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On the Numerical Evaluation for Studying the Time Fractional Partial Delay Differential Equations

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Abstract. An important challenge in mathematical modeling is to find a model that captures the fundamental physics of a system and is simple enough to allow for mathematical analysis. In the physical sciences, physiology, ecology, and other practical research domains, fractional delay differential equations (FDDEs) are frequently used. Most fractional delay differential equations can only be solved numerically because they lack analytic solutions. In contrast to the Adomian decomposition method, a new method for solving delay differential equations called the new Fractional Novel Analytical Scheme (FNAS) is presented in this paper. A fractional novel analytical scheme is built on the fractional Taylor series. The calculation is done using the Caputo derivative. Three well-known physical models such as advection-dispersion equation of fractional-order, non-linear gas-dynamics equation of fractional-order and convection-diffusion equation of fractional-order with proportional time-delay are solved by using the proposed technique to demonstrate the performance and efficiency of the FNAS. By graphing absolute error values and contrasting results to numerous existing solutions, the correctness of the proposed technique is presented. In addition to being straightforward, the suggested strategy is accurate and logical in the difficulties it solves.

AMS (MOS) Subject Classification Codes: 26A33; 35R11
Key Words: Fractional Calculus, Fractional Partial Delay Differential Equations, Fractional Taylor Series, Fractional Novel Analytical Scheme (FNAS).

1. Introduction

Applications for DDEs (Delay Differential Equations) in engineering and research are numerous. The DDE, which is data-dependent and applied to physical systems, simplifies the ordinary differential equation. Nowadays, FDDEs are the subject of greater research than differential equations since even a slight delay may have a big effect. Fractional derivatives and temporal delays are used in FDDEs. Time-delays reveal the past of a previous state, although fractional derivatives are non-local and can imitate memory effects. Fractional derivatives and delays can be used to represent real-world circumstances better. Numerous other domains, including population dynamics, bioengineering, electrochemistry control systems, and physics, also use FDDEs [3, 7, 4, 5, 6, 9]. There are several pertinent studies in the [15, 33, 37, 36, 26, 27, 28] series. The importance of fractional-order delay models has increased as a result, and in recent years, this field of study has become increasingly interdisciplinary. Many academics apply numerical and analytical techniques to analyze the FDDE.

Neutral functional proportional DDEs were solved using the replicating kernel Hilbert space method by Lv and Gao [16]. The fractional-order delay model was studied using Jacobi polynomials in [23]. The authors in [21] used Bernoulli wavelets to solve the DDEs. The Hermite wavelet and Chebyshev wavelet were used to examine the approximate solution of the fractional delay model in [22, 10], respectively. Haar wavelets were utilized to solve the delay model system [14]. Ali et al. employed the spectral collocation approach to solve the fractional delay model [2]. In [32], Wang solved the FDDE problem utilizing the Runge-Kutta type approach. Later, artificial neural networks were employed to address this issue [39]. The publications listed in [8, 17, 30, 31, 18, 11] provide additional insights and strategies for solving fractional delay models.

Partial functional differential equations represent a particular category of proportional delay partial differential equations. Various contexts, including biology, medicine, population ecology, control systems, and climate models, use equations of this type [35]. Their independent variables are time t and one or more dimensional variables s, which frequently refer to a cell's size, position in space, or other characteristics. The answers might be the substitutes for voltage, temperature, or the densities of other particles, such as chemicals, cells, and so on. Partial delay differential equations (PDDEs) can be solved numerically using relatively novel methods. Zubik-Kowal [38] solved differential-functional parabolic equations and linear differential equations using the Chebyshev pseudospectral method. Zubik-Kowal and Jackiewicz [12] employed waveform relaxation and spectral collocation methods to solve nonlinear PDDEs. In [1], authors introduced the two-dimensional differential transform method and its condensed variant as a solution for PDDEs. Tanthanuch [29] utilized the group analysis method to resolve the Burgers DDE. Polyanin and Zhurov [20] proposed employing the functional constraints technique as a means to generate exact solutions for the reaction-diffusion delay equations.

The current study applies the Fractional Novel Analytical Scheme (FNAS) to obtain approximate solutions to FDDEs, both linear and nonlinear cases. This approach was first examined by [34] in her PhD thesis. Wiwatwanich intended to modify the truncated Taylor series to solve ordinary and partial differential equations. Authors of [25, 24] developed this method for fractional differential equations by using the Fractional Taylor Series, which is an effective tool for solving nonlinear equations in this technique. The suggested approach yields Taylor Series solutions by combining the linear and nonlinear parts and then proceeds using the calculus of many variables. Compared to the Adomian Decomposition Method (ADM), this new variation is believed to be more straightforward to grasp. The classical ADM splits a function into two parts, linear and nonlinear parts, which can be approximated using Adomian polynomials. However, this process often involves complex computations. This study proposes a method to approximate the entire operator if $\mathscr{K}[\phi(s,t)]$ is analytical in its arguments. The operator is treated as an implicit function of t and expanded its Taylor series around $t = 0$. This approach improves the efficiency of the ADM process.

On the other hand, a comprehensive survey indicates that the DDEs of fractional-order has not yet been examined using this approach. FNAS avoids iterative procedures and matrix computations commonly required in other numerical methods. It significantly reduces computation time, offering quick convergence with just a few terms in the series, even for complex and highly nonlinear systems. These are the motivations for the current study to explore solving advection-dispersion equation of fractional-order, nonlinear gas-dynamics equation of fractional-order and convection-diffusion equation of fractional-order with proportional time-delay. This research includes a variety of test problems. This strategy has shown to be incredibly successful because of its great accuracy, convergence, and adaptability. Notably, it does not require linearization or other type of modification. However, despite its merits, the approach may have limited utility. It might not be suited for certain types of nonlinear Fractional Partial Differential Equations (FPDEs), particularly those with singularities, discontinuities, or specified boundary conditions. It is imperative to use caution when applying this strategy to different FPDE settings due to its potential limitations.

The rest of the paper is organized as follows: Section 2 contains preliminary information and notations about fractional calculus. Section 3 describes a new, revolutionary numerical approach (FNAS) for FDDEs, and its error analysis is discussed in Section 4. Section 5 provides several exemplary instances with a discussion of results in Section 6. Finally, Section 7 concludes the research paper.

2. Preliminaries and Notations

In this part, we present fundamental definitions and properties of fractional calculus.

Definition 1. The fractional integral J_t^γ is defined as, $\gamma > 0$

$$J_t^\gamma \phi(s,t) = \frac{1}{\Gamma\gamma}\int_0^t (t-\eta)^{\gamma-1}\phi(s,\eta)d\eta;\ \eta > 0, n-1 < \gamma \leq n, n \in \mathbb{N}. \tag{1}$$

Definition 2. The Caputo time-fractional derivative of ϕ is defined as:

$$D_t^\gamma \phi(s,t) = \left\{ \begin{array}{ll} \frac{1}{\Gamma[n-\gamma]}\int_0^t (t-\eta)^{n-\gamma-1}\frac{\partial^n \phi(s,\eta)}{\partial \eta^n}d\eta = J_t^{n-\gamma}D_t^n\phi(s,t), & n-1<\gamma<n \\ \frac{\partial^n \phi(s,t)}{\partial t^n}, & \gamma = n \end{array} \right\} \tag{2}$$

here $\frac{\partial^n \phi(s,t)}{\partial t^n}$ denotes n^{th}-order partial derivatives, for further details, see [19].

Attributes of CFD (Caputo Fractional Derivative):
For CFD the following properties exist.

(i) $D_t^\gamma t^n = \frac{\Gamma(1+n)}{\Gamma(1+n-\gamma)}t^{n-\gamma}, \quad n > 0;$

(ii) $D_t^\gamma\left(c\phi(s,t)\right) = c\left(D_t^\gamma \phi(s,t)\right);$
(iii) $D_t^\gamma\left(a\phi(s,t) + b\psi(s,t)\right) = aD_t^\gamma \phi(s,t) + bD_t^\gamma \psi(s,t);$
(vi) $D_t^\gamma c = 0,$

where a, b and c are constants in these equations. Caputo fractional derivative will be used in this research.

