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Generalized CRF Geometry on Matsuki Orbits in Complex Grassmannians

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Abstract. From the ambient complex-type generalized complex (GC) structure on $Z = \mathrm{Gr}_k(\mathbb{C}^n)$, we prove an Inheritance Theorem establishing that each Matsuki orbit $M = (G_0 \cdot z) \cap (K \cdot z)$ inherits a canonical generalized CRF structure (in the sense of Vaisman) $\Phi_{\mathcal{M}}$ via eigenbundle restriction; the construction is intrinsic and invariant under closed B-fields. We establish a *restricted Darboux normal form* splitting $(\mathcal{M}, \Phi_{\mathcal{M}})$ locally into a complex block and a symplectic transverse factor, yielding the invariants *type* and *transverse symplectic rank*. Thus we obtain a natural generalized CRFK structure on each Matsuki orbit in the Grassmannian, together with explicit formulas for its type and transverse symplectic rank. Our results fit into the project of describing the complex geometry of lower-dimensional G_0-orbits in complex flag manifolds. For $G_0 = \mathrm{SU}(p,q)$ these are computed directly from the signature data (a,b,r) of $h|_W$, with detailed calculations for $\mathrm{SU}(2,2) \curvearrowright \mathrm{Gr}_2(\mathbb{C}^4)$; in purespinor terms a generator is $\varphi_{\mathcal{M}} = \eta^{t_{\mathrm{CR}},0} \wedge e^{\mathrm{i}\omega_T}$ with ω_T the transverse Levi form. These results connect classical CR geometry of real orbits with generalized complex/Dirac geometry and provide the first explicit homogeneous examples of generalized CRFK manifolds from real group actions, opening avenues for deformation, cohomology, and quantization.

AMS (MOS) Subject Classification Codes: 32V20; 32D20; 32M10; 22E46.
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1. Introduction

Let $Z = \mathrm{Gr}_k(\mathbb{C}^n)$ be a complex Grassmannian endowed with its natural Kähler structure. The corresponding generalized complex (GC) structure of complex type on the Courant bundle $\mathbb{T}Z = TZ \oplus T^*Z$ will be denoted by $\mathcal{J}_Z$. Generalized complex geometry unifies complex and symplectic geometries within the Courant–Dirac framework and admits a Darboux-type local normal form [5, 1] with further developments including generalized contact structures [7] and interactions with T-duality [3]. Generalized CRF structures, introduced by Vaisman, extend the classical CR- and F-structures to this setting [8].

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Fix a real form $G_0 \subset \mathrm{SL}(n, \mathbb{C})$ with maximal compact subgroup $K_0 \subset G_0$, and let $K = (K_0)^{\mathbb{C}}$. For $z \in Z$, the *Matsuki orbit* through z is defined by

$$\mathcal{M}(z) := (G_0 \cdot z) \cap (K \cdot z),$$

the clean intersection of a real and a complex orbit. Hence $\mathcal{M}(z)$ is a natural real-analytic CR submanifold of Z. We refer to [9, 4, 6] for background on the geometry of G_0- and K-orbits and the Matsuki duality between them.
Goals and contributions.

- **Inheritance theorem.** If $(Z, \mathcal{J}_Z)$ is GC of complex type and $\mathcal{M} \subset Z$ is a Matsuki orbit, then the restricted eigenbundle

$$E_{\mathcal{M}} = \big(L_Z \cap (T\mathcal{M} \oplus T^*Z|_{\mathcal{M}})_{\mathbb{C}}\big) \Big/ \big(L_Z \cap \mathrm{ann}(T\mathcal{M})\big)$$

  defines a generalized CRF structure $\Phi_{\mathcal{M}}$ on $\mathcal{M}$, natural under closed B-field transforms.
- **Restricted Darboux normal form.** Intersecting a GC Darboux chart of Z with a linear model of $\mathcal{M}$ yields a product model $\mathbb{C}^{t_{\mathrm{CR}}} \times (\mathbb{R}^{2s-2t_{\mathrm{CR}}}, \omega_0)$ whenever the type is locally constant. This provides explicit *type* and *symplectic rank* formulas.
- **Explicit computations on Grassmannians.** For $G_0 = \mathrm{SU}(p, q)$, we compute $\dim_{\mathbb{C}} T^{1,0}\mathcal{M}$ by analyzing the block decomposition of $T^{1,0}\mathrm{Gr}_k(\mathbb{C}^n) \cong \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W)$ with respect to the signature data (a, b, r) of the restricted Hermitian form. A complete table is given for $\mathrm{SU}(2, 2) \curvearrowright \mathrm{Gr}_2(\mathbb{C}^4)$.
- **Pure spinors and CRFK packages.** We describe invariant pure spinors associated to $\Phi_{\mathcal{M}}$; on open orbits, the resulting generalized CRFK structures arise from the homogeneous metric of Vaisman [8].

