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Designing by constructing Cubic Trigonometric Nu B-Spline with Arabic Font

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Abstract. In stark contrast to the Latin script used in English, Arabic script is a cursive writing system that is written from right to left. The structure of the Arabic script and the relationships between the characters must be thoroughly understood while designing Arabic fonts. In this research, a novel freeform spline method is built for the designing and modeling of Arabic characters and letters, with practical interpolative and approximation features. Authors have constructed a new trigonometric nu

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B-spline technique which is different in construction from the simple B-spline and contains shape parameters that help in designing. As in this paper, Arabic fonts are formatted by using suggested spline techniques. The proposed technique possesses geometric continuity of order 2 and satisfies B-spline properties such as property of local support, property of positivity, and property of unity. Moreover, different Arabic typefaces are used to demonstrate geometric features such as affine invariance, variation decreasing, and convex hull. Likewise, shape properties are successfully employed at points, intervals, and each side by using a single parameter family. The goal of smoothness and flexibility has also been achieved by the shape features. Numerous data points are collected through human contact, which takes a little bit longer than scanning illustrations yet leads to excellent results.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Trigonometric B-spline; Blending Bezier basis; Single tension parameter; Fonts; Cubic Trigonometric Nu B-Spline; Geometric Continuity; Arabic Typeface Modeling..

1. Introduction

Arabic script is a cursive writing style that is written from right to left, which is a sharp contrast to the Latin script that is used in English. When developing Arabic typefaces, it is crucial to have a full understanding of the Arabic script's structure and the relationships between its letters. Arabic calligraphy, script, and graphic design fundamentals must all be understood to create Arabic typefaces. Additionally, graphic design and typography expertise are very necessary. Effective use of typography and visual design is also essential. Arabic symbols include complex curves and multiple overlapping strokes. These curves must be produced smoothly and consistently, which spline interpolation delivers, for the characters to be legible and visually attractive. Piecewise polynomials known as spline play a significant part in computer modeling and geometric design. They are introduced because of interpolation and approximation. An approximation develops when the piecewise curve does not traverse every data point, whereas an interpolative spline emerges when the curve traverses every control point. Generally, Bezier curves behave as an interpolative spline in which a complete curve changes its path by altering the position of a single point, and the curve through basis functions is an approximate curve in which a change will occur only in the position of the shifting data point. A transformation from a basis curve to a blending Bezier basis enhances the ability to control the designed curve locally and globally in an appropriate way. The cubic spline is a valuable tool in computer graphics and geometry for modeling, but its effectiveness is limited when it comes to handling freeform curves. This may have discouraged the authors from working with these splines. Therefore, to get a freeform approach all over the curve, the researcher may have to work on basis functions that will get local as well as global approach all over the curve and adore all the ultimate feathers of basis functions. These basis functions are classified as B-splines. This local support basis contains the same continuity constraints just like the continuity of their cubic spline. These local basis functions have the characteristic of being progressive everywhere. B-splines are an essential tool for approximation problems due to local support. A transformation from a B-spline basis to a blending basis gives freedom to effectively control the shape locally and globally as needed. It has various applications in the computer using technologies such as font designing, route networks, scanners, textiles, garments, etc. Modeling of fonts is an appreciated implementation in the world of freeform splines. Modeling fonts is an appreciated implementation in the world of freeform splines. Many researchers have performed their work in this area using different splines. Various schemes of designing fonts are introduced in the given literature. An elementary and effective method fulfilling qualities like those of a B-spline with a control parameter has been proposed in this study. The developed scheme offers aesthetically pleasing Arabic fonts and allows for user adjustment whenever necessary. In this regard, further changes as required for user modification in different shapes of Arabic characters regarding points, intervals, and complete continuous portions of a model are allowed using a single parameter family of shape control. Therefore, the technique ensures good freeform shape aspects of the tension to accomplish the geometric properties (convex hull, variation diminishing, and invariance) and shape properties (point tension, interval tension, and global tension). This paper provides an effort to accomplish the design using Arabic fonts taken from Human interaction and using a non-uniform freeform designing technique to compute the different shapes. Samreen et al. [10] illustrated a new rational cubic spline with beneficial achievements at local and global modification. The process recovers the fine geometric aspects of the shapes with a piecewise parametric smoothness of order 2. Sarfraz et al. [12] developed a weighted trigonometric spline with local tension shape effects. The strategy involved the characteristics of local support bases like B-splines and is a substitute scheme of cubic B-spline with the default second-order parametric continuity. Amat et al. [2] composed a comparative analysis between two Bezier methods for the rehabilitation of outlining ideas for different fonts. In this work, the authors designed the shapes of "lam alif", "kha" and "ka" to build the boundary and perceive the corner points of these shapes. Samreen et al. [11] derived a C2 Quadratic Trigonometric Spline (QTS) with an error bound of order 3 for curve modeling. The authors also presented a comparative study of Quadratic Trigonometric Spline (QTS) and Cubic Polynomial Spline (CPS) to determine the best technique for curve modeling. They proved in their comparative analysis that the proposed QTS is computationally faster, robust, and ideal geometrically. Ibrahim and Albergail [6] generated a rational cubic Bezier curve with Ball's basis functions for designing different Arabic shapes. Zainudin et al. [19] offered a new algorithm named butterfly optimization algorithm using a ball cubic curve and implemented it on sketching Arabic fonts broadly. Shah et al. [15] originated a technique in which Urdu handwritten words are detected through a machine. Razali et al. [9] designed a reconstructive Chinese font by using cubic spline with B-spline-like basis functions along with a variable parameter. Ahmed et al. [1] formed a quadratic Bezier scheme for the design of Arabic fonts and modified the shapes with the Arabic font "dal" for rational and non-rational Bezier curves using two shape parameters without changing the control points. Han et al. [5] proposed the cubic trigonometric Bezier function (T-Bezier) which accomplishes all the properties of the Beizer function and can be adapted for CAD/CAM systems. Li [7] provided an efficient and simple way to automatic interpolation of data points by defining C2 cubic trigonometric functions. The paper [8] introduced a cubic B-spline collocation method for wave propagation simulation, achieving high accuracy with fourth-order convergence while maintaining linear computational complexity, ensuring efficient and precise forward modeling for inversion imaging. The study [20] developed innovative B-spline honeycombs (BSHs) optimized for superior energy absorption using deep learning and genetic algorithms, achieving a maximum specific energy absorption of 13.4 J/g through a systematic design framework. The study [17] proposed a disk B-spline representation for Poisson's ratio metamaterial design, enabling precise shape and thickness control, achieving tunable Poisson's ratios from -0.1 to -0.9, and extending to 3D structures with validated accuracy. The study [13] enhanced V-shaped rib turbulators by adding a central spline, reducing stress concentration, improving heat transfer by up to 8.2The study [18] developed a non-uniform spline finite strip method (N-u SFSM) for efficient and precise buckling analysis of structures with local abnormalities, improving accuracy and computational efficiency in structural design. The study [14] applied a nonparametric cubic B-spline model with penalization for clustering longitudinal data, demonstrating its effectiveness through simulations and real-world analysis of kidney failure patients' dialysis needs. This paper [16] uses deep reinforcement learning to enhance jet actuator placement for reducing drag and lift on square cylinders, showing leading-edge control significantly outperforms trailing-edge control across Reynolds numbers. This study [4] acquires and appraises co-operative machine learning models to precisely calculate shear strength of reinforced concrete deep beams, outpacing conventional mechanics-based methods. In [3], Gerald Farin's Curves and Surfaces for Computer Aided Geometric Design (Third Edition) offers an accessible introduction to CAGD, focusing on Bernstein-Bezier techniques and ideal for advanced undergraduate or graduate students.

