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Hilfer-Hadamard Volterra Fractional Integro-Differential Equations with Initial Conditions

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*Abduljawad K. Anwar
Department of Mathematics, College of Science, University of Duhok, Duhok-42001, Iraq

Shayma Adil Murad
Department of Mathematics, College of Science, University of Duhok, Duhok-42001, Iraq

Abstract. The focal point of this research is the investigation of Hilfer-Hadamard. Volterra fractional integro-differential equations subject to nonlocal initial conditions. The study addresses the analytical challenges in establishing the existence and uniqueness of solutions under appropriate assumptions. To this end, Krasnoselskii's fixed point theorem and the Banach contraction principle are employed to guarantee the well-posedness of the proposed system. Furthermore, the Ulam-Hyers and Ulam-Hyers-Rassias stability concepts are utilized to analyze the stability properties of the obtained solutions, offering deeper insights into the robustness of the model. Finally, several illustrative examples are provided to substantiate the theoretical results and to demonstrate the accuracy, applicability, and efficiency of the proposed approach.

AMS (MOS) Subject Classification Codes: 34A12; 26A33; 47F10; 34K20
Key Words: Fractional Differential Equations; Volterra integro-differential equations $\mathcal{VIDEs}$; Hilfer-Hadamard fractional derivative$\mathcal{HH}$; fixed point theorem; Ulam-Hyers stability ($\mathcal{UH}$)..

1. Introduction

Recently, fractional differential equations are regarded as one of the most important subjects for mathematicians because it is widely used in various branches of science such as engineering, mathematics, physics, see[8, 16, 20, 30, 32, 38]. At the same time, many researchers explained important theoretical results of the fractional calculus and differential equations (DEs) with arbitrary order, one can consult monographs for clarification, Kilbas et al.[25], Samko et al.[35].

Several researchers have contributed to the study of fractional differential equations with various types of boundary conditions and analytical frameworks. Anwar [9] investigate the existence and uniqueness of Atangana-Baleanu solutions in Caputo sense fractional differential equations with boundary conditions. Murad and Hadid [29] the existence and uniqueness of solutions for fractional differential equations subject to integral boundary conditions, establishing key foundational results in this area. Song and Cu [36] analyzed

*Corresponding Author: abduljawad.anwar@uod.ac

integral boundary value problems for mixed fractional differential equations under resonance, extending the discussion to cases where standard existence conditions fail. Abdo et al. [2] examined the existence of positive solutions for fractional differential equations with integral boundary conditions, highlighting the role of positivity in applied models. Anwar and Murad [10] focused on Caputo-Hadamard fractional differential equations, proving existence and Ulam stability results, thereby linking stability theory with fractional dynamics. Panda et al. [31] developed a novel fixed point approach to study Atangana-Baleanu fractional and (L^p)-Fredholm integral equations, which strengthened the analytical tools for modern fractional models. In [5], the authors established existence results for fractional functional differential equations, emphasizing the functional dependence in fractional systems.

Abbas et al. [1] extended the framework of Caputo-Hadamard fractional differential equations to Banach spaces, ensuring the existence and uniqueness of solutions in a generalized functional setting.

Furthermore, many researchers attracted to study the stability of solutions because it is check the behavior of solutions over the time. Ibrahim [22] and Khan et al. [24] investigated Ulam?Hyers stability $(\mathcal{UH})$ for boundary value and singular-delay fractional problems. Benchohra and Lazreg [13], together with Almalahi et al. [7], studied existence and Ulam-type stability in systems with Hadamard and weighted fractional operators. Murad [28] focused on the stability analysis of nonlinear two-point boundary value problems involving the Caputo fractional derivative, while Dai et al. [14] examined Ulam-Hyers-Rassias stability $(\mathcal{UHR})$ for a wider class of fractional equations.

On the other hand, attractive results for various classes of fractional integro-differential equations $(FIDEs)$ have been reported. Wang et al. [41] and Sousa [37] studied Ulam-Hyers stability, Dai and Liu [15] analyzed stability using weighted spaces, Mardanov et al. [27], and Baleanu et al. [12] focused on the existence and uniqueness of solutions for nonlinear equations with integral or nonlocal boundary conditions.

Hamoud et al. [19] investigated Hadamard-Caputo implicit fractional integro-differential equations with boundary fractional conditions, Ivaz et al. [23] studied Hilfer fractional Volterra-Fredholm integro-differential equations, and Hussain [21] presented new existence and uniqueness results for Caputo fractional integro-differential equations.

Elahi et al. [17] used Laguerre polynomials to solve linear Fredholm integro-differential equations, achieving high accuracy and efficiency. Elahi et a1. [18] applied Laguerre polynomials to a linear system of Fredholm integral equations, transforming it into linear algebraic equations solvable by standard methods. Akram et al. [6] employed Laguerre polynomials to efficiently solve linear systems of differential equations, providing accurate approximate solutions.

The existence of solutions for the nonlinear fractional integro-differential equations (FIDEs) studied by Ahmad and Sivasundaram [4] is investigated by considering the following nonlinear fractional integro-differential equation with four-point nonlocal boundary conditions.

$$\begin{cases} {}^{c}D^{q}u(\eth) = f(\eth, u(\eth), (\phi u)(\eth), (\psi u)(\eth)), & 0 < \eth < 1,\ 1 < q \leq 2, \\ u'(0) + au(\eta_1) = 0, \quad bu'(1) + u(\eta_2) = 0, & 0 < \eta_1 \leq \eta_2 < 1, \end{cases}$$

where ${}^{c}D$ is the Caputo?s fractional derivative, $f : [0,1] \times X \times X \times X \to X$ is continuous, and for $\gamma, \delta : [0,1] \times [0,1] \to [0,\infty)$, with $a, b \in (0,1)$.

$$(\phi u)(\eth) = \int_0^{\eth} \gamma(\eth, \nabla) u(\nabla)\, d\nabla, \qquad (\psi u)(\eth) = \int_0^{\eth} \delta(\eth, \nabla) u(\nabla)\, d\nabla.$$

