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Architecting Resilient Communal Habitats: A Treatise on the Application of Wheel Graphs to Security Optimization with Codes

Abdul Aleem Mughal
Department of Mathematics,
University of Management and Technology, Lahore, Pakistan

*Raja Noshad Jamil
Department of Mathematics,
University of Management and Technology, Lahore, Pakistan

Muhammad Reza Farahani
Department of Mathematics and Computer Science,
In University of Science and Technology (IUST), Narmak, Tehran, 16844, Iran

*Corresponding Author
noshad.jamil@umt.edu.pk

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Abstract. Secure housing is a fundamental pillar of stable and thriving communities. It encompasses the provision of safe and reliable shelter, where individuals and families can live without fear of harm, theft, or intrusion. Security plan of a housing society can be constructed in many ways. In this article, authors have discovered a plan to strengthen security of a housing society by using face irregularity strength on wheel graphs which contains vertices, edges and faces. The vertices will act like houses, edges will be connections between houses and faces will represent the area among the houses. Mathematically, this research includes irregularity strength of face, vertex-face, edge-face and vertex-edge-face under $\Upsilon-$labelling of class (σ, φ, ψ). An algorithm is established which will lead to the real life security of housing society.

AMS (MOS) Subject Classification Codes: 68R10; 81Q30; 90C35
Key Words: Practical Applications of Irregularity Strength, Algorithmic Codes, Everyday Life Problems.

1. Introduction

Everyone dreams a secured living. The increasing rates of robberies and other security threats, have warned the people to strengthen their security. Now a promising security and realtime protection of our loved ones is a basic need of every house. All housing societies always try to make a strong security structure but improved steps can be taken any time. Application of mathematics can be seen everywhere around us, for example, the numbers of houses, structures of houses, maps of houses etc. In this study, we are applying a sub topic of mathematics, named as 'face irregularity strength in graph theory' on the housing societies.

Whatever the model we design, it should meet the reality to maximum level. Some future imaginations which an author observes, can be added but again it should exist in nearby future. If a person living in a house calls the word 'drone' with a different accent and the drone appears then it is possible in nearby future but if a person thinks about 'drone' and drone appears, not possible in nearby future. Minor improvements in models can be handled later by presenting some counter examples which is technically named as 'fixing a bug in the design of system'. In our model, we are providing equal rights to almost everyone who is living in the society. A device is being used in our security model named as 'AleemTech' which will act as a cordless phone and will have a dual SIM property. The device 'AleemTech' will have future properties as well. In future, if people want to observe their home activities from cloud then they can activate the cloud feature from the device.

In this model, if security personal are absent then security rights will be shifted to the next level. The next level will be the security outside the society. Model is designed in such a way that negligence of one person will shift the command to the next relevant person. By this methodology, everyone living inside will feel themselves secure in all respects. Today, graph techniques are very common to apply on real life situations. We are focussing on those applications of graph theory which are mostly based on graphical diagrams. In this model, the nodes represent houses and edges represent communication among houses. A combination of three houses form a face which have its own unique value, known as weight. A model can be shaped to any structure which have a definite beginning and a meaningful constructive ending. Models are not the things which exist and have meaning or sense, but models are human imaginations which reflect a human mind into real life application. Sometimes, this real life application restricts to one mind only but most of the times, this application spreads all over the universe.

Security systems can be handled by many intelligent ways but basic security measures, which are foundation of a security system, are almost same for every intelligent way. We discuss here some important factors which can be helpful for both, layman and professional. The first basic necessary product is CCTV (closed-circuit television) cameras. These cameras always base on latest technologies. A CCTV camera should have night vision capturing capacity and should be HD (high definition). They can be fixed in all over the society in order to cover every moment. This will make a stronger security system. Drone cameras have vital importance in the field of security. Sensor based drone cameras can be linked with CCTV cameras to record every irregular movement. Memory of CCTV recordings can be increased by implanting higher gigabyte hard disks. Trained staff is very important for a developed housing society. Degrees in criminology, social security, cyber security, network security etc. are very popular in all over the world. These degree holders are highly suitable for the job of security staff. The other people who have no degree in security can also be appointed on the basis that 'practical is more powerful than theory'. Security alarms are necessary in a housing project. Alarms should be connected with each other and should be based on rechargeable technology so that they could be live in 24/7. These alarms can be attached with some powerful lithium batteries and batteries can be connected with automatic solar system. The security personals often rest during duty hours but alarm system should be designed in a way that it should never rest. Automatic solar sensor street lights, having 12 volts capacity, can be implanted which should automatically run after the sunset. Most of the countries issue an identity card to its citizens. No entry can be made until a person submits his/her identity. This action will reduce the security risk in a society. Professionals in the security desk should be fully responsible to enter every person in some proper way. A person without identity is dangerous, not only for a particular society, but for the whole country. Every housing society should have a proper boundary wall. A concrete wall is the best blockage in all the environments. A blockage with some non-electric galvanized wires is not a good boundary. Face irregularity strength can be applied to any society whether it has a boundary wall or not, but security measures should be on top priority.

In this article, wheel graphs have been applied on a housing society project. The main motive behind this project is to establish a more secure environment for a housing society. Authors have tried to make a real life user-friendly project, which could be handled even by children. This project can be framed on a housing society containing at least 5 buildings including the security room. Authors will estimate the minimum label given to the houses, then this label will be used to calculate $\Upsilon-$labelling of buildings, then this $\Upsilon-$lableing will be used to estimate entire face irregularity strength of this project. Applications of graph theory are somehow, someway found in everything around us. Wheel graphs have important significance in graph theory as they include vertices, edges and faces. These graphs can be applied to the fields of computer networking, housing societies, Facebook networking etc. Many authors have examined values for irregularity strength of graphs by using different methods but no real life applications regarding housing security have been discovered yet.