Definition 3. A fractional product rule [19] can be defined as:

$$D_t^\gamma\left[\phi(s,t)\psi(s,t)\right] = \sum_{n=0}^{\infty}\binom{\gamma}{n}\left[D_t^{\gamma-n}\phi(s,t)\right]\left[D_t^\gamma \psi(s,t)\right], \tag{3}$$

when $n > \gamma$, the term $D_t^{\gamma-n}$ is called a fractional integral and is also represented by $J_t^{n-\gamma}$.

3. Proposed Scheme for Fractional Partial Delay Differential Equation

We will discuss the general fractional partial delay differential equation to be written as

$$D_t^\gamma(\phi(s,t)) = \mathscr{K}\left(s, \phi(q_0 s, k_0 t), \frac{\partial}{\partial s}\phi(q_1 s, k_1 t), \cdots, \frac{\partial^n}{\partial s^n}\phi(q_n s, k_n t)\right), \tag{4}$$

with initial condition

$$\phi(s,0) = \phi_0(s). \tag{5}$$

Here $D_t^\gamma(\phi(s,t))$ is the Caputo derivative of order γ, $n-1 < \gamma < n$, s is space variable, t is the time (s and t are independent variables), ϕ_0 is a primary value, $q_p, k_p \in (0,1) \forall p \in \mathbb{N} \cup 0$ indicates delay parameter and $\mathscr{K}$ is the differential operator. Equation (4) can be written as

$$D_t^\gamma(\phi(s,t)) = \mathscr{K}[\phi(s,t)],$$

where $\mathscr{K}[\phi(s,t)] = \mathscr{K}\left(s, \phi(q_0 s, k_0 t), \frac{\partial}{\partial s}\phi(q_1 s, k_1 t), \cdots, \frac{\partial^n}{\partial s^n}\phi(q_n s, k_n t)\right)$. By taking fractional integral J_t^γ on both sides of (4), the equation will be

$$\phi(s,t) = \phi_0(s) + J_t^\gamma \mathscr{K}[\phi(s,t)], \tag{6}$$

The fractional Taylor series is extended for $\mathscr{K}[\phi(s,t)]$ about the point $t = t_0$, which is

$$\mathscr{K}[\phi(s,t)] = \mathscr{K}[\phi(s,t_0)] + \sum_{p=1}^{\infty} D_t^{p\gamma}\mathscr{K}[\phi(s,t_0)]\frac{(t-t_0)^{p\gamma}}{\Gamma(p\gamma+1)}. \tag{7}$$

Put $t_0 = 0$, then (7) can be written as

$$\begin{aligned}\mathscr{K}[\phi(s,t)] =& \mathscr{K}[\phi(s,0)] + D_t^\gamma \mathscr{K}[\phi(s,0)]\frac{t^\gamma}{\Gamma(\gamma+1)} + D_t^{2\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \\ & D_t^{3\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots + D_t^{p\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{p\gamma}}{\Gamma(p\gamma+1)} + \cdots. \end{aligned} \tag{8}$$

Substituting (8) by (6), we get

$$\begin{aligned}\phi(s,t) = \phi_0(s) + J_t^\gamma\Big[& \mathscr{K}[\phi(s,0)] + D_t^\gamma \mathscr{K}[\phi(s,0)]\frac{t^\gamma}{\Gamma(\gamma+1)} + D_t^{2\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \\ & D_t^{3\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots + D_t^{p\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{p\gamma}}{\Gamma(p\gamma+1)} + \cdots\Big].\end{aligned}$$

Taking fractional integration J_t^γ and solving then we get

$$\begin{aligned}\phi(s,t) =&\phi_0(s) + \mathscr{K}[\phi(s,0)]\frac{t^\gamma}{\Gamma(\gamma+1)} + D_t^\gamma \mathscr{K}[\phi(s,0)]\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + D_t^{2\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \\ &+ \cdots + D_t^{(p-1)\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{p\gamma}}{\Gamma(p\gamma+1)} + D_t^{p\gamma}\mathscr{K}[\phi(s,0)]\frac{t^{(p+1)\gamma}}{\Gamma((p+1)\gamma+1)} + \cdots,\end{aligned} \tag{9}$$

$$\begin{aligned}\phi(s,t) = a_0 + a_1\frac{t^\gamma}{\Gamma(\gamma+1)} + a_2\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + a_3\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots + a_p\frac{t^{p\gamma}}{\Gamma(p\gamma+1)} \\ + a_{(p+1)}\frac{t^{(p+1)\gamma}}{\Gamma((p+1)\gamma+1)} + \cdots,\end{aligned} \tag{10}$$

where

$$\begin{aligned} a_0 &= \phi_0(s) = \phi(s,0), \\ a_1 &= \mathscr{K}[\phi(s,t_0)] = D_t^\gamma \phi(s,0), \\ a_2 &= D_t^\gamma \mathscr{K}[\phi(s,t_0)] = D_t^{2\gamma}\phi(s,0), \\ a_3 &= D_t^{2\gamma}\mathscr{K}[\phi(s,t_0)] = D_t^{3\gamma}\phi(s,0), \\ &\vdots \\ a_p &= D_t^{(p-1)\gamma}\mathscr{K}[\phi(s,t_0)] = D_t^{p\gamma}\phi(s,0)\end{aligned} \tag{11}$$

such that p is the highest derivative of ϕ. Hence, the desired solution will be obtained by substituting these values in (10).

4. Convergence Analysis of Proposed Technique

To prove the convergence of the resultant series.

$$\sum_{i=0}^{p}\phi_i = \sum_{i=0}^{p} a_i\frac{t^{i\gamma}}{\Gamma(i\gamma+1)}, \tag{12}$$

with exact solution $\phi(s,t)$. The investigated equation written in the form

$$D_t^\gamma(\phi(s,t)) = \mathscr{K}\left(s, \phi(q_0 s, k_0 t), \frac{\partial}{\partial s}\phi(q_1 s, k_1 t), \cdots, \frac{\partial^n}{\partial s^n}\phi(q_n s, k_n t)\right), \ 0 < \gamma \le 1, \tag{13}$$

Theorem 4. *Let $\mathscr{K}$ be an operator from $\chi \to \chi$, where χ is the Hilbert space. Let the exact solution of* (13) *be ϕ. The estimated solution* (12) *is converging to ϕ, when there exists a constant δ, here $0 < \delta \le 1$ in which $\|\phi_{p+1}(s,t)\| \le \delta\|\phi_p(s,t)\|$ for all $p \in \mathbb{N}\cup\{0\}$.*

Proof of Theorem 4: To prove that $|\phi_i|_{i=0}^{\infty}$ is converging Cauchy sequence,

$$\|\phi_{i+1} - \phi_i\| = \|\phi_{i+1}\| \le \delta\|\phi_i\| \le \delta^2\|\phi_{i-1}\| \le \cdots \le \delta^i\|\phi_1\| \le \delta^{i+1}\|\phi_0\|, \tag{14}$$

now for $i, m \in \mathbb{N}, i > m$, we obtain

$$\begin{aligned}
\|\phi_i - \phi_m\| &= \|(\phi_i - \phi_{i-1}) + (\phi_{i-1} - \phi_{i-2}) + \cdots + (\phi_{m+1} - \phi_m)\| \\
&\leq \|\phi_i - \phi_{i-1}\| + \|\phi_{i-1} - \phi_{i-2}\| + \cdots + \|\phi_{m+1} - \phi_m\| \\
&\leq \delta^i \|\phi_0(s)\| + \delta^{i-1}\|\phi_0(s)\| + \cdots + \delta^{m+1}\|\phi_0(s)\| \\
&\leq \left(\delta^i + \delta^{i-1} + \cdots + \delta^{m+1}\right) \|\phi_0(s)\| \\
&= \delta^{m+1} \frac{1 - \delta^{i-m}}{1 - \delta} \|\phi_0(s)\|.
\end{aligned} \tag{15}$$

Left hand side of equation (15) approaches to zero as $i, m \to \infty$.

Hence $|\phi_i|_{i=0}^{\infty}$ is a convergent Cauchy sequence in χ.