The remainder of this paper has been set as follows: A cubic trigonometric nu spline has been constructed in section 2. Local support bases like B splines have been formed in section 3 with a brief introduction of properties of local support, partition of unity, and positivity. In Section 4, a method for generating freeform curves has been established. Geometric properties such as affine invariance, convex hull, and variation diminishing have been demonstrated in Section 5. In Section 6, shape properties like point, interval, and global tension have been implemented on different fonts and a summary of geometric properties has been given in Section 7 and Section 8 describes the advantages of proposed spline techniques with the help of a table. Section 9 concludes this research.

2. Development of cubic trigonometric spline

Consider a trigonometric piecewise polynomial $Z(t)$ with knots

$$t_i, \qquad i = 0, 1, 2, \ldots, n-1,$$

ordered as

$$t_0 < t_1 < t_2 < \cdots < t_{n-1},$$

defined on the interval $[t_i,\, t_{i+1}]$. Let $U_i, V_i, W_i \in \mathbb{R}^n$ be the control points. Then $Z(t)$ on $t \in [t_i,\, t_{i+1}]$ is given by equation (2.1).

$$Z(t) \equiv Z_i(t) = (1-\sin\theta)^3 U_i + \left(3\sin\theta - 4\sin^2\theta + \sin^3\theta\right) V_i + \left(3\cos\theta - 4\cos^2\theta + \cos^3\theta\right) W_{i+1} + (1-\cos\theta)^3 U_{i+1}, \tag{2. 1}$$

where

$$\theta = \theta(t) = \frac{t - t_i}{h_i}\,\frac{\pi}{2}, \qquad h_i = t_{i+1} - t_i, \qquad 0 \le \theta \le \frac{\pi}{2}, \qquad i = 0, 1, 2, \ldots, n-1,$$

with

$$V_i = U_i + \frac{2h_i}{3\pi} G_i, \qquad W_i = U_{i+1} - \frac{2h_i}{3\pi} G_{i+1}. \tag{2. 2}$$

Equation (2.2) is the first-order continuity condition satisfying equation (2.5). Now, rewriting (1) in a new form as shown in equation (2.3),

$$Z_i(t) = D_0(\theta)\,U_i + D_1(\theta)\,V_i + D_2(\theta)\,W_i + D_3(\theta)\,U_{i+1}. \tag{2. 3}$$

For which

$$D_0(\theta) = (1 - \sin\theta)^3,$$

$$D_1(\theta) = 3\sin\theta - 4\sin^2\theta + \sin^3\theta,$$

$$D_2(\theta) = 3\cos\theta - 4\cos^2\theta + \cos^3\theta,$$

$$D_3(\theta) = (1 - \cos\theta)^3.$$

Subsequently, the sum of these basis functions is unity, as written in equation (2.4):

$$\sum_{j=0}^{3} D_j(\theta) = 1. \tag{2. 4}$$

Also, we have

$$Z_i(t_i) = U_i, \qquad Z_i(t_{i+1}) = U_{i+1}, \qquad Z_i'(t_i) = G_i, \qquad Z_i'(t_{i+1}) = G_{i+1}. \tag{2. 5}$$

The above shows that the interpolated curve holds a Hermite-like form. Now, to evaluate second-order continuity at the links, we require

$$Z_i''(t_i) = Z_{i-1}''(t_i). \tag{2. 6}$$

Furthermore, continuity for nu spline is given by the following matrix in equation (2.7).

$$\begin{bmatrix} Z(t_{i+}) \\ Z_i'(t_{i+}) \\ Z_i''(t_{i+}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & v_i & 1 \end{bmatrix} \begin{bmatrix} Z(t_{i-}) \\ Z_i'(t_{i-}) \\ Z_i''(t_{i-}) \end{bmatrix} \tag{2. 7}$$

Equation (2.7) After some computations, we have

$$\frac{1}{3h_{i-1}} G_{i-1} + \left[\frac{4}{3h_i} + \frac{4}{3h_{i-1}} + \frac{v_i}{\pi}\right] G_i + \frac{1}{3h_i} G_{i+1} = \frac{\pi}{2h_{i-1}} \Delta_{i-1} + \frac{\pi}{2h_i} \Delta_i \tag{2. 8}$$

where

$$\Delta_i = \frac{U_{i+1} - U_i}{h_i}, \quad i = 0, 1, 2, \ldots, n-1.$$

The equation (2.8) represents a diagonally dominant system with unknowns G_i's and a shape parameter $v_i \ge 0$. The values of unknowns can be intended from a suitable method by signifying suitable settings.