A *wheel graph*, denoted as $\mathcal{W}_n$ for $n \geq 3$, is a type of graph that consists of a cycle of $(n-1)$ vertices with an additional central vertex connected to all other $(n-1)$ vertices. Formally, a wheel graph $\mathcal{W}_n$ can be constructed from a cycle graph C_{n-1} by adding a new vertex (central vertex) and connecting it to every vertex of C_{n-1}. The number of edges in $\mathcal{W}_n$ is given by:

$$E(\mathcal{W}_n) = 2(n-1). \tag{1. 1}$$

The total number of vertices in $\mathcal{W}_n$ is n. The minimum degree of $\mathcal{W}_n$ is 3 (for cycle vertices), while the maximum degree is $(n-1)$ (for the central vertex). A wheel graph is always planar. It contains a Hamiltonian cycle for all $n \geq 4$. The diameter of $\mathcal{W}_n$ is at most 2. Wheel graphs appear in various real-world applications such as network topology and routing algorithms, graph-based social network analysis, computational geometry and pattern recognition, decision-making models etc.

In this article, assume that $\mathcal{W}_n$ is a wheel graph where $n \geq 5$. Now suppose that Υ is a positive integer and T is a mapping from graph elements into Z^+, that is, $\mathrm{T} : (V, E, F) \to \{1, 2, 3, \ldots, \Upsilon\}$. Let $(\sigma, \varphi, \psi) \in \{0, 1\}$ then a $\Upsilon-$labelling T of class (σ, φ, ψ) is the least positive integer Υ on which the graph has unique weights. [17, 23, 13, 31, 32].

Note that, a $\Upsilon-$labelling T of class (σ, φ, ψ) can be defined as face irregular $\Upsilon-$labelling of class (σ, φ, ψ) if for every two distinct faces $f, g \in \mathcal{W}_n$, we have, $W_{\mathrm{T}(\sigma,\varphi,\psi)}(f) \neq W_{\mathrm{T}(\sigma,\varphi,\psi)}(g)$. Further details on the relevant labelling can be studied in [25].

If the domain is set of vertices then the labelling is called vertex $\Upsilon-$lableing of class $(1, 0, 0)$. If the domain is edge set, face set, vertex-edge set, vertex-face set, edge-face set or vertex-edge-face set then the labelling is called edge $\Upsilon-$labelling of class $(0, 1, 0)$, face $\Upsilon-$labelling of class $(0, 0, 1)$, total $\Upsilon-$labelling of class $(1, 1, 0)$, vertex-face $\Upsilon-$labelling of class $(1, 0, 1)$, edge-face $\Upsilon-$labelling of class $(0, 1, 1)$ or entire $\Upsilon-$labelling of class $(1, 1, 1)$. Note that the case $(0, 0, 0)$ is irrelevant for discussion [18, 2, 3].

The types of labelling under discussion in this research are $(0, 0, 1), (1, 0, 1), (0, 1, 1)$ and $(1, 1, 1)$. The weight of any vertex v in a wheel graph is the addition of the label of vertex itself and the labels of edges connecting this vertex [15]. Mathematically, we have

$$W_{\mathrm{T}(\sigma,\varphi,\psi)}(v) = \sigma\mathrm{T}(v) + \varphi \sum_{e\sim v} \mathrm{T}(e). \tag{1. 2}$$

The weight of any edge e in a wheel graph is the addition of label of edge itself and the labels of vertices around this edge [14]. Mathematically, we have

$$W_{\mathrm{T}(\sigma,\varphi,\psi)}(e) = \sigma \sum_{v\sim e} \mathrm{T}(v) + \varphi\mathrm{T}(e). \tag{1. 3}$$

The weight of any face f of the wheel graph is simply an addition of the labels of vertices, edges and faces (itself) surrounding that face [6]. Mathematically, we have

$$W_{\mathrm{T}(\sigma,\varphi,\psi)}(f) = \sigma \sum_{v\sim f} \mathrm{T}(v) + \varphi \sum_{e\sim f} \mathrm{T}(e) + \psi\mathrm{T}(f). \tag{1. 4}$$

In 2007, Martin Bača et al. published a research on total labelling of plane graphs [6]. They explained it as a modification of vertex irregular and edge irregular total labellings of graphs. The face irregularity strength of class (σ, φ, ψ) of the plane graph $\mathcal{W}_n$, written as $fs_{(\sigma,\varphi,\psi)}(\mathcal{W}_n)$, is the least positive integer Υ such that $\mathcal{W}_n$ satisfies face irregular $\Upsilon-$labelling. For further details on irregularity strength of graphs, reader can go through [18, 27, 2]. Kamran et al. worked on total edge irregularity strength of hexagonal grid graphs, disjoint union of sun graphs, generalized prism and wheel graphs [12, 28, 30]. Sohan et al. worked on the metric dimension of generalized wheel graphs [20, 21]. The application of different fields of mathematics in real life can be studied in [29, 24, 26].

In 2021, Aleem et al. published a research on grid graphs [2]. In 2023, Aleem et al. examined some real life application under tight face irregularity strength and applied on an internet structure [3]. In 2024, Aleem et al. applied graph theory on choosing best friends in our circle [4]. Now, in this article, Aleem and co-authors worked on applications of graph theory and applied to improve the security strength of housing societies.