  Finally, we briefly outline the structure of the paper. In Sections 3–5 we establish the Inheritance Theorem and the restricted Darboux normal form for generalized CRF structures on Matsuki orbits. Sections 6–7 are devoted to explicit computations for the prototype action

$$SU(2, 2) \curvearrowright \mathrm{Gr}_2(\mathbb{C}^4),$$

  including a detailed analysis of all nondefinite orbit types. In Section 8 we adopt the pure-spinor viewpoint and describe the induced CRFK structures, while Section 9 formulates a general scheme for $SU(p, q)$ acting on $\mathrm{Gr}_k(\mathbb{C}^{p+q})$. Section 10 concludes with a discussion of the geometric significance of the invariants $(t_{\mathrm{CR}}, \rho_{\mathrm{symp}})$ and possible directions for future work.

2. Preliminaries on Generalized Geometry

2.1. The Courant bracket and B-field action.

Let M be a smooth manifold, and denote by

$$\mathbb{T}M := TM \oplus T^*M$$

its *Courant (or generalized tangent) bundle*. The bundle $\mathbb{T}M$ carries a natural nondegenerate symmetric bilinear form

$$\langle X + \alpha,\, Y + \beta \rangle := \tfrac{1}{2}\big(\beta(X) + \alpha(Y)\big), \qquad X, Y \in TM,\ \alpha, \beta \in T^*M,$$

and a bracket operation on its smooth sections called the *Courant bracket*:

$$[X+\alpha,\, Y+\beta]_C = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \tfrac{1}{2}\,\mathrm{d}\big(\beta(X)-\alpha(Y)\big). \tag{2. 1}$$

The Courant bracket and the pairing together define the *standard Courant algebroid structure* on $\mathbb{T}M$ [5].

A closed 2-form $B \in \Omega^2(M)$ acts on $\mathbb{T}M$ by the B*-field transform*

$$e^B(X+\alpha) := X + \alpha + \iota_X B. \tag{2. 2}$$

The map $e^B : \mathbb{T}M \to \mathbb{T}M$ is orthogonal with respect to $\langle\cdot,\cdot\rangle$, and if $\mathrm{d}B = 0$ it preserves the Courant bracket:

$$[e^B u,\, e^B v]_C = e^B[u,v]_C.$$

Hence, closed B-fields act as Courant algebroid automorphisms [5, 1].

Remark 2.1. The formula (2. 1) is the *skew-symmetrized Courant bracket*. The nonskew-symmetrized version defines the Dorfman bracket, which is often used in Dirac and generalized geometry literature. Both brackets determine the same integrability conditions.

2.2. **Generalized complex structures.** A *generalized almost complex structure* on M is an orthogonal bundle endomorphism $\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$ satisfying $\mathcal{J}^2 = -\mathrm{Id}$. Equivalently, its $+$i-eigenbundle

$$L := \{\, u - \mathrm{i}\,\mathcal{J}u : u \in \mathbb{T}M \,\} \subset (\mathbb{T}M)_{\mathbb{C}}$$

is a complex, maximally isotropic subbundle satisfying $L \cap \overline{L} = \{0\}$. The pair $(M, \mathcal{J})$ is a *generalized almost complex manifold*. It is *integrable* (a *GC manifold*) if the space of sections $\Gamma(L)$ is closed under the Courant bracket (2. 1).

Generalized complex manifolds locally decompose as products of a complex and a symplectic factor. More precisely, each point admits a neighborhood that, after a closed B-field transform, is Courant-isomorphic to $(\mathbb{C}^t, J_{\mathrm{std}}) \times (\mathbb{R}^{2m-2t}, \omega_0)$, where t is the local *type* [5, 1].

Example 2.2 (Complex and symplectic types)**.** If J is an ordinary complex structure on M, then

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

is a GC structure of *complex type*. If ω is a symplectic form, then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

is a GC structure of *symplectic type*.

2.3. **Generalized CRF structures.** A *generalized almost CRF structure* is an orthogonal skew endomorphism $\Phi : \mathbb{T}M \to \mathbb{T}M$ satisfying the cubic relation

$$\Phi^3 + \Phi = 0. \tag{2. 3}$$

Equivalently, it is specified by a complex, maximally isotropic subbundle

$$E \subset (\mathbb{T}M)_{\mathbb{C}}$$

with the property $E \cap \overline{E}^{\perp} = \{0\}$. If, moreover, the space of sections $\Gamma(E)$ is closed under the Courant bracket (2. 1), then (M, Φ) is a *generalized CRF manifold* [8]. The class of generalized CRF structures includes both classical CR structures (complex type) and F-structures (symplectic type) as particular cases.

Remark 2.3 (Pure spinor interpretation). Let $\mathrm{Cl}(T \oplus T^*)$ denote the Clifford algebra bundle associated with the natural pairing. A *pure spinor line* $K \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ is annihilated by E under Clifford action:

$$(X+\alpha)\cdot\varphi := \iota_X\varphi + \alpha\wedge\varphi = 0 \quad \text{for all } X+\alpha \in E.$$

Integrability is equivalent to the existence of a local generator φ of K satisfying

$$\mathrm{d}\varphi = (X+\alpha)\cdot\varphi$$

for some local section $X+\alpha \in \Gamma(\mathbb{T}M)$ [5, 8]. This characterization extends the pure-spinor formalism of generalized complex geometry to the CRF setting.

