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Mathematical Study of Plant Disease Model using Atangana-Baleanu Fractional Operators with Beddington-DeAngelis Incidence

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Abstract. This study discusses vector-borne plant epidemics through the Atangana-Baleanu type fractional model, considering the Beddington-DeAngelis functional response. A unique global solution has been developed through the Picard-Lindelof method. A numerical scheme for obtaining the solutions of plant disease model has been developed. Several graphical interpretations expressing the obtained solutions have been discussed, and many novel results have been observed through the variation of fractional order. This work leads to the idea of application of fractional derivatives in the field of plant epidemiology. The use of the Atangana-Baleanu derivative is novelty of this work, which explores many features that are missed by using the ordinary derivative.

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1. Introduction

The study of numerous diseases in different species of plants is called plant epidemiology. Cellular inborn immunity only protects plants against infections as they don't have mobile protection [11]. Vector-borne diseases of plants called plant viral epidemics insisted that scientists study these diseases through mathematical modeling because these models arose as the efficient tools for explaining the dynamics of disease transmission among host plants [27].

Many mathematicians have utilized mathematical models to comprehend the disease transmission among plants. For instance, Smith and Walker [29] presented a simple mathematical model to study the disease that appears in the roots of Trifolium subterraneum. Vesicular-arbuscular mycorrhizal fungi cause this infection. Brassett and Gilligan [8] developed a theoretical model for botanical epidemics by considering two types of infection, primary and secondary. The basic disease arises from the reservoir of the surviving inoculum, whereas other infections arise from infected tissue. This model matches the data for damping off the cress caused by the fungus. Jeger *et al.* [18] developed the SEIR model for host plants and the SEI model for explaining the transmission process. Gilligan *et al.* [13] formulated a mathematical model to analyze a stem canker infection assuming susceptible, infected, and recovered classes. The infection of potatoes and soil-borne fungus are the main causes of this disease. They fitted the model to field data and analyzed it. Bailey and Gilligan [5] discussed the wheat disease caused by take-all fungus using experimentation and mathematical modeling. Gubbins *et al.* [14] developed a theory for the generic model considering two aspects of plant-parasite interplay: the first is the dual source of inoculum, and the second is the infection load appearing due to host species. Venturino *et al.* [31] formulated a mathematical model and explored the dynamical features of mosaic virus disease, which appear in Jatropha plants. Buonomo & Cerasuolo [9] studied a plant-pathogen interaction through a mathematical model. They performed bifurcation analysis using the threshold value. Cunniffe *et al.* [10] assumed time-dependent infectivity in the compartmental model of plant disease. They first time used multiple exposed and infected compartments as an extension of the SEIR model in the form of the SEmInR and used it for the mathematical study of plant disease. More interested readers for various diseases modeling can see [15, 20, 30].

Fractional Calculus has become a very popular phenomenon in mathematical modeling. Many researchers have discussed a lot of theories and applications involving fractional order derivatives [6, 19, 23, 25]. Baleanu *et al.* [7] designed a model for the deep analysis of tumor-immune phenomena having fractional order derivative. The authors studied the interaction among the immune system and various tumor cell populations through differential equations having fractional order derivative. Owolabi and Atangana [24] used the Atangana-Baleanu fractional operator to study a nonlinear competition model describing the interaction among three species. Prakasha *et al.* [26] analyzed the fractional model, using the Atangana-Baleanu derivative, for analyzing Hepatitis E viral disease. Jajarmi and Baleanu [16] investigated the relationship among HIV and $CD4^{+}$ T-cells using the fractional calculus. Kumar *et al.* analyzed nonlinear fractional mathematical models of a mosaic epidemic in plants caused by begomovirus, which is distributed by white flies to plants. Al-Basir *et al.* studied vector-borne plant disease through a mathematical model. The authors assumed the resistance factor of plants and the overcrowding effect by using Beddington-DeAngelis incidence [1]. Recently, Kumar *et al.* [22] studied the plant disease model having fractional derivative in Caputo form.

The incorporation of fractional derivatives into the vector-borne disease model introduces an innovative dimension by accounting for memory and hereditary properties inherent in biological systems. Fractional order equations are important to elaborate on how fractional dynamics provide a more realistic framework for capturing the long-term interactionas and latent effects in plant-vector-pathogen systems. Unlike the classical integer order models, fractional derivatives introduce non-locality, meaning that the current state of the system depends not only on its present conditions but also on its historical evolution. This is especially relevant for plant-pathogen interactions where disease progression, vector behavior, and plant immune responses may exhibit delayed effects over time.

In order to gain the more accurate findings about the infections in plants and designing preventive policies, Jan *et al.* [17] presented the fractional framework. The impact of illness produced by virus in plants, Farman *et al.* [12] designed a system of fractional differential equations through fractal fractional operator. Shukla *et al.* [28] proposed a compartmental model of four classes for the dynamics of age-structured pests and plants. The authors used the fractional derivative by creating the memory effects which made the model more realistic. The impact of curative and preventive control measures for the epidemic in plants, Ali *et al.* [2] presented a novel mathematical model based on fractional differential equations.

The Atangana-Baleanu (AB) fractional derivative offers notable advantages over traditional derivatives when modeling infectious diseases, primarily due to its ability to incorporate memory and non-local effects through its non-singular Mittag-Leffler kernal. Unlike the power law kernels used in Caputo and Riemann-Liouville (RL) derivatives, the exponential-type nature of the Mittag-Leffler function results in a smoother and more realistic memory response [3]. This characteristics allows it to more accurately capture key epidemiological features such as latency , incubation periods,and immunity waning, which are often oversimplified in classical models. Additionally, AB-based models tend to fit real epidemic data more effectively, providing improved realism and flexibility in capturing complex dynamics like multiple outbreak waves or behavioral changes. The smooth nature of the AB kernel also enhances numerical stability, making simulations more robust. However, these benefits come with limitations such as the analytical solutions are rare, numerical implementation is more complex, and fractional parameters can be difficult to interpret biologically. Furthermore, AB models require historical data

for initial conditions, which can be challenging to obtain, and the method is not yet widely supported by standardized tools or software. Despite these challenges, the AB derivative remains a powerful tool for advancing the theoretical understanding of disease spread, particularly in systems where memory and history play a critical role. Consequently, the AB derivative was found to offer the most appropriate framework for modeling rabies transmission within a fractional-order system [21].
Motivating from the above ideas, we develop an SEIR plant disease model using the Beddington-DeAngelis incidence function and studied through the Atangana-Baleanu fractional derivative. The remaining sections are overviewed: Section 2 describes important definitions of said derivative having forms of Caputo and Riemann-Liouville. Section 3 states qualities of this derivative. Section 4 comprises the complete development of a plant infection model, and Section 5 contains the proof of existence and uniqueness of the model's outcomes. Section 6 is devoted for the development of numerical scheme. Graphical results are shown and explained in Section 7, and the conclusion of the work is given at the end.

2. Basic Definitions

Here, we present the definition of a fractional derivative and analyze its behavior, employing a non-local Mittag-Leffer kernel that does not exhibit singularity [3, 4].

Definition 2.1. *Suppose $f \in H^1(c,d), c < d$ and $0 \leq \xi \leq 1$. The Atangana-Baleanu derivative, having Caputo form, is described as*

$$ {}^{ABC}_{\ c}D_t^{\xi} f(t) = \frac{N(\xi)}{1-\xi} \int_c^t f'(\theta)\, E_{\xi}\left[-\xi \frac{(t-\theta)^{\xi}}{1-\xi}\right] d\theta, \tag{2. 1} $$

where the function $N(\xi)$ is the normalization having unit values at 0 and 1.

Definition 2.2. *Assume that $f(t)$ is any non differentiable function belonging to $H^1(c,d,$ for $d > c$ and $\xi \in [0,1]$. The Atangana-Baleanu derivative of fractional order, having Riemann-Liouville form, is stated as*

$$ {}^{ABR}_{\ c}D_t^{\xi} f(t) = \frac{N(\xi)}{1-\xi} \frac{d}{dt} \int_c^t f(\theta)\, E_{\xi}\left[-\xi \frac{(t-\theta)^{\xi}}{1-\xi}\right] d\theta. \tag{2. 2} $$

Definition 2.3. *The fractional integral with order ξ, for the above derivative, is defined as*

$$ {}^{AB}_{\ c}I_t^{\xi} f(t) = \frac{1-\xi}{G(\xi)\,\Gamma(\xi)} \int_c^t f(h)\,(t-h)^{\xi-1}\, dh. \tag{2. 3} $$

For $\xi = 0$, we get the initial function and ordinary integral is attained for the unit value of ξ.

3. About Atangana-Baleanu Derivative

The definitions defined above is very helpful in designing the models expressing real world problems. The relationship among Laplace transform and above defined definitions, for $n = 1$, may be established as:

$$ L\left\{{}^{ABR}_{\ 0}D_t^{\xi} f(t)\right\}(p) = \frac{G(\xi)}{1-\xi} \frac{p^{\xi} L\{f(t)\}(p)}{p^{\xi} + \frac{\xi}{1-\xi}}, \tag{3. 4} $$

$$ L\left\{{}^{ABC}_{\ 0}D_t^{\xi} f(t)\right\}(p) = \frac{G(\xi)}{1-\xi} \frac{p^{\xi} L\{f(t)\}(p) - p^{\xi-1} f(0)}{p^{\xi} + \frac{\xi}{1-\xi}}. $$

Theorem 3.1. *Assuming a function f to be continuous on the closed interval $[c,d]$, we obtain the following inequality :*

$$ \left\| {}^{ABR}_{\ 0}D_t^{\xi} f(t) \right\| < \frac{G(\xi)}{1-\xi} \|f(x)\|. \tag{3. 5} $$

The norm function of $f(x)$ has values as $\max_{c \leq x \leq d} |f(x)|$.