3. Establishment of a local basis for CTNS

Consider a local function $g_k(t)$ to build local basis functions for cubic spline and the additional knots described at the ends of the interval

$$[t_0, t_n] \quad \text{with} \quad t_{-3} < t_{-2} < t_{-1} < t_0 \quad \text{and} \quad t_n < t_{n+1} < t_{n+2} < t_{n+3}.$$
$$Then, g_k(t) \quad \text{for } k = -1, \ldots, n-2 \text{ is given by} \tag{3. 9}$$

$$g_k(t) = \begin{cases} 0, & t < t_{k-2}, \\ 1, & t \geq t_{i+1}. \end{cases} \tag{3. 10}$$

For $i = k-2, k-1, k$ on $[t_i, t_{i+1}]$, the local function $g_k(t)$ follows:

$$g_k(t) = (1-\sin\theta)^3\,\hat{U}_{k,i} + \left(3\sin\theta - 4\sin^2\theta + \sin^3\theta\right)\hat{V}_{k,i} + \left(3\cos\theta - 4\cos^2\theta + \cos^3\theta\right)\hat{W}_{k,i} + (1-\cos\theta)^3\,\hat{U}_{k,i+1}. \tag{3. 11}$$

$$\hat{U}_{k,i} = g_k(t_i), \quad \hat{V}_{k,i} = g_k(t_i) + \frac{2h_i}{3\pi} g'_k(t_i), \quad \hat{W}_{k,i} = g_k(t_{i+1}) - \frac{2h_i}{3\pi} g'_k(t_{i+1}). \tag{3. 12}$$

Now, by applying continuity to $g_k(t)$, we have:

$$g_k(t_{k-2}) = 0, \quad g'_k(t_{k-2}) = 0, \quad g''_k(t_{k-2}) = 0, g_k(t_{k-1}) = \rho_{k-1}, \quad g'_k(t_{k-1}) = \hat{\rho}_{k-1}, g_k(t_k) = 1-\eta_k, \quad g'_k(t_k) = \hat{\eta}_k. \tag{3. 13}$$

where

$$\eta_k = \frac{2h_k}{3\pi}\,\hat{\eta}_k(t_k), \qquad \rho_{k-1} = \frac{2h_{k-2}}{3\pi}\,\hat{\rho}_{k-1}(t_k),$$

$$\hat{\eta}_k(t_k) = \frac{v_3u_1 - v_1u_3}{v_2u_1 - v_1u_2}, \qquad \hat{\rho}_{k-1}(t_k) = \frac{u_3}{u_1} - \frac{u_2}{u_1}\left(\frac{v_3u_1 - v_1u_3}{v_2u_1 - v_1u_2}\right),$$

$$u_1 = \frac{h_{k-2}}{3h_{k-1}^2} + \frac{4}{3h_{k-1}} + \frac{1}{h_{k-2}} + \frac{v_{k-1}}{\pi},$$

$$u_2 = \frac{h_k}{3h_{k-1}^2} + \frac{1}{3h_{k-1}}, \qquad u_3 = \frac{1}{2h_{k-1}^2},$$

Now the freeform local support basis $S_k(t)$ for $k = -1, \ldots, n+1$ is obtained from the difference of local functions as

$$S_k(t) = g_k(t) - g_{k+1}(t), \quad k = -1, \ldots, n+1. \tag{3. 14}$$

It is worth mentioning that both $g_k(t)$ and $g_{k+1}(t)$ satisfy the piecewise cubic trigonometric spline for every $[t_i, t_{i+1}]$. Therefore, the local support basis $S_k(t)$ for $[t_i, t_{i+1}]$ fulfills the following Hermite properties:

$$S_k(t) = D_0(\theta)U_{k,i} + D_1(\theta)V_{k,i} + D_2(\theta)W_{k,i} + D_3(\theta)U_{k,i+1}. \tag{3. 15}$$

where

$$S_k(t_i) = U_{k,i}, \quad V_{k,i} = S_k(t_i) + \frac{2h_i}{3\pi} S'_k(t_i), \quad W_{k,i} = S_k(t_{i+1}) - \frac{2h_i}{3\pi} S'_k(t_{i+1}). \tag{3. 16}$$

The local support basis can be expressed as

$$S_k(t) = (1-\sin\theta)^3 S_k(t_i) + \left(3\sin\theta - 4\sin^2\theta + \sin^3\theta\right)\left(S_k(t_i) + \frac{2h_i}{3\pi}S'_k(t_i)\right) + \left(3\cos\theta - 4\cos^2\theta + \cos^3\theta\right)\left(S_k(t_{i+1}) - \frac{2h_i}{3\pi}S'_k(t_{i+1})\right) + (1-\cos\theta)^3 S_k(t_{i+1}). \tag{3. 17}$$

where

$$S_k(t_i) = 0, \quad S'_k(t_i) = 0, \quad \text{for } i \neq k-1, k, k+1, \tag{3. 18}$$

$$S_k(t_{k-1}) = \rho_{k-1}, \quad S'_k(t_{k-1}) = \hat{\rho}_{k-1}, \tag{3. 19}$$

$$S_k(t_k) = 1 - \eta_k - \rho_k, \quad S'_k(t_k) = \hat{\eta}_k - \hat{\rho}_k, \tag{3. 20}$$

$$S_k(t_{k+1}) = \eta_{k+1}, \quad S'_k(t_{k+1}) = -\hat{\rho}_{k+1}. \tag{3. 21}$$

Proposition 1. The proposed B-spline-like local basis functions hold the following properties:

- **Local support property:**

$$S_k(t) = 0, \quad \text{for } t \notin (t_{k-2}, t_{k+2}). \tag{3. 22}$$

- **Partition of unity property:**

$$\sum_{k=-1}^{n+1} S_k(t) = 1, \quad \text{for } t \in [t_0, t_n]. \tag{3. 23}$$

- **Property of positivity:**

$$S_k(t) \geq 0, \quad \text{for all values of } t. \tag{3. 24}$$

4. Transformation to Bezier curve

For freeform curve designing, the following form will be constructed:

$$Z(t) = \sum_{k=-1}^{i+1} S_k(t)\, Z_k, \tag{4. 25}$$

where $Z_k \in \mathbb{R}^n$ indicates the control points. The local basis functions in terms of linear combination with shape governor points can be represented as

$$Z(t) = \sum_{k=i-1}^{i+2} S_k(t)\, Z_k, \quad t \in [t_i, t_{i+1}],\ i = 0, 1, \ldots, n-1. \tag{4. 26}$$

