq 2^{p-1}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1^{p}(s)+v_2^{p}(s))), \tag{2. 20}$$

and

$$\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)+v_2(s))^{q}\leq 2^{q-1}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1^{q}(s)+v_2^{q}(s)). \tag{2. 21}$$

Hence the required inequality can be obtained by combining inequalities (2. 16), (2. 18) and (2. 20). $\square$

Remark 2.12. *Special cases of the Young-type inequality (2. 14) follow from the substitutions in the remark after Theorem 2.2. The principal reductions are $\gamma = 0$, $\gamma = \jmath = 0$ with $\nu = 1$, and the Riemann–Liouville specialization $\varpi = 0$, $\chi \to \chi - 1$. We refer to that remark for the full hierarchy of specialized operators.*

We now obtain a Minkowski-type inequality under separate absolute bounds on v_1 and v_2, rather than a ratio condition; the proof reduces this setting to the ratio framework used in Theorem 2.2.

Theorem 2.13. *For $p \geq 1$, let $v_1, v_2 \in \mathcal{U}_p(\alpha, s)$. If $0 < \hbar \leq v_1(\varkappa) \leq \hbar$ and $0 < \aleph \leq v_2(\varkappa) \leq \Im$ for $\hbar, \hbar, \aleph, \Im \in R^+, \forall \varkappa \in [\alpha, s]$, then the following inequality holds.*

$$\begin{aligned}&(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} + (\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}}\\ &\leq \frac{\hbar(\hbar + \Im) + \Im(\hbar + \aleph)}{(\aleph + \hbar)(\hbar + \Im)} (\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} (v_1(s) + v_2(s))^p))^{\frac{1}{p}} \qquad (2.\ 22)\end{aligned}$$

Proof. Using the assumptions $0 < \aleph \leq v_2(\varkappa) \leq \Im$ and $\hbar \leq v_1(\varkappa) \leq \hbar$, we observe that

$$\frac{\hbar}{\Im} \leq \frac{v_1(\varkappa)}{v_2(\varkappa)} \leq \frac{\hbar}{\aleph}$$

Using $\frac{\hbar}{\Im} \leq \frac{v_1(\varkappa)}{v_2(\varkappa)}$, we have

$$v_2^p(\varkappa) \leq \left(\frac{\Im}{(\hbar + \Im)}\right)^p (v_1(\varkappa) + v_2(\varkappa))^p$$

After making some necessary settings, we get

$$\begin{aligned}&\int_\alpha^s (s-\varkappa)^\chi F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat} (\varpi(s-\varkappa)^\varrho) v_2^p(\varkappa) d\varkappa\\ &\leq \left(\tfrac{\Im}{(\hbar+\Im)}\right)^p \int_\alpha^s (s-\varkappa)^\chi F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat} (\varpi(s-\varkappa)^\varrho)(v_1(\varkappa) + v_2(\varkappa))^p d\varkappa\end{aligned}$$

This can be written as

$$(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \leq \frac{\Im}{(\hbar + \Im)} (\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} (v_1(s) + v_2(s))^p)^{\frac{1}{p}}. \qquad (2.\ 23)$$

Similarly, now using $\frac{v_1(\varkappa)}{v_2(\varkappa)} \leq \frac{\hbar}{\aleph}$, we have that

$$(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} \leq \frac{\hbar}{\hbar + \aleph} (\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} (v_1(s) + v_2(s))^p)^{\frac{1}{p}}. \qquad (2.\ 24)$$

Adding inequalities (2. 23) and (2. 24), we obtain the required inequality (2. 22), which completes the proof. □

Remark 2.14. *Special cases of inequality (2. 22) are generated by the same parameter choices as in the remark after Theorem 2.2. In particular, the Riemann–Liouville reduction $\varpi = 0$ with χ replaced by $\chi - 1$ gives the corresponding Minkowski-type bound for $F^{\beta}_{\alpha^+}$ with the absolute bounds $\hbar \leq v_1 \leq \hbar$, $\aleph \leq v_2 \leq \Im$.*

The following sandwich inequality bounds the fractional integral of the product $v_1 v_2$ between terms involving $(v_1 + v_2)^2$, under the same ratio hypothesis as in Theorem 2.2.