Theorem 3.2. *Fractional derivative defined by Atangana and Baleanu, in Caputo as well as Riemann-Liouville form, admits following Lipschitiz condition:*

$$\left\| {}_0^{ABR}D_t^{\xi} g(t) - {}_0^{ABR} D_t^{\xi} f(t) \right\| \leq H \left\| g(t) - f(t) \right\|, \tag{3. 6}$$

$$\left\| {}_0^{ABC}D_t^{\xi} g(t) - {}_0^{ABC} D_t^{\xi} f(t) \right\| \leq H \left\| g(t) - f(t) \right\|.$$

Theorem 3.3. *The fractional ordinary DE, with respect to time, is given below,*

$${}_0^{ABC}D_t^{\xi} g(t) = u(t). \tag{3. 7}$$

The application of inverse Laplace transform with utilization of convolution theorem [14], help us to obtain the unique solution in the following form:

$$g(t) = \frac{1-\xi}{G(\xi)} u(t) + \frac{\xi}{G(\xi)\Gamma(\xi)} \int_c^t u(h)(t-h)^{\xi-1}\, dh. \tag{3. 8}$$

The proofs of above theorems can be seen in [3].

4. Mathematical Model

The plant population is represented as $\mathsf{S}(t)$, $\mathsf{E}(t)$ and $\mathsf{I}(t)$, representing susceptible plants, exposed plants and infectious plants, respectively. The vector population is symbolized as $\mathsf{V}(t)$. Susceptible plants grow though logistic growth representing with rate r. The carrying capacity is denoted by K. The incidence function is represented as $\frac{\lambda \mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)}$, where λ is the maximum contact rate. The rate at which plants resist to the infection is denoted by a. The crowding effect of vectors is also assumed and is denoted by c. Exposed plants move to the infectious class with the rate m and d is death rate of infectious plants. The recruitment of insect vectors is directly proportional to the plants having infection and increase with the rate b. The mortality rate of the vectors is denoted by the parameter μ. Keeping in mind the stated assumptions, the fractional model in the form of differential equations can be described as

$$\begin{aligned}
\frac{d\mathsf{S}(t)}{dt} &= r\mathsf{S}(t)\left[1 - \frac{\mathsf{N}(t)}{K}\right] - \frac{\lambda \mathsf{S}(t)\mathsf{V}(t)}{1 + a\mathsf{S}(t) + c\mathsf{V}(t)}, && (4.\ 9)\\
\frac{d\mathsf{E}(t)}{dt} &= \frac{\lambda \mathsf{S}(t)\mathsf{V}(t)}{1 + a\mathsf{S}(t) + c\mathsf{V}(t)} - m\mathsf{E}(t), && (4.\ 10)\\
\frac{d\mathsf{I}(t)}{dt} &= m\mathsf{E}(t) - d\mathsf{I}(t), \\
\frac{d\mathsf{V}(t)}{dt} &= b\mathsf{I}(t) - \mu \mathsf{V}(t). && (4.\ 11)
\end{aligned}$$

5. Solutions of the Plant Disease Problem

This work is aimed to study the non linear problem of plants infection. It is not possible to find the analytical solution of the problem but, under certain reasonable conditions, the existence of exact solutions is guaranteed. It is proved as follows:

Suppose $L(I)$ is the Banach space which contains functions, having real values in continuity, on interval I. Furthermore defining $Q = L(I) * L(I)$, the norm can be expressed in the following way

$$\|\mathsf{S}, \mathsf{E}, \mathsf{I}, \mathsf{V}\| = \|\mathsf{S}\| + \|\mathsf{E}\| + \|\mathsf{I}\| + \|\mathsf{V}\|. \tag{5. 12}$$

The norm of S is defined as $\|\mathsf{S}\| = \{\sup\{|\mathsf{S}(t)|\} : t \in I\}$.. All remaining variables E, I and V have same norm functions. The expression of model (4. 11), using time fractional derivative with order $\xi \in [0, 1]$, has the following form

$$\begin{aligned}
{}_{0}^{ABC}D_t^{\xi}\mathsf{S}(t) &= r\mathsf{S}(t)\left[1-\frac{\mathsf{N}(t)}{K}\right]-\frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)},\\
{}_{0}^{ABC}D_t^{\xi}\mathsf{E}(t) &= \frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)}-m\mathsf{E}(t),\\
{}_{0}^{ABC}D_t^{\xi}\mathsf{I}(t) &= m\mathsf{E}(t)-d\mathsf{I}(t),\\
{}_{0}^{ABC}D_t^{\xi}\mathsf{V}(t) &= b\mathsf{I}(t)-\mu\mathsf{V}(t),
\end{aligned} \tag{5. 13}$$

having initial values

$$(\mathsf{S}(0),\mathsf{E}(0),\mathsf{I}(0),\mathsf{V}(0)) = (\mathsf{S}_0,\mathsf{E}_0,\mathsf{I}_0,\mathsf{V}_0). \tag{5. 14}$$

Modified version of the above stated system of equations, as Volterra type integral equations, by using Atangana-Baleanu fractional integral can be obtained. The application of Theorem (3.3) transform the model as

$$\begin{aligned}
\mathsf{S}(t)-\mathsf{S}(0) &= \frac{1-\xi}{G(\xi)}\left\{r\mathsf{S}(t)\left[1-\frac{\mathsf{N}(t)}{K}\right]-\frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)}\right\}\\
&\quad+\frac{\xi}{G(\xi)\Gamma(\xi)}\int_0^t(t-\mathsf{z})^{-(1-\xi)}\left[r\mathsf{S}(z)\left[1-\frac{\mathsf{N}(z)}{K}\right]-\frac{\lambda\mathsf{S}(z)\mathsf{V}(z)}{1+a\mathsf{S}(z)+c\mathsf{V}(z)}\right]\mathsf{dz},\\
\mathsf{E}(t)-\mathsf{E}(0) &= \frac{(1-\xi)}{G(\xi)}\left\{\frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)}-m\mathsf{E}(t)\right\}\\
&\quad+\frac{\xi}{G(\xi)\Gamma(\xi)}\int_0^t(t-\mathsf{z})^{-(1-\xi)}\left[\frac{\lambda\mathsf{S}(z)\mathsf{V}(z)}{1+a\mathsf{S}(z)+c\mathsf{V}(z)}-m\mathsf{E}(z)\right]\mathsf{dz},\\
\mathsf{I}(t)-\mathsf{I}(0) &= m\mathsf{E}(t)-d\mathsf{I}(t)+\frac{\xi}{G(\xi)\Gamma(\xi)}\int_0^t(t-\mathsf{z})^{-(1-\xi)}\left[m\mathsf{E}(z)-d\mathsf{I}(z)\right]\mathsf{dz},\\
\mathsf{V}(t)-\mathsf{V}(0) &= b\mathsf{I}(t)-\mu\mathsf{V}(t)+\frac{\xi}{G(\xi)\Gamma(\xi)}\int_0^t(t-\mathsf{z})^{-(1-\xi)}\left[b\mathsf{I}(z)-\mu\mathsf{V}(z)\right]\mathsf{dz},
\end{aligned} \tag{5. 15}$$

For the easy calculations, we suppose

$$\begin{aligned}
\mathbb{L}_1(t,\mathsf{S}) &= r\mathsf{S}(t)\left[1-\frac{\mathsf{N}(t)}{K}\right]-\frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)},\\
\mathbb{L}_2(t,\mathsf{E}) &= \frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)}-m\mathsf{E}(t),\\
\mathbb{L}_3(t,\mathsf{I}) &= m\mathsf{E}(t)-d\mathsf{I}(t),\\
\mathbb{L}_4(t,\mathsf{R}) &= b\mathsf{I}(t)-\mu\mathsf{V}(t).
\end{aligned} \tag{5. 16}$$

Theorem 5.1. *The kernels, $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, and $\mathbb{L}_4$, will satisfy Lipschitz Condition and contraction for the following inequalities:*

$$\begin{aligned}
0 &\leq \delta_1 \leq 1,\\
0 &\leq \delta_2 \leq 1,\\
0 &\leq \delta_3 \leq 1,\\
0 &\leq \delta_4 \leq 1.
\end{aligned} \tag{5. 17}$$

Proof. We prove the first inequality by taking the kernel $\mathbb{L}_1(t,\mathsf{S}) = r\mathsf{S}(t)\left[1-\frac{\mathsf{N}(t)}{K}\right]-\frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)}$.
Assuming two functions S and S_1, we have

$$\begin{aligned}\left\|\mathbb{L}_1(t,\mathsf{S})-\mathbb{L}_1(t,\mathsf{S}_1)\right\| &= \left\|r\mathsf{S}(t)\left[1-\frac{\mathsf{N}(t)}{K}\right]-\frac{\lambda\mathsf{S}(t)\mathsf{V}(t)}{1+a\mathsf{S}(t)+c\mathsf{V}(t)}-r\mathsf{S}_1(t)\left[1-\frac{\mathsf{N}_1(t)}{K}\right]-\frac{\lambda\mathsf{S}_1(t)\mathsf{V}_1(t)}{1+a\mathsf{S}_1(t)+c\mathsf{V}_1(t)}\right\|\\ &\le (2r+\lambda)\left\|\mathsf{S}-\mathsf{S}_1\right\|.\end{aligned} \tag{5. 18}$$