The Bernstein–Bézier interpolation form of the above equation is given by

$$Z_i(t) = (1-\sin\theta)^3 U_i + \left(3\sin\theta - 4\sin^2\theta + \sin^3\theta\right)V_i + \left(3\cos\theta - 4\cos^2\theta + \cos^3\theta\right)W_i + (1-\cos\theta)^3 U_{i+1}. \tag{4. 27}$$

where

$$U_i = \eta_i Z_{i-1} + (1 - \eta_i - \rho_i) Z_i + \rho_i Z_{i+1}, \tag{4. 28}$$

$$U_{i+1} = \eta_{i+1} Z_i + (1 - \eta_{i+1} - \rho_{i+1}) Z_{i+1} + \rho_{i+1} Z_{i+2}, \tag{4. 29}$$

$$V_i = \left(\eta_i - \frac{2h_i}{3\pi}\hat{\eta}_i\right) Z_{i-1} + \left(1 - \eta_i - \rho_i + \frac{2h_i}{3\pi}(\hat{\eta}_i - \hat{\rho}_i)\right) Z_i + \left(\rho_i + \frac{2h_i}{3\pi}\hat{\rho}_i\right) Z_{i+1}, \tag{4. 30}$$

$$W_i = \left(\eta_{i+1} + \frac{2h_i}{3\pi}\hat{\eta}_{i+1}\right) Z_i \quad + \left(1 - \eta_{i+1} - \rho_{i+1} - \frac{2h_i}{3\pi}(\hat{\eta}_{i+1} - \hat{\rho}_{i+1})\right) Z_{i+1} \quad + \left(\rho_{i+1} - \frac{2h_i}{3\pi}\hat{\rho}_{i+1}\right) Z_{i+2}. \tag{4. 31}$$

TABLE 1. Comparison of Basis Types for Curve Design

Feature	Classical B-spline	NURBS	Trigonometric Nu B-spline
Circular Arc Representation	Poor	Good (with rational weights)	Excellent (intrinsic, no weights)
Periodic/Oscillatory Shape Support	Limited	Moderate	Strong
Shape Parameter Control	None	Limited	Flexible (via ν parameter)
Geometric Continuity (G^2)	Moderate	High (with careful construction)	Natural and smooth
Local Shape Modification	Moderate	Moderate	Strong
Rational Weights Required	No	Yes	No
Curve Compactness	Requires more segments/knots	Moderate	Compact representation
Computational Complexity	Low	High (due to weight handling)	Moderate
Suitability for Arabic Font Design	Limited	Possible	Ideal

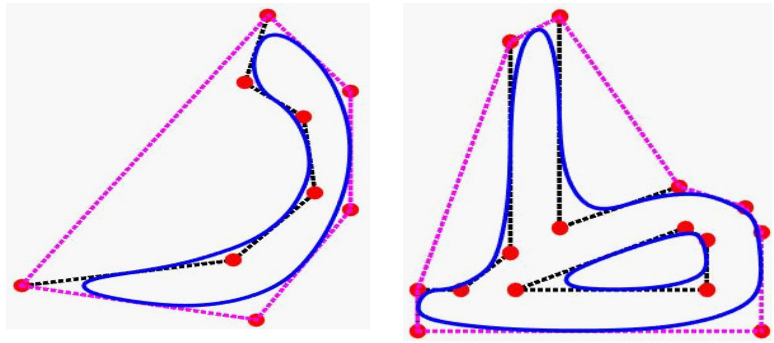
Table 2. shows the comparison of proposed bases with the existing.

5. GEOMETRIC PROPERTIES FOR CTNS

Proposition 2 (Property of Convex Hull).
The CTNS curve for numerous data points must exist within the structure defined by the control points $[U_i, V_i, W_i, U_{i+1}]$.

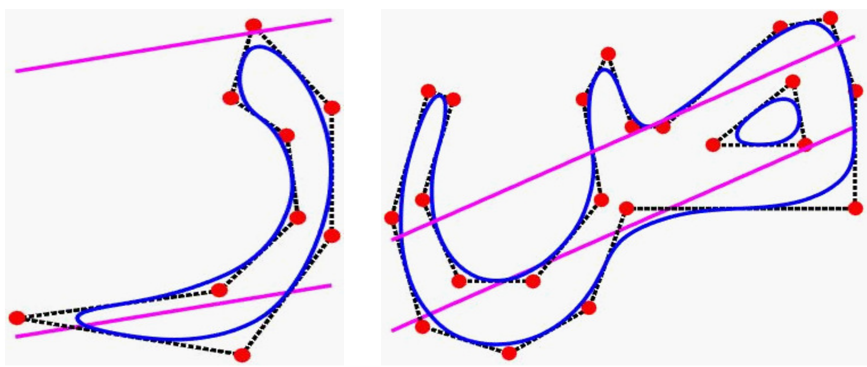
Proof.
The property of positivity (23) and the partition of unity (22) together establish the convex hull property. Since each local basis function $S_k(t)$ is non-negative and the sum of all basis functions equals one, the curve is expressed as a convex combination of its control points. Therefore, the CTNS curve lies entirely within the convex hull of the control points $[U_i, V_i, W_i, U_{i+1}]$. **Proposition 3 (Property of Variation Diminishing).**
Any hyperplane intersects the curve defined by the proposed CTNS less than or equal to the intersections with the control polygon through the control points $[U_i, V_i, W_i, U_{i+1}]$.

Proof.
This property follows directly from the convex hull property (Proposition 2). Since the CTNS curve is contained within the convex hull of its control points, any hyperplane



(a) (b)

FIGURE 1. . This figure satisfies the convex hull property for Arabic letters, they should be listed as (a) "raa" and (b) "taa" exhibits convex hull property.



(a) (b)

FIGURE 2. This figure satisfies the variation diminishing property for Arabic letters, they should be listed as: (a) ”raa” and (b) "saad” demonstrates the property of variation diminishing.

can intersect the curve no more times than it intersects the control polygon formed by $[U_i, V_i, W_i, U_{i+1}]$. Thus, the variation diminishing property is satisfied.