5. Numerical Application of the Proposed Method

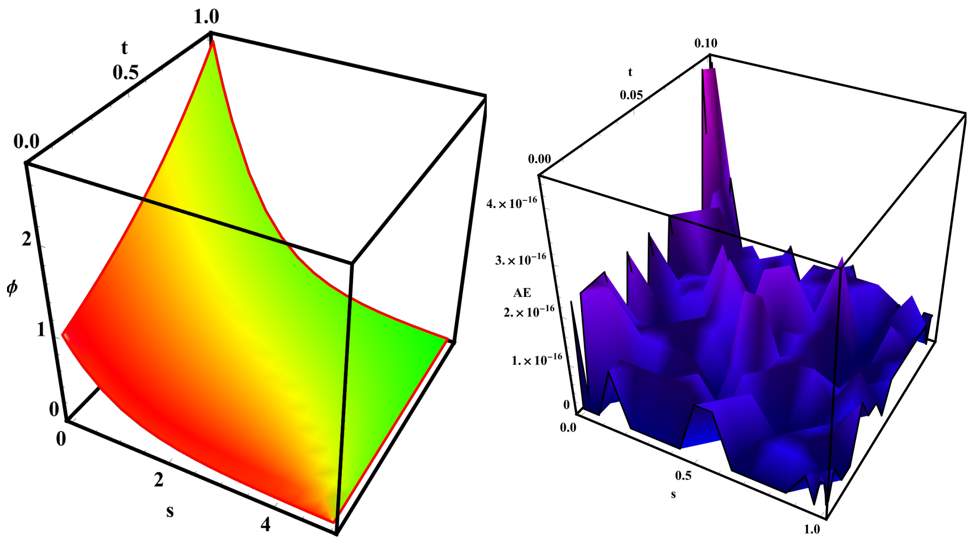
In this section, three well-known physical models such as fractional advection-dispersion equation, nonlinear fractional gas-dynamics equation and fractional convection-diffusion equation with proportional time-delay are solved by using the proposed technique to demonstrate the performance and efficiency of the FNAS. Wolfram Mathematica 13 has been used for numerical calculations and graphics in all examples.

Example 1. The fractional advection-dispersion equation is employed in groundwater hydrology and is a reliable method to simulate the transport of passive tracers carried. Consider the following IVP known as proportional time-delay advection-dispersion equation of fractional-order [13]:

$$\frac{\partial^\gamma \phi(s, kt)}{\partial t^\gamma} + v\frac{\partial \phi(s,t)}{\partial s} - d\frac{\partial^2 \phi(s,t)}{\partial s^2} = 0, \tag{16}$$

with initial condition $\phi(s, 0) = E_1(-s)$, where ϕ is the solute concentration, k is a proportional time-delay parameter, the positive constants v and d are the average fluid velocity and the dispersion coefficient, respectively, E_1 is the Mittag-Leffler function, s represent the spatial domain, and t is the time. The specific procedure for achieving the objectives that are wanted is given in Section 3. By thoroughly following these steps, we are able to acquire a comprehensive series of solutions. Each step builds upon the preceding one, offering a systematic approach to problem-solving. This technique ensures that every component of the process is covered, leading to exact and effective results.

$$\begin{aligned}
\phi(s,t) &= e^{-s} + \frac{t^\gamma (de^{-s}k^{-\gamma} + ve^{-s}k^{-\gamma})}{\Gamma(1+\gamma)} + \frac{t^{2\gamma}}{\Gamma(1+2\gamma)}\Big(\frac{d^2 e^{-s}k^{-2\gamma}t^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{2dve^{-s}k^{-2\gamma}t^{1-\gamma}}{\Gamma(2-\gamma)} \\
&+ \frac{v^2 e^{-s}k^{-2\gamma}t^{1-\gamma}}{\Gamma(2-\gamma)}\Big) + \frac{t^{3\gamma}}{\Gamma(1+3\gamma)}\Big(\frac{d^3 e^{-s}k^{-3\gamma}t^{3-3\gamma}}{\Gamma(4-3\gamma)} + \frac{3vd^2 e^{-s}k^{-3\gamma}t^{3-3\gamma}}{\Gamma(4-3\gamma)} \\
&+ \frac{3dv^2 e^{-s}k^{-3\gamma}t^{3-3\gamma}}{\Gamma(4-3\gamma)} + \frac{v^3 e^{-s}k^{-3\gamma}t^{3-3\gamma}}{\Gamma(4-3\gamma)}\Big) + \frac{t^{4\gamma}}{\Gamma(1+4\gamma)}\Big(\frac{d^4 e^{-s}k^{-4\gamma}t^{6-6\gamma}}{\Gamma(7-6\gamma)} \\
&+ \frac{4vd^3 e^{-s}k^{-4\gamma}t^{6-6\gamma}}{\Gamma(7-6\gamma)} + \frac{6d^2v^2 e^{-s}k^{-4\gamma}t^{6-6\gamma}}{\Gamma(7-6\gamma)} + \cdots\Big) + \cdots.
\end{aligned} \tag{17}$$



(A) 3D-view of obtained solution by FNAS to solve (16) at $k = \gamma = v = 1$ and $d = \frac{1}{100}$.

(B) Absolute error of (16) is presented in this figure at $\gamma = v = k = 1$ and $d = \frac{1}{100}$.

FIGURE 1. Three-dimensional graphical analysis of time-fractional advection-dispersion equation is depicted here. Also the AE graph is presented to show the increase in accuracy of FNAS.

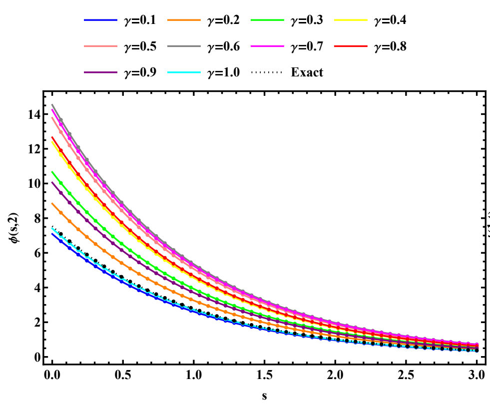
It is noteworthy to remark here that in the integer–space that is $\gamma \to 1$ and dismissing the effect of the time-delay that is $k \to 1$, we obtain the exact solution $\phi_E(s,t) = e^{-s+(v+d)t}$ at $\gamma = 1$.

Example 2. The conservation laws that apply to engineering procedures, such as the conservation of mass, the conservation of momentum, the conservation of energy, etc., are mathematically expressed in the gas-dynamics equation. Three different types of nonlinear waves, including shock fronts, rarefactions, and contact discontinuities, can be described using the nonlinear equations of ideal gas dynamics. Consider the following IVP known as nonlinear proportional time-delay gas-dynamics equation of fractional-order [13]:

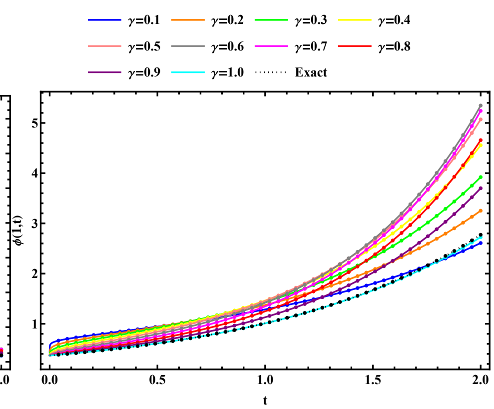
$$\frac{\partial^\gamma \phi(s,kt)}{\partial t^\gamma} + \phi(s,t)\frac{\partial \phi(s,t)}{\partial s} = \phi(s,t)\left(1 - \phi(s,t)\right), \tag{18}$$

with initial condition $\phi(s,0) = E_1(-s)$, here E_1 is the Mittag-Leffler function. To implement the suggested method, the numerical approximation of FDDE in the form of series, is as follows.