The concept of elements of wheel graphs and calculation of their weights under $\Upsilon-$labelling of class $(1, 1, 1)$ can be explained by the following diagram. This will help reader to understand the basic concept of this research.

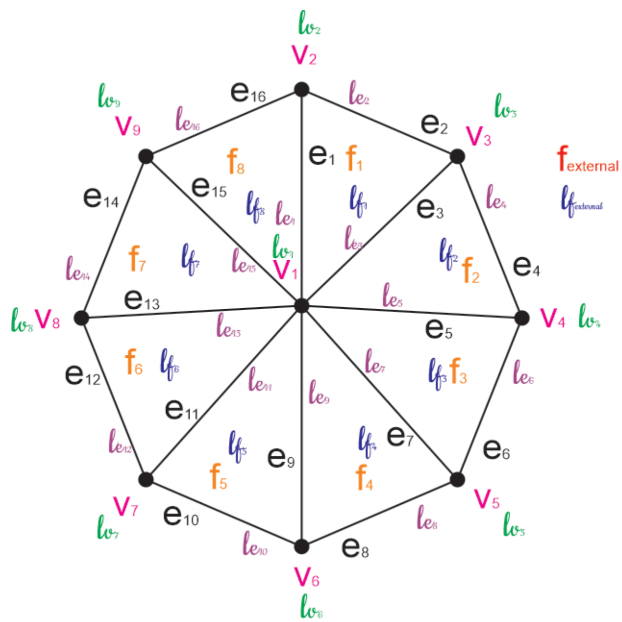

FIGURE 1. **Entire FIS Model of $\mathcal{W}_9$ under $\Upsilon-$lablleing of Class $(1, 1, 1)$.**

In figure 1, wheel graph $\mathcal{W}_9$ has 9 vertices $\{v_1, v_2, \dots, v_9\}$ where v_1 is the internal vertex which is joined with all other vertices on the boundary of this wheel. This graph has 16 edges (calculated by $2(n-1)$), written as $\{e_1, e_2, \dots, e_{16}\}$. Any wheel graph $\mathcal{W}_n$ has n faces, so this wheel has 9 faces where 8 faces are internal and one is external, written as $\{f_1, f_2, \dots, f_8, f_{\text{external}}\}$. The letters l_v, l_e and l_f represent the label of vertex, label of edge and the label of face respectively. If one observes the colors of labels in the diagram, that does not mean that all same color labels have same value. Labels have different integer values which depend on the class of labelling. There are many types of labelling in graph theory and there are many other ways of $\Upsilon-$labelling which can be adopt by the readers if they study deeply. Labels can not be zero but they can be one or more. Weight of vertex v_2 can be measured by adding $l_{v_2} + l_{e_{16}} + l_{e_1} + l_{e_2}$. Weights of other vertices can be measured in the same way. Weight of edge e_2 can be measured by adding $l_{e_2} + l_{v_2} + l_{v_3}$. Other edges can be weighted similarly. Weight of face 1 can be measured by the commutative addition of $l_{v_1} + l_{v_2} + l_{v_3} + l_{e_1} + l_{e_2} + l_{e_3} + l_{f_1}$. Similarly, weights of other faces can be calculated in a similar way. No matter, we label vertices, edges or faces of the graph, ultimately, we have to calculate weights of faces of the graph. The maximum number among the labels which will be minimum among the integers, will be known as Υ or the face irregularity strength of this graph.

In this article, authors have calculated face irregularity strength of wheel graphs under entire $\Upsilon-$labelling $\Tau$. This article is especially designed for real life applications of face irregularity strength of graphs. The structure of graph is represented in a way that vertex v_1 is always placed on the center of the graph. If reader replaces order of vertices then labelling might have little change. Before moving to the main results, here are some basic definitions which will be helpful for the reader to understand this research.

2. Notations and Preliminaries

A simple graph is a mathematical structure comprising vertices connected by edges, with no loops or multiple connections between vertices [10]. The figure 2 represents a simple graph.

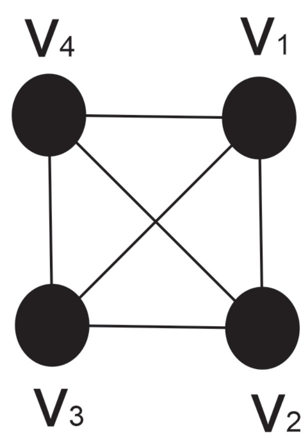

FIGURE 2. **Simple Graph**

A connected graph is a graph where there exists a path between every pair of vertices [15]. The figure 3 represents a connected graph.

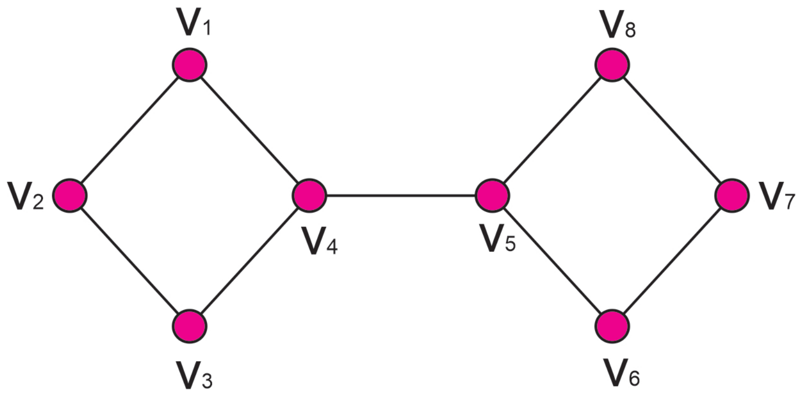

FIGURE 3. **Connected Graph**

A regular graph is a graph where each vertex has the same degree [10]. The figure 4 shows a 2-regular graph.