Taking, $\delta_1 = (2r+\lambda)$, and suppose that $a_1 = max_{t\in\mathsf{S}}\|\mathsf{S(t)}\|$, $a_2 = max_{t\in\mathsf{E}}\|\mathsf{E(t)}\|$, $a_3 = max_{t\in\mathsf{I}}\|\mathsf{I(t)}\|$, and $a_4 = max_{t\in\mathsf{V}}\|\mathsf{V(t)}\|$, are bounded functions. With these assumptions, we get the inequality as

$$\left\|\mathbb{L}_1(t,\mathsf{S})-\mathbb{L}_1(t,\mathsf{S}_1)\right\| \le \delta_1\left\|\mathsf{S}-\mathsf{S}_1\right\|. \tag{5. 19}$$

Thus the kernel $\mathbb{L}_1$ meet the criteria of Lipschitz condition. Moreover, the contraction is also obtained by restricting δ_1 as $0\le\delta_1\le 1$,. The same procedure gives the Lipschitz condition for the remaining kernels as:

$$\begin{aligned}\left\|\mathbb{L}_2(t,\mathsf{E})-\mathbb{L}_2(t,\mathsf{E}_1)\right\| &\le \delta_2\left\|\mathsf{E}-\mathsf{E}_1\right\|.\\ \left\|\mathbb{L}_3(t,\mathsf{I})-\mathbb{L}_3(t,\mathsf{I}_1)\right\| &\le \delta_3\left\|\mathsf{I}-\mathsf{I}_1\right\|,\\ \left\|\mathbb{L}_4(t,\mathsf{V})-\mathbb{L}_4(t,\mathsf{V}_1)\right\| &\le \delta_4\left\|\mathsf{V}-\mathsf{V}_1\right\|.\end{aligned} \tag{5. 20}$$

By using the kernels given in the equation (5. 19) and (5. 20), Eq. (5. 15) will be of the form

$$\begin{aligned}\mathsf{S}(t) &= \mathsf{S}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_1(t,\mathsf{S})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{-(1-\xi)}\left[\mathbb{L}_1(z,\mathsf{S})\right]dy,\\ \mathsf{E}(t) &= \mathsf{E}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_2(t,\mathsf{E})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{-(1-\xi)}\left[\mathbb{L}_2(z,\mathsf{E})\right]dy,\\ \mathsf{I}(t) &= \mathsf{I}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_3(t,\mathsf{I})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{-(1-\xi)}\left[\mathbb{L}_3(z,\mathsf{I})\right]dy,\\ \mathsf{V}(t) &= \mathsf{V}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_4(t,\mathsf{V})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{-(1-\xi)}\left[\mathbb{L}_4(z,\mathsf{V})\right]dy.\end{aligned} \tag{5. 21}$$

The recursion relation can be established as follows:

$$\begin{aligned}\mathsf{S}_n(t) &= \mathsf{S}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_1(t,\mathsf{S}_{n-1})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{(\xi-1)}\left[\mathbb{L}_1(z,\mathsf{S}_{n-1})\right]dz,\\ \mathsf{E}_n(t) &= \mathsf{E}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_2(t,\mathsf{E}_{n-1})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{-(1-\xi)}\left[\mathbb{L}_2(z,\mathsf{E}_{n-1})\right]dz,\\ \mathsf{I}_n(t) &= \mathsf{I}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_3(t,\mathsf{I}_{n-1})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{-(1-\xi)}\left[\mathbb{L}_3(z,\mathsf{I}_{n-1})\right]dz,\\ \mathsf{V}_n(t) &= \mathsf{V}(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_4(t,\mathsf{V}_{n-1})+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t(t-z)^{-(1-\xi)}\left[\mathbb{L}_4(z,\mathsf{V}_{n-1})\right]dz.\end{aligned} \tag{5. 22}$$

The initial conditions are also given as

$$\begin{aligned}\mathsf{S}_0(0) &= \mathsf{S}_0,\\ \mathsf{E}_0(0) &= \mathsf{E}_0,\\ \mathsf{I}_0(0) &= \mathsf{I}_0,\\ \mathsf{V}_0(0) &= \mathsf{V}_0.\end{aligned} \tag{5. 23}$$

Successive terms may differ in the following way:

$$\begin{aligned}
\gamma_n(t) &= \mathsf{S}_n(t) - \mathsf{S}_{n-1}(t) = \frac{(1-\xi)}{G(\xi)}\left[\mathbb{L}_1(t,\mathsf{S}_{n-1}) - \mathbb{L}_1(t,\mathsf{S}_{n-2})\right] \\
&\quad + \frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-z)^{-(1-\xi)}\left[\mathbb{L}_1(z,\mathsf{S}_{n-1}) - \mathbb{L}_1(z,\mathsf{S}_{n-2})\right]dz, \\
\Phi_n(t) &= \mathsf{E}_n(t) - \mathsf{E}_{n-1}(t) = \frac{(1-\xi)}{G(\xi)}\left[\mathbb{L}_2(t,\mathsf{E}_{n-1}) - \mathbb{L}_2(t,\mathsf{E}_{n-2})\right] \\
&\quad + \frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-z)^{-(1-\xi)}\left[\mathbb{L}_1(z,\mathsf{E}_{n-1}) - \mathbb{L}_1(z,\mathsf{E}_{n-2})\right]dz, \\
\Psi_n(t) &= \mathsf{I}_n(t) - \mathsf{I}_{n-1}(t) = \frac{(1-\xi)}{G(\xi)}\left[\mathbb{L}_3(t,\mathsf{I}_{n-1}) - \mathbb{L}_3(t,\mathsf{I}_{n-2})\right] \\
&\quad + \frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-z)^{-(1-\xi)}\left[\mathbb{L}_3(z,\mathsf{I}_{n-1}) - \mathbb{L}_3(z,\mathsf{I}_{n-2})\right]dz, \\
\varepsilon_n(t) &= \mathsf{V}_n(t) - \mathsf{V}_{n-1}(t) = \frac{(1-\xi)}{G(\xi)}\left[\mathbb{L}_4(t,\mathsf{V}_{n-1}) - \mathbb{L}_4(t,\mathsf{V}_{n-2})\right] \\
&\quad + \frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-z)^{-(1-\xi)}\left[\mathbb{L}_4(z,\mathsf{V}_{n-1}) - \mathbb{L}_4(z,\mathsf{V}_{n-2})\right]dz. \qquad (5.\ 24)
\end{aligned}$$

It is worth noticing that

$$\begin{aligned}
\mathsf{S}_n(t) &= \sum_{i=1}^{n}\gamma_i(t), \\
\mathsf{E}_n(t) &= \sum_{i=1}^{n}\Phi_i(t), \\
\mathsf{I}_n(t) &= \sum_{i=1}^{n}\Psi_i(t), \\
\mathsf{V}_n(t) &= \sum_{i=1}^{n}\varepsilon_i(t). \qquad (5.\ 25)
\end{aligned}$$

Considering equality (5. 24), applying the norm together with the triangular inequality, we have

$$\begin{aligned}
\|\gamma_n(t)\| &= \|\mathsf{S}_n(t) - \mathsf{S}_{n-1}(t)\| \\
&\leq \frac{(1-\xi)}{G(\xi)}\delta_1\|\mathsf{S}_{n-1}(t) - \mathsf{S}_{n-2}(t)\| \qquad (5.\ 26)\\
&\quad + \frac{\xi}{G(\xi)+\Gamma(\xi)}\left\|\int_0^t (t-z)^{-(1-\xi)}\left[\mathbb{L}_1(z,\mathsf{S}_{n-1}) - \mathbb{L}_1(z,\mathsf{S}_{n-2})\right]dz\right\|.
\end{aligned}$$

As the Kernal satisfy the Lipschitz condition, so

$$\begin{aligned}
\|\mathsf{S}_n - \mathsf{S}_{n-1}\| &\leq \frac{(1-\xi)}{G(\xi)}\delta_1\|\mathsf{S}_{n-1}(t) - \mathsf{S}_{n-2}(t)\| \qquad (5.\ 27)\\
&\quad + \frac{\xi}{G(\xi)+\Gamma(\xi)}\delta_1\int_0^t (t-z)^{-(1-\xi)}\|\mathsf{S}_{n-1}(t) - \mathsf{S}_{n-2}(t)\|\,dz,
\end{aligned}$$

for which we have

$$\|\gamma_n(t)\| \leq \frac{(1-\xi)}{G(\xi)}\delta_1 \|\gamma_{n-1}(t)\| + \frac{\xi}{G(\xi)+\Gamma(\xi)}\delta_1 \int_0^t (t-z)^{(\xi-1)} \|\gamma_{n-1}(z)\| \, dz. \tag{5. 28}$$

The following relations are obtained in similar fashion:

$$\begin{aligned}
\|\Phi_n(t)\| &\leq \frac{(1-\xi)}{G(\xi)}\delta_2 \|\Phi_{n-1}(t)\| + \frac{\xi}{G(\xi)+\Gamma(\xi)}\delta_2 \int_0^t (t-z)^{(\xi-1)} \|\Phi_{n-1}(z)\| \, dz, && (5.\ 29)\\
\|\Psi_n(t)\| &\leq \frac{(1-\xi)}{G(\xi)}\delta_3 \|\Psi_{n-1}(t)\| + \frac{\xi}{G(\xi)+\Gamma(\xi)}\delta_3 \int_0^t (t-z)^{(\xi-1)} \|\Psi_{n-1}(z)\| \, dz,\\
\|\varepsilon_n(t)\| &\leq \frac{(1-\xi)}{G(\xi)}\delta_4 \|\varepsilon_{n-1}(t)\| + \frac{\xi}{G(\xi)+\Gamma(\xi)}\delta_4 \int_0^t (t-z)^{(\xi-1)} \|\varepsilon_{n-1}(z)\| \, dz. && (5.\ 30)
\end{aligned}$$

□

With the aid of above results, it is convenient to describe and prove the subsequent result.