In[39] authors apply the Schaefer's fixed point theorem to seek for the existence of solutions of fractional Hilfer-Hadamard IDEs and using ($\mathcal{UH}$) analyze the stability. Additionally, the authors in [3] establish the existence of solutions for Hilfer-Hadamard ($\mathcal{HH}$) fractional boundary value problems ($\mathcal{BVP}$s) by employing fixed point theorems, specifically those of Krasnoselskii and Schaefer, along with the Leray-Schauder nonlinear alternative. Regarding recent findings related to the Hilfer-Hadamard fractional derivative, several studies have investigated boundary value problems, existence, uniqueness, and dynamical behavior. Moreover, Hilfer-Hadamard?Volterra fractional integro-differential equations provide an effective framework for modeling real-world phenomena with memory and hereditary effects. Such equations naturally arise in physics, biology, control theory, and epidemiology. The use of local or nonlocal initial conditions enables the modeling of systems with either well-defined starting states or time-distributed interactions, offering a more realistic representation of complex dynamical processes.

Vivek et al. [40] conducted a dynamical analysis of fractional pantograph equations using successive approximation. Bachira et al. [11] focused on existence and attractivity, while Saengthong et al. [34] analyzed existence and uniqueness for systems with two-point boundary conditions.

Promsakon et al. [33] extended the study to nonlocal integro-multipoint fractional bound-ary value problems, demonstrating the flexibility of the Hilfer-Hadamard derivative in modeling complex fractional systems.

Inspired by the aforementioned research, we investigate the existence and stability properties of solutions to the following Hilfer-Hadamard($\mathcal{HH}$) Volterra fractional integro-differential equations ($\mathcal{VFIDE}s$):

$$ {}^{HH}D_{1+}^{\beth,\omega} u(\eth) = \phi(\eth, u(\eth), (\theta u)(\eth), (\psi u)(\eth)), \quad \eth \in \mathfrak{T} = [1, e], \quad \beth \in (1,2], \quad \omega \in [0,1], \tag{1. 1}$$

with initial conditions

$$\begin{aligned} {}^{H}D^{\mho-2} u(1) &= 0, \\ {}^{H}D^{\mho-1} u(1) &= {}^{H}I^{\mho-1} u(\mu), \end{aligned} \tag{1. 2}$$

where $\mho = \beth + n\omega - \beth\omega$, ${}^{H}D^{\mho-2}$ is Hadamard fractional derivative of order $\mho - 2$,${}^{H}I^{\mho-1}$ is Hadamard fractional integral of order $\mho - 1$ and ${}^{HH}D_{1+}^{\beth,\omega}$, is Hilfer-Hadamard fractional derivative with $\phi : \mathfrak{T} \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is continuous with the property

$$(\theta u)(\eth) = \int_1^{\eth} y(\eth, \nabla) u(\nabla) d\nabla, \qquad (\psi u)(\eth) = \int_1^{\eth} z(\eth, \nabla) u(\nabla) d\nabla. \tag{1. 3}$$

when and $\int_1^{\eth} y(\eth, \nabla) u(\nabla) d\nabla < \infty$, and $\int_1^{\eth} z(\eth, \nabla) u(\nabla) d\nabla < \infty$, where y, z are continuous functions on $\mathfrak{T} \times \mathfrak{T}$.

This paper is organized as follows. Section 2 provides preliminary results that will be utilized in subsequent sections. In Section 3, we establish the existence and uniqueness of a viable integral solution to the problem (1. 1)-(1. 2) by apply standard fixed

point techniques. Section 4 discusses the Ulam-Hyers ($\mathcal{UH}$) stability and Ulam-Hyers-Rassias ($\mathcal{UHR}$) stability of the solutions. Finally, Section 5 presents illustrative examples to demonstrate the practical applicability of the proposed results.

2. Notations and Preliminaries

In this section, we present some basic definitions and lemmas that are essential and will be used throughout this work.

Definition 2.1. [25] Let $\gamma \in \mathcal{R}$, then the Hadamard fractional integral:

$$I^{\gamma}_{a+}\phi(x) := \frac{1}{\Gamma(\beth)} \int_{a+}^{x} \Big(\ln(\frac{x}{\tau})\Big)^{\beth-1} \frac{\phi(\tau)}{\tau}\, d\tau. \tag{2. 4}$$

Definition 2.2. [25] Let $\gamma \in \mathcal{R}$, with $n = [\beth] + 1$ and $\delta = (x\frac{d}{dx})$; then the Hadamard derivative of fractional of a continuous function ϕ is defined as:

$$D^{\gamma}_{a+}\phi(x) := \frac{1}{\Gamma(n-\beth)}\delta^n \int_{a+}^{x} \Big(\ln(\frac{x}{\tau})\Big)^{n-\beth-1} \frac{\phi(\tau)}{\tau} d\tau, \tag{2. 5}$$

where Γ is the gamma function and $[\gamma]$ denotes the integer part of the real number γ.

Definition 2.3. [20] Let $\phi : [0, +\infty) \to \mathbb{R}$ be a differentiable function. The Hilfer-Hadamard fractional derivative of order $n-1 < \beth < n$ and type $0 \le \omega \le 1$ is defined as

$$({}^{HH}D^{\beth,\omega}\phi)(t) = \big({}^{H}I^{\omega(n-\beth)}\delta^n\, {}^{H}I^{(n-\beth)(1-\omega)}\phi\big)(t) = \big({}^{H}I^{\omega(n-\beth)}\delta^n\, {}^{H}I^{(n-\mu)}\phi\big)(t) = \big({}^{H}I^{\omega(n-\beth)}\, {}^{H}D^{\mu}\phi\big)(t),$$

where

$$\mu = \beth + n\omega - \beth\omega, \quad \delta = \frac{d}{dt},$$

and ${}^{H}I^{(\cdot)}$ and ${}^{H}D^{(\cdot)}$ denote the Hadamard fractional integral and derivative, respectively.