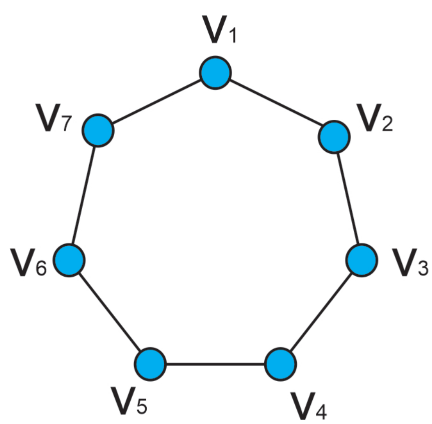

FIGURE 4. **Regular Graph**

An irregular graph is a graph where the vertices have varying degrees [10]. The figure 5 represents an irregular graph.

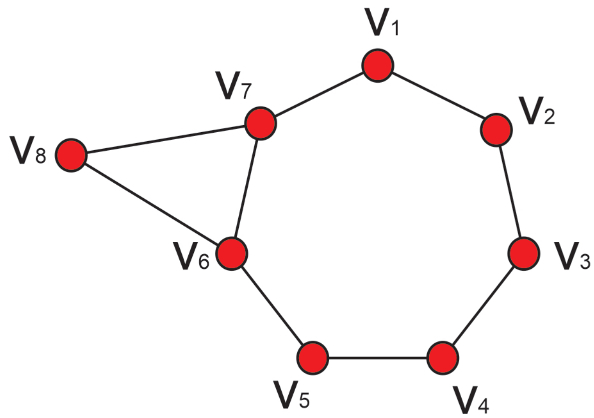

FIGURE 5. **Irregular Graph**

A planar graph is a graph that can be drawn on a plane without any edges crossing [15]. The figure 6 shows a planar graph.

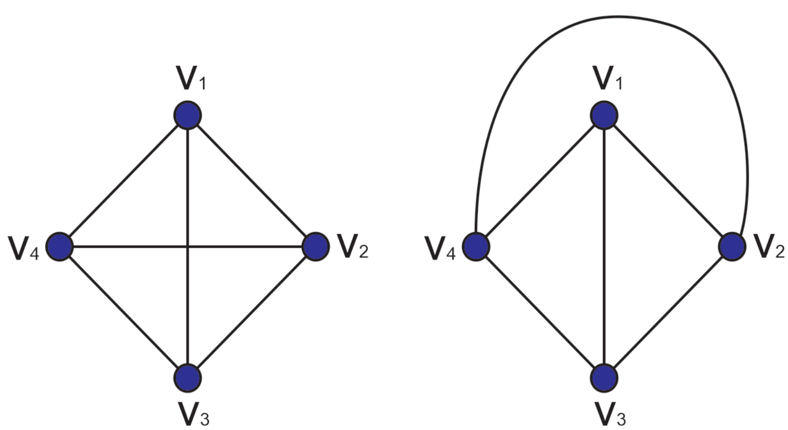

FIGURE 6. **Planar Graphs**

Graph labeling is the assignment of numerical values to the vertices, edges, or faces of a graph according to certain rules or constraints [14]. The figure 7 represents labelling of a graph.

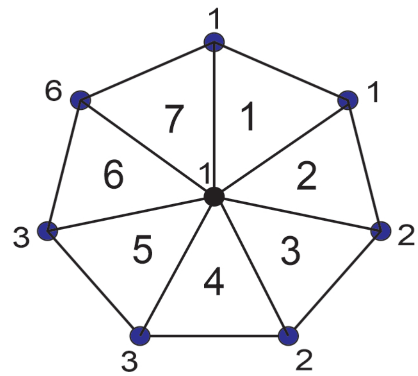

FIGURE 7. **Labelling of Graph**

The vertices and edges of a wheel graph can be generalized as
$V(\mathcal{W}_n) = \{v_i\ ;\ i = 1, 2, 3, \ldots, n\}$.
$E(\mathcal{W}_n) = \{v_1 v_i\ ;\ i = 2, 3, \ldots, n\} \cup \{v_i v_{i+1}\ ;\ i = 2, 3, \ldots, n-1\} \cup \{v_n v_2\}$.

Let G be a wheel graph with n vertices containing v_1 as center vertex which is connected to all other vertices v_i of G for $2 \leq i \leq n$ and $n \geq 5$. Then the following theorems will hold regarding the graph G.