Theorem 5.2. *Plant infection problem, represented by (5. 13), has a solution satisfying the criteria*

$$\frac{1-\xi}{G(\xi)}\delta_i + \frac{t_{\max}^{\xi}}{G(\xi)\Gamma(\xi)}\delta_i < 1, fori = 1,2,3,4. \tag{5. 31}$$

Proof. Assuming the boundedness of functions presented in the system (5. 13) keeping kernels with the property of Lipschitz condition, the succeeding terms can be obtained, from (5. 28), as

$$\begin{aligned}
\|\gamma_n(t)\| &\leq \|\mathsf{S}(0)\| \left[\frac{1-\xi}{G(\xi)}\delta_1 + \frac{t_{\max}^{\xi}}{\Gamma(\xi)G(\xi)}\delta_1\right]^n,\\
\|\Phi_n(t)\| &\leq \|\mathsf{E}(0)\| \left[\frac{1-\xi}{G(\xi)}\delta_3 + \frac{t_{\max}^{\xi}}{G(\xi)\Gamma(\xi)}\delta_3\right]^n,\\
\|\Psi_n(t)\| &\leq \|\mathsf{I}(0)\| \left[\frac{1-\xi}{G(\xi)}\delta_4 + \frac{t_{\max}^{\xi}}{G(\xi)\Gamma(\xi)}\delta_4\right]^n,\\
\|\epsilon_n(t)\| &\leq \|\mathsf{R}(0)\| \left[\frac{1-\xi}{G(\xi)}\delta_5 + \frac{t_{\max}^{\xi}}{G(\xi)\Gamma(\xi)}\delta_5\right]^n. && (5.\ 32)
\end{aligned}$$

Thus expressions represented in (5. 25) exist and defined functions are smooth. After proving the existence of solutions for model (5. 13), we express them in the form of above functions. Let

$$\begin{aligned}
\mathsf{S}(t) &= \mathsf{S}(0) + \mathsf{S}_n(t) - a_n(t),\\
\mathsf{E}(t) &= \mathsf{E}(0) + \mathsf{E}_n(t) - c_n(t),\\
\mathsf{I}(t) &= \mathsf{I}(0) + \mathsf{I}_n(t) - d_n(t),\\
\mathsf{V}(t) &= \mathsf{V}(0) + \mathsf{V}_n(t) - e_n(t). && (5.\ 33)
\end{aligned}$$

We have to prove that $\|a_\infty(t)\| \to 0$ ultimately. It is proceeded as

$$\|a_n(t)\| \leq \left\| \frac{1-\xi}{G(\xi)}\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_{n-1}) + \frac{\xi}{G(\xi)\Gamma(\xi)} \int_0^t (t-z)^{\xi-1} (\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_{n-1}))\, dz \right\|, \tag{5. 34}$$

$$\begin{aligned}
\|a_n(t)\| &\leq \frac{1-\xi}{G(\xi)} \|\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_{n-1})\| + \frac{\xi}{G(\xi)\Gamma(\xi)} \int_0^t (t-z)^{\xi-1} \|(\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_{n-1}))\| \, dz\\
&\leq \frac{1-\xi}{G(\xi)}\delta_1 \|\mathsf{S}-\mathsf{S}_{n-1}\| + \frac{t^{\xi}}{G(\xi)\Gamma(\xi)}\delta_1 \|\mathsf{S}-\mathsf{S}_{n-1}\|. && (5.\ 35)
\end{aligned}$$

Repeating this process recursively, we obtain

$$\|a_n(t)\| \le \left[\frac{1-\xi}{G(\xi)} + \frac{t^{\xi}}{G(\xi)\Gamma(\xi)}\right]^{n+1} \delta_1^n M. \tag{5. 36}$$

Then at $t_{\max}$ we have

$$\|a_n(t)\| \le \left[\frac{1-\xi}{G(\xi)} + \frac{t_{\max}^{\xi}}{G(\xi)\Gamma(\xi)}\right]^{n+1} \delta_1^n M. \tag{5. 37}$$

Thus $\|a_\infty(t)\| \to 0$ as $n- > \infty$, which completes the assertion. □

5.1. **Existence of Unique Solution.** The Picard-Lindelof theorem tells us that solutions to differential equations not only exist but are also unique-at least in a small range. But putting this into practice is not always smooth. These methods can run into trouble, like becoming unstable, taking too long to reach at a result, or being overly sensitive to where one starts. Things get even trickier with stiff or very nonlinear equations. Often, we have to take tiny steps to keep things accurate, which means using more time and computer power.
In order to prove the existence of unique solution, another one say $(\mathsf{S}_1, \mathsf{E}_1, \mathsf{I}_1, \mathsf{V}_1)$, exists. Then we have,

$$\|\mathsf{S}(t) - \mathsf{S}_1(t)\| \le \frac{1-\xi}{G(\xi)}\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_1) + \frac{\xi}{G(\xi)\Gamma(\xi)}\int_0^t (t-z)^{\xi-1}\|(\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_1))\|\,dz. \tag{5. 38}$$

The norm help us in obtaining the following result:

$$\|\mathsf{S}(t) - \mathsf{S}_1(t)\| \le \frac{1-\xi}{G(\xi)}\|\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_1)\| + \frac{\xi}{G(\xi)\Gamma(\xi)}\int_0^t (t-z)^{\xi-1}\|\mathbb{L}_1(t,\mathsf{S}) - \mathbb{L}_1(t,\mathsf{S}_1)\|\,dz. \tag{5. 39}$$

The Lipschitz condition help us in getting the following form

$$\|\mathsf{S}(t) - \mathsf{S}_1(t)\| \le \frac{1-\xi}{G(\xi)}\delta_1\|\mathsf{S}(t) - \mathsf{S}_1(t)\| + \frac{\delta_1 t^{\xi}}{G(\xi)\Gamma(\xi)}\|\mathsf{S}(t) - \mathsf{S}_1(t)\|. \tag{5. 40}$$

This gives

$$\|\mathsf{S}(t) - \mathsf{S}_1(t)\|\left(1 - \frac{1-\xi}{G(\xi)}\delta_1 - \frac{t^{\xi}\delta_1}{G(\xi)\Gamma(\xi)}\right) \le 0. \tag{5. 41}$$

According to Theorem (5.2), $\left(1 - \frac{1-\xi}{G(\xi)}\delta_1 + \frac{t^{\xi}\delta_1}{G(\xi)\Gamma(\xi)}\right) > 0$, for $\xi \in [0,1]$ and $\delta_1 \in [0,1]$. Thus

$$\|\mathsf{S}(t) - \mathsf{S}_1(t)\| = 0.$$

Thus, we have

$$\mathsf{S}(t) - \mathsf{S}_1(t). \tag{5. 42}$$

Applying the same procedure, it can be shown that $E(t) = E_1(t), I(t) = I_1(t), V(t) = V_1(t)$. Hence the outcomes of model (5.2) are unique.