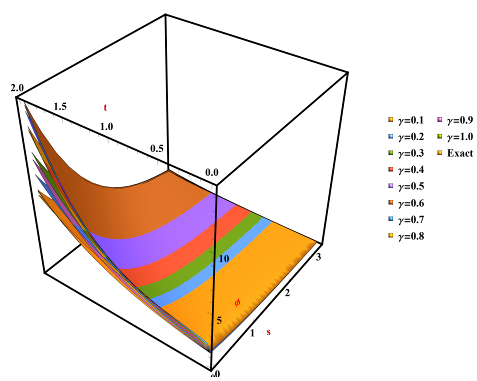
$$\begin{aligned}\phi(s,t) = e^{-s} &+ \frac{e^{-s}k^{-\gamma}t^\gamma}{\Gamma(1+\gamma)} + \frac{e^{-s}k^{-2\gamma}t^{1+\gamma}}{\Gamma(2-\gamma)\Gamma(2\gamma+1)} + \frac{e^{-s}k^{-3\gamma}t^3}{\Gamma(4-3\gamma)\Gamma(1+3\gamma)} \\ &+ \frac{e^{-s}k^{-4\gamma}t^{6-2\gamma}}{\Gamma(7-6\gamma)\Gamma(1+4\gamma)} + \frac{e^{-s}k^{-5\gamma}t^{10-5\gamma}}{\Gamma(11-10\gamma)\Gamma(1+5\gamma)} + \cdots .\end{aligned} \tag{19}$$



(A) 2D-graphical representation of fractional behaviour of (16) with delay at $k = 1, t = 2, v = 1$ and $d = \frac{1}{100}$.



(B) 2D-graphical view of fractional behaviour of (16) with delay at $k = s = v = 1$ and $d = \frac{1}{100}$.

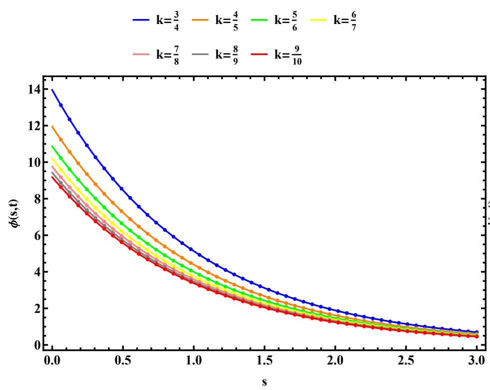


(C) 3D-view of fractional behaviour of (16) with delay at $k = s = v = 1$ and $d = \frac{1}{100}$.

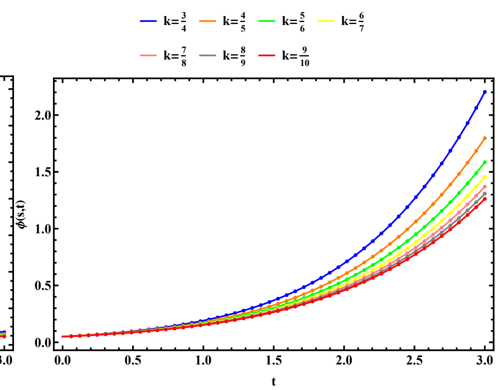
FIGURE 2. Two- and three-dimensional graphical analysis of fractional behavior for fractional advection-dispersion equation with proportional time-delay in (16) at different γ values between 0 and 1.

It is important to note that if we do not take into account the time-delay effect by setting k approach to 1, we obtain the exact solution $\phi_E(s,t) = e^{-s+t}$ with $\gamma = 1$.

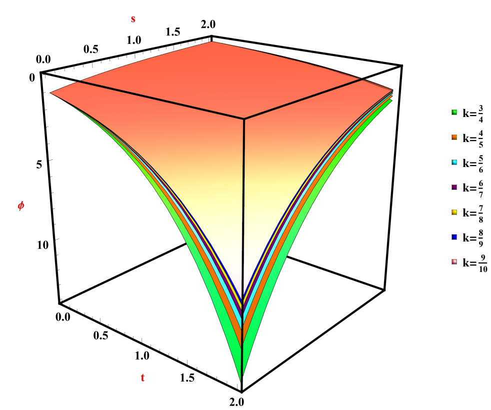
Example 3. The convection-diffusion equation is a mathematical representation of physical phenomena in which energy,particles, or other physical quantities are transported inside a physical system due to the combined effects of diffusion and convection. Consider the following IVP known as nonlinear proportional time-delay convection-diffusion equation



(A) 2D-graphical representation of (16) with varying delay term that is for different values of k between zero and one with $t = 2, \gamma = v = 1, d = \frac{1}{100}$, which clearly shows the variation in time-delay of the numerical solution.



(B) 2D-graphical view of (16) with varying delay term that is for different values of k between zero and one at $s = 3, \gamma = v = 1, d = \frac{1}{100}$. This obviously displays the numerical solution's time-delay's fluctuations.



(C) 3D-view of view of (16) with varying delay term i.e., for different values of k between 0 and 1 at $\gamma = v = 1, d = \frac{1}{100}$, which demonstrates the numerical solution's fluctuating time-delay.

FIGURE 3. Two- and three-dimensional graphical analysis of fractional advection-dispersion equation with varying delay terms is illustrated here.

TABLE 1. Numerical Comparison of 5^{th} iteration $\phi_5(s,t)$ and 8^{th} iteration $\phi_8(s,t)$ of proposed technique with exact solution $\phi_E(s,t)$ and its absolute error $\phi_{AE}(s,t)$ is also calculated to prove the efficiency of FNAS at $t = 2$, $\gamma = v = 1, d = \frac{1}{100}$ for advection-dispersion equation of fractional-order with proportional time-delay.

s	$\phi_E(s,t)$	$\phi_5(s,t)$	$(\phi_5)_{AE}$ *	$\phi_8(s,t)$	$(\phi_8)_{AE}$ *
0	7.54E+00	7.41E+00	1.30E-01	7.54E+00	1.92E-03
10	3.42E-04	3.36E-04	5.92E-06	3.42E-04	8.73E-08
20	1.55E-08	1.53E-08	2.69E-10	1.55E-08	3.96E-12
30	7.05E-13	6.93E-13	1.22E-14	7.05E-13	1.80E-16
40	3.20E-17	3.15E-17	5.54E-19	3.20E-17	8.17E-21
50	1.45E-21	1.43E-21	2.51E-23	1.45E-21	3.71E-25
60	6.60E-26	6.49E-26	1.14E-27	6.60E-26	1.68E-29
70	3.00E-30	2.94E-30	5.18E-32	3.00E-30	7.65E-34
80	1.36E-34	1.34E-34	2.35E-36	1.36E-34	3.47E-38
90	6.18E-39	6.07E-39	1.07E-40	6.18E-39	1.58E-42
100	2.80E-43	2.76E-43	4.85E-45	2.80E-43	7.16E-47

* represents the absolute errors of 5^{th} and 8^{th} iterations respectively.

of fractional-order [13]:

$$\frac{\partial^\gamma \phi(s,kt)}{\partial t^\gamma} - \frac{\partial^2 \phi(s,t)}{\partial s^2} + \frac{\partial \phi(s,t)}{\partial s} - \phi(s,t)\frac{\partial^2 \phi(s,t)}{\partial s^2} = \phi(s,t)\left(1 - \phi(s,t)\right), \tag{20}$$

with initial condition $\phi(s,0) = E_1(s)$, here E_1 is the Mittag-Leffler function. By adhering to these steps in section 3, we obtain subsequent sequence of solution as

$$\phi(s,t) = e^s + \frac{e^s k^{-\gamma} t^\gamma}{\Gamma(1+\gamma)} + \frac{e^s k^{-2\gamma} t^{1+\gamma}}{\Gamma(2-\gamma)\Gamma(1+2\gamma)} + \frac{e^s k^{-3\gamma} t^3}{\Gamma(4-3\gamma)\Gamma(1+3\gamma)} + \frac{e^s k^{-4\gamma} t^{6-2\gamma}}{\Gamma(7-6\gamma)\Gamma(1+4\gamma)} + \frac{e^s k^{-5\gamma} t^{10-5\gamma}}{\Gamma(11-10\gamma)\Gamma(1+5\gamma)} + \cdots. \tag{21}$$

Without considering the time-delay and memory effects by letting $\gamma, k \to 1$, the series solution (21) converges to the exact solution $\phi_E(s,t) = e^{s+t}$.