Proposition 4 (Property of Affine Invariance).
Assume an affine transformation is defined as

$$M(x, y) = (hx + ly + m,\ nx + py + q),$$

and let

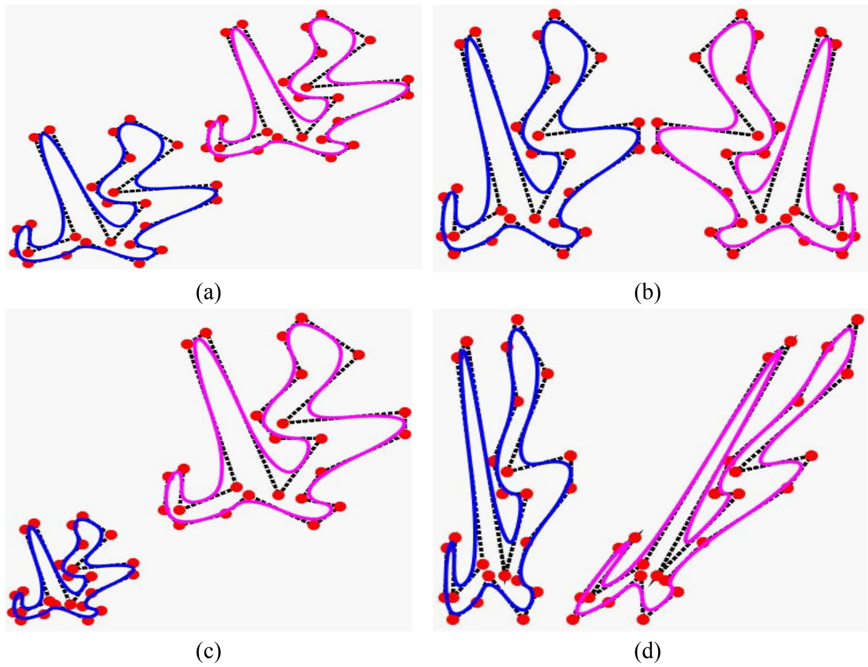


FIGURE 3. Types of Autonomous Agents

$$Z(t) = \sum_{k=0}^{3} S_k(t)\, Z_k$$

be the proposed scheme having the control points

$$Z_k = [U_i, V_i, W_i, U_{i+1}].$$

Then, for the control points $Z_k = (a_k, b_k),\ j = 0, \ldots, 3$, the CTNS satisfies the following transformation:

$$M(Z(t)) = \sum_{k=0}^{3} S_k(t)\, M(Z_k).$$

The proposition is well depicted in Figure 3.

6. Shape Properties

The only parameter v_i is used to represent the shape attributes for different data points illustrated by the CTNS scheme.

Proposition 5 (Point Tension).
Since $v_i \to \infty$ is bounded, then by fixing $i = m$, $v_m \to \infty$ holds and we have

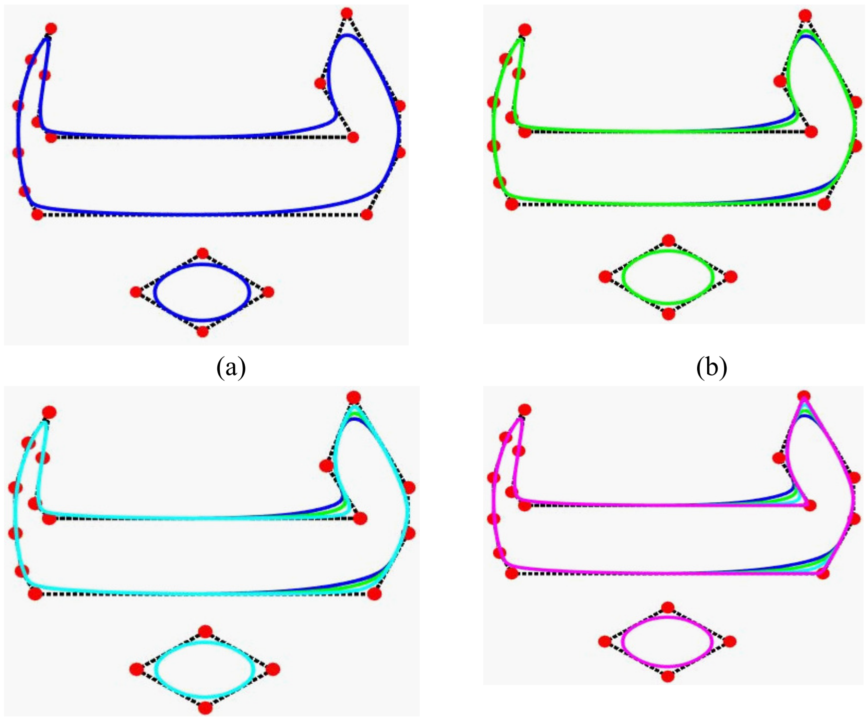

(a) (b)
(c) (d)

FIGURE 4. (a), (b), (c) and (d) illustrate point tension property at three points of "baa" by varying $v_m = 0, 3, 11, 100$.

$$\lim_{v_m \to \infty} Z(t_m) = Z_m, \quad 1 \leq m \leq n-1.$$

That is, a particular portion of the default CTNS curve is dragged in the direction of the corresponding control point.

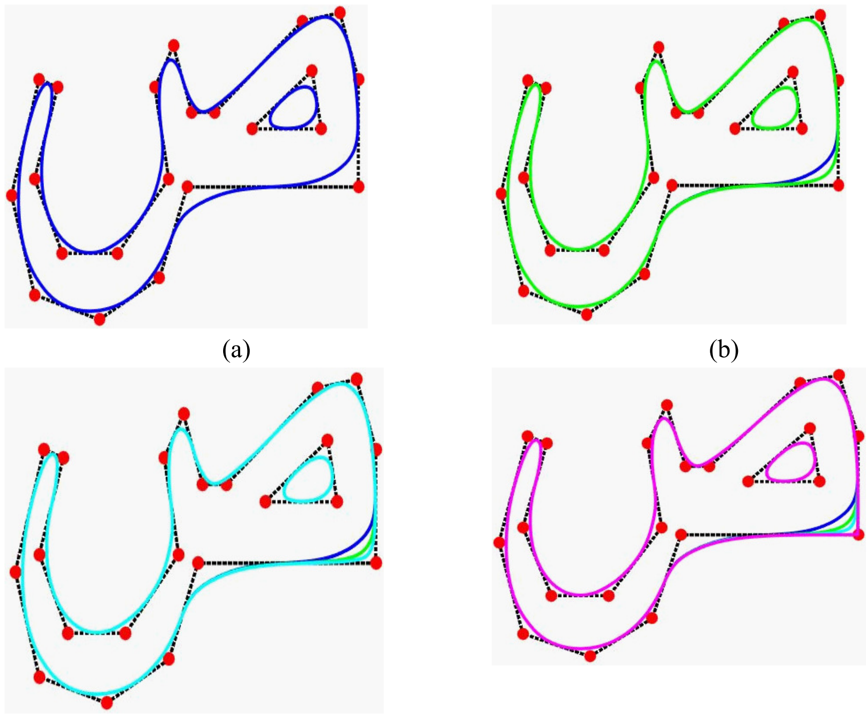

(a) (b)
(c) (d)

FIGURE 5. Tension applied at one point taking $v_m = 0, 5, 12, 300$.