6. Numerical Scheme

Recently explored scheme by Toufik and Atangana [19] is very helpful for solving the problems having the rate of change with the additional property of non-local kernel in the absence of singularity. This work expresses two salient features, one is quick convergence and other is almost very accurate results. Method applied by these authors can be easily understand by assuming the following non linear form of ordinary differential equation

$$\begin{aligned} {}_0^{ABC}D_t^{\xi} v(t) &= f(t, v(t)) \\ v(0) &= v_0. \end{aligned} \tag{6. 43}$$

The equivalency of initial value problem (6. 43) with the fractional integral may be expressed as

$$v(t) - v(0) \leq \frac{1-\xi}{G(\xi)} f(t, v(t)) + \frac{\xi}{\Gamma(\xi) G(\xi)} \int_0^t f(\eta, v(\eta)) (t-z)^{\xi-1} \, d\eta. \tag{6. 44}$$

The integral equation (6. 44), for $t = t_{n+1}$, $n = 0, 1, 2...$, can be written as

$$v(t_{n+1}) - v(0) \leq \frac{1-\xi}{G(\xi)} f(t_n, v(t_n)) + \frac{\xi}{G(\xi)\Gamma(\xi)} \int_0^{t_{n+1}} f(z, v(z)) (t_{n+1} - z)^{\xi-1} \, dz. \tag{6. 45}$$

Considering the interval $[t_k, t_{k+1}]$ and applying two step Lagrange polynomial interpolation on $f(\eta, v(\eta))$ gives the following form

$$\begin{aligned} p_k(z) &= f(z, v(z)) \\ &= \frac{z - t_{k-1}}{t_k - t_{k-1}} f(t_k, v(t_k)) - \frac{z - t_k}{t_k - t_{k-1}} f(t_{k-1}, v(t_{k-1})) \\ &= \frac{f(t_k, v(t_k))}{h} (z - t_{k-1}) - \frac{f(t_{k-1}, v(t_{k-1}))}{h} (z - t_k) \\ &\simeq \frac{f(t_k, v_k)}{h} (z - t_{k-1}) - \frac{f(t_{k-1}, v_{k-1})}{h} (z - t_k) \end{aligned} \tag{6. 46}$$

Again assuming eq. (6. 45) for $f(\eta, v(\eta))$ (45) and the Lagrange polynomial interpolation as well, we will arrive at the following result

$$v_{n+1} = v(0) + \frac{1-\xi}{G(\xi)} f(t_n, v(t_n)) + \frac{\xi}{G(\xi)\Gamma(\xi)} \sum_{k=0}^{n} \left(\begin{array}{l} \frac{f(t_k, v_k)}{h} \int_{t_k}^{t_{k+1}} (z - t_{k-1}) (t_{n+1} - z)^{\xi-1} \, dz \\ -\frac{f(t_{k-1}, v_{k-1})}{h} \int_{t_k}^{t_{k+1}} (z - t_k) (t_{n+1} - z)^{\xi-1} \, dz \end{array} \right), \tag{6. 47}$$

The calculation of integral given in (6. 47) leads to the following result

$$\begin{aligned} v_{n+1} &= v(0) + \frac{1-\xi}{G(\xi)} f(t_n, v(t_n)) + \frac{\xi}{G(\xi)} \sum_{k=0}^{n} \frac{h^{\xi} f(t_k, v_k)}{\Gamma(\xi+2)} (n+1-k)^{\xi+1} (n-k+2+\xi) \\ &- \frac{\xi}{G(\xi)} \sum_{k=0}^{n} \frac{h^{\xi} f(t_k, v_k)}{\Gamma(\xi+2)} (n-k)^{\xi} (n-k+2+2\xi) \\ &- \frac{h^{\xi} f(t_{k-1}, v_{k-1})}{\Gamma(\xi+2)} \left((n+1-k)^{\xi+1} - (n-k+1+\xi)(n-k)^{\xi} \right) + R_n^{\xi}, \end{aligned} \tag{6. 48}$$

where R_n^{ξ} is remainder term. Its expression has the form

$$R_n^{\xi} = \frac{\xi}{G(\xi)\Gamma(\xi)} \sum_{k=0}^{n} \int_{t_k}^{t_{k-1}} \frac{(z - t_{k-1})(z - t_k)}{2} . \frac{\partial^2}{\partial z^2} [f(z, y(z))]_{z=\varepsilon z} (t_{n+1} - z)^{\xi-1} \, dz. \tag{6. 49}$$

Upper bounds for error may be seen in [19] . Taking in consideration the plant disease model (5. 13), and applying Atangana-Baleanu fractional integral, we can express in the form of kernels which is as follows:

$$\begin{aligned} S(t) &= S(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_1(t,S(t))+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-z)^{-(1-\xi)}\mathbb{L}_1(z,S(z))\,dz,\\ E(t) &= E(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_2(t,E(t))+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-z)^{-(1-\xi)}\mathbb{L}_2(z,E(z))\,dz,\\ I(t) &= I(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_3(t,I(t))+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-z)^{-(1-\xi)}\mathbb{L}_3(z,I(z))\,dz,\\ V(t) &= V(0)+\frac{(1-\xi)}{G(\xi)}\mathbb{L}_4(t,V(t))+\frac{\xi}{G(\xi)+\Gamma(\xi)}\int_0^t (t-\eta)^{-(1-\xi)}\mathbb{L}_4(\eta,V(\eta))\,d\eta. \end{aligned} \tag{6. 50}$$

Also, the initial conditions

$$S(0)=E(0)=I(0)=V(0)=0. \tag{6. 51}$$

$$\begin{aligned} S_{n+1} &= S_0+\frac{1-\xi}{G(\xi)}\mathbb{L}_1(t_n,S(t_n))\\ &\quad+\frac{\xi}{G(\xi)}\sum_{k=0}^{n}\left(\frac{h^{\xi}\mathbb{L}_1(t_k,S_k)}{\Gamma(\xi+2)}\left(\ (n+1-k)^{\xi}(n-k+2+\xi)-(n-k)^{\xi}(n-k+2+2\xi)\ \right)\right.\\ &\quad-\frac{h^{\xi}\mathbb{L}_1(t_{k-1},S_{k-1})}{\Gamma(\xi+2)}\Big((n+1-k)^{\xi+1}-(n-k)^{\xi}(n-k+1+\xi)\Big)+{}^1R_n^{\xi}, \end{aligned} \tag{6. 52}$$

$$\begin{aligned} E_{n+1} &= E_0+\frac{1-\xi}{G(\xi)}\mathbb{L}_2(t_n,E(t_n))+\frac{\xi}{G(\xi)}\sum_{k=0}^{n}\left(\frac{h^{\xi}\mathbb{L}_2(t_k,E_k)}{\Gamma(\xi+2)}\left(\begin{array}{c}(n-k+2+\xi)(n+1-k)^{\xi}\\ -(n-k)^{\xi}(n-k+2+2\xi)\end{array}\right)\right.\\ &\quad-\frac{h^{\xi}\mathbb{L}_2(t_{k-1},E_{k-1})}{\Gamma(\xi+2)}\Big((n+1-k)^{\xi+1}-(n-k)^{\xi}(n-k+1+\xi)\Big)+{}^2R_n^{\xi}, \end{aligned} \tag{6. 53}$$

$$\begin{aligned} I_{n+1} &= I_0+\frac{1-\xi}{G(\xi)}\mathbb{L}_3(t_n,I(t_n))+\frac{\xi}{G(\xi)}\sum_{k=0}^{n}\left(\frac{h^{\xi}\mathbb{L}_3(t_k,I_k)}{\Gamma(\xi+2)}\left(\begin{array}{c}(n+1-k)^{\xi}(n-k+2+\xi)\\ -(n-k)^{\xi}(n-k+2+2\xi)\end{array}\right)\right.\\ &\quad-\frac{h^{\xi}\mathbb{L}_3(t_{k-1},I_{k-1})}{\Gamma(\xi+2)}\Big((n+1-k)^{\xi+1}-(n-k)^{\xi}(n-k+1+\xi)\Big)+{}^3R_n^{\xi}, \end{aligned} \tag{6. 54}$$

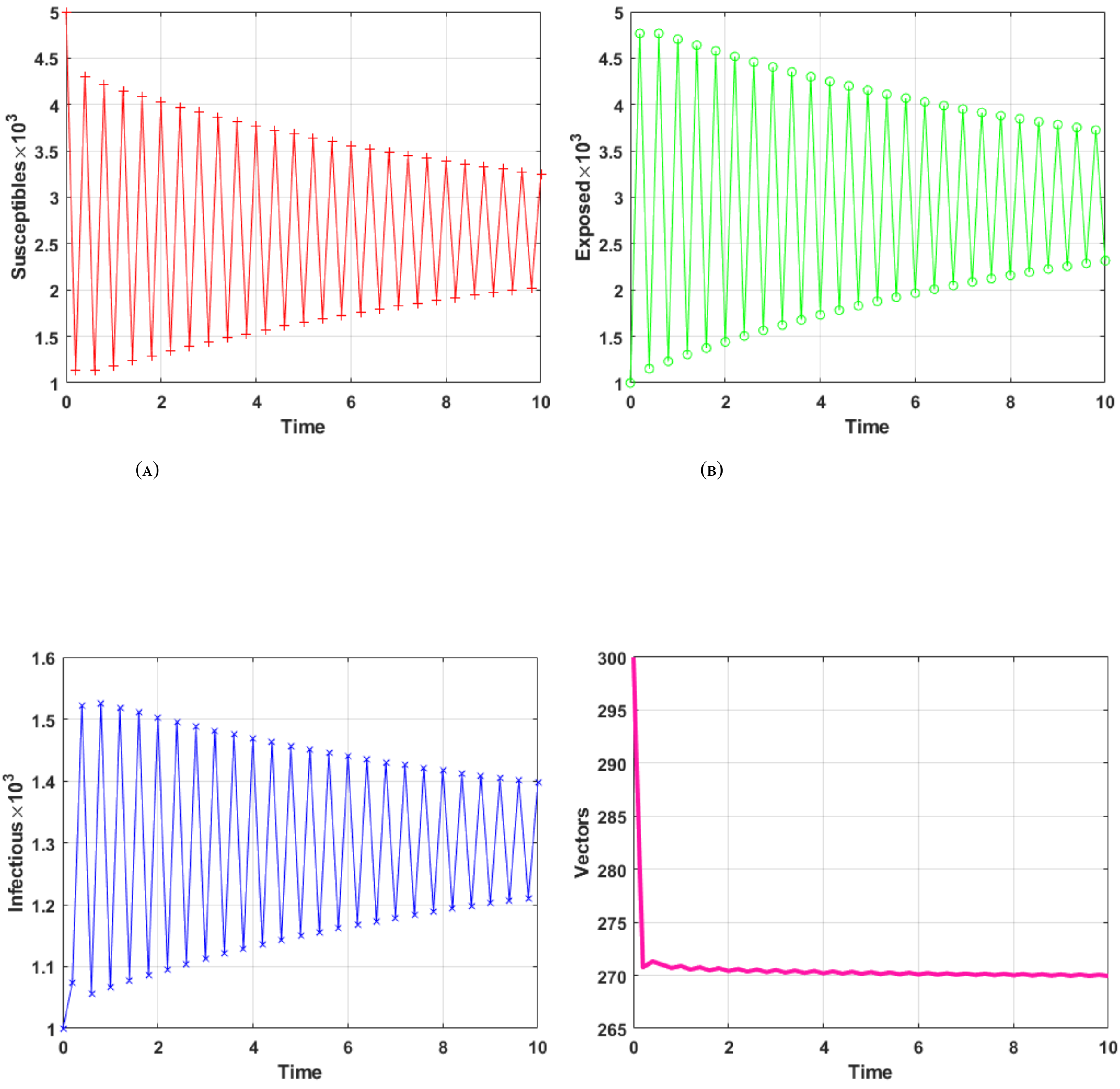
$$\begin{aligned} V_{n+1} &= V_0+\frac{1-\xi}{G(\xi)}\mathbb{L}_4(t_n,V(t_n))+\frac{\xi}{G(\xi)}\sum_{k=0}^{n}\left(\frac{h^{\xi}\mathbb{L}_4(t_k,V_k)}{\Gamma(\xi+2)}\left(\begin{array}{c}(n+1-k)^{\xi}(n-k+2+\xi)\\ -(n-k)^{\xi}(n-k+2+2\xi)\end{array}\right)\right.\\ &\quad-\frac{h^{\xi}K_4(t_{k-1},V_{k-1})}{\Gamma(\xi+2)}\Big((n+1-k)^{\xi+1}-(n-k)^{\xi}(n-k+1+\xi)\Big)+{}^4R_n^{\xi}, \end{aligned} \tag{6. 55}$$