3. Matsuki orbits in Grassmannians

Let $Z = \mathrm{Gr}_k(\mathbb{C}^n)$ be the complex Grassmannian of k-planes in $\mathbb{C}^n$. Fix a real form $G_0 \subset \mathrm{SL}(n,\mathbb{C})$ with maximal compact subgroup $K_0 \subset G_0$, and write $K := (K_0)^{\mathbb{C}}$ for its complexification (which acts holomorphically on Z). For any point $z \in Z$ the *Matsuki orbit* through z is the intersection

$$\mathcal{M}(z) \;:=\; (G_0\cdot z) \;\cap\; (K\cdot z).$$

The following facts are standard for complex flag manifolds (in particular for Grassmannians); we summarize them here with proofs or references for the reader.

Proposition 3.1 (Tangent sum and clean intersection)**.** *For any $z \in Z$ one has the tangent-space equality*

$$T_zZ \;=\; T_z(G_0\cdot z) \;+\; T_z(K\cdot z). \tag{3. 1}$$

Consequently $(G_0\cdot z)$ and $(K\cdot z)$ intersect cleanly *in Z, so $\mathcal{M}(z)$ is a smooth real submanifold of Z whose tangent space is*

$$T_z\mathcal{M}(z) \;=\; T_z(G_0\cdot z)\cap T_z(K\cdot z).$$

Moreover, $\mathcal{M}(z)$ is a real-analytic CR submanifold of Z.
(The tangent equality expresses that G_0- and K-orbits form a transverse pair in the Matsuki setting.)

Proof. The equality (3. 1) is a standard consequence of the general structure of real and complex orbits in complex flag manifolds; see, e.g., Wolf [9], Matsuki [6] and Fels–Huckleberry–Wolf [4]. We sketch the elementary Lie-theoretic reason specialized to the flag (homogeneous) setting.

Write $Z = G/Q$ where $G = \mathrm{SL}(n,\mathbb{C})$ and Q is a parabolic stabilizer of z. The tangent space T_zZ identifies with $\mathfrak{g}/\mathfrak{q}$, where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{q} = \mathrm{Lie}(Q)$. The tangent to the (real) G_0-orbit is the image of $\mathfrak{g}_0 := \mathrm{Lie}(G_0)$ in $\mathfrak{g}/\mathfrak{q}$, and the tangent to the complex K-orbit is the image of $\mathfrak{k} := \mathrm{Lie}(K)$. Because $g = \mathfrak{g}_0 + \mathfrak{k}$ at the Lie algebra level, their images span g/q, yielding (3. 1); see, for example, Matsuki [6], and Fels–Huckleberry–Wolf [4] for full accounts of the orbit structure and Matsuki duality in the flag (homogeneous) setting.

Since the tangent spaces sum to the ambient tangent space, the intersection is clean (the tangent-space intersection has constant dimension equal to the expected value). Smoothness and real-analyticity of the intersection follow from the homogeneous-algebraic nature of the orbits (intersections of real-analytic submanifolds in an analytic homogeneous space remain real-analytic). □

Corollary 3.1 (Dimension count)**.** *At* $z \in Z$*,*

$$\dim_{\mathbb{R}} \mathcal{M}(z) \;=\; \dim_{\mathbb{R}} G_0 \cdot z \;+\; \dim_{\mathbb{R}} K \cdot z \;-\; \dim_{\mathbb{R}} Z.$$

Proof. For any two linear subspaces A, B of a finite-dimensional vector space, $\dim(A \cap B) = \dim A + \dim B - \dim(A+B)$. Apply this with $A = T_z(G_0 \cdot z)$ and $B = T_z(K \cdot z)$, and use Proposition 3.1 which gives $A + B = T_z Z$. □

CR structure on $\mathcal{M}(z)$**.** Because $K \cdot z$ is a complex submanifold of the complex manifold Z, its holomorphic tangent at z is the $(1,0)$-subspace $T_z^{1,0}(K \cdot z) \subset T_z^{1,0} Z$. The G_0-orbit is a real submanifold; complexifying its tangent yields $T_z(G_0 \cdot z) \otimes \mathbb{C} \subset T_z Z \otimes \mathbb{C}$. From these remarks we obtain:

Proposition 3.2. *The CR-holomorphic tangent space of the Matsuki orbit* $\mathcal{M}(z)$ *at* z *is*

$$T_z^{1,0}\mathcal{M}(z) \;=\; T_z^{1,0}(K \cdot z) \;\cap\; \big(T_z(G_0 \cdot z) \otimes \mathbb{C}\big) \;\subset\; T_z^{1,0} Z. \tag{3. 2}$$

Equivalently, using the Grassmann model $z = [W]$ *with* $W \subset \mathbb{C}^n$ *a* k*-plane,*

$$T_z^{1,0} Z \;\cong\; \mathrm{Hom}_{\mathbb{C}}\big(W, \mathbb{C}^n/W\big), \tag{3. 3}$$

and $T_z^{1,0}\mathcal{M}(z)$ *is obtained by imposing the linear conditions coming from the infinitesimal actions of* $\mathrm{Lie}(K)$ *and* $\mathrm{Lie}(G_0)$*.*

Proof. The first identity (3. 2) follows directly from the definition $T_z^{1,0}\mathcal{M} = T_z\mathcal{M} \otimes \mathbb{C} \cap T_z^{1,0} Z$, together with $T_z\mathcal{M} = T_z(G_0 \cdot z) \cap T_z(K \cdot z)$ from Proposition 3.1.