Definition 2.4. [41] The problem (1. 1) is known as ($\mathcal{UH}$) stable if $\grave{q}$, any positive constant such that for each $\epsilon > 0$, when $w \in C(\mathfrak{T}, \mathcal{R})$, is a solution of the below inequality

$$|{}^{HH}D^{\beth,\omega}_{1+}u(\grave{\eth}) - \phi(\grave{\eth}, u(\grave{\eth}), (\theta u)(\grave{\eth}), (\psi u)(\grave{\eth}))| \le \epsilon, \quad \grave{\eth} \in \mathfrak{T}, \tag{2. 6}$$

and $\exists$ an additional solution $y \in C(\mathfrak{T}, \mathcal{R})$, to equation (1. 1) that satisfies the following inequality

$$|u(\grave{\eth}) - y(\grave{\eth})| \le \grave{q}\epsilon, \quad \grave{\eth} \in \mathfrak{T}. \tag{2. 7}$$

Definition 2.5. [41] The problem (1. 1) is said to possess ($\mathcal{UHR}$) stability with respect to a function $\beta \in C(\mathfrak{T}, \mathcal{R})$ and let V be a positive. Specifically, for any $\epsilon > 0$ and any function $u_1 \in C(\mathfrak{T}, \mathcal{R})$ that satisfies the inequality

$$|{}^{HH}D^{\beth,\omega}_{1+}u(\grave{\eth}) - \phi(\grave{\eth}, u(\grave{\eth}), (\theta u)(\grave{\eth}), (\psi u)(\grave{\eth}))| \le \epsilon\beta(\grave{\eth}), \quad \grave{\eth} \in \mathfrak{T}, \tag{2. 8}$$

thus, there exist a solution $S \in C(\mathfrak{T}, \mathcal{R})$ of the problem (1. 1) satisfied

$$|u(\grave{\eth}) - S(\grave{\eth})| \le V\epsilon\beta(\grave{\eth}), \quad \grave{\eth} \in \mathfrak{T}. \tag{2. 9}$$

Theorem 2.6. [35] Let $\beth > 0$, $0 \leq \omega \leq 1$,$\mho = \beth + n\omega - \beth\omega$, $n = [\beth] + 1$. If $f \in L^1(a,b)$ and $({}^{H}I^{n-\mho}\phi)(\eth) \in AC^n_\delta[a,b]$, then

$$
\begin{aligned}
{}^{H}I^{\beth}_{a}({}^{HH}D^{\beth,\omega}\phi)(\eth) &= {}^{H}\ I^{\mho}_{a}({}^{HH}D^{\mho}\phi)(\eth) \\
&= \phi(\eth) - \sum_{k=0}^{n-1} \frac{(\delta^{n-k-1}({}^{H}I^{n-\mho}_{a}\phi))(a)}{\Gamma(\mho-k)} \left(\ln(\frac{\eth}{a})\right)^{\mho-k-1}.
\end{aligned}
$$

Clearly that $\Gamma(\mho-k)$ exists for all $k = 1, 2, ..., n-1$ for $\mho \in [\beth, n]$.

Theorem 2.7. [26] (Krasnoselskii Fixed Point Theorem).
Consider a Banach space E and a subset $z \subseteq E$ be a closed convex and nonempty. Suppose that A_1 and A_2 are two operators satisfying the following conditions:

(1) $A_1 z_1 + A_2 z_2 \in Z$, whenever $z_1, z_2 \in Z$,
(2) A_1 is compact and continuous;
(3) A_2 is a contraction.

Then there exists $x \in Z$ such that $x = A_1 x + A_2 x$.

Lemma 2.8. For any $u(\eth) \in C(\mathfrak{T}, \mathcal{R})$, $1 < \beth \leq 2$, then the IVP (1. 1)-(1. 3) has a solution

$$
u(\eth) = \frac{\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b} \int_1^{\mu} \frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)} h(\nabla)\frac{d\nabla}{\nabla} + \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} h(\nabla)\frac{d\nabla}{\nabla}.
$$

Proof. Applying the Hadamard fractional integral on (1. 1) and apply Theorem (2.6), we obtain

$$
u(\eth) - \frac{\delta({}^{H}I^{2-\mho}u)(1)}{\Gamma(\mho)}(\ln(\eth))^{\mho-1} - \frac{({}^{H}I^{2-\mho}u)(1)}{\Gamma(\mho-1)}(\ln(\eth))^{\mho-2} = {}^{H}I^{\beth}_{1}h(\eth),
$$

$$
u(\eth) = \frac{\delta({}^{H}I^{2-\mho}u)(1)}{\Gamma(\mho)}(\ln(\eth))^{\mho-1} + \frac{({}^{H}I^{2-\mho}u)(1)}{\Gamma(\mho-1)}(\ln(\eth))^{\mho-2} + {}^{H}I^{\beth}_{1}h(\eth),
$$

which can be rewritten as follows:

$$
u(\eth) = c_0(\ln(\eth))^{\mho-1} + c_1(\ln(\eth))^{\mho-2} + \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} h(\nabla)\frac{d\nabla}{\nabla}.
$$

Next, we will use the boundary conditions specified in (1. 2) to determine the values of the constants c_0 and c_1

$$
D^{\mho-2}u(\eth) = \Gamma(\mho)c_0(\ln(\eth)) + c_1 + \int_1^{\eth} \frac{(\ln((\frac{\eth}{\nabla})))^{n-\mho-3}}{\Gamma(\mho-2)} \left(\int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} h(\nabla)\frac{d\nabla}{\nabla}\right)\frac{d\nabla}{\nabla},
$$

$$
D^{\mho-2}u(1) = \Gamma(\mho)c_0(\ln(1)) + c_1 + \int_1^{1} \frac{(\ln((\frac{\eth}{\nabla})))^{n-\mho-3}}{\Gamma(\mho-2)} \left(\int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} h(\nabla)\frac{d\nabla}{\nabla}\right)\frac{d\nabla}{\nabla},
$$

we get $c_1 = 0$, which can be rewritten as follows:

$$
u(\eth) = c_0(\ln(\eth))^{\mho-1} + \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} h(\nabla)\frac{d\nabla}{\nabla}.
$$