Proposition 6 (Interval Tension).
Suppose an interval $t \in [t_m, t_{m+1}],\ i = 0, 1, 2, \ldots, n-1$ for varying parameter $v_m \geq 0$. Let X_m and X_{m+1} be two distinct points on the line linking Z_m and Z_{m+1} of the control polygon:

$$X_m = (1-\rho)C_m + \rho C_{m+1}, \qquad X_{m+1} = \eta C_m + (1-\eta)C_{m+1},$$

with

$$\rho = \lim_{v_m \to \infty} \rho_m, \qquad \eta = \lim_{v_m \to \infty} \eta_{m+1}.$$

Then, the freeform CTWNS (24) converges uniformly to $X(t)$ for $v_m \to \infty$ on $[t_m, t_{m+1}]$, where

$$X(t) = (1-\theta)X_m + \theta X_{m+1}, \qquad \theta(t) = \frac{t - t_m}{h_m}.$$

Proof.
From (12), we know that $\rho + \eta < 1$ and X_{m+1} lies after X_l. Also, as $v_m \to \infty$, we have $\hat{\eta}_{m+1} \to \infty$, thus $\hat{\rho}_m \to \infty$.

Hence,

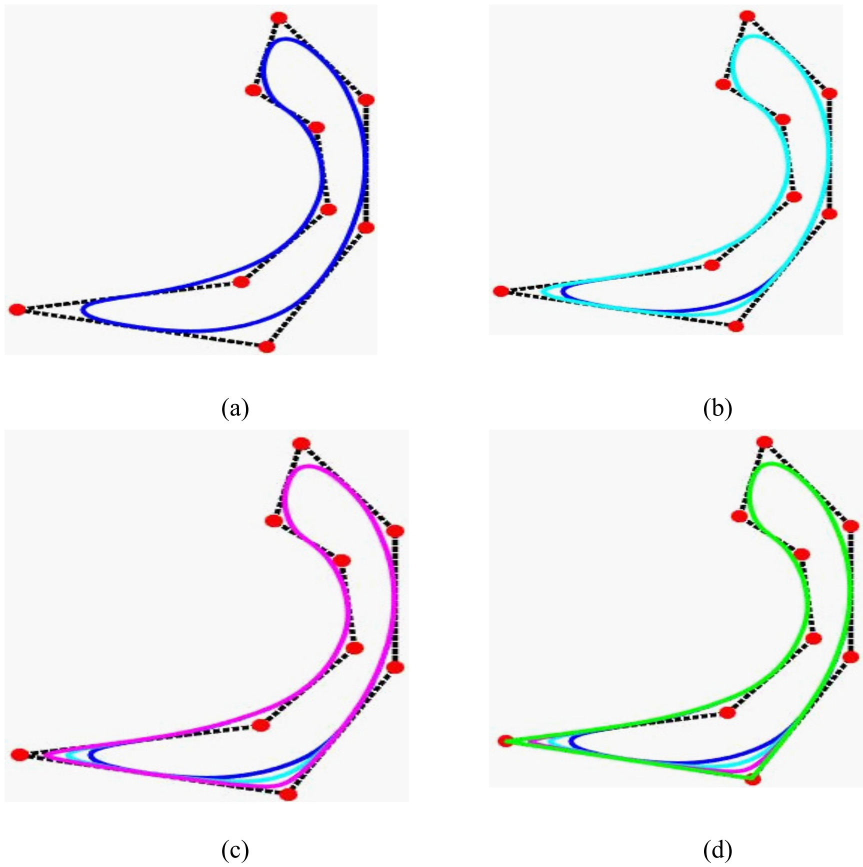


FIGURE 6. In (a)-(d), an interval tension property is revealed in the shape "dal" using $v_m = 0, 4, 14, 300$.

$$\lim_{v_m \to \infty} \rho_m = 0, \qquad \lim_{v_m \to \infty} \eta_{m+1} = 0.$$

Now, from (27)–(30), we obtain

$$\lim_{v_m \to \infty} U_m = X_m, \qquad \lim_{v_m \to \infty} U_{m+1} = X_{m+1}.$$

Moreover, the Bernstein–Bézier form (26) can be expressed as

$$Z(t) = p_m(t) + e_m(t; r_m), \qquad t \in [t_m, t_{m+1}].$$

where

$$p_m(t) = (1-\theta)U_m + \theta U_{m+1}, \qquad \theta(t) = \frac{t - t_m}{h_m},$$

and

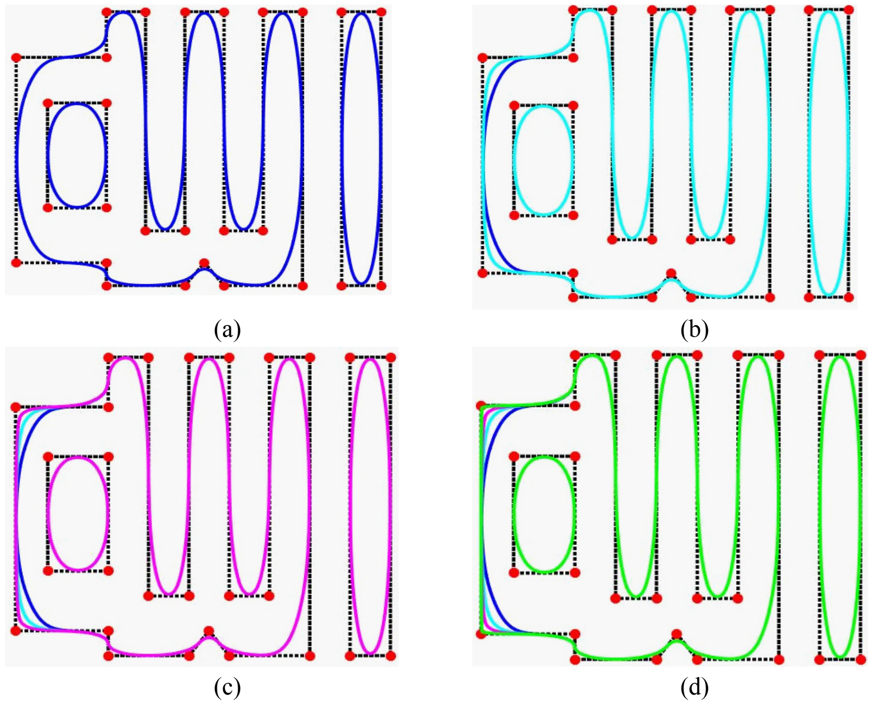


FIGURE 7. Interval tension property in (a), (b), (c) and (d) using $v_m = 0, 6, 17, 100$.

$$\lim_{v_m \to \infty} \|e_m\| = 0.$$

Now we have

$$\lim_{v_m \to \infty} \|X_m - p_m\| \leq \lim_{v_m \to \infty} \|X - Z\| + \lim_{v_m \to \infty} \|e_m\| = 0, \qquad t \in [t_m, t_{m+1}],$$

which finalizes the proof.