The identification (3. 3) is classical: given a local holomorphic family of k-planes $W(t)$ with $W(0) = W$, the infinitesimal variation is given by a linear map $W \to \mathbb{C}^n/W$ whose graph describes the first-order deformation of W. Thus the tangent space of the Grassmannian at $[W]$ is canonically $\mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W)$. Under this identification the infinitesimal action of an element $X \in \mathrm{Lie}(G) = \mathfrak{sl}(n, \mathbb{C})$ is the map

$$W \xrightarrow{X|_W} \mathbb{C}^n \xrightarrow{\pi} \mathbb{C}^n/W, \qquad \text{so} \qquad X \mapsto \pi \circ X|_W \in \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W).$$

Hence the spaces $T_z^{1,0}(K \cdot z)$ and $T_z(G_0 \cdot z) \otimes \mathbb{C}$ are exactly the images of $\mathrm{Lie}(K)$ and $\mathrm{Lie}(G_0) \otimes \mathbb{C}$ under this map, and their intersection yields $T_z^{1,0}\mathcal{M}$. □

Practical computation in the $\mathrm{SU}(p,q)$ **case.** In applications (and in Section §6) we work with the classical real form $G_0 = \mathrm{SU}(p,q) \subset \mathrm{SL}(n, \mathbb{C})$ (with $p + q = n$), defined using a nondegenerate Hermitian form h of signature (p,q). Choose a Witt (orthogonal) decomposition

$$\mathbb{C}^n \;=\; V_+ \oplus V_-, \qquad h|_{V_+} > 0,\; h|_{V_-} < 0,$$

so that $K_0 \cong S(U(p) \times U(q))$ and $K = \mathrm{S}(GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$.

For a k-plane $W \subset \mathbb{C}^n$ one records its signature data (a, b, r) where

$$a = \dim(W \cap V_+), \qquad b = \dim(W \cap V_-), \qquad r = \dim\big(\mathrm{Rad}(h|_W)\big),$$

so that $a + b + r = k$. These integers are invariants of the K-orbit of W (and they play a central role in Matsuki orbit classification). Using the identification $T_z^{1,0}Z \cong \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W)$ one computes $\dim_{\mathbb{C}} T_z^{1,0}(K \cdot z)$ and $\dim_{\mathbb{C}}(T_z(G_0 \cdot z) \otimes \mathbb{C})$ by writing the block decomposition of W and $\mathbb{C}^n/W$ relative to $V_+ \oplus V_-$ and imposing the linear conditions coming from the respective Lie algebras. Concretely:

- choose bases adapted to the decomposition $\mathbb{C}^n = V_+ \oplus V_-$ and to W;
- write a general element $\phi \in \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W)$ as a block matrix relative to these bases;
- the infinitesimal image of $\mathrm{Lie}(K)$ (respectively $\mathrm{Lie}(G_0)$) consists of those blocks satisfying certain linear relations (unitary / skew-Hermitian type conditions for $\mathrm{Lie}(K_0)$, and signature-preserving linear constraints for $\mathrm{Lie}(G_0)$);
- intersecting these images yields $T_z^{1,0}\mathcal{M}$.

We therefore obtain an explicit finite-dimensional linear algebra problem: computing $\dim_{\mathbb{C}} T_z^{1,0}\mathcal{M}$ reduces to linear algebra on block matrices. We carry out these computations in detail for the case $\mathrm{SU}(2,2) \curvearrowright \mathrm{Gr}_2(\mathbb{C}^4)$ in Section §6.

Remark 3.2 (Matsuki duality and minimal/maximal orbits). Matsuki duality pairs G_0-orbits with K-orbits in a manner that controls closure relations and combinatorics of orbit types; see [6, 4]. In particular, open G_0-orbits (flag domains) intersect unique compact K-orbits, while minimal (closed) G_0-orbits intersect open K-orbits, etc. These duality properties are often convenient for understanding which Matsuki intersections are nonempty and for organizing the linear-algebra computations above.

4. Inheritance of generalized CRF structures from the ambient GC

Let $(Z, \mathcal{J}_Z)$ be a generalized complex manifold of complex type, with $+$i-eigenbundle $L_Z \subset (\mathbb{T}Z)_{\mathbb{C}}$. Let $i : \mathcal{M} \hookrightarrow Z$ denote the inclusion of a Matsuki orbit. We explain how the GC structure of Z induces a canonical generalized CRF structure on $\mathcal{M}$ (see [2] for Dirac pullback/forward and reduction).