Theorem 2.1. *The face irregularity strength of G under $\Upsilon-$labelling T of class (σ, φ, ψ) is $n-1$.*

Proof. Let us consider the labelling of faces in the graph. We assign labels to the faces according to the relation $f_i = 1 + i$ for $1 \leq i \leq n-1$. This means the first face gets the label of 1, the second face gets a label 2, and so on up to the $(n-1)th$ face getting a label of $n-1$. Since there are n faces in total, including the external face, the internal faces will be $n-1$. The label of the first face is 1, and the label of the $(n-1)th$ face will be $n-1$. Therefore, the maximum label any face in graph G can attain is $n-1$, indicating that $FS(G) = n-1$. This proof demonstrates that regardless of the specific labelling strategy chosen, the face irregularity strength of the wheel graph G will always be $n-1$ given the specified conditions. □

Theorem 2.2. *The vertex-face irregularity strength of G under $\Upsilon-$labelling T of class (σ, φ, ψ) is $n-2$.*

Proof. Considering G as a wheel graph with n vertices, let us define the vertex-face $\Upsilon-$labelling T of class $(1, 0, 1)$ as follows:

- $\mathrm{T}(v_i) = 1$ for $i = 1$
- $\mathrm{T}(v_i) = \lfloor \frac{i-1}{2} \rfloor$ for $i = 2, 3, 4, \ldots, n-1$
- $\mathrm{T}(v_i) = n-2$ for $i = n$
- $\mathrm{T}(f_i) = 1$ for $i = 1, 2, 3, 4, \ldots, n-1$.

Now let us determine the weights of the faces:

- For $i = 1, 2, 3, 4, \ldots, n-3$, we have
  $W_{T(\sigma,\varphi,\psi)}(f_i) = 3 + i$
- For $i = n-2$, we have
  $W_{T(\sigma,\varphi,\psi)}(f_{n-2}) = 3 + (n-3) + \lfloor \frac{n}{2} \rfloor - 1$
- For $i = n-1$, we have
  $W_{T(\sigma,\varphi,\psi)}(f_{n-1}) = 3 + (n-3) + 1$

We will prove this theorem using mathematical induction:

For $n = 5$, the vertices and faces will have the following labels

- $\mathrm{T}(v_i) = 1$ for $i = 1$
- $\mathrm{T}(v_i) = \lfloor \frac{i-1}{2} \rfloor$ for $i = 2, 3, 4$
- $\mathrm{T}(v_i) = 3$ for $i = 5$
- $\mathrm{T}(f_i) = 1$ for $i = 1, 2, 3, 4, 5$.

The corresponding weights satisfy the conditions, as all face weights are distinct.

Assume the theorem holds true for $n = k$. Then for $n = k + 1$, the vertex-face labelling T can be expressed similarly with $\mathrm{T}(v_i)$ and $\mathrm{T}(f_i)$.

By following the same reasoning, we find that the results are valid for $n = k + 1$.

Thus by induction, we have shown that the vertex-face irregularity strength of the wheel graph G under $\Upsilon-$labelling T of class (σ, φ, ψ) is $n - 2$. This completes the proof. □

Theorem 2.3. *The edge-face irregularity strength of $\mathcal{W}_n$ under $\Upsilon-$labelling T of class (σ, φ, ψ) be $n - 1$.*

Proof. Let $\mathcal{W}_n$ be a wheel graph with n vertices, and consider the $\Upsilon-$labelling T of class (σ, φ, ψ). We aim to prove that the face irregularity strength of $\mathcal{W}_n$ under this labelling is $n - 1$.
Assume, for the sake of contradiction, that the edge-face irregularity strength of $\mathcal{W}_n$ under T of class (σ, φ, ψ) is not equal to $n - 1$, but instead $n - 2$. Under the assumption, let us examine the weights of the faces. We find that $W_{\mathrm{T}(\sigma,\varphi,\psi)}(f_{n-1}) = 1{+}1{+}1{+}n{-}2 = n{+}1$, a repetition, because the weight of $f_{(n-2)}$ is already $n + 1$. So, our supposition is wrong. Now again suppose on contrary that $E - FS(G) = n$. Then weight of the face $n - 1$ will be $1 + 1 + 1 + n = n + 3$, possible only if the face label of $f_{(n-1)}$ is n. So, the face weights of the graph are distinct for the labels n and $n - 1$. We know that the face irregularity strength of graphs is the least positive integer. Hence $E - FS(G) = n - 1$. □

Theorem 2.4. *The entire face irregularity strength of G under $\Upsilon-$labelling T of class (σ, φ, ψ) is $n - 2$.*

Proof. Let G be a wheel graph with n vertices, and consider the $\Upsilon-$labelling T of class (σ, φ, ψ). We aim to prove that the entire face irregularity strength of G under this labelling is $n - 2$. Firstly, note that the external face is a special case, as it represents the boundary of the graph. Let $f_{external}$ denote the external face. Now, let's consider the weights of the internal faces of the graph. We denote the internal faces as $f_1, f_2, f_3, \ldots, f_{n-1}$. The

weight of each internal face f_i can be measured by adding the labels of the vertices, edges, and the face itself surrounding that face, according to the labelling T.
Considering the labelling T of class (σ, φ, ψ), the weight of each internal face f_i can be expressed as $W_{\mathrm{T}(\sigma,\varphi,\psi)}(f_i) =$ label sum of f_i. Now, observe that for $i = 1, 2, 3, \ldots, n-2$, the weight of each internal face f_i is uniquely determined by its position and the labelling scheme. Therefore, there are $n-2$ distinct weights among the internal faces. Additionally, note that the weight of the external face $f_{external}$ is also distinct from the weights of the internal faces since it represents the boundary. Hence, the total number of distinct face weights, including both internal and external faces, is $(n-2)+1 = n-1$. However, the entire face irregularity strength of the graph is defined as the least positive integer such that the graph satisfies face irregular $\Upsilon-$labelling. Therefore, the maximum number of distinct face weights is $n-2$. Thus, we conclude that the entire face irregularity strength of G under $\Upsilon-$labelling T of class (σ, φ, ψ) is $n-2$, completing the proof. □

3. Resident's Handbook to Ethical Living in a Housing Society

In the context of graph theory, a housing society can be represented as a graph where each house or unit is a vertex (node), and connections or edges between these vertices represent relationships or proximity between the houses. Each house or unit in the housing society is represented as a vertex (node) in the graph. One can assign a unique identifier or label to each vertex, typically using numbers or names. Edges in the graph represent relationships or proximity between the houses. If two houses are next to each other or share a common boundary, one can draw an edge between the corresponding vertices to indicate their adjacency. If houses share common facilities (e.g., a swimming pool, playground, or clubhouse), we can draw edges between the vertices representing these houses and the vertices representing the shared facilities. If houses are connected by roads or pathways within the housing society, we can represent these connections as edges in the graph. We can assign weights to edges to represent the distance or proximity between houses.