where ${}^iR_n^{\xi}$, $i=1,2,3,4$ are remainders. These have the forms

$$\begin{aligned} {}^1R_n^{\xi} &= \frac{\xi}{G(\xi)\Gamma(\xi)}\sum_{k=0}^{n}\int_{t_k}^{t_{k-1}}\frac{(z-t_{k-1})(z-t_k)}{2}\cdot\frac{\partial^2}{\partial z^2}[\mathbb{L}_1(z,S(z))]_{z=\varepsilon z}(t_{n+1}-z)^{\xi-1}\,dz,\\ {}^2R_n^{\xi} &= \frac{\xi}{G(\xi)\Gamma(\xi)}\sum_{k=0}^{n}\int_{t_k}^{t_{k-1}}\frac{(z-t_{k-1})(z-t_k)}{2}\cdot\frac{\partial^2}{\partial z^2}[\mathbb{L}_2(z,E(z))]_{z=\varepsilon z}(t_{n+1}-z)^{\xi-1}\,dz,\\ {}^3R_n^{\xi} &= \frac{\xi}{G(\xi)\Gamma(\xi)}\sum_{k=0}^{n}\int_{t_k}^{t_{k-1}}\frac{(z-t_k)(z-t_{k-1})}{2}\cdot\frac{\partial^2}{\partial z^2}[\mathbb{L}_3(z,I(z))]_{z=\varepsilon z}(t_{n+1}-z)^{\xi-1}\,dz,\\ {}^4R_n^{\xi} &= \frac{\xi}{G(\xi)\Gamma(\xi)}\sum_{k=0}^{n}\int_{t_k}^{t_{k-1}}\frac{(z-t_{k-1})(z-t_k)}{2}\cdot\frac{\partial^2}{\partial z^2}[\mathbb{L}_4(z,V(z))]_{z=\varepsilon z}(t_{n+1}-z)^{\xi-1}\,dz, \end{aligned} \tag{6. 56}$$

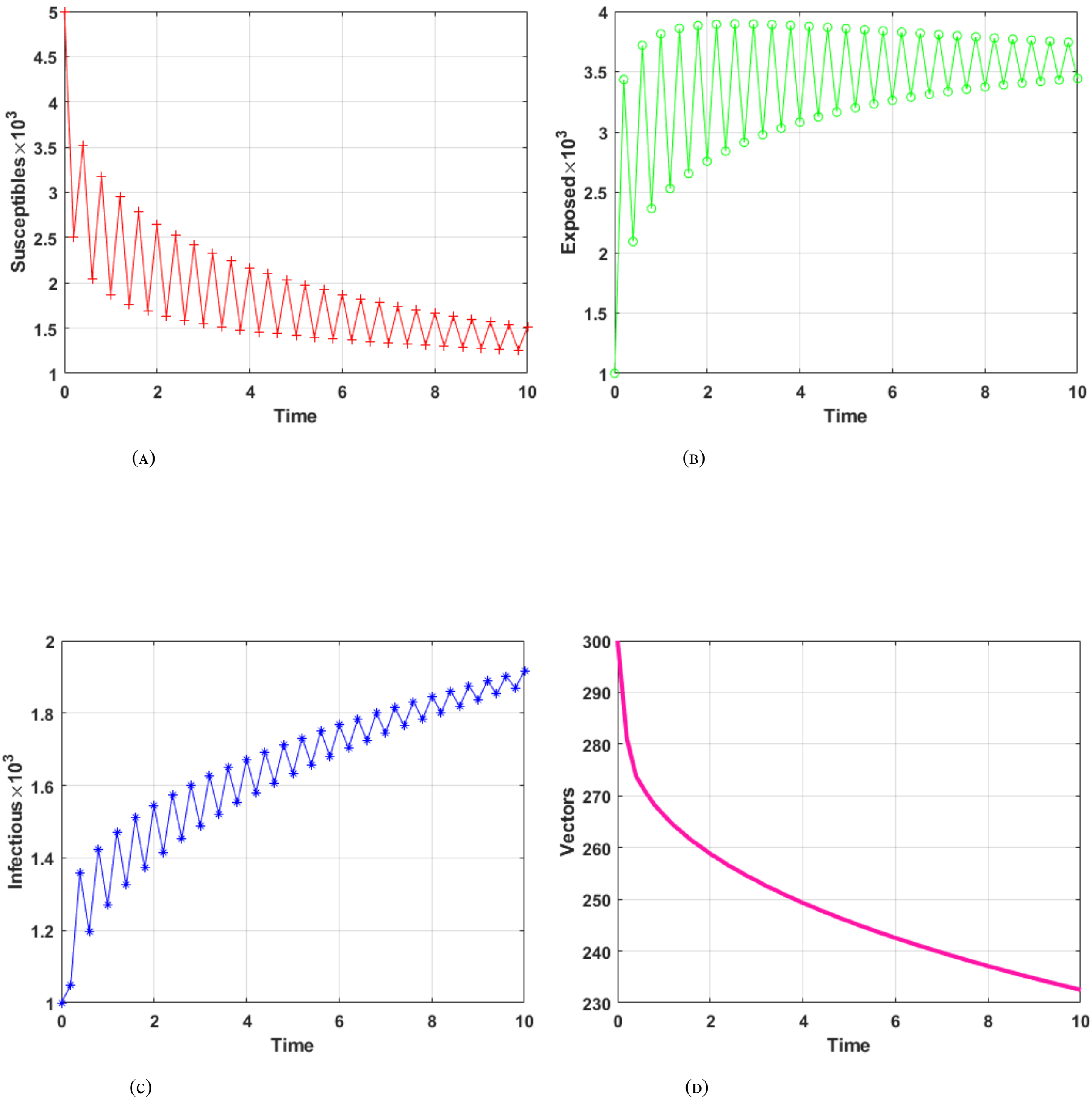
7. Numerical Simulation

We apply the newly developed numerical scheme to the system of equations (5. 13) representing plant disease. We assume different fractional order values for getting the system's solution (5. 13). Initial values are taken as $S(0) = 500, E(0) = 100, I(0) = 10, V(0) = 300.$, and parameter values used for the numerical illustration are given in Table 1. The period for the solutions is ten months. Figure (1) shows the solution of system (5. 13) for $\xi = 0.1$. It can be observed that the plant population shows oscillatory behavior. The osculations are large initially but reduce over time. However, the vector population shows minor oscillation. Similar oscillatory behavior, for $\xi = 0.5$, can be observed in the plant population as shown in Figure (2). It is noted that susceptible plants decrease, showing the oscillation, and seem to approach the constant level. In contrast, exposed and infectious plants increase and approach the constant level in the long run. Vector population decreases exponentially, and the phenomena of sudden declination and approach to the constant level, as we see in the previous case, disappear for this value of ξ. Figure (3) shows the solution of plant and vector population taking $\xi = 1$. We can see that the susceptible population decreases exponentially, and infectious plants grow where, whereas the exposed plants decrease after approaching the maximum level. The vector population declines almost linearly. However, oscillation does not occur in any class of the plant population. Figures (4), (5) and (6) show the plants and vector population for different initial values taking $\xi = 0.1, 0.5$ and 1, respectively. Figure (4) shows that all compartments of plants and vector population approach the constant level irrespective of initial conditions. Solutions of the susceptible and exposed classes show more oscillations than the infectious class of the plants and vector class. Different behavior of the solutions can be observed as we take $\xi = 0.5$ and $\xi = 1$. All the above graphical results, we conclude that for smaller value of the fractional order ξ (ξ=1), strong oscillation occurs taking the system longer time to stabilize. Increasing the value of ξ to 0.5, the oscillations are milder and decay faster showing faster stabilization. For the integral value of fractional order, all the oscillations vanish and the system show exponential type trends. When different initial conditions are introduced, the magnitude and shape of the trajectories vary, but the qualitative behavior remains consistent for a given ξ. This implies that while initial population size affect outbreak intensity, the intrinsic system dynamics, particularly oscillatory behavior controlled by ξ. It plays a dominant role in determining whether the model exhibits oscillatory epidemics or smooth convergence to equilibrium.