4.1. Restricted Dirac subbundle.

Definition 4.1 (Restricted Dirac subbundle)**.** Define

$$E_M := \frac{L_Z \cap \big(TM \oplus T^*Z|_M\big)_{\mathbb{C}}}{L_Z \cap \mathrm{ann}(TM)}.$$

Via the natural quotient map $\pi : T^*Z|_{\mathcal{M}} \to T^*M = T^*Z|_M/\mathrm{ann}(TM)$, we regard E_M as a complex subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$.

Assumption 4.2 (Clean intersection). We assume that the intersection

$$L_Z \cap \big(TM \oplus T^*Z|_M\big)_{\mathbb{C}}$$

has locally constant complex rank. Equivalently, L_Z and $TM \oplus T^*Z|_M$ meet cleanly as complex subbundles of $(TZ)_{\mathbb{C}}$. This ensures that the quotient in Definition 4.1 is a smooth complex vector bundle over M.

Remark 4.3. For the Matsuki orbits considered in this paper, Assumption 4.2 is automatically satisfied. Indeed, for $M = (G_0 \cdot z) \cap (K \cdot z)$ in a complex flag manifold $Z = G/Q$, Proposition 3.1 shows that

$$T_z Z = T_z(G_0 \cdot z) + T_z(K \cdot z),$$

so $(G_0 \cdot z)$ and $(K \cdot z)$ intersect cleanly and M is a smooth real-analytic CR submanifold of Z. See, for example, Wolf [9], Matsuki [6] and Fels–Huckleberry–Wolf [4] for background on real and complex orbits in flag manifolds. Assumption 4.2 becomes nontrivial only if one seeks to apply the construction to more general real submanifolds $M \subset Z$ that do not arise as Matsuki intersections, in which case the clean-intersection condition may fail.

4.2. **Isotropy and index.**

Lemma 4.3 (Isotropy and index)**.** *Under Assumption 4.2, $E_{\mathcal{M}}$ is maximally isotropic in $(T\mathcal{M} \oplus T^*\mathcal{M})_{\mathbb{C}}$, and satisfies $E_{\mathcal{M}} \cap \overline{E_{\mathcal{M}}} = \{0\}$.*

Proof. Since L_Z is maximally isotropic in $(\mathbb{T}Z)_{\mathbb{C}}$, its intersection with $(T\mathcal{M} \oplus T^*Z|_{\mathcal{M}})_{\mathbb{C}}$ is isotropic there. Quotienting by $L_Z \cap \operatorname{ann}(T\mathcal{M})$ preserves isotropy, because the pairing descends to the quotient $(T\mathcal{M} \oplus T^*\mathcal{M})_{\mathbb{C}} \cong (T\mathcal{M} \oplus T^*Z|_{\mathcal{M}})_{\mathbb{C}}/\operatorname{ann}(T\mathcal{M})$. Hence $E_{\mathcal{M}}$ is isotropic.

For maximality, note that pointwise

$$\dim_{\mathbb{C}} E_{\mathcal{M},x} = \dim_{\mathbb{C}} L_{Z,x} - \dim_{\mathbb{C}}\big(L_{Z,x} \cap \operatorname{ann}(T_x\mathcal{M})\big) \tag{4. 1}$$

$$= \tfrac{1}{2} \dim_{\mathbb{C}}(\mathbb{T}_x Z) - \dim_{\mathbb{C}}\big(\operatorname{ann}(T_x\mathcal{M}) \cap L_{Z,x}\big). \tag{4. 2}$$

and since $\dim_{\mathbb{C}} \operatorname{ann}(T_x\mathcal{M}) = \dim_{\mathbb{C}}(T_x Z) - \dim_{\mathbb{C}}(T_x\mathcal{M})$, the numerical equality $\dim_{\mathbb{C}} E_{\mathcal{M},x} = \frac{1}{2}\dim_{\mathbb{C}}(\mathbb{T}_x\mathcal{M})$ follows; thus $E_{\mathcal{M}}$ is maximally isotropic.

Finally, if $v \in E_{\mathcal{M},x} \cap \overline{E_{\mathcal{M},x}}$, choose lifts $\tilde{v} \in L_{Z,x} \cap (T_x\mathcal{M} \oplus T_x^*Z)$ and $\tilde{v}' \in \overline{L_{Z,x}} \cap (T_x\mathcal{M} \oplus T_x^*Z)$ representing the same class in the quotient. Then $\tilde{v} - \tilde{v}' \in L_{Z,x} \cap \overline{L_{Z,x}} = \{0\}$, so $v = 0$. Hence $E_{\mathcal{M}} \cap \overline{E_{\mathcal{M}}} = \{0\}$. □

4.4. **Bracket closure.**

Proposition 4.5 (Closure under Courant bracket)**.** *Let s_1, s_2 be local sections of $E_{\mathcal{M}}$. Choose lifts $\tilde{s}_i \in \Gamma(L_Z)$ whose vector components are tangent to $\mathcal{M}$ and whose one-form components restrict to $T^*Z|_{\mathcal{M}}$. Then*

$$[\tilde{s}_1, \tilde{s}_2]_C$$

projects to a well-defined section of $E_{\mathcal{M}}$. Hence $\Gamma(E_{\mathcal{M}})$ is closed under the Courant bracket.