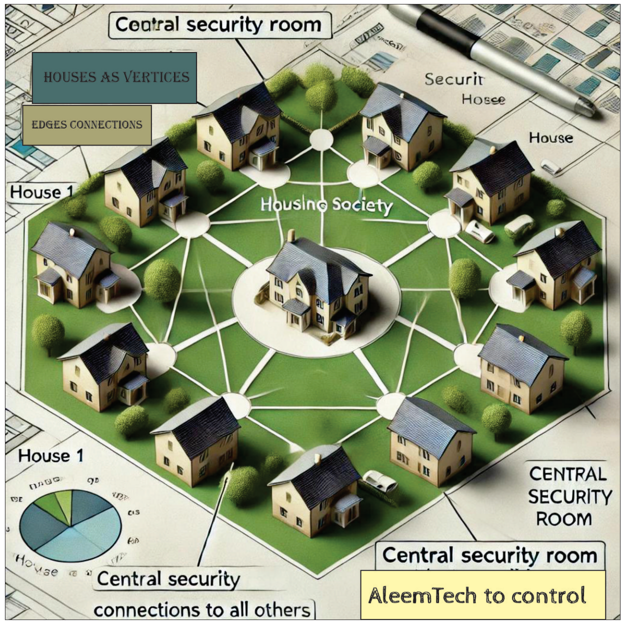

FIGURE 8. **Model of a Housing Plan.**

Living in a housing society often involves sharing common spaces and resources with neighbors, so it is important to take certain precautions to ensure a harmonious and safe living environment. Maintain a respectful and friendly attitude towards your neighbors. Keep noise levels within acceptable limits, especially during quiet hours. Be mindful of cultural and social differences among neighbors. Lock your doors and windows when leaving your home. Use peepholes or intercoms to verify visitors before allowing them into the building. Report any suspicious activity or security concerns to the housing society management or security personnel. Follow the rules and regulations regarding the use of common areas such as the clubhouse, pool, gym, and parking facilities. Clean up after using common spaces and return equipment to their designated places. Park your vehicles only in designated parking areas and follow the parking regulations. Avoid double parking or blocking other residents' vehicles. Dispose of trash and recyclable in the designated bins or areas. Avoid littering common areas or hallways. If pets are allowed, follow the community's pet rules and clean up after your pets. Ensure your pets are not causing disturbances or posing a danger to others. Inform the housing society management or security when you have guests or visitors staying over for an extended period. Ensure your guests follow community rules and guidelines. Attend housing society meetings and engage in discussions about community matters and decisions. Be informed about community rules and regulations. Familiarize yourself with emergency evacuation procedures and the location of fire extinguishers and first-aid kits. Be prepared for natural disasters or emergencies with necessary supplies. Maintain open and respectful communication with your neighbors and the housing society management. Use official communication channels or platforms established by the housing society for sharing information and updates. Promptly report any maintenance issues or repairs needed in your unit or common areas to the management. Cooperate with scheduled maintenance work and access requests. Abide by the rules and bylaws of the housing society, which may cover issues like noise, alterations to your unit, or use of common areas. Consider participating in community activities, events, or committees to foster a sense of community and cooperation. Take steps to secure your personal information and valuables. Install appropriate security measures in your home, such as locks and alarms.

By following these precautions and being a considerate and responsible member of your housing society, one can contribute to a pleasant and safe living environment for yourself and your neighbors. Additionally, staying informed about the rules and guidelines established by your housing society is essential for a smooth communal living experience.

4. Algorithm

Initialization: Begin with a housing society represented as a graph, where vertices represent houses, edges represent connections between houses, and faces represent areas among the houses.
Labelling: Apply a labeling scheme called $\Upsilon-$labelling to the graph. This labeling assigns positive integer values to graph elements. There are different types of labeling depending on the domain (vertices, edges, faces, etc.).
Calculation of Irregularity Strength: Calculate the irregularity strength which will be the least positive integer on which the face weights of the graph are distinct. Weights can be calculated by using the formula

$$W_{\mathrm{T}(\sigma,\varphi,\psi)}(f) = \sigma \sum_{v\sim f} \mathrm{T}(v) + \varphi \sum_{e\sim f} \mathrm{T}(e) + \psi \mathrm{T}(f)$$

where σ, φ and ψ are associated with vertices, edges and faces.
Security Protocol: Implement a security protocol based on the irregularity strength. For example, if residents of a house feel unsatisfied with internal security, they can press an exterior weight to call security from outside. Similarly, if all security measures fail or are absent, a designated house with a high irregularity strength label can activate a system-wide security alert.
Neighborhood Watch Program: Establish a neighborhood watch program utilizing the introduced device, "AleemTech," which is a cordless phone with dual SIM properties. This program aims to involve residents in the security of their community.
Future Enhancements: Make model according to your plans, wishes, ideas, strategies and most of all security proposals. Consider potential future enhancements, such as integrating motion-activated drones into the security system based on the entire $\Upsilon-$labelling of the graph.