6. Graphical Analysis of Obtained Numerical Results and Discussion

This section presents various graphs illustrating the relationship between all parameters of the solution, presented in (16), (18), and (20) in both two-dimensional and three-dimensional formats.

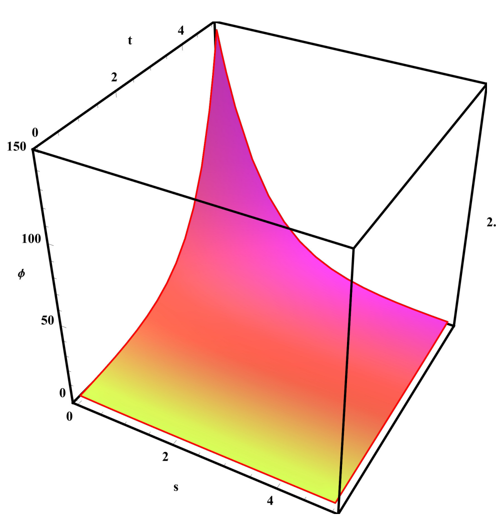
6.1. Graphical Analysis of advection-dispersion equation of fractional-order with proportional time-delay.

FNAS solved the advection-dispersion equation of fractional-order with proportional time-delay numerically and achieved a remarkable solution that is very close to the exact solution, see Figure 1a at $s \in [0,5], t \in [0,1]$ with $\gamma = k = v = 1$ and $d = \frac{1}{100}$. Figure 1b illustrates the Absolute Error (AE) graph of the obtained numerical result with the exact solution of (16) at $\gamma = v = k = 1, s \in [0,1], t \in [0,0.1]$ and $d = \frac{1}{100}$, which validates

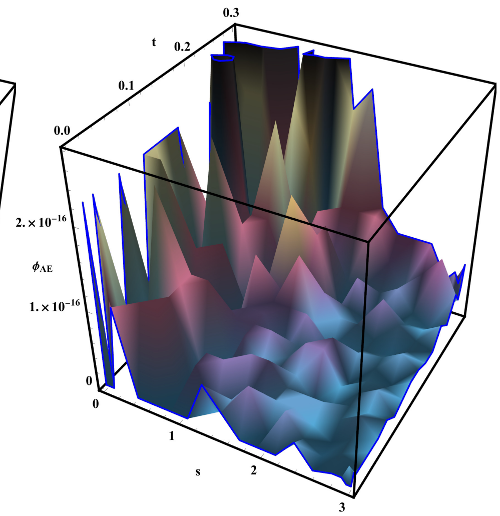
TABLE 2. Numerical results of (16) for different γ values is presented in this table at $k=1$. Comparison of numerical results by FNAS and exact solution $\phi_E(s,t)$ at $v=1, d=\frac{1}{100}$ is given for advection-dispersion equation of fractional-order with proportional time-delay.

s	t	$\phi_E(s,t)$	$\gamma=0.25$	$\gamma=0.50$	$\gamma=0.75$	$\gamma=1.0$
0.2	0.0	0.818731	0.818731	0.818731	0.818731	0.818731
	0.2	1.002000	2.505610	1.482970	1.154130	1.002000
	0.4	1.226300	3.530450	2.011100	1.483390	1.226300
	0.6	1.500800	4.576270	2.611770	1.869310	1.500800
	0.8	1.836750	5.677880	3.320200	2.332700	1.836750
	1.0	2.247910	6.845560	4.163720	2.894100	2.247910
0.4	0.0	0.670320	0.670320	0.670320	0.670320	0.670320
	0.2	0.820370	2.051420	1.214150	0.944926	0.820370
	0.4	1.004010	2.890480	1.646550	1.214500	1.004010
	0.6	1.228750	3.746740	2.138340	1.530460	1.228750
	0.8	1.503810	4.648650	2.718350	1.909860	1.503810
	1.0	1.840430	5.604670	3.408960	2.369490	1.840430
0.6	0.0	0.548812	0.548812	0.548812	0.548812	0.548812
	0.2	0.671662	1.679560	0.994063	0.773640	0.671662
	0.4	0.822012	2.366530	1.348080	0.994348	0.822012
	0.6	1.006020	3.067570	1.750720	1.253030	1.006020
	0.8	1.231210	3.806000	2.225600	1.563660	1.231210
	1.0	1.506820	4.588720	2.791020	1.939970	1.506820
0.8	0.0	0.449329	0.449329	0.449329	0.449329	0.449329
	0.2	0.549910	1.375110	0.813870	0.633403	0.549910
	0.4	0.673007	1.937550	1.103710	0.814103	0.673007
	0.6	0.823658	2.511510	1.433370	1.025900	0.823658
	0.8	1.008030	3.116090	1.822170	1.280220	1.008030
	1.0	1.233680	3.756920	2.285100	1.588320	1.233680
1.0	0.0	0.367879	0.367879	0.367879	0.367879	0.367879
	0.2	0.450229	1.125840	0.666341	0.518586	0.450229
	0.4	0.551011	1.586330	0.903644	0.666531	0.551011
	0.6	0.674354	2.056250	1.173550	0.839934	0.674354
	0.8	0.825307	2.551230	1.491860	1.048150	0.825307
	1.0	1.010050	3.075910	1.870880	1.300400	1.010050

FNAS accuracy. The fractional behavior of fractional advection-dispersion equation with proportional time-delay at different values of s and t with $k=v=1, d=\frac{1}{100}$ was another point view to understudy of this work. The fractional behavior of the numerical fractional solution obtained by FNAS can be seen in Figure 2 with $k=v=1$, and $d=\frac{1}{100}$ at different values of s,t and γ. For $\gamma \in (0,1]$, the numerical solution has a smaller variation and appears similar to the exact solution. Another aspect of this work is to study the change that occurs in the fractional advection-dispersion equation numerical solution due to the change in the delay term values. In Figure 3 there emerges observation that there is a smaller variation in the numerical solution for the k between 0 and 1. In Table 1, a comparison of the fifth and eighth iterations of the numerical solution by the proposed technique



(A) Graphical view of obtained solution by FNAS for (18) at $\gamma = k = 1$.



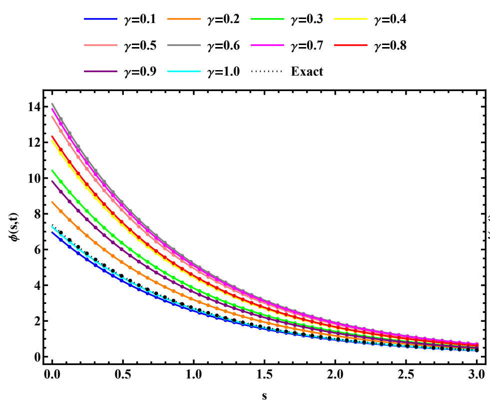
(B) Graphical representation of absolute error for (18) at $\gamma = k = 1$.

FIGURE 4. The FNAS solution for the time-fractional gas-dynamics problem with proportional time-delay is graphically analyzed in three dimensions, and the validity of the suggested scheme is demonstrated by the AE graph.