Now, take the derivative $D^{\mho-1}$, to both sides

$$D^{\mho-1}u(\eth) = D^{\mho-1}c_0(\ln(\eth))^{\mho-1} + D^{\mho-1}\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}h(\nabla)\frac{d\nabla}{\nabla},$$

$$D^{\mho-1}u(1) = \Gamma(\mho)c_0 + \int_1^1\frac{(\ln((\frac{\eth}{\nabla})))^{n-\mho-2}}{\Gamma(\mho-1)}\bigg(\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}h(\nabla)\frac{d\nabla}{\nabla}\bigg)\frac{d\nabla}{\nabla},$$

$$D^{\mho-1}u(1) = \Gamma(\mho)c_0.$$

But, from the condition $D^{\mho-1}u(1) = I^{\mho-1}u(\mu)$, implies that

$$D^{\mho-1}u(\eth) =_1^{\mu} I^{\mho-1}c_0(\ln(\eth))^{\mho-1} + I^{\mho-1}\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}h(\nabla)\frac{d\nabla}{\nabla},$$

$$\Gamma(\mho)c_0 - c_0\frac{\Gamma(\mho)(\ln(\eth))^{2\mho-2}}{\Gamma(2\mho-1)} = \int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho-2}}{\Gamma(\mho-1)}\bigg(\int_1^{\nabla}\frac{(\ln\frac{\nabla}{r})^{\beth-1}}{\Gamma(\beth)}h(r)\frac{dr}{r}\bigg)\frac{d\nabla}{\nabla}.$$

Let $b = \frac{\Gamma(\mho)(\ln(\eth))^{2\mho-2}}{\Gamma(2\mho-1)}$, then $c_0 = \frac{1}{\Gamma(\mho)-b}\Upsilon$, where

$$\Upsilon = \int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho-2}}{\Gamma(\mho-1)}\bigg(\int_1^{\nabla}\frac{(\ln\frac{\nabla}{r})^{\beth-1}}{\Gamma(\beth)}h(r)\frac{dr}{r}\bigg)\frac{d\nabla}{\nabla},$$

by using Fubini?s theorem, we get

$$\Upsilon = \int_1^{\mu}\int_r^{\mu}\frac{(\ln(\mu)-\ln(\nabla))^{\mho-2}}{\Gamma(\mho-1)}\frac{(\ln(\nabla)-\ln(r))^{\beth-1}}{\Gamma(\beth)}\frac{d\nabla}{\nabla}h(r)\frac{dr}{r}. \tag{2. 10}$$

Suppose that $\ln(\nabla) = \ln(r)+u(\ln(\mu)-\ln(r))$, when $s = r \to u = 0$ and $s = \mu \to u = 1$, $\frac{d\nabla}{\nabla} = (\ln(\frac{\mu}{r}))du$. From (2. 10), we obtain

$$\Upsilon = \int_1^{\mu}\frac{(\ln(\frac{\mu}{r}))^{\mho+\beth-2}}{\Gamma(\mho-1)\Gamma(\beth)}\bigg(\int_0^1(1-u)^{\mho-2}u^{\beth-1}du\bigg)h(r)\frac{dr}{r},$$

By using beta function we have

$$u(\eth) = \frac{1}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}h(\nabla)\frac{d\nabla}{\nabla}(\ln(\eth))^{\mho-1} + \int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}h(\nabla)\frac{d\nabla}{\nabla}.$$

□

3. Existence and Uniqueness Results

In this section, we investigate the existence of solutions for the problem (1. 1)-(1. 2), and then we prove uniqueness. Let ϕ be a continuous function, and $C(\mathfrak{T},\mathcal{R})$ be the Banach space consisting of all continuous real-valued functions defined on

$$\|u\| = sup\{\Big|u(\eth)\Big|;\eth\in\mathfrak{T}\}.$$

In order to establish our findings, we use the below assumptions:
($\mathbf{F1}$) Let $L > 0$, be a constant such that

$$\Big|\phi(\eth,u(\eth),(\theta u)(\eth),(\psi u)(\eth))\Big| \le L.$$

(**F2**) There exists positive functions $k_1(\eth), k_2(\eth), k_3(\eth)$, such that

$$\left|\phi(\eth, u(\eth), (\theta u)(\eth), (\psi u)(\eth))-\phi(\eth, w(\eth), (\theta w)(\eth), (\psi w)(\eth))\right| \leq k_1(\eth)|u-w|+k_2(\eth)|\theta u-\theta w|+k_3(\eth)|\psi u-\psi w|.$$

For all $\eth \in \mathfrak{T}$.

$$y_0 = \sup_{\eth\in\mathfrak{T}} \int_1^{\eth} y(\eth, \nabla)u(\nabla)d\nabla, \qquad z_0 = \sup_{\eth\in\mathfrak{T}} \int_1^{\eth} z(\eth, \nabla)u(\nabla)d\nabla,$$

$d_2 = sup_{\eth\in[1,e]}\{I^{\beth}k_1(\eth), I^{\beth}k_2(\eth), I^{\beth}k_3(\eth)\}$, $\quad d_1 = \max\{I^{\mho+\beth-1}k_1(\mu), I^{\mho+\beth-1}k_2(\mu), I^{\mho+\beth-1}k_3(\mu)\}$.

(**F3**) The function $\beta \in C(\mathfrak{T}, R_+)$ is increasing and there exist $\mho_\beta > 0$, such that, for each $\eth \in \mathfrak{T}$, we have

$$I^{\beth}\beta(\eth) < \mho_\beta\beta(\eth).$$

Theorem 3.1. The IVP (1. 1)-(1. 2) has at least one solution if $\phi : \mathfrak{T} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is continuous and satisfies (**F1**)-(**F2**).