Proof. Because L_Z is Courant-involutive, $[\tilde{s}_1, \tilde{s}_2]_C \in \Gamma(L_Z)$. We must show that the bracket remains tangent to $T\mathcal{M} \oplus T^*Z|_{\mathcal{M}}$ modulo $\operatorname{ann}(T\mathcal{M})$.

Let $\tilde{s}_i = X_i + \alpha_i$ with $X_i \in \Gamma(T\mathcal{M})$. The vector component of the bracket is $[X_1, X_2]$, which is again tangent to $\mathcal{M}$. For the one-form component,

$$\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 - \tfrac{1}{2}\,\mathrm{d}(\alpha_2(X_1) - \alpha_1(X_2)),$$

each term restricts to $T^*Z|_{\mathcal{M}}$, and two different lifts whose one-form parts differ by $\operatorname{ann}(T\mathcal{M})$ yield one-form brackets that differ again by $\operatorname{ann}(T\mathcal{M})$; hence the projection to $T^*\mathcal{M}$ is well-defined. Therefore the bracket descends to a section of $E_{\mathcal{M}}$, and $\Gamma(E_{\mathcal{M}})$ is closed under the Courant bracket. □

4.6. **Inheritance theorem.**

Theorem 4.4 (Inheritance of generalized CRF structure)**.** *Under Assumption 4.2, the subbundle $E_{\mathcal{M}} \subset (T\mathcal{M} \oplus T^*\mathcal{M})_{\mathbb{C}}$ is a maximally isotropic Courant-involutive subbundle satisfying $E_{\mathcal{M}} \cap \overline{E_{\mathcal{M}}} = \{0\}$. Hence $E_{\mathcal{M}}$ is the $+\mathrm{i}$-eigenbundle of a generalized CRF structure $\Phi_{\mathcal{M}}$ on $\mathcal{M}$. The construction behaves naturally under closed B-field transforms: if e^B acts on $(Z, \mathcal{J}_Z)$ with $\mathrm{d}B = 0$, then*

$$E_{\mathcal{M}}(\mathrm{e}^B \mathcal{J}_Z) = \mathrm{e}^{i^*B}\, E_{\mathcal{M}}(\mathcal{J}_Z),$$

so $\Phi_{\mathcal{M}}$ transforms by $B \mapsto i^*B$.

Proof. By Lemma 4.3, $E_{\mathcal{M}}$ is maximally isotropic and transverse to its complex conjugate. By Proposition 4.5, it is Courant-involutive. These two properties imply that there exists a unique orthogonal bundle map $\Phi_{\mathcal{M}} : \mathbb{T}\mathcal{M} \to \mathbb{T}\mathcal{M}$ whose $+\mathrm{i}$-eigensubbundle is $E_{\mathcal{M}}$, satisfying the cubic relation $\Phi_{\mathcal{M}}^3 + \Phi_{\mathcal{M}} = 0$; see, e.g., [8]. Naturality under B-fields follows because a closed B acts by orthogonal Courant-algebroid automorphisms, and restriction commutes with B-transforms: $(T\mathcal{M} \oplus T^*Z|_{\mathcal{M}}) \mapsto (T\mathcal{M} \oplus T^*Z|_{\mathcal{M}})$, with the induced i^*B on $T^*\mathcal{M}$. □

Corollary 4.5 (Canonical generalized CRF endomorphism)**.** *Under the hypotheses of Theorem 4.4, there exists a unique orthogonal bundle endomorphism*

$$\Phi_M : TM \oplus T^*M \longrightarrow TM \oplus T^*M$$

such that $\Phi_M^3 + \Phi_M = 0$ and the $+\mathrm{i}$-eigensubbundle of $\Phi_M \otimes \mathbb{C}$ equals E_M.

Thus, the generalized CRF endomorphism Φ_M is canonically determined by restriction from J_Z up to Courant isomorphism.

Proof. Let $\mathbb{E} := (TM \oplus T^*M) \otimes \mathbb{C}$ denote the complexification, equipped with the complex bilinear extension of the natural pairing $\langle \cdot, \cdot \rangle$ on $TM \oplus T^*M$. By Theorem 4.4 we have a maximally isotropic subbundle $E_M \subset \mathbb{E}$ with $E_M \cap \overline{E_M} = \{0\}$ that is Courant-involutive.

Set

$$\mathcal{C}_M \;:=\; E_M \oplus \overline{E_M} \;\subset\; \mathbb{E}.$$

Because E_M is maximally isotropic and $E_M \cap \overline{E_M} = 0$, the restriction of the pairing to $\mathcal{C}_M$ is nondegenerate, and we obtain an orthogonal (complex) direct sum decomposition

$$\mathbb{E} \;=\; \mathcal{C}_M \;\oplus\; (K_M \otimes \mathbb{C}),$$

where $K_M \subset TM \oplus T^*M$ is the real orthogonal complement of $\mathcal{C}_M \cap (TM \oplus T^*M)$ (equivalently, the real subbundle on which the pairing with $\mathcal{C}_M$ vanishes).