Theorem 2.15. *Let the assumptions of the Theorem 2.2 be satisfied. Then the given inequality holds:*

$$\begin{aligned}&\frac{1}{w_2}\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)v_2(s))\\ &\leq \frac{1}{(w_1+1)(w_2+1)}\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}((v_1(s)+v_2(s))^2)\\ &\leq \frac{1}{w_1}\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)v_2(s))\end{aligned} \tag{2. 25}$$

Proof. Assumption $w_1 \leq \frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$ follows that

$$(w_1+1)v_2(\varkappa) \leq (v_1(\varkappa)+v_2(\varkappa)) \leq (w_2+1)v_2(\varkappa). \tag{2. 26}$$

Also we have that $\frac{1}{w_1} \geq \frac{v_2(\varkappa)}{v_1(\varkappa)} \geq \frac{1}{w_2}$. This results that

$$\frac{w_2+1}{w_2}v_1(\varkappa) \leq (v_1(\varkappa)+v_2(\varkappa)) \leq \frac{w_1+1}{w_1}v_1(\varkappa). \tag{2. 27}$$

Combining inequalities (2. 26)and(2. 27), we have

$$\frac{1}{w_2}(v_1(\varkappa)v_2(\varkappa)) \leq \frac{1}{(w_1+1)(w_2+1)}(v_1(\varkappa)+v_2(\varkappa))^2 \leq \frac{1}{w_1}(v_1(\varkappa)v_2(\varkappa)). \tag{2. 28}$$

Multiplying by $(s-\varkappa)^{\chi}F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})$ on both sides integrating with respect to $\varkappa$ on $[\alpha,s]$, we get

$$\begin{aligned}&\int_{\alpha}^{s}(s-\varkappa)^{\chi}F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})\tfrac{1}{w_2}v_1(\varkappa)v_2(\varkappa)d\varkappa\\ &\leq \int_{\alpha}^{s}(s-\varkappa)^{\chi}F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})\tfrac{1}{(w_1+1)(w_2+1)}(v_1(\varkappa)+v_2(\varkappa))^2d\varkappa\\ &\leq \int_{\alpha}^{s}(s-\varkappa)^{\chi}F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})\tfrac{v_1(\varkappa)v_2(\varkappa)}{w_1}d\varkappa.\end{aligned}$$

This results

$$\begin{aligned}\frac{1}{w_2}\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}\big(v_1(s)v_2(s)\big) &\leq \frac{1}{(w_1+1)(w_2+1)}\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}\big((v_1(s)+v_2(s))^2\big)\\ &\leq \frac{1}{w_1}\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}\big(v_1(s)v_2(s)\big).\end{aligned}$$

This is the required inequality (2. 25), which completes the proof. $\square$

Remark 2.16. *Special cases of the sandwich inequality (2. 25) follow from the substitutions listed in the remark after Theorem 2.2. Representative choices are $\gamma=0$ and the Riemann–Liouville limit $\varpi=0$, $\chi\to\chi-1$, which yields the corresponding three-term bound for $F^{\beta}_{\alpha^+}(v_1v_2)$.*

Introducing the parameter $\wp<0$, we next compare the fractional integral of the shifted combination $(v_1-\wp v_2)^p$ with those of v_1^p and v_2^p.

Theorem 2.17. *Let the assumptions of the Theorem 2.2 be satisfied. Then the given inequality holds:*

$$\begin{aligned}&\frac{w_2+1}{w_2-\wp}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)-\wp v_2(s))^p)^{\frac{1}{p}}\\&\leq(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_1^p(s))^{\frac{1}{p}}+(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s))^{\frac{1}{p}}\\&\leq\frac{w_1+1}{w_1-\wp}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)-\wp v_2(s))^p)^{\frac{1}{p}}.\end{aligned}\tag{2. 29}$$

Proof. Using the assumption $w_1\leq w_2$, we have

$$(w_2+1)(w_1-\wp)\leq(w_1+1)(w_2-\wp).$$

It yields that

$$\frac{w_2+1}{w_2-\wp}\leq\frac{w_1+1}{w_1-\wp}.\tag{2. 30}$$

Also using $w_1\leq\frac{v_1(\varkappa)}{v_2(\varkappa)}\leq w_2$, we have

$$w_1-\wp\leq\frac{v_1(\varkappa)-\wp v_2(\varkappa)}{v_2(\varkappa)}\leq w_2-\wp.$$

This results

$$(\frac{1}{w_2-\wp})^p(v_1(\varkappa)-\wp v_2(\varkappa))^p\leq v_2^p(\varkappa)\leq(\frac{1}{w_1-\wp})^p(v_1(\varkappa)-\wp v_2(\varkappa))^p.$$