Proof. : Let the set $G_r = \{u \in C : ||u|| < r\}$, then G_r is a closed, bounded, and convex set of $\mathfrak{T}$.
Now, let the operators A and B defined as:

$$(Au)(\eth) = \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla))\frac{d\nabla}{\nabla}$$

,

$$(Bu)(\eth) = \frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu} \frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla))\frac{d\nabla}{\nabla}.$$

For any $u \in G_r$ and $\eth \in \mathfrak{T}$, we have

$$|(Au)(\eth)| = \left|\int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla))\frac{d\nabla}{\nabla}\right|,$$

$$\leq \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}|\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u))(\nabla)-\phi(\nabla, 0, 0, 0)|\frac{d\nabla}{\nabla}+\int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}|\phi(\nabla, 0, 0, 0)|\frac{d\nabla}{\nabla},$$

$$\leq (I^{\beth}k_1(\eth) + y_0I^{\beth}k_2(\eth) + z_0I^{\beth}k_3(\eth))||u|| + \frac{q}{\Gamma(\beth+1)}(\ln(\eth))^{\beth},$$

$$\leq d_2(1 + y_0 + z_0)||u|| + \frac{q}{\Gamma(\beth+1)}.$$

Hence, we obtain

$$||Au|| \leq d_2(1 + y_0 + z_0)||u|| + \frac{q}{\Gamma(\beth+1)}. \tag{3. 11}$$

By the same way, we have

$$|(Bu_1)(\eth)| = \left|\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu} \frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla, u_1(\nabla), (\theta u_1)(\nabla), (\psi u_1)(\nabla))\frac{d\nabla}{\nabla}\right|,$$

$$\leq \frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}|\phi(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla))-\phi(\nabla,0,0,0)|\frac{d\nabla}{\nabla}$$
$$+\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}|\phi(\nabla,0,0,0)|\frac{d\nabla}{\nabla}.$$
$$\leq \frac{(\ln(\eth))^{\mho-1}||u_1||}{\Gamma(\mho)-b}(I^{\mho+\beth-1}k_1(\mu)+y_0I^{\mho+\beth-1}k_2(\mu)+z_0I^{\mho+\beth-1}k_3(\mu))+\frac{q(\ln(\mu))^{\mho+\beth-1}}{\Gamma(\mho+\beth)}\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b},$$
$$||Bu_1||\leq\frac{d_1}{\Gamma(\mho)-b}\left((1+y_0+z_0)||u_1||+\frac{q}{\Gamma(\mho+\beth)}\right). \qquad (3.\ 12)$$

Now, from (3. 11) and (3. 12) , we get

$$||Au+Bu_1||\leq||Au||+||Bu_1||,$$
$$||Au+Bu_1||\leq d_2(1+y_0+z_0)||u||+\frac{q}{\Gamma(\beth+1)}+\frac{d_1}{\Gamma(\mho)-b}((1+y_0+z_0)||u_1||+\frac{q}{\Gamma(\mho+\beth)}),$$

then $||Au+Bu_1||\leq R$.

To show that the operator B is a contraction, let $u_1,u_2\in G_r$, we have the following:

$$|(Bu_1)(\eth)-(Bu_2)(\eth)|=\left|\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla))\frac{d\nabla}{\nabla}\right.$$
$$\left.-\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla,u_2(\nabla),(\theta u_2)(\nabla),(\psi u_2)(\nabla))\frac{d\nabla}{\nabla}\right|,$$
$$\leq\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\left|\phi(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla))-\phi(\nabla,u_2(\nabla),(\theta u_2)(\nabla),(\psi u_2)(\nabla))\right|\frac{d\nabla}{\nabla},$$
$$\leq\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}(k_1(\nabla)|u_1-u_2|+k_2(\nabla)|\theta u_!-\theta u_2|+k_3(\eth)|\psi u_1-\psi u_2|)\frac{d\nabla}{\nabla},$$
$$||Bu_1-Bu_2||\leq\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}||u_1-u_2||\left(I^{\mho+\beth-1}k_1(\mu)+y_0I^{\mho+\beth-1}k_2(\mu)+z_0I^{\mho+\beth-1}k_3(\mu)\right),$$
$$||Bu_1-Bu_2||\leq\frac{d_1}{\Gamma(\mho)-b}(1+y_0+z_0)||u_1-u_2||.$$

It follows from $\frac{d_1}{\Gamma(\mho)-b}(1+y_0+z_0)<1$ that the operator B is a contraction.

To establish that A is compact and continuous, we show that it is both continuous and bounded on G_r. Continuity of A follows directly from the continuity of the function f. Moreover, A is uniformly bounded on G_r, as evidenced by equation (3. 11):

$$||Au||\leq d_2(1+y_0+z_0)||u||+\frac{q}{\Gamma(\beth+1)}\leq d_2(1+y_0+z_0)r+\frac{q}{\Gamma(\beth+1)}.$$

Let $u\in G_r$ and $\eth_1<\eth_2$,

$$|(Au)(\eth_2)-(Au)(\eth_1)|=\left|\int_1^{\eth_2}\frac{(\ln\frac{\eth_2}{\nabla})^{\beth-1}}{\Gamma(\beth)}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla))\frac{d\nabla}{\nabla}\right.$$
$$\left.-\int_1^{\eth_1}\frac{(\ln\frac{\eth_1}{\nabla})^{\beth-1}}{\Gamma(\beth)}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla))\frac{d\nabla}{\nabla}\right|,$$

$$\leq \left| \frac{1}{\Gamma(\beth)} \int_1^{\eth_1} \left((\ln \frac{\eth_2}{\nabla})^{\beth-1} - (\ln \frac{\eth_1}{\nabla})^{\beth-1} \right) \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla)) \frac{d\nabla}{\nabla} \right.$$
$$\left. + \int_{\eth_1}^{\eth_2} \frac{(\ln \frac{\eth_2}{\nabla})^{\beth-1}}{\Gamma(\beth)} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla)) \frac{d\nabla}{\nabla} \right|,$$
$$\leq \frac{L}{\Gamma(\beth+1)} \left[(\ln \frac{\eth_2}{\nabla})^{\beth} - (\ln \frac{\eth_1}{\nabla})^{\beth} \right]_1^{\eth_1} + \frac{L}{\Gamma(\beth+1)} \left[(\ln \frac{\eth_2}{\nabla})^{\beth} \right]_{\eth_1}^{\eth_2},$$
$$= \frac{L}{\Gamma(\beth+1)} \left[(\ln \frac{\eth_2}{\eth_1})^{\beth} \right] = \frac{L}{\Gamma(\beth+1)} \left[(\ln(\eth_2) - \ln(\eth_1))^{\beth} \right].$$

Hence, as $\eth_2 \to \eth_1$, the difference $|Au(\eth_2) - Au(\eth_1)| \to 0$ tends to zero. This establishes that $A$is equicontinuous. By the profound Arzelá-Ascoli theorem, it follows that A is a compact operator on G_r. Consequently, A is completely continuous. Applying the robust framework of Krasnoselskii's fixed point theorem, so we obtain that the boundary value problem described by (1. 1)-(1. 2) has at least one solution. □

Now, for the uniqueness theorem, the Banach fixed point theorem is applied.