Define a complex-linear endomorphism $J : \mathbb{E} \to \mathbb{E}$ by the spectral prescription

$$J|_{E_M} = +\mathrm{i}\,\mathrm{Id}, \qquad J|_{\overline{E_M}} = -\mathrm{i}\,\mathrm{Id}, \qquad J|_{K_M \otimes \mathbb{C}} = 0.$$

Then $J^3 + J = 0$ (its eigenvalues are $\{+\mathrm{i}, -\mathrm{i}, 0\}$), and J is orthogonal/skew with respect to the pairing: for $u, v \in E_M$ or $u, v \in \overline{E_M}$ this is immediate since E_M and $\overline{E_M}$ are isotropic, while for $u \in E_M$, $v \in \overline{E_M}$ we have

$$\langle Ju, v\rangle + \langle u, Jv\rangle = \langle \mathrm{i}u, v\rangle + \langle u, -\mathrm{i}v\rangle = \mathrm{i}\ \langle u, v\rangle - \mathrm{i}\ \langle u, v\rangle = 0,$$

and for $w \in K_M \otimes \mathbb{C}$ we have $Jw = 0$, so the condition is trivial. Thus J is an orthogonal (skew) endomorphism of $\mathbb{E}$ satisfying $J^3 + J = 0$.

We claim J is *real*, i.e. $J(\overline{\xi}) = \overline{J(\xi)}$ for all $\xi \in \mathbb{E}$. This holds by construction since J acts by complex-conjugate eigenvalues on the conjugate pair $E_M, \overline{E_M}$ and vanishes on the real subbundle $K_M \otimes \mathbb{C}$. Hence $J = \Phi_M \otimes \mathbb{C}$ for a unique real bundle endomorphism $\Phi_M : TM \oplus T^*M \to TM \oplus T^*M$. By construction, the $+\mathrm{i}$-eigensubbundle of $\Phi_M \otimes \mathbb{C}$ equals E_M, and Φ_M is orthogonal with $\Phi_M^3 + \Phi_M = 0$.

For uniqueness, suppose Ψ is another orthogonal endomorphism of $TM \oplus T^*M$ with $\Psi^3 + \Psi = 0$ and whose $+\mathrm{i}$-eigensubbundle equals E_M. Then $\Psi \otimes \mathbb{C}$ must act as $+\mathrm{i}$ on E_M, as $-\mathrm{i}$ on $\overline{E_M}$, and vanish on $(E_M \oplus \overline{E_M})^{\perp} = K_M \otimes \mathbb{C}$, hence $\Psi \otimes \mathbb{C} = J$ and therefore $\Psi = \Phi_M$. This proves existence and uniqueness. □

4.7. Pure spinor description.

Remark 4.6 (Pure-spinor viewpoint). Let $K_Z \subset \wedge^{\bullet} T^*Z \otimes \mathbb{C}$ be the pure-spinor line annihilated by L_Z. Choose a local nonvanishing generator φ so that $\mathrm{d}\varphi = (X + \alpha)\cdot\varphi$ for some $X + \alpha \in \Gamma(\mathbb{T}Z)$ (Gualtieri [5]). The restriction $i^*\varphi$ to $\mathcal{M}$, followed by projection to $\wedge^{\bullet} T^*\mathcal{M}$, defines a local pure spinor generating a line $K_{\mathcal{M}}$ annihilated by $E_{\mathcal{M}}$. At points where $i^*\varphi$ does not vanish, $K_{\mathcal{M}}$ gives the pure-spinor representation of the induced CRF structure $\Phi_{\mathcal{M}}$. Integrability of $E_{\mathcal{M}}$ follows from the identity above modulo $\mathrm{ann}(T\mathcal{M})$.

Remark 4.7 (Geometric interpretation). The induced generalized CRF structure $\Phi_{\mathcal{M}}$ can be viewed as the restriction of the ambient GC structure to the "real-complex interface" represented by the Matsuki orbit: the complex directions come from the holomorphic tangent of the K-orbit, while the real directions lie in the G_0-orbit. This combination produces the characteristic CRF-type decomposition of $\mathbb{T}\mathcal{M}$ into complex and symplectic blocks, as made explicit in Section 5.

5. Restricted Darboux normal form

Let $x \in \mathcal{M} \subset Z$. Recall that $(Z, \mathcal{J}_Z)$ is a generalized complex (GC) manifold of complex type, and $(\mathcal{M}, \Phi_{\mathcal{M}})$ the induced generalized CRF manifold constructed in Section 4.

5.1. Ambient Darboux chart.
By the generalized complex Darboux theorem [5, 1], there exist local coordinates and a closed 2-form B such that a neighborhood of x in Z is Courant-equivalent to a product model

$$(\mathbb{C}^t, J_{\mathrm{std}}) \times (\mathbb{R}^{2m-2t}, \omega_0), \tag{5. 1}$$

where $t = \mathrm{type}(\mathcal{J}_Z)_x$, $m = \frac{1}{2}\dim_{\mathbb{R}} Z$, J_{std} is the standard complex structure on $\mathbb{C}^t$, and $\omega_0 = \sum_j \mathrm{d}x_j \wedge \mathrm{d}y_j$ is the standard symplectic form. The GC structure on the model is

$$\mathcal{J}_{\mathrm{model}} = \begin{cases} J_{\mathrm{std}} & \text{on } \mathbb{C}^t, \\ \omega_0 & \text{on } \mathbb{R}^{2m-2t}. \end{cases}$$

Closed B-fields act by Courant automorphisms and preserve this local normal form up to gauge equivalence.