Making some necessary settings, we have

$$\begin{aligned}&(\frac{1}{w_2-\wp})^p\int_\alpha^s(s-\varkappa)^\chi F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^\varrho)(v_1(x)-\wp v_2(\varkappa))^pd\varkappa\\\leq\ &\int_\alpha^s(s-\varkappa)^\chi F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^\varrho)v_2^p(\varkappa)d\varkappa\\\leq\ &(\frac{1}{w_1-\wp})^p\int_\alpha^s(s-\varkappa)^\chi F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^\varrho)(v_1(\varkappa)-\wp v_2(\varkappa))^pd\varkappa.\end{aligned}$$

We can write as

$$\begin{aligned}&\frac{1}{w_2-\wp}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)-\wp v_2(s))^p)^{\frac{1}{p}}\\&\leq(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}v_2^p(s))^{\frac{1}{p}}\\&\leq\frac{1}{w_1-\wp}(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)-\wp v_2(s))^p)^{\frac{1}{p}}.\end{aligned}\tag{2. 31}$$

Similarly using $\frac{1}{w_1} \geq \frac{v_2(\varkappa)}{v_1(\varkappa)} \geq \frac{1}{w_2}$, we have that

$$\begin{aligned}
& (\frac{w_2}{w_2-\wp})^p \int_{\alpha}^{s} (s-\varkappa)^{\chi} F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})(v_1(\varkappa)-\wp v_2(\varkappa))^p d\varkappa \\
\leq \quad & \int_{\alpha}^{s} (s-\varkappa)^{\chi} F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho}) v_1^p(\varkappa) d\varkappa \\
\leq \quad & (\frac{w_1}{w_1-\wp})^p \int_{\alpha}^{s} (s-\varkappa)^{\chi} F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho})(v_1(\varkappa)-\wp v_2(\varkappa))^p d\varkappa.
\end{aligned}$$

This results

$$\begin{aligned}
& \frac{w_2}{w_2-\wp}(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)-\wp v_2(s))^p)^{\frac{1}{p}} \\
\leq \quad & (\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} \\
\leq \quad & \frac{w_1}{w_1-\wp}(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+}(v_1(s)-\wp v_2(s))^p)^{\frac{1}{p}} \qquad (2.\ 32)
\end{aligned}$$

Adding inequalities (2. 31) and (2. 32) results the required inequality (2. 29), which completes the proof. □

Remark 2.18. *Special cases of the shifted Minkowski inequality (2. 29) are obtained by the parameter reductions in the remark after Theorem 2.2. The Riemann–Liouville specialization $\varpi = 0$, $\chi \to \chi - 1$ gives the analogous bound for $F^{\beta}_{\alpha^+}$ applied to $(v_1 - \wp v_2)^p$.*

Finally, we majorize the sum of the two fractional integral terms by $2\beth U^p$, where U is an auxiliary function built from v_1 and v_2 under the ratio bounds of Theorem 2.2.

Theorem 2.19. *Let the assumptions of the Theorem 2.2 be satisfied. Then the given inequality holds.*

$$\begin{aligned}
& \left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s)\right)^{\frac{1}{p}} + \left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s)\right)^{\frac{1}{p}} \\
& \leq 2\left(\beth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} U\big(v_1(s), v_2(s)\big)\right)^{\frac{1}{p}}.
\end{aligned} \qquad (2.\ 33)$$

where

$$U(v_1(s), v_2(s)) = \max\left\{w_2[(1+\frac{w_2}{w_1})v_1(s) - w_2 v_2(s)], \frac{(w_1+w_2)v_2(s)-v_1(s)}{w_1}\right\}.$$

Proof. Using the assumption $w_1 \leq \frac{v_1(\varkappa)}{v_2(\varkappa)}$, we get

$$w_1 + w_2 - \frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2. \qquad (2.\ 34)$$

Also using $\frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$, we have

$$v_2(\varkappa) \leq \frac{(w_1 + w_2 v_2(\varkappa) - v_1(\varkappa))}{w_1}. \qquad (2.\ 35)$$

From the inequalities (2. 34) and (2. 35), we obtain

$$v_2(\varkappa) \leq \frac{(w_1 + w_2 v_2(\varkappa) - v_1(\varkappa))}{w_1} \leq U(v_1(\varkappa), v_2(\varkappa)),$$

where

$$U(v_1(\varkappa), v_2(\varkappa)) = \max\left\{ w_2[(1 + \frac{w_2}{w_1})v_1(\varkappa) - w_2 v_2(\varkappa)], \frac{(w_1 + w_2)v_2(\varkappa) - v_1(\varkappa)}{w_1} \right\}.$$