Theorem 3.2. Assume that $\phi : \mathfrak{T} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is continuous and satisfies conditions **(F1)**–**(F2)**, and let $\sup_{\eth \in [1,e]} \phi(\eth, 0, 0, 0) = q$. Then, the problem (1. 1)-(1. 2) has a unique solution.

Proof. Define the operator $B : C(\mathfrak{T}, \mathcal{R}) \to C(\mathfrak{T}, \mathcal{R})$ as the following

$$\begin{aligned}(Tu)(\eth) =& \frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho) - b} \int_1^{\mu} \frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla)) \frac{d\nabla}{\nabla} \\ &+ \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla)) \frac{d\nabla}{\nabla}.\end{aligned} \tag{3. 13}$$

Our objective is to demonstrate that the operator T admits a fixed point within the set G_r, which coincides with a solution to the initial value problem specified by equations (1. 1) and (1. 2).

First, we establish the invariance of G_r under T, namely, that $T(G_r) \subseteq G_r$. To this end, consider an arbitrary $u \in G_r$. where, $G_r = \{u \in C : ||u|| < r\}$. For $u \in G_r$. The operator T is bounded set into the bounded sets in $C(\mathfrak{T}, \mathcal{R})$.

$$|(Tu)(\eth)| = \left| \frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho) - b} \int_1^{\mu} \frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)} (\phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla)) - \phi(\nabla, 0, 0, 0) + \phi(\nabla, 0, 0, 0)) \frac{d\nabla}{\nabla} \right.$$
$$\left. + \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} \phi(\nabla, u(\nabla), (\theta u)(\nabla), (\psi u)(\nabla) - \phi(\nabla, 0, 0, 0) + \phi(\nabla, 0, 0, 0)) \frac{d\nabla}{\nabla} \right|,$$
$$\leq \frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho) - b} \int_1^{\mu} \frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)} ((k_1(\nabla)|u| + k_2(\nabla)|\theta u| + k_3(\eth)|\psi u|) + q) \frac{d\nabla}{\nabla}$$
$$+ \int_1^{\eth} \frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)} ((k_1(\nabla)|u| + k_2(\eth)|\theta u| + k_3(\eth)|\psi u|) + q) \frac{d\nabla}{\nabla},$$

$$\leq \frac{1}{\Gamma(\mho)-b}\left(d_1(1+y_0+z_0)+\frac{q}{\Gamma(\mho+\beth)}\right)+d_2(1+y_0+z_0)+\frac{q}{\Gamma(\beth+1)},$$

$$||Tu|| \leq (\frac{d_1}{\Gamma(\mho)-b}+d_2)(1+y_0+z_0)+q(\frac{1}{(\Gamma(\mho)-b)\Gamma(\mho+\beth)}+\frac{1}{\Gamma(\beth+1)}).$$

To prove that B is a contraction mapping, let $u, u_1 \in G_r$ and for each $\eth \in \mathfrak{T}$, the following inequality holds:

$$\begin{aligned}|(Tu)(\eth)-(Tu_1)(\eth)| =& \Big|\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}(\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla))\\ &-\phi(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla)))\frac{d\nabla}{\nabla}+\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}(\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla))\\ &-\phi(\nabla,u_1(\nabla),(\theta u_1)(\nabla),(\psi u_1)(\nabla)))\frac{d\nabla}{\nabla}\Big|,\end{aligned}$$

$$\begin{aligned}&\leq \frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}(k_1(\nabla)|u-u_1|+y_0k_2(\nabla)|\theta u-\theta u_1|+z_0k_3(\nabla)|\psi u-\psi u_1|)\frac{d\nabla}{\nabla}\\ &+\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}(k_1(\nabla)|u-u_1|+y_0k_2(\nabla)|\theta u-\theta u_1|+z_0k_3(\nabla)|\psi u-\psi u_1|)\frac{d\nabla}{\nabla},\end{aligned}$$

$$||Tu-Tu_1|| \leq (\frac{d_1}{\Gamma(\mho)-b}+d_2)(1+y_0+z_0)||u-u_1||,$$

where $\left(\frac{d_1}{\Gamma(\mho)-b}+d_2\right)(1+y_0+z_0)<1$. Therefore, the operator T is a contraction mapping, and hence the problem (1. 1)-(1. 2) has a unique solution. □

4. Stability Theorems

In this section, we analyze the stability of our result, focusing on Ulam-Hyers stability and Ulam-Hyers-Rassias stability.

Theorem 4.1. Assume that the assumption (**F2**) holds. Then, the fractional integro-differential equation (1. 1), subject to the initial condition (1. 2), is Ulam-Hyers ($\mathcal{UH}$) stable.