5. Decision Making on Security System of a Housing Society by using Face Irregularity Strength of Wheel Graphs

Face irregularity strength of graphs with class (σ, φ, ψ) can be applied on security system of a house or society to make it more secured. In case of a home or housing society, the minimum number of rooms or houses should be five respectively. The following figure 2 shows a security plan for a housing society under entire $19-$labelling of class $(1, 1, 1)$.

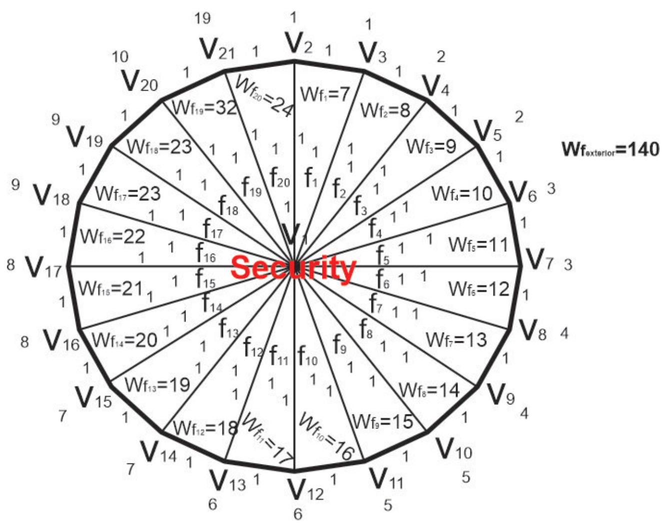

FIGURE 9. **EFIS of $19-$lablleing for Security Plan of a Housing Society.**

In figure 2, the vertices $v_1, v_2, v_3, \ldots, v_{21}$ represent the number of houses. The house v_1 is taken as the security room which will protect all the society from some bad happening. This will be a user friendly security system. Security room v_1 will also be attached with the security outside the society. Every two houses are given a same number(label) except the last one because the last house has special rights due to possessing the face irregularity strength. Every three houses are linked with a face label(code). This code is planned to alert two houses at the same time. The security device which is named as AleemTech will be introduced in this system which will look like old button mobile phones but it will have all latest facilities. AleemTech will have a dual SIM slot so that people can call anywhere, it will also act like a cordless phone. Simply, we can say that AleemTech will be a friendly solution to all the security problems.

In figure 2, $W_{f_1}, W_{f_2}, \ldots, W_{f_{20}}, W_{f_{external}}$ are the weights of all the faces including the external face. This weight will be a common security code for houses to handle any bad situation. If house number 2 feels something insecure from the people inside the house or outside the house then they will press the number face irregularity strength (19) to run a siren for help. This siren will be listened by house number 3 and security room, because if security officers are absent or not available then house number 3 can help house number 2 by calling security from outside the society or by using some other means. The weight of the external face will be a number that can be used by any house to call outside Police in any panic situation. Now, in response, security room will blow a horn by pressing the weight of face 1 which includes security room, house 2 and house 3. This horn will be 20 percent audible by all the houses inside the society and 100 percent audible by house numbers 2 and 3. If house 3 or other houses in the society will not be interested in hearing this alarm then security will have option on its device to be connected with only house 2. If house 2 wants to inform house 3 about any kind of danger then house 2 will press the weight of face 1, which is covering house 2, house 3 and security room. But if house 2 presses the weight of face 1 then it can only inform to house 3, but not to the security room. Let us repeat the procedure for those who love applications of graph theory. If house 3 faces any danger or insecurity then they can press the number (face irregularity strength) to put a siren in the security room. If house 3 feels something irregular in house 4 or house 2 then they will press the weight numbers involving house 4 or house 2 respectively. If residents of house 3 feel unsatisfied from the security inside then they will press the exterior weight to call the security from outside. The same process can be observed among all the houses in the society. This security plan is actually fulfilling the neighbours rights. Weight of the exterior face will be used as a security code to connect other departments if any panic situation happens inside the housing society. Note that the house which contains the label of irregularity strength will have approximate same rights like the security personals. What will happen if all security is absent, all houses in society are not responding or any other dark situation happens, then house 21 which contains the label of face irregularity strength will press the 'sum weight' (addition of all internal face weights) to inform the main security of that state or province. This security system is fully connected. The device, 'AleemTech' which is introduced in this system can have many registered copies. All the people in any house will have AleemTech devices. These devices will be friendly and will make home protection quite easy. Every human who knows 'how to call' can use this device easily. We have introduced neighbourhood watch program, which have never been activated before. This will be a vital step in the fields of security system. This system is planned to attach motion activated drones which can be activated in future by using entire $\Upsilon-$labelling of class (σ, φ, ψ) of graphs.

6. Conclusion

In this article, authors worked on calculation of face irregularity strength of wheel graphs which are an extension of cycle graphs. Face labelling of graphs have many applications in our real life. Some applications have been discussed in this research article. There are many more applications and most of them are linked with edge $\Upsilon-$lablleing of class $(0, 1, 0)$ because an edge represents a relation between any two objects and everything in this universe is related to at least one other thing by a strong edge. This research on security models in housing societies, grounded in graph theory, contributes significantly to the field of security studies. These models offer housing societies the tools to comprehensively assess vulnerabilities, optimize security operations, and foster safer, more resilient communities. As housing societies continue to evolve in response to modern challenges, the insights gained from this research will remain invaluable for ensuring the safety and well-being of their residents.