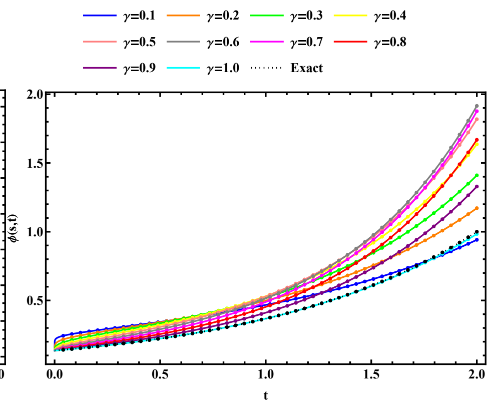
TABLE 3. Numerical Comparison of 5^{th} iteration $\phi_5(s,t)$ and 8^{th} iteration $\phi_8(s,t)$ of proposed technique with exact solution $\phi_E(s,t)$ and absolute error ϕ_{AE} is also calculated to prove the efficiency of FNAS at $t = 0.1$ for nonlinear gas-dynamics equation of fractional-order with proportional time-delay.

s	$\phi_E(s,t)$	$\phi_5(s,t)$	$(\phi_5)_{AE}$ *	$\phi_8(s,t)$	$(\phi_8)_{AE}$ *
0	1.10517	1.10517	8.47E-08	1.10517	2.89E-15
10	5.02E-05	5.02E-05	3.85E-12	5.02E-05	1.02E-19
20	2.28E-09	2.28E-09	1.75E-16	2.28E-09	9.51E-24
30	1.03E-13	1.03E-13	7.93E-21	1.03E-13	4.04E-28
40	4.70E-18	4.70E-18	3.60E-25	4.70E-18	1.85E-32
50	2.13E-22	2.13E-22	1.63E-29	2.13E-22	7.99E-37
60	9.68E-27	9.68E-27	7.42E-34	9.68E-27	3.87E-41
70	4.39E-31	4.39E-31	3.37E-38	4.39E-31	1.49E-45
80	1.99E-35	1.99E-35	1.53E-42	1.99E-35	6.41E-50
90	9.06E-40	9.06E-40	6.94E-47	9.06E-40	2.94E-54
100	4.11E-44	4.11E-44	3.15E-51	4.11E-44	1.29E-58

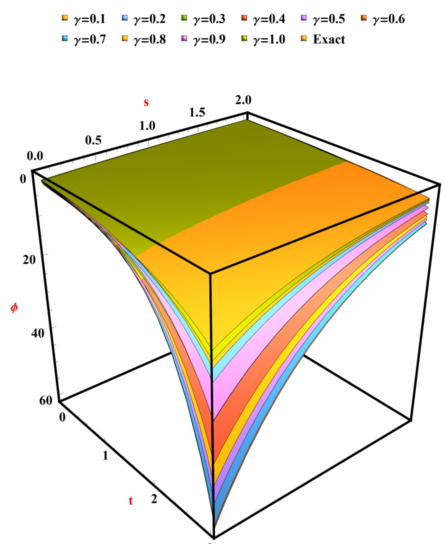
* represents the absolute errors of 5^{th} and 8^{th} iterations respectively.



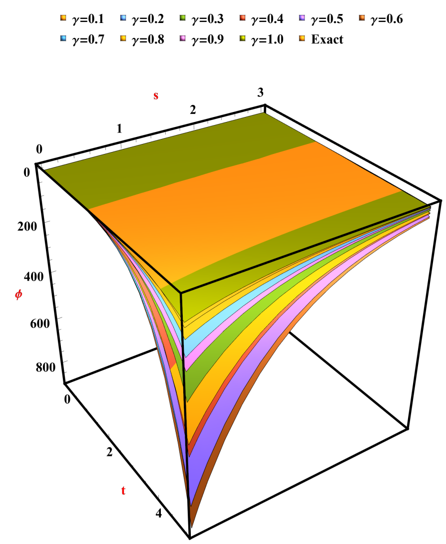
(A) 2D-view of fractional behaviour of (18) at $t = 2, k = 1$.



(B) 2D-representation of fractional behaviour of (18) at $s = 2, k = 1$.



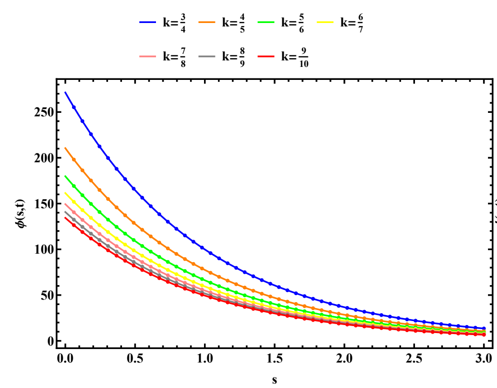
(C) 3D graphical representation of fractional behaviour of (18) at $k = 1$.



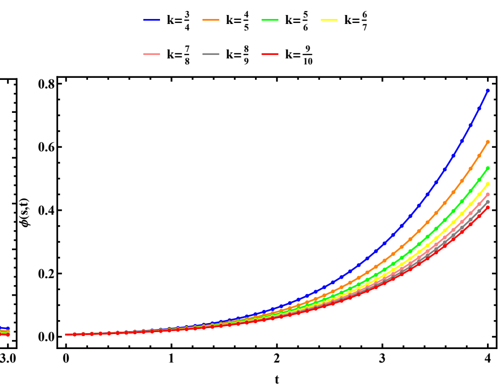
(D) 3D graphical view of fractional behaviour of (18) at $k = 1$.

FIGURE 5. Graphical investigation of fractional behavior in two and three dimensions for gas-dynamics fractional-order equation is displayed at various γ values between 0 and 1.

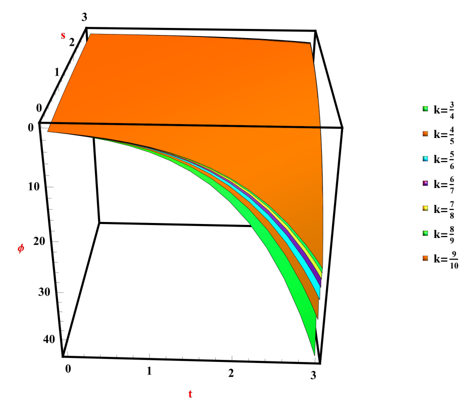
with an exact solution at $t = 2$ is given, which shows clear accuracy. A detailed analysis of the numerical solution obtained by FNAS for fractional advection-dispersion equation at different values of γ i.e., $\gamma = 0.25, 0.50, 0.75, 1.0$ is given in Table 2.



(A) Graphical representation of (18) with varying delay terms i.e., for different k values lie between 0 and 1 with $t = 5$, which shows the variation in the time-delay of the obtained solution.



(B) Graphical view of (18) with varying delay terms i.e., for different values of k lie between 0 and 1 with $s = 5$, which display the variation in the time-delay of the solution obtained by FNAS.



(C) 3D-graphical representation of (18) with varying delay terms i.e., for different k values.

FIGURE 6. A visual representation in two and three dimensions of the fractional gas-dynamics equation with various delay terms (k) is showcased.

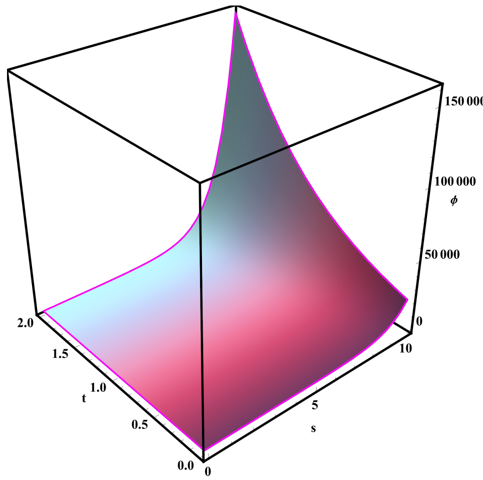
6.2. **Graphical Analysis of nonlinear gas-dynamics equation of fractional-order with proportional time-delay.**

The nonlinear gas-dynamics problem of fractional-order with proportional time-delay in

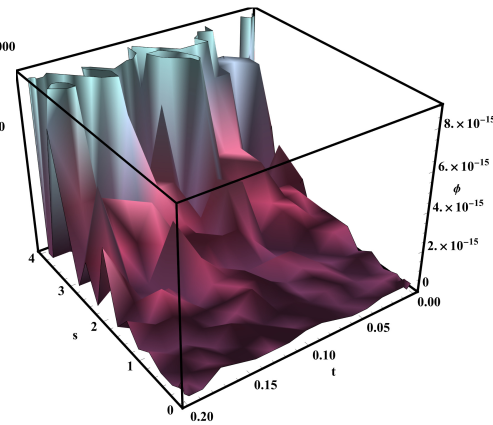
TABLE 4. The table shows the numerical results of (18) for various γ values at $k = 1$. This paper compares numerical results using FNAS to the precise solution $\phi_E(s,t)$ for a nonlinear fractional gas-dynamics equation.