Proof. Let $w \in C(\mathfrak{T},\mathcal{R})$ be a solution of the inequality (2. 6) i.e,

$$|^{HH}D_{1+}^{\beth,\omega}w(\eth)-\phi(\eth,w(\eth),(\theta w)(\eth),(\psi w)(\eth))| \leq \epsilon, \quad \eth \in \mathfrak{T}. \tag{4. 14}$$

If $u \in C(\mathfrak{T},\mathcal{R})$ is defined as the unique solution of the problem (1. 1)-(1. 2). Since u and w are continuous functions on $C(\mathfrak{T},\mathcal{R})$, and from lemma (2.8) we get

$$\begin{aligned}w(\eth) =& \frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla))\frac{d\nabla}{\nabla}\\ &+\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla))\frac{d\nabla}{\nabla}.\end{aligned}$$

From (4. 14), we get

$$|w(\eth)-\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla))\frac{d\nabla}{\nabla}$$
$$-\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi u)(\nabla))\frac{d\nabla}{\nabla}|\leq\frac{\epsilon(\ln(\eth))^{\beth}}{\Gamma(\beth+1)}.$$

On the other hand we have

$$\left|w(\eth)-u(\eth)\right|=\left|w(\eth)-\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla))\frac{d\nabla}{\nabla}\right.$$
$$\left.-\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla))\frac{d\nabla}{\nabla}\right|,$$

by (**F2**), we obtain

$$\leq\frac{\epsilon(\ln(\eth))^{\beth}}{\Gamma(\beth+1)}+\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}(k_1(\nabla)+y_0k_2(\nabla)+z_0k_3(\nabla))|w-u|\frac{d\nabla}{\nabla}+$$
$$\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\left(I^{\mho+\beth-1}k_1(\mu)+y_0I^{\mho+\beth-1}k_2(\mu)+z_0I^{\mho+\beth-1}k_3(\mu)\right)||w-u||,$$

$$||w-u||\leq\frac{\epsilon(\ln(\eth))^{\beth}}{\Gamma(\beth+1)}+\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}(k_1(\nabla)+y_0k_2(\nabla)+z_0k_3(\nabla))|w-u|\frac{d\nabla}{\nabla}$$
$$+\frac{vd_1(1+y_0+z_0)}{\Gamma(\mho)-b}||w-u||.$$

Let $\xi=\frac{vd_1(1+y_0+z_0)}{\Gamma(\mho)-b}$, then we obtain

$$||w-u||\leq\frac{\epsilon(\ln(\eth))^{\beth}}{\Gamma(\beth+1)(1-\xi)}e^{d_2(1+y_0+z_0)}.$$

Hence, the initial value problem (1. 1)-(1. 2) is Ulam-Hyers stable. □

Theorem 4.2. Suppose that the assumptions (**F1**)-(**F3**) are satisfied. Then, the problem (1. 1) with the initial condition (1. 2) is Ulam-Hyers-Rassias stable.

Proof. Let $w\in C(\mathfrak{T},\mathcal{R})$ be a solution of the inequality (2. 6) i.e,

$$|{}^{HH}D_{1+}^{\beth,\omega}w(\eth)-\phi(\eth,w(\eth),(\theta w)(\eth),(\psi w)(\eth))|\leq\epsilon\beta(\eth),\quad\eth\in\mathfrak{T}.\tag{4. 15}$$

From (4. 15), the following obtained

$$|w(\eth)-\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla))\frac{d\nabla}{\nabla}$$
$$-\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi u)(\nabla))\frac{d\nabla}{\nabla}|\leq\epsilon\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\beta(\eth)\frac{d\nabla}{\nabla}=\epsilon I^{\beth}\beta(\eth),$$

$$|w(\eth)-\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla))\frac{d\nabla}{\nabla}$$
$$-\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi u)(\nabla))\frac{d\nabla}{\nabla}|\leq\epsilon\mho_\beta\beta(\eth).$$

On the other hand we have

$$\Big|w(\eth)-u(\eth)\Big|\leq\epsilon\mho_\beta\beta(\eth)+\Big|\frac{(\ln(\eth))^{\mho-1}}{\Gamma(\mho)-b}\int_1^{\mu}\frac{(\ln(\frac{\mu}{\nabla}))^{\mho+\beth-2}}{\Gamma(\mho+\beth-1)}(\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla))$$
$$-\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla)))\frac{d\nabla}{\nabla}$$
$$+\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}(\phi(\nabla,w(\nabla),(\theta w)(\nabla),(\psi w)(\nabla))-\phi(\nabla,u(\nabla),(\theta u)(\nabla),(\psi u)(\nabla)))\frac{d\nabla}{\nabla}\Big|,$$

$$||w-u||\leq\epsilon\mho_\beta\beta(\eth)+\frac{vd_1(1+y_0+z_0)}{\Gamma(\mho)-b}||w-u||+\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}(k_1(\nabla)+y_0k_2(\nabla)+z_0k_3(\nabla))|w-u|\frac{d\nabla}{\nabla},$$

$$||w-u||\leq\frac{\epsilon\mho_\beta\beta(\eth)}{(1-\xi)}+\frac{1}{(1-\xi)}\int_1^{\eth}\frac{(\ln(\frac{\eth}{\nabla}))^{\beth-1}}{\Gamma(\beth)}(k_1(\nabla)+y_0k_2(\nabla)+z_0k_3(\nabla))|w-u|\frac{d\nabla}{\nabla},$$

where $\xi=\frac{vd_1(1+y_0+z_0)}{\Gamma(\mho)-b}$.

$$||w-u||\leq\frac{\epsilon\mho_\beta\beta(\eth)}{(1-\xi)}e^{d_2(1+y_0+z_0)}.$$

Hence, the initial value problem (1. 1)-(1. 2) is Ulam-Hyers-Rassias stable. □

5. Examples

Example 5.1. Consider the following problem

$$^{HH}D^{\frac{3}{2},\frac{1}{2}}u(\eth)=\frac{\sqrt{\ln(\eth)}}{10}\frac{|x|}{1+|x|}+\frac{(\ln(\eth))^2\sin(y)}{20}+\frac{\ln(\eth)}{10}\frac{1}{1+|z|},\tag{5. 16}$$

with the conditions

$$D^{\frac{-1}{4}}u(1)=0,$$
$$D^{\frac{3}{4}}u(1)=I^{\frac{3}{4}}u(\frac{5}{3}),\tag{5. 17}$$

where $\beth=\frac{3}{2}$, $\omega=\frac{1}{2}$, $\mho=\frac{7}{4}$, and $\mu=\frac{5}{3}$, $\Gamma(\frac{7}{4})=0.919$, $b=\frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{2})}=0.691$.