(A) (B) (C) (D)

FIGURE 1. Solution of the system (5. 13) for $\xi = 0.1$.



(A) (B) (C) (D)

FIGURE 2. Solution of the system (5. 13) for $\xi = 0.5$.

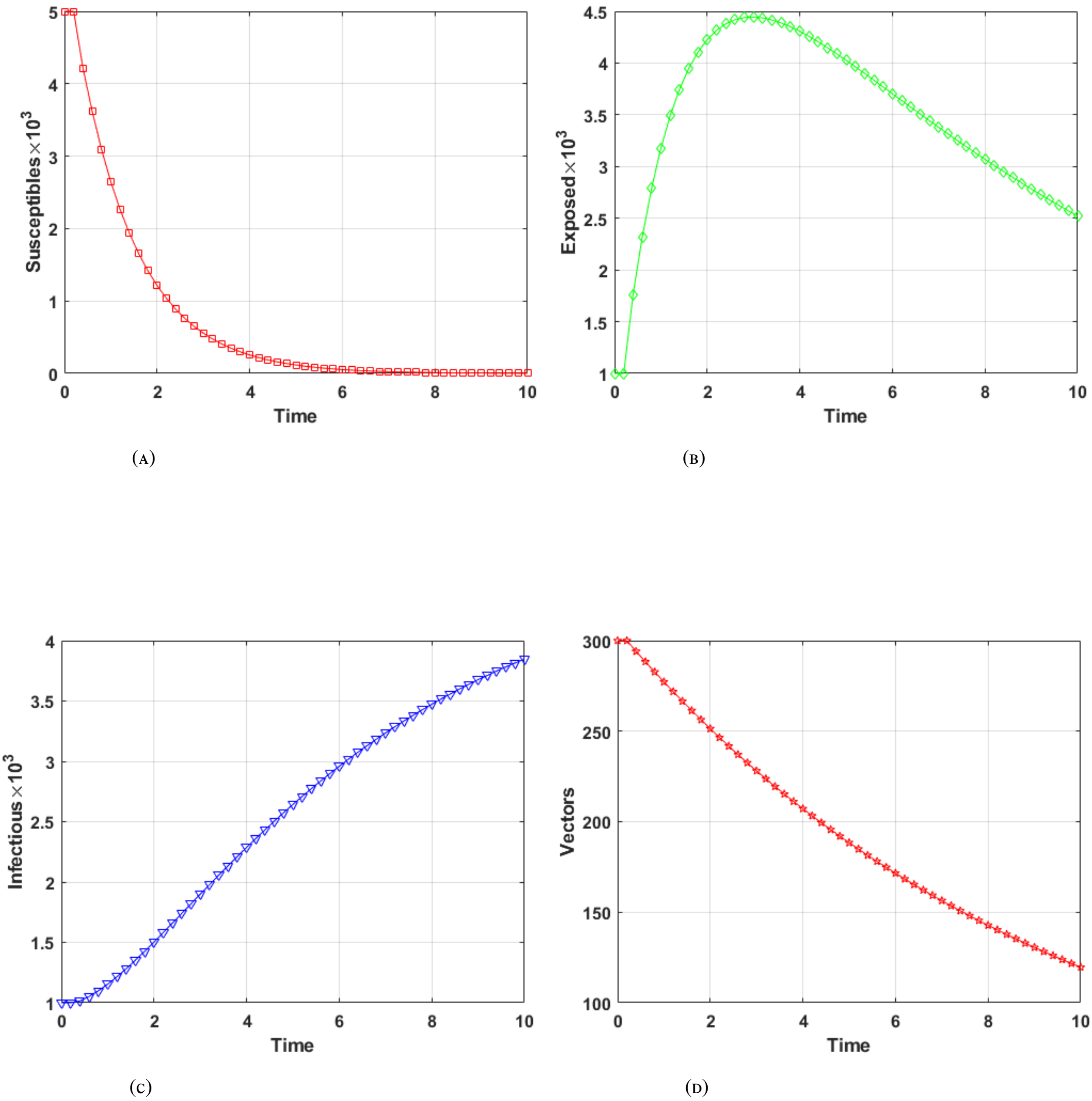


FIGURE 3. Solution of the system (5. 13) for $\xi = 1$.

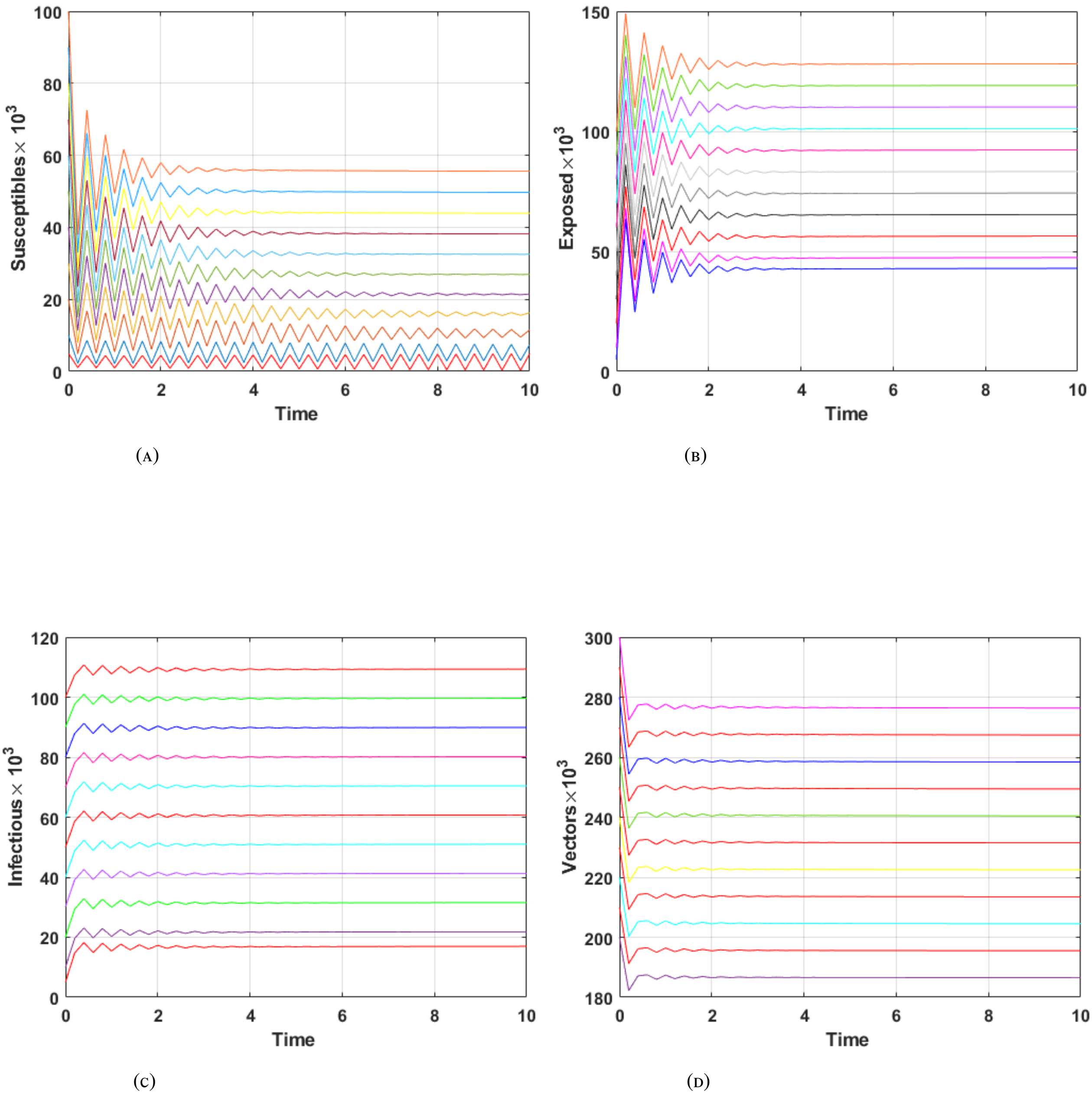


Figure 4. Solution of the system (5. 13) for $\xi = 0.1$ with respect to different initial conditions.