s	t	$\phi_E(s,t)$	$\gamma = 0.25$	$\gamma = 0.50$	$\gamma = 0.75$	$\gamma = 1.0$
0.2	0.0	0.818731	0.818731	0.818731	0.818731	0.818731
	0.2	1.000000	2.464670	1.472900	1.150050	1.000000
	0.4	1.221400	3.450520	1.989200	1.474090	1.221400
	0.6	1.491820	4.451290	2.573550	1.852710	1.491820
	0.8	1.822120	5.501740	3.259810	2.306020	1.822120
	1.0	2.225540	6.612330	4.073870	2.853650	2.225540
0.4	0.0	0.670320	0.670320	0.670320	0.670320	0.670320
	0.2	0.818731	2.017900	1.205910	0.941583	0.818731
	0.4	1.000000	2.825050	1.628620	1.206880	1.000000
	0.6	1.221400	3.644410	2.107050	1.516870	1.221400
	0.8	1.491820	4.504440	2.668900	1.888010	1.491820
	1.0	1.822120	5.413720	3.335400	2.336370	1.822120
0.6	0.0	0.548812	0.548812	0.548812	0.548812	0.548812
	0.2	0.670320	1.652120	0.987312	0.770903	0.670320
	0.4	0.818731	2.312950	1.333400	0.988111	0.818731
	0.6	1.000000	2.983790	1.725100	1.241910	1.000000
	0.8	1.221400	3.687930	2.185110	1.545770	1.221400
	1.0	1.491820	4.432380	2.730800	1.912860	1.491820
0.8	0.0	0.449329	0.449329	0.449329	0.449329	0.449329
	0.2	0.548812	1.352640	0.808343	0.631162	0.548812
	0.4	0.670320	1.893680	1.091700	0.808997	0.670320
	0.6	0.818731	2.442920	1.412400	1.016790	0.818731
	0.8	1.000000	3.019420	1.789020	1.265570	1.000000
	1.0	1.221400	3.628930	2.235790	1.566110	1.221400
1.0	0.0	0.367879	0.367879	0.367879	0.367879	0.367879
	0.2	0.449329	1.107450	0.661815	0.516752	0.449329
	0.4	0.548812	1.550420	0.893807	0.662350	0.548812
	0.6	0.670320	2.000090	1.156370	0.832475	0.670320
	0.8	0.818731	2.472090	1.464730	1.036160	0.818731
	1.0	1.000000	2.971110	1.830510	1.282230	0.999999

equation (18) was numerically solved using the FNAS. The solution obtained was remarkable, as it perfectly matched the precise solution. This can be observed in Figures 4a, which depict the solution for different values of s in the range of 0 to 5, and t in the range of 0 to 5, with γ and k both equal to 1. Figure 4b shows the absolute error graph of the numerical result produced using the precise solution of equation (18) with $\gamma = k = 1$, s ranging from 0 to 3, and t ranging from 0 to 0.3. This graph confirms the correctness of FNAS. The Figure 5 displays the fractional behaviour of the numerical fractional solution derived by FNAS for different values of γ, s and t with $k = 1$. The numerical solution has a reduced level of fluctuation, resembling the precise solution across various γ values. The main component of this solution is the alteration of the delay term values. Figure 6 also



(A) Graphical representation of solution obtained by FNAS of (20) at $\gamma = k = 1$.



(B) Absolute error graphical representation of (20) at $\gamma = k = 1$.

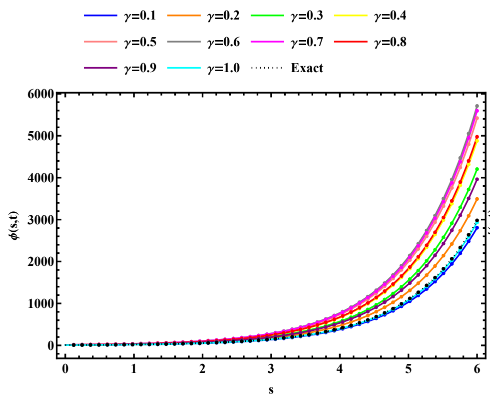
FIGURE 7. Three-dimensional graphical analysis of the FNAS solution for convection-diffusion equation of fractional-order with proportional time-delay is depicted here and the AE graph is also display to show the accuracy of proposed technique.

TABLE 5. Numerical Comparison of 8^{th} iteration $\phi_8(s,t)$ and 10^{th} iteration $\phi_{10}(s,t)$ of proposed technique with exact solution $\phi_E(s,t)$ and its absolute error ϕ_{AE} is also calculated to prove the efficiency of FNAS at $t = 0.01$ for nonlinear convection?diffusion equation of fractional-order.

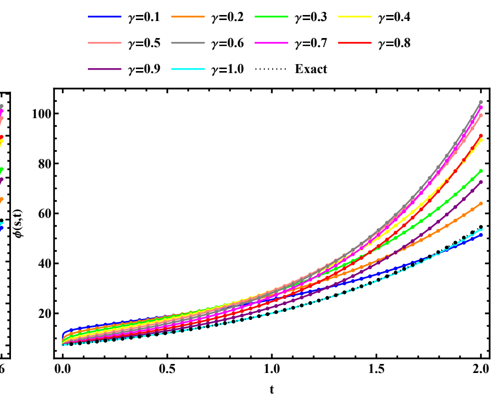
s	$\phi_E(s,t)$	$\phi_8(s,t)$	$(\phi_8)_{AE}$ *	$\phi_{10}(s,t)$	$(\phi_{10})_{AE}$ *
0	1.01E+00	1.01E+00	2.22E-16	1.01E+00	2.22E-16
3	2.03E+01	2.03E+01	7.11E-15	2.03E+01	7.11E-15
6	4.07E+02	4.07E+02	5.68E-14	4.07E+02	5.68E-14
9	8.18E+03	8.18E+03	2.73E-12	8.18E+03	1.82E-12
12	1.64E+05	1.64E+05	2.91E-11	1.64E+05	2.91E-11
15	3.30E+06	3.30E+06	9.31E-10	3.30E+06	9.31E-10
18	6.63E+07	6.63E+07	1.12E-07	6.63E+07	1.12E-07
21	1.33E+09	1.33E+09	2.15E-06	1.33E+09	1.91E-06
24	2.68E+10	2.68E+10	4.20E-05	2.68E+10	4.20E-05
27	5.37E+11	5.37E+11	8.54E-04	5.37E+11	8.54E-04
30	1.08E+13	1.08E+13	1.76E-02	1.08E+13	1.76E-02

* represents the absolute errors of 8^{th} and 10^{th} iterations respectively.

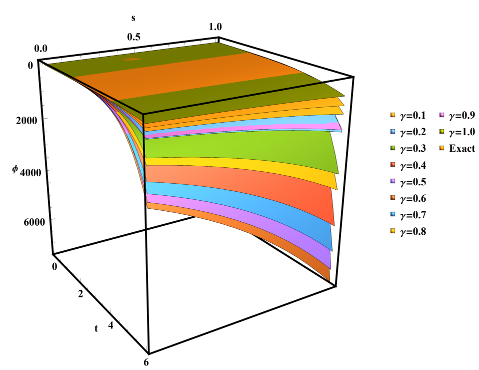
indicates that the numerical solution is affected by the change in time-delay, which varies



(A) Graphical representation of fractional behaviour of (20) $k = 1$ and $t = 2$.



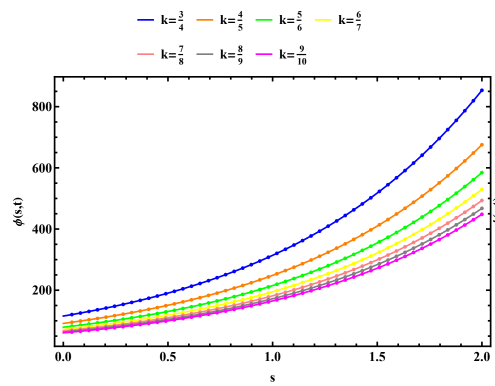
(B) Graphical Representation of fractional behaviour of (20) $k = 1$ and $s = 2$.