Proposition 7 (Global Tension).

For $v_i \geq v \geq 0,\ i = 1, 2, 3, \ldots, n$, the proposed CTNS curve converges uniformly to the control polygon as $v_i \to \infty$.

Proof.

Suppose $v_i = v\ \forall i$. Then,

$$\lim_{v \to \infty} Z'(t_i) = 0.$$

Also,

$$\lim_{v \to \infty} \hat{\rho}_i = 0 = \lim_{v \to \infty} \hat{\eta}_i, \quad \forall i.$$

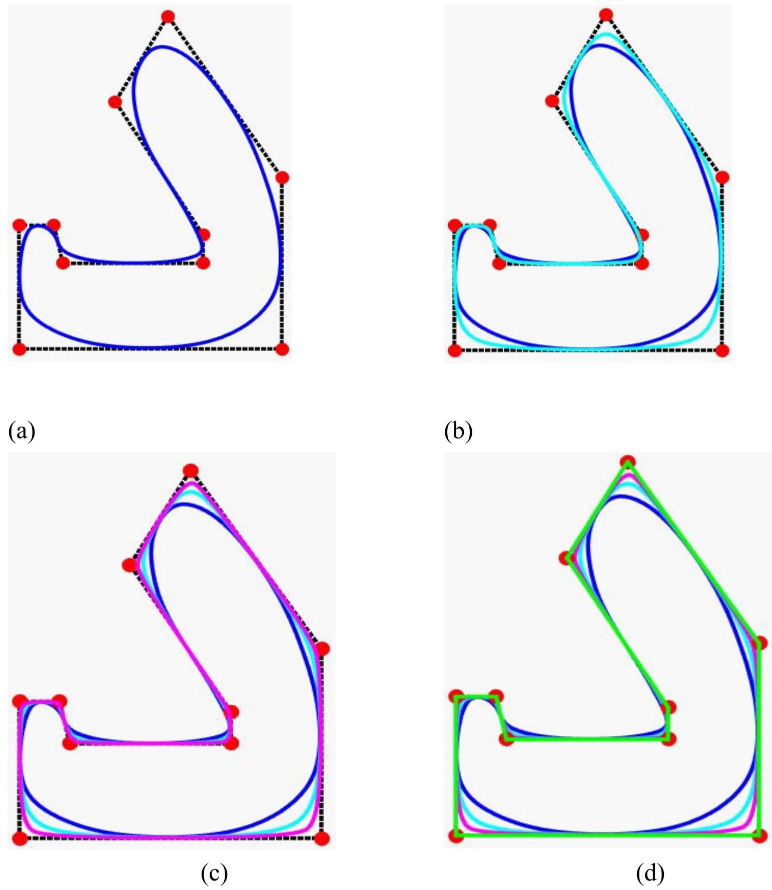


FIGURE 8. Property of global tension setting $v_i = 0, 6, 17, 1000$.

This implies that

$$\lim_{v\to\infty} U_i = Z_i, \quad \forall i, \qquad \lim_{v\to\infty} V_i = Z_i, \quad \forall i,$$

$$\lim_{v\to\infty} U_{i+1} = Z_{i+1}, \quad \forall i, \qquad \lim_{v\to\infty} W_i = Z_{i+1}, \quad \forall i.$$

It is evident from the above equations that the cubic trigonometric Nu-spline converges to the control polygon as the tension parameter approaches infinity. Hence, the generalized result can be written as

$$\lim_{v_i\to\infty} Z(t_i) = Z_i, \quad \forall i.$$

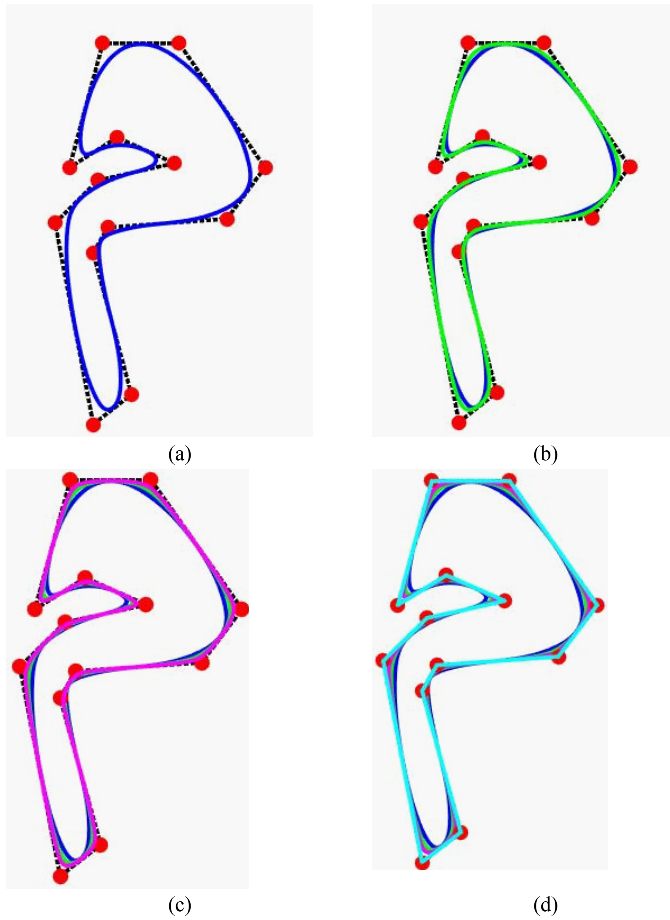


FIGURE 9. Global tension property in (a)-(d) taking $v_i = 0, 4, 10, 400$, respectively.