This implies that

$$v_2^p(\varkappa) \leq U^p(v_1(\varkappa), v_2(\varkappa)).$$

Making necessary settings, we have

$$\begin{aligned} & \int\limits_{\alpha}^{s} (s-\varkappa)^{\chi} F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho}) v_2^p(\varkappa) d\varkappa \\ \leq & \int\limits_{\alpha}^{s} (s-\varkappa)^{\chi} F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho}) U^p(v_1(\varkappa), v_2(\varkappa)) d\varkappa. \end{aligned}$$

This yields that

$$(\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_2^p(s))^{\frac{1}{p}} \leq (\sqsupset^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} U^p(v_1(s), v_2(s)))^{\frac{1}{p}}. \tag{2. 36}$$

From the assumption$w_1 \leq \frac{v_1(\varkappa)}{v_2(\varkappa)} \leq w_2$, we have

$$\frac{1}{w_2} \leq \frac{v_2(\varkappa)}{v_1(\varkappa)} \leq \frac{1}{w_1}.$$

Using the inequality $\frac{1}{w_2} \leq \frac{v_2(\varkappa)}{v_1(\varkappa)}$, we have

$$w_1(\frac{1}{w_1} + \frac{1}{w_2})v_1(\varkappa) - w_1 v_2(\varkappa) \leq v_1(\varkappa). \tag{2. 37}$$

Now using $\frac{v_2(\varkappa)}{v_1(\varkappa)} \leq \frac{1}{w_1}$, we get

$$v_1(\varkappa) \leq (\frac{w_2}{w_1} + 1)v_1(\varkappa) - w_2 v_2(\varkappa). \tag{2. 38}$$

From the inequalities (2. 37) and (2. 38), we obtain that

$$v_1(\varkappa) \leq (\frac{w_2}{w_1} + 1)v_1(\varkappa) - w_2 v_2(\varkappa) \leq U(v_1(\varkappa), v_2(\varkappa))$$

This implies that

$$v_1^p(\varkappa) \leq U^p(v_1(\varkappa), v_2(\varkappa)).$$

Making necessary settings, we have

$$\begin{aligned} & \int\limits_{\alpha}^{s} (s-\varkappa)^{\chi} F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho}) v_1^p(\varkappa) d\varkappa \\ \leq & \int\limits_{\alpha}^{s} (s-\varkappa)^{\chi} F^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat}(\varpi(s-\varkappa)^{\varrho}) U^p(v_1(\varkappa), v_2(\varkappa)) d\varkappa. \end{aligned}$$

This results

$$(\daleth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} v_1^p(s))^{\frac{1}{p}} \leq \daleth^{\varrho,\beta,\zeta,\mu,\nu,\lambda}_{\chi,\kappa,\gamma,\ell,\jmath,\flat,\omega;\alpha^+} U^p(v_1(s), v_2(s))^{\frac{1}{p}}. \tag{2. 39}$$

Adding inequalities (2. 36)and(2. 39), we obtained the required inequality (2. 33), which completes the proof. □

Remark 2.20. *Special cases of inequality (2. 33) follow from the substitutions in the remark after Theorem 2.2. In particular,* $\gamma = 0$ *and the Riemann–Liouville choice* $\varpi = 0$*,* $\chi \to \chi - 1$ *produce the corresponding majorization bound for* $\digamma^{\beta}_{\alpha^+}$ *with the auxiliary function* $U(v_1, v_2)$.

3. Conclusion

In this paper we have established Minkowski-type and related integral inequalities for the fractional operator with EVBMF kernel defined in (1. 2). Under ratio or absolute bounds on the functions v_1 and v_2, we obtained eight main results (Theorems 2.2–2.19): a base Minkowski inequality, a product refinement, Hölder- and Young-type bounds, a Minkowski estimate with absolute hypotheses, a sandwich inequality for $\daleth(v_1 v_2)$, a shifted Minkowski bound, and a majorization result via an auxiliary function. The full hierarchy of operator specializations is recorded in the remark following Theorem 2.2. Example 2.6 provides a numerical illustration of the proposed operator in the Riemann–Liouville case, showing finite computed outputs and a strict, non-trivial Minkowski gap on explicit data.

Possible directions for future work include sharpness and extremal function analysis for the constants appearing in these inequalities, extensions to reversed or weighted forms, and application of the bounds to concrete fractional integral and differential models with Bessel–Maitland kernels.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Sajid Iqbal: Investigation, methodology, writing original draft, Writing-review & editing and supervision.
Muhammad Samraiz: Data curation, formal analysis, writing the original draft.
Muhammad Waseem Riaz: Conceptualization, software, writing review & editing
Asfand Fahad: Validation, writing review & editing
Artion Kashuri: Resources, software, writing review & editing. All authors read and approved the final manuscript.

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