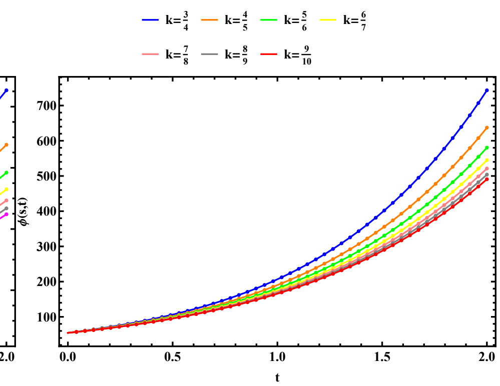
(C) 3D-graphical representation of fractional behaviour of (20) $k = 1$ and different values of γ.

FIGURE 8. Graphical examination of fractional behavior in two and three dimensions for the fractional-order convection-diffusion equation is presented.

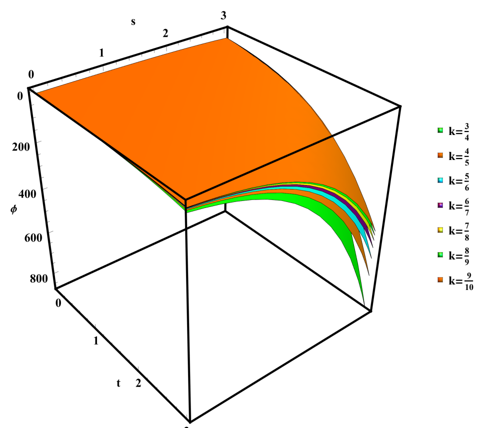
for different values of k. At $t = 0.1$, the numerical solution obtained by FNAS is compared to the precise solution in Table 3 for the 5^{th} and 8^{th} iterations. Empirical evidence demonstrates that as the number of iterations grows, the outcome converges more. Table 4 provides a detailed examination of the numerical solution found by FNAS for different values of γ $(0.25, 0.50, 0.75, 1.0)$ in the fractional gas-dynamics equation. The analysis focusses on studying the variations in the equation's numerical results at different values of fractional derivative.



(A) Graphical representation of (20) with varying delay terms i.e., for different k values lie between 0 and 1 with $t = 4$, which shows the variation in the time-delay of the obtained solution.



(B) Visual representation of equation (20) with variable delay terms, i.e., for various values of k lie between 0 and 1 with $s = 4$, showing the fluctuation in the solution's time-delay produced by FNAS.



(C) 3D-graphical representation of (20) with varying delay terms i.e., for different k values.

FIGURE 9. A graphical analysis of the fractional convection-diffusion equation in both two and three dimensions, considering various delay factors represented by the variable k.

6.3. **Graphical Analysis of nonlinear convection-diffusion equation of fractional-order with proportional time-delay.**

Nonlinear convection-diffusion equation of fractional-order with proportional time-delay

TABLE 6. The table displays the numerical outcomes of equation (20) for various γ values when k is equal to 1. The numerical results obtained via FNAS are compared to the precise solution $\phi_E(s,t)$ for a nonlinear convection-diffusion problem of fractional-order.

s	t	$\phi_E(s,t)$	$\gamma=0.25$	$\gamma=0.50$	$\gamma=0.75$	$\gamma=1.0$
0.2	0.0	1.221400	1.221400	1.221400	1.221400	1.221400
	0.2	1.491820	2.327290	1.966590	1.683770	1.491820
	0.4	1.822120	2.803600	2.487000	2.129690	1.822110
	0.6	2.225540	3.313270	3.086760	2.670920	2.225450
	0.8	2.718280	3.904150	3.828080	3.360060	2.717780
	1.0	3.320120	4.613230	4.773830	4.260660	3.318140
0.4	0.0	1.491820	1.491820	1.491820	1.491820	1.491820
	0.2	1.822120	2.842560	2.402000	2.056560	1.822120
	0.4	2.225540	3.424320	3.037630	2.601220	2.225530
	0.6	2.718280	4.046830	3.770180	3.262270	2.718180
	0.8	3.320120	4.768540	4.675620	4.103980	3.319500
	1.0	4.055200	5.634620	5.830770	5.203980	4.052790
0.6	0.0	1.822120	1.822120	1.822120	1.822120	1.822120
	0.2	2.225540	3.471910	2.933800	2.511890	2.225540
	0.4	2.718280	4.182470	3.710170	3.177130	2.718270
	0.6	3.320120	4.942810	4.604910	3.984550	3.319990
	0.8	4.055200	5.824310	5.710820	5.012610	4.054450
	1.0	4.953030	6.882140	7.121720	6.356150	4.950090
0.8	0.0	2.225540	2.225540	2.225540	2.225540	2.225540
	0.2	2.718280	4.240600	3.583360	3.068030	2.718280
	0.4	3.320120	5.108480	4.531610	3.880560	3.320100
	0.6	4.055200	6.037160	5.624450	4.866740	4.055040
	0.8	4.953030	7.113820	6.975210	6.122420	4.952120
	1.0	6.049650	8.405860	8.698490	7.763420	6.046050
1.0	0.0	2.718280	2.718280	2.718280	2.718280	2.718280
	0.2	3.320120	5.179480	4.376720	3.747300	3.320120
	0.4	4.055200	6.239520	5.534920	4.739720	4.055180
	0.6	4.953030	7.373810	6.869720	5.944250	4.952840
	0.8	6.049650	8.688840	8.519540	7.477940	6.048530
	1.0	7.389060	10.266900	10.624400	9.482270	7.384670

TABLE 7. The table displays the evaluation time of three well-known physical models.

	Example-1	**Example-2**	**Example-3**
CPU Time (in seconds)	11.53	3.05	3.59

in (20) solved by FNAS numerically attained an implausible solution that is similar to the exact solution, this can be observed in Figures 7a at different values of $s \in [0,10]$ and $t \in [0,2]$ with $\gamma = k = 1$. Figure 7b illustrates the absolute error graph of the obtained numerical result with the exact solution of (20) at $\gamma = k = 1, s \in [0,4]$ and $t \in [0,0.2]$, which validates FNAS accuracy. The fractional behavior of the numerical fractional solution obtained by FNAS can be seen in Figure 8 with $k = 1$, $t \in [0, 6], s \in [0, 1]$ and at different values of γ. The numerical solution has smaller deviations, and the solution tends to be close to the exact solution for $\gamma \in (0, 1]$. Table 5 compares the exact solution with numerical approximation by the proposed method at $t = 0.01$. We take our numerical solution on the eighth and tenth iterations to show the accuracy of the method. A detailed analysis of the numerical solution obtained by FNAS for $\gamma = 0.25, 0.50, 0.75, 1.0$ is given inTable 6. Table 7 shows the CPU time of a FNAS reflects its computational efficiency and indicates the speed of convergence.

7. Concluding Remarks

The suggested approach produces good results when applied to various fractional linear and nonlinear DDEs. Iterative outcomes are thoroughly explored in both tabular and graphical formats. For varied fractional-order and delay, the approximate answer varies continually. Additionally, it is found that only a few numbers of iterations are necessary to reach the needed precision, proving the method's excellent efficiency. To demonstrate the usefulness and simplicity of the FNAS, we presented three numerical experiments and sketched the solutions for various values of γ. Fractional Taylor series often converge only in a small neighborhood around the expansion point (typically $t_0 = 0$). This makes it difficult to use for global solutions or for large time domains. The approach to solving nonlinear fractional partial differential equations (FPDEs) may have limitations, especially for singularities, discontinuities, or specific boundary conditions. Future study should focus on improving this strategy by addressing and overcoming its limitations, hence increasing its applicability and efficacy. We will extend the given approach and its convergence analysis for other kinds of fractional delay issues including fractional integral equations and nonlinear fractional delay integro-differential equations.

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Authors' contributions

Mariam Sultana: Conceptualization, Methodology, Validation, Formal analysis, Investigation; Uroosa Arshad: Formal analysis, Investigation, Methodology. All authors read and approved the final manuscript.

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