$y(\eth,s)=\frac{t\ln(\nabla)}{10s}$, $\quad z(\eth,s)=\frac{\sqrt{t+s}}{40}$. $y_0=\frac{e}{15}=0.1812$, $z_0=\frac{(2e)^{\frac{3}{2}}-(e+1)^{\frac{3}{2}}}{60}=0.0917$.

$|\phi(\eth,x_1,y_1,z_1)-\phi(\eth,x_2,y_2,z_2))|=|\frac{\sqrt{\ln(\eth)}}{10}|x_1-x_2|+\frac{(\ln(\eth))^2}{20}|y_1-y_2|+\frac{\ln(\eth)}{10}|z_1-z_2|$

$d_2=\sup_{\eth\in[1,e]}\{\frac{\sqrt{\pi}}{40}(\ln(\eth))^2,\frac{32}{105\sqrt{\pi}}(\ln(\eth))^{\frac{7}{2}},\frac{8}{150\sqrt{\pi}}(\ln(\eth))^{\frac{5}{2}}\}$,

$d_2=\sup_{\eth\in[1,e]}\{0.0443,0.1719,0.0301\}=0.1719$.

$d_1=\max\{\frac{(\ln(\frac{5}{3}))^{\frac{11}{4}}\Gamma(\frac{3}{2})}{10\Gamma(\frac{15}{4})},\frac{2(\ln(\frac{5}{3}))^{\frac{17}{4}}}{\Gamma\frac{21}{4}},\frac{(\ln(\frac{5}{3}))^{\frac{13}{4}}}{10\Gamma\frac{17}{4}}\}$,

$d_1 = \max\{0.003, 0.0016, 0.00136\} = 0.003.$
The result becomes

$$|(Tu)(\eth) - (Tu_1)(\eth)| \leq \left(\frac{d_1}{\Gamma(\mho) - b} + d_2\right)(1 + y_0 + z_0),$$

$$|(Tu)(\eth) - (Tu_1)(\eth)| \leq \left(\frac{0.003}{0.919 - 0.691} + 0.1719\right)(1.2729) = 0.2355.$$

Since

$$\left(\frac{d_1}{\Gamma(\mho) - b} + d_2\right)(1 + y_0 + z_0) < 1.$$

Then, the problem (5. 16)?(5. 17) has a unique solution.

Example 5.2. Consider the following problem

$$^{HH}D^{\frac{4}{3},\frac{1}{2}}u(\eth) = \frac{\ln(\eth)}{10(1+|u|)} + \frac{\eth}{10}\int_1^{\eth}\frac{1}{10}\left(\ln\left(\frac{\eth}{s}\right)\right)^{2-1}u(s)\,\frac{ds}{s} + \frac{1}{15}\int_1^{\eth}\left(\ln\left(\frac{\eth}{s}\right)\right)^{2+1}u(s)\,\frac{ds}{s}, \tag{5.3}$$

with the conditions

$$D^{-\frac{1}{3}}u(1) = 0, \qquad D^{\frac{2}{3}}u(1) = I^{\frac{2}{3}}u\left(\frac{5}{4}\right), \tag{5.4}$$

where $\eth \in [1, e]$, $j = \frac{3}{4}$, $\omega = \frac{1}{2}$, $\eth_0 = \frac{5}{3}$, and $\mu = \frac{7}{4}$. $\Gamma\left(\frac{3}{4}\right) = 0.903$, $b = \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{3}{5}\right)} = 0.758$.

$$y(\eth, s) = \frac{1}{10s}\left(\ln\left(\frac{\eth}{s}\right)\right)^{2-1}, \qquad z(\eth, s) = \left(\ln\left(\frac{\eth}{s}\right)\right)^{2+1}.$$

$$y_0 = \frac{3}{10}, \qquad z_0 = \frac{3}{10}.$$

$$d_2 = \sup_{\eth\in[1,e]}\left\{I^2\ln(\eth), I^{\frac{1}{10}}(\eth), I^{\frac{1}{2}}(\eth)\right\}.$$

$$d_2 = \sup_{\eth\in[1,e]}\left\{\frac{\Gamma(\frac{3}{4})}{10^2}\left(\ln(\eth)\right)^{\frac{1}{2}}, \frac{\eth}{4}\ln(\eth), \frac{\ln(\eth)}{15\Gamma(\frac{5}{3})}\right\}.$$

$$d_2 = \sup_{\eth\in[1,e]}\{0.0942,\ 0.1338,\ 0.1866\} = 0.0942.$$

$$d_1 = \max\left\{\frac{0.0011}{10^2}, \frac{0.0027}{10^2}, \frac{0.0017}{10^2}\right\} = 0.0027.$$

The result becomes

$$|(Tu)(\eth) - (Tu_1)(\eth)| \leq \frac{d_1 + d_2}{\Gamma(\frac{5}{3}) - b}(1 + y_0 + z_0),$$

$$|(Tu)(\eth) - (Tu_1)(\eth)| \leq \left(\frac{0.0027}{0.903 - 0.758} + 0.0942\right)(1.375) = 0.1551.$$

Since

$$\left(\frac{d_1}{\Gamma(\frac{5}{3})-b}+d_2\right)(1+y_0+z_0)<1.$$

Then, the problem (5.3)-(5.4) has a unique solution.

6. Conclusion

In this study, we have examined the existence and uniqueness of solutions for Hilfer-Hadamard fractional integro-differential equations with fractional initial conditions. Krasnoselskii's fixed point theorem has been employed to establish the existence of solutions, while the Banach contraction principle is utilized to secure their uniqueness. Moreover, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the problem have been investigated. To illustrate the applicability of the obtained results, a concrete example is provided.

In addition to the theoretical results, the proposed Hilfer-Hadamard Volterra fractional integro-differential model can be extended to various practical applications. For instance, this framework can be applied to describe anomalous diffusion processes, viscoelastic materials, and population dynamics with memory effects, where nonlocal and hereditary properties play a crucial role. Moreover, the current approach could be utilized to investigate biological growth models and heat transfer problems involving complex media. These potential directions demonstrate the flexibility and applicability of the present work and may inspire further research in applied sciences and engineering.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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