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9. CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

10. DATA AVAILABILITY

Data available on request from the authors.

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Appendix

A. Python Code for Graph Representation (for $n \geq 5$)

Below is the Python code for constructing and visualizing a wheel graph W_n for $n \geq 5$ using the `networkx` and `matplotlib` libraries.

```
import networkx as nx
import matplotlib.pyplot as plt

def create_wheel_graph(n):
    """Creates a wheel graph Wn with n
    vertices (n >= 5)."""
    if n < 5:
        raise ValueError("A wheel graph
```

```
        must have at least 5 vertices.")
    return nx.wheel_graph(n)

def draw_wheel_graph(G):
    """Draws the wheel graph with labeled vertices."""
    pos = nx.shell_layout(G)
    nx.draw(G, pos, with_labels=True,
    node_color='lightblue',
    node_size=700, font_size=10)
    plt.title("Wheel Graph")
    plt.show()

if __name__ == "__main__":
    wheel_graph = create_wheel_graph(10)
    # Example for n = 10
    draw_wheel_graph(wheel_graph)
```

A1. Explanation of the Code

- **Graph Creation:** The function `create_wheel_graph(n)` generates a wheel graph W_n, ensuring that $n \geq 5$.
- **Graph Visualization:** The function `draw_wheel_graph(G)` arranges nodes in a circular layout and plots the graph.
- **Execution:** The script creates a wheel graph for $n = 10$ and visualizes it.

B. Python Code for Algorithm Implementation

The following code implements an algorithm for face irregularity strength calculations in wheel graphs.

```
import networkx as nx
import matplotlib.pyplot as plt

def create_wheel_graph(n):
    if n < 4:
        raise ValueError("A wheel graph must
```

```
        have at least 4 vertices.")
    return nx.wheel_graph(n)

def assign_labels(G, sigma=1, phi=1, psi=1):
    """Assigns \Upsilon-labeling to the vertices,
    edges, and faces."""
    labels = {'vertices': {}, 'edges': {}, 'faces': {}}

    for i, node in enumerate(G.nodes()):
        labels['vertices'][node] = sigma * (i + 1)

    for i, edge in enumerate(G.edges()):
        labels['edges'][edge] = phi * (i + 1)

    num_faces = len(G.nodes())
    for i in range(num_faces):
        labels['faces'][f"f{i+1}"] = psi * (i + 1)

    return labels

def calculate_face_weights
(G, labels, sigma=1, phi=1, psi=1):
    """Computes the weights of the
    faces based on \Upsilon-labeling."""
    face_weights = {}
    for face_id in labels['faces'].keys():
        weight = (
            sigma * sum(labels['vertices'].values()) +
            phi * sum(labels['edges'].values()) +
            psi * labels['faces'][face_id]
        )
        face_weights[face_id] = weight
    return face_weights

def implement_security_protocol(G, labels, face_weights):
    """Implements the security protocol based on
```

```
    face irregularity strength."""
    print("\n=== Security Protocol Implementation ===")
    for face_id, weight in face_weights.items():
        print(f"Face {face_id} has weight {weight}.")

    max_weight_face = max(face_weights, key=face_weights.get)
    print(f"\nAlert: Face {max_weight_face} with weight
    {face_weights[max_weight_face]}
    requires immediate attention!")

def draw_wheel_graph_with_labels(G, labels):
    """Draws the wheel graph with labeled
    vertices and edges."""
    pos = nx.shell_layout(G)
    nx.draw(G, pos, with_labels=True, node_color='lightblue',
    node_size=700, font_size=10)

    vertex_labels = {node: f"{label}"
    for node, label in labels['vertices'].items()}
    nx.draw_networkx_labels
    (G, pos, labels=vertex_labels, font_color="black")

    edge_labels = {edge: f"{label}" for edge,
    label in labels['edges'].items()}
    nx.draw_networkx_edge_labels
    (G, pos, edge_labels=edge_labels, font_color="red")

    plt.title("Wheel Graph with \Upsilon-labeling")
    plt.show()

if __name__ == "__main__":
    n = 9
    wheel_graph = create_wheel_graph(n)
    labels = assign_labels
    (wheel_graph, sigma=1, phi=2, psi=3)
    draw_wheel_graph_with_labels(wheel_graph, labels)
```

```
face_weights = calculate_face_weights
(wheel_graph, labels, sigma=1, phi=2, psi=3)
implement_security_protocol
(wheel_graph, labels, face_weights)
```

B1. Explanation of the Code

- **Graph Construction:**
  - `create_wheel_graph(n)`: Ensures a valid wheel graph with at least 4 vertices.
  - `assign_labels(G, sigma, phi, psi)`: Assigns numerical labels to vertices, edges, and faces.
- **Weight Calculation:**
  - `calculate_face_weights(G, labels, sigma, phi, psi)`: Computes face weights based on assigned labels.
- **Security Protocol Implementation:**
  - `implement_security_protocol(G, labels, face_weights)`: Identifies the most critical face weight for security monitoring.
- **Visualization:**
  - `draw_wheel_graph_with_labels(G, labels)`: Displays the graph with labeled elements.
- **Execution:**
  - The script constructs a wheel graph for $n = 9$, assigns labels, visualizes it, calculates weights, and implements security protocols based on the highest face irregularity strength.