More generally, for $v_i \geq v \geq 0,\ i = 1, 2, 3, \ldots, n$, it can be presented that

$$\max_i |\hat{\rho}_i| \leq w(v), \qquad \max_i |\hat{\eta}_i| \leq j(v),$$

with

$$\lim_{v\to\infty} w(v) = 0 = \lim_{v\to\infty} j(v).$$

This verifies the global tension property.

7. DEMONSTRATION

Several kinds of 2D Arabic typefaces are created using the proposed CTNS system. Every shape construct involves a default form of CTNS at $v_i = 0$. The Arabic fonts “baa”, "raa", "dal", "saad", "taa", "miim", "Allah", and "Muhammad" are modeled and designed to satisfy the defined properties. The following discussion describes the ideology behind the discovery of fonts built from cubic trigonometric Nu-spline to fulfill the mathematically proved properties.

Figure 1 elaborates on the *convex hull property*, in which Arabic fonts "raa" and "taa" built from the constructed cubic trigonometric Nu-spline exist inside the polygon structured by the involved control points. The *variation diminishing property*, shown in Figure 2, displays that the lines drawn at "raa" and "saad" cut the cubic trigonometric Nu-spline curve less than or equal to the control polygon, thus accomplishing the property.

Various affine transformations such as translation, rotation, scaling, and shear for the cubic trigonometric Nu-spline have been employed on data points of the Arabic font "Muhammad", with graphical results shown in Figures 3(a)–3(d).

The *point tension property* verifies that the cubic trigonometric Nu-spline curve moves towards a single point for increasing arbitrary values of the tension parameter. Figures 4 and 5 demonstrate the same point tension strategy at "baa" on two points and at "saad" on one point with varying values of $v_m = 0, 3, 11, 100$ and $v_m = 0, 5, 12, 300$, respectively. It can be seen from these figures that the CTNS curve is dragged towards the control points wherever the tension is applied.

In the same manner, Figures 6 and 7 illustrate the *interval tension property* with Arabic fonts "raa" and "Allah" under the influence of the tension parameter on two consecutive points at $v_m = 0, 4, 14, 300$ and $v_m = 0, 6, 17, 3700$, correspondingly. Interval tension property is observed by applying the tension at two joining control data points on the line segment of a curve.

A *global tension* is achieved in Figure 8 for the Arabic letter "dal" for parametric successive values $v_i = 0, 6, 17, 1000$, which demonstrates the strategy of tension at each point simultaneously. The same strategy is followed by the letter "miim" in Figure 9 by controlling each point of CTNS at $v_i = 0, 4, 10, 400$. Global tension property is satisfied by setting a random value of the tension parameter at each point of the cubic trigonometric Nu-spline curve.

The above discussion has explored the illustrated font design for the evaluated properties of the cubic trigonometric Nu-spline. Moreover, it highlights the discovery of a satisfactory effort of a tension factor for numerous properties of geometry and shape.

8. Conclusion

The research conducted offers a freeform proficient graphical approximation and numerical technique using a cubic trigonometric Nu-spline that contains B-spline-like basis functions and local support. The presented cubic trigonometric Nu-spline has a smoothness of GC^2. The proposed B-spline-like local basis functions satisfy the properties of local support, partition of unity, and positivity. A transformation from B-spline to Bernstein–Bézier form is also presented.

This local support curve methodology consists of two shape parameters and plays a vital role in computer-aided geometric design for the designing and modeling of fonts. Different 2D Arabic letters have been constructed to justify the geometric and shape properties by controlling the values of a single parameter at points, intervals, and entire curves. It has

TABLE 2. Advantages of the Proposed Scheme as Comparison to Existing Splines

#	**Advantages of the Proposed Scheme**
1	Authors have constructed a new trigonometric Nu B-spline technique which is different in construction from the simple B-spline and contains shape parameters that are helpful in shape modifications locally and globally according to one's desire.
2	Arabic fonts are formatted using the suggested spline techniques. In stark contrast to the Latin script used in English, Arabic script is a cursive writing system written from right to left. The structure of the Arabic script and the relationships between the characters must be thoroughly understood while designing Arabic fonts.
3	Most existing schemes in the literature focus on capturing outlines of typefaces. In contrast, our scheme provides an algorithm for modeling and designing Arabic typefaces. A novel cubic Bézier Nu trigonometric spline has been constructed for this purpose. The Nu B-spline contains shape parameters to control the shape accordingly, which is different from existing B-splines.
4	The method supports the cubic B-spline's advantageous characteristics.
5	The built scheme contains the characteristics of both B-spline and Nu-spline.
6	This method has a geometrical smoothness of order 2, i.e., GC^2. So, it is geometrically flexible.
7	It satisfies the properties of local support, partition of unity, and positivity.
8	It recovers all the geometric properties like convex hull, variation diminishing, and affine invariance.
9	It also covers the shape properties of point tension, interval tension, and global tension.
10	It has one family of a shape control parameter which provides various shape attributes as explained in the paper.
11	The cubic trigonometric Nu-spline utilizes positively varying values of the shape controller.
12	The curve strategy is computationally convenient due to conversion from local basis functions to Bernstein basis functions.
13	The authors are in the process of extending this work to Nastaleeq style.

been found that flexibility in shape increases by producing tension in the curve wherever required.

The developed scheme has freeform geometric characterization due to the conversion of local support basis to piecewise blending Bernstein basis. These transformed blending functions behave like the basis functions utilized to construct the Bézier curve. In short, it is a robust geometric approach for both calculation and approximation to achieve elegant designs.

Features of the Proposed Cubic Trigonometric Nu-spline.

- Supports the advantageous characteristics of cubic splines.
- Contains the combined characteristics of B-spline and Nu-spline.
- Provides geometrical smoothness of order 2 (GC^2), ensuring flexibility.

- Satisfies local support, partition of unity, and positivity.
- Recovers geometric properties such as convex hull, variation diminishing, and affine invariance.
- Covers shape properties of point tension, interval tension, and global tension.
- Offers one family of shape control parameters providing diverse shape attributes.
- Utilizes positively varying values of the shape controller.
- Ensures computational convenience through conversion from local basis functions to Bernstein basis functions.
- Incorporates geometric continuity (G^2) in Arabic font design, making curves smoother and more pleasant.
- Introduces a parameter-controlled basis blending shape flexibility with B-spline properties in a consistent framework.
- Demonstrates adaptability to various Arabic typefaces with distinct structural geometries.

Availability of Supporting Data. All the data used in this study is presented within the paper.

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