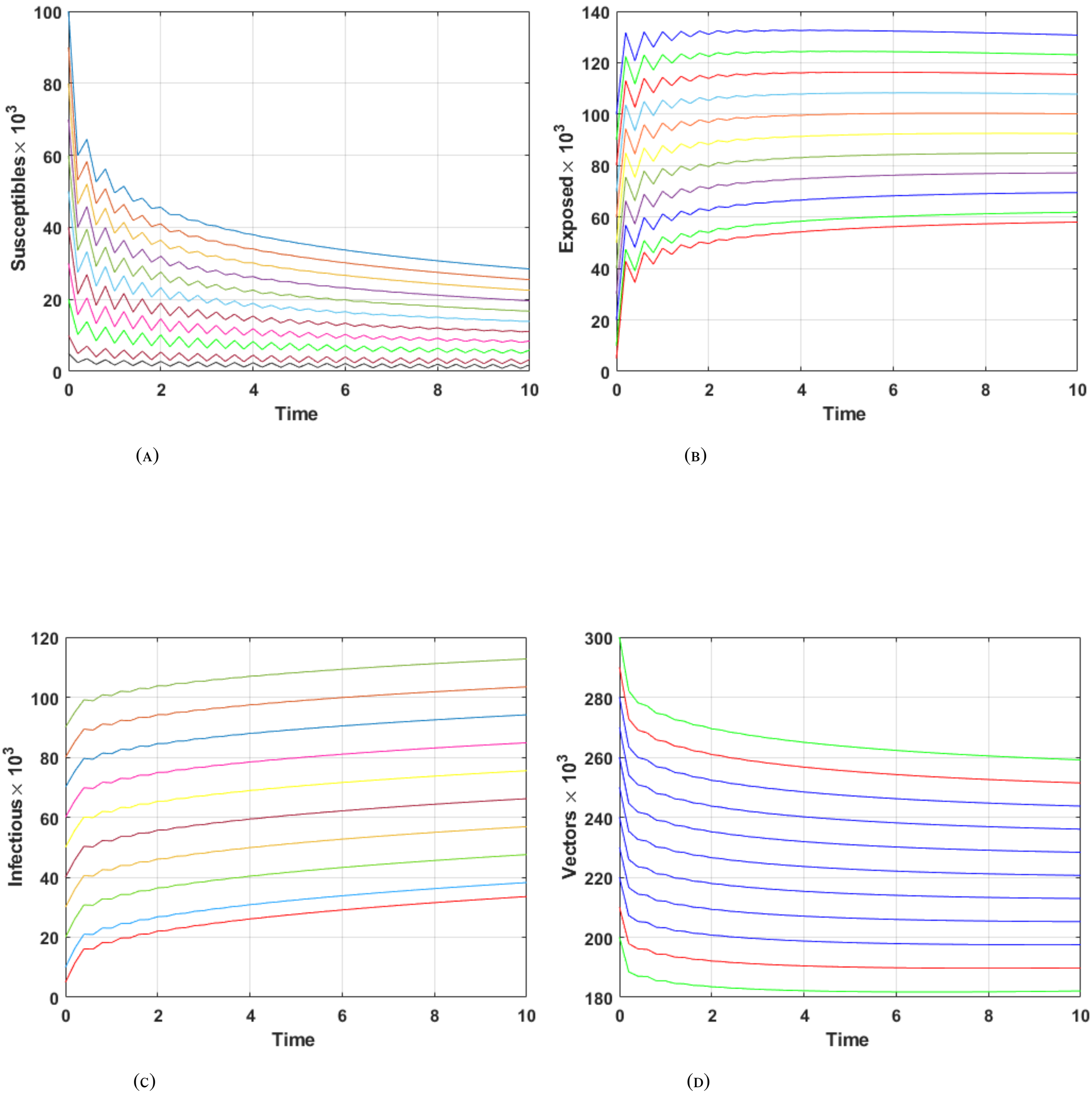


FIGURE 5. Solution of the system (5. 13) for $\xi = 0.5$ with respect to different initial conditions.

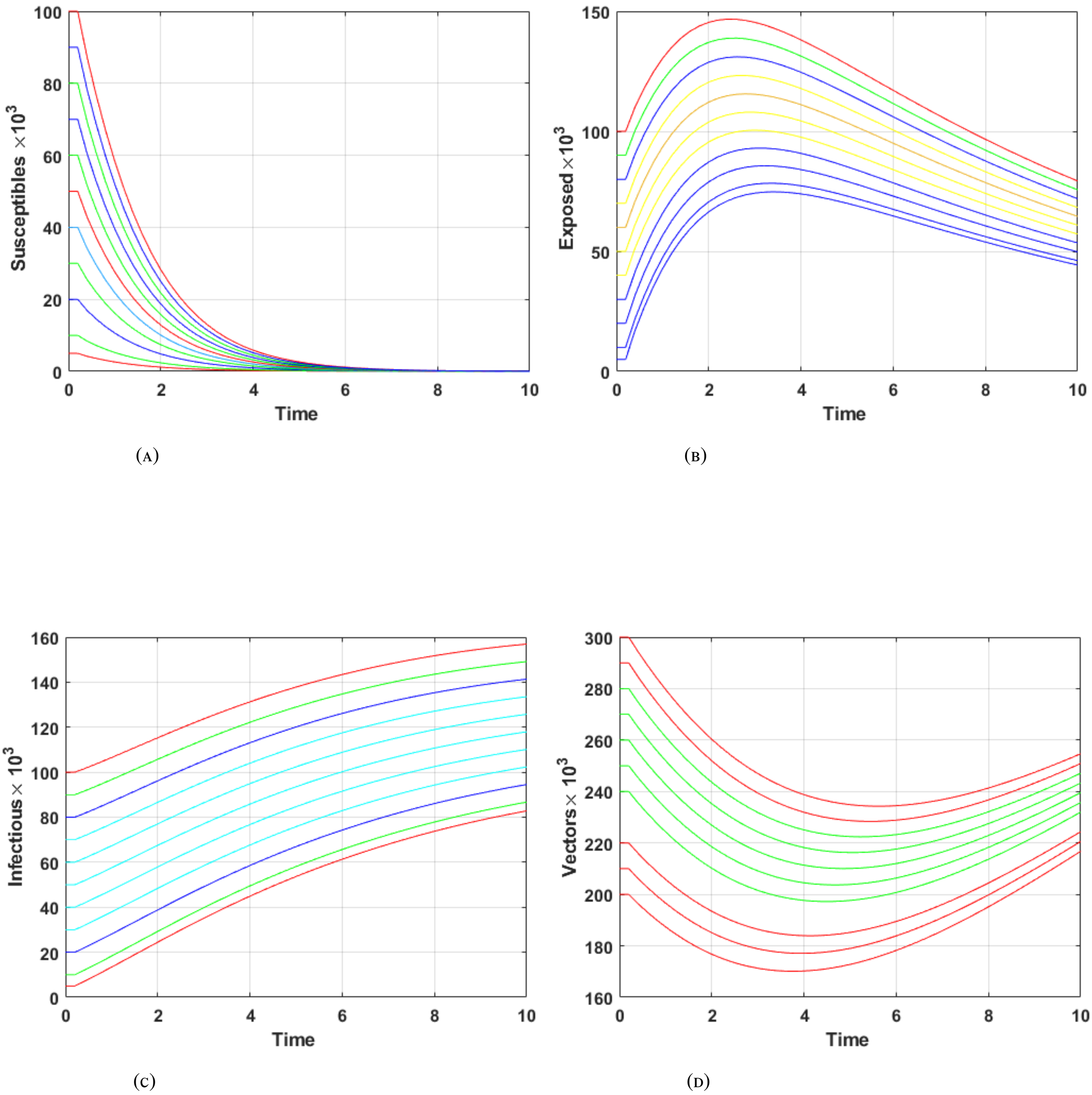


FIGURE 6. Solution of the system (5. 13) for $\xi = 1$ with respect to different initial conditions.

8. CONCLUSION

In this work, we studied a vector-borne plant disease model with the Beddington-DeAngelis Incidence function using the technique of fractional differentiation. This study relies on the non-local kernel without singularity assumption recently offered by Atangana and Baleanu. The existence of solutions has been proved by using the existing method as the fixed-point theorem for solving the nonlinear ordinary differential equations with non-local fading memory. Furthermore, we developed a numerical scheme developed by Atangana and Toufik. The development of a numerical scheme took place while considering the stated assumptions. The effectiveness of the numerical scheme has been displayed through the numerical simulations. We have shown that the results obtained through the numerical simulations verify the analytical ones derived in the existence and uniqueness sections. It can be seen that the variation of the fractional variable ξ or the initial values of the variable show the different behavior of solutions. Furthermore, the oscillatory behavior of solutions was observed using the fractional derivative, which is missing in the case of ordinary derivatives. The parameter ξ, which influences the oscillatory behavior of the epidemic model, plays a critical role in determining the stability of the system. Its lower values represent persistent oscillations in disease compartments, representing delayed behavioral responses, weak health interventions and time-lagged immunity. Its higher values result in smooth, monotonic trends toward equilibrium indicating that timely and effective interventions can remove oscillatory behavior of the solutions. These insights suggest that enhancing ξ through faster detection, efficient quarantine and vector control can significantly reduce epidemic fluctuations and reduce long-term disease burden.

Declarations

Ethical approval: The authors affirm their commitment to ethical standards.

Conflict of Interest: There is no conflicts of interest to declare.

Data Availability: All the data used during this study is accessible within the manuscript.

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