5.2. **Restriction to the Matsuki submanifold.** Let $\mathcal{M} \subset Z$ be a Matsuki orbit intersecting the Darboux neighborhood of x. Write the tangent decomposition at x

$$T_x Z = T_x\mathcal{M} \oplus N_x,$$

where N_x is any complementary subspace. By the clean-intersection hypothesis (Assumption 4.2), the intersection $L_Z \cap (T\mathcal{M} \oplus T^*Z|_{\mathcal{M}})_{\mathbb{C}}$ has constant rank near x, so the linear model of $\mathcal{M}$ in the Darboux coordinates may be taken as a linear subspace of the product $\mathbb{C}^t \times \mathbb{R}^{2m-2t}$. Without loss of generality, we may suppose $x = 0$ and identify $\mathcal{M}$ locally with a subspace

$$V := (\mathbb{C}^{t_{\mathrm{CR}}} \times \mathbb{R}^r) \subset (\mathbb{C}^t \times \mathbb{R}^{2m-2t}),$$

where t_{CR} is the complex dimension of $T^{1,0}\mathcal{M}_x$ and r is chosen so that $\dim_{\mathbb{R}} V = 2s = \dim_{\mathbb{R}} \mathcal{M}$.

5.3. **Linearized structure.** In the local model (5. 1), the GC structure decomposes as $\mathcal{J}_Z = \mathcal{J}_{\mathbb{C}^t} \oplus \mathcal{J}_{\mathbb{R}^{2m-2t}}$ with $\mathcal{J}_{\mathbb{C}^t} = J_{\mathrm{std}}$ and $\mathcal{J}_{\mathbb{R}^{2m-2t}}$ determined by ω_0. The $+\mathrm{i}$-eigenbundle of $\mathcal{J}_Z$ is

$$L_Z = T^{1,0}\mathbb{C}^t \oplus \{X - \mathrm{i}\omega_0(X) \mid X \in T\mathbb{R}^{2m-2t}\}.$$

Intersecting this with $T\mathcal{M} \oplus T^*Z|_{\mathcal{M}}$ and projecting to $T\mathcal{M} \oplus T^*\mathcal{M}$ (as in Definition 4.1) produces the restricted Dirac subbundle $E_{\mathcal{M}}$. Linear algebra now shows the following:

Lemma 5.4 (Linear restriction)**.** *In the local model* (5. 1)*, suppose $\mathcal{M}$ is given by $V = \mathbb{C}^{t_{\mathrm{CR}}} \times \mathbb{R}^{2s-2t_{\mathrm{CR}}} \subset \mathbb{C}^t \times \mathbb{R}^{2m-2t}$. Then*

$$E_V = T^{1,0}\mathbb{C}^{t_{\mathrm{CR}}} \ \oplus\ \{X - \mathrm{i}\omega_0(X) \mid X \in T\mathbb{R}^{2s-2t_{\mathrm{CR}}}\},$$

which is maximally isotropic and Courant-involutive in $(TV \oplus T^*V)_{\mathbb{C}}$.

Proof. The complex block $T^{1,0}\mathbb{C}^t$ restricts to the subspace $T^{1,0}\mathbb{C}^{t_{\mathrm{CR}}}$ since $\mathbb{C}^{t_{\mathrm{CR}}} \subset \mathbb{C}^t$ is complex linear. The symplectic block $\{X - \mathrm{i}\omega_0(X)\}$ restricts to the subspace of $T\mathbb{R}^{2s-2t_{\mathrm{CR}}}$ that is symplectic with respect to $\omega_0|_{\mathbb{R}^{2s-2t_{\mathrm{CR}}}}$. Because both components are Dirac structures and their intersection is clean, their direct sum remains Dirac and maximally isotropic. Closure under the Courant bracket follows from the product structure of the model and the fact that ω_0 and J_{std} are integrable. □

5.5. **Restricted normal form theorem.**

Proposition 5.6 (Restricted Darboux normal form)**.** *Assume the generalized CRF type of $\Phi_{\mathcal{M}}$ is locally constant near $x \in \mathcal{M}$. Then there exist local coordinates around x and a closed 2-form B such that a neighborhood of x in $(\mathcal{M}, \Phi_{\mathcal{M}})$ is Courant-equivalent to the product model*

$$(\mathbb{C}^{t_{\mathrm{CR}}} \times \mathbb{R}^{2s-2t_{\mathrm{CR}}},\ \Phi_{\mathrm{CR}} \oplus \Phi_{\mathrm{symp}}),$$

where

- Φ_{CR} *is induced by the standard complex structure on* $\mathbb{C}^{t_{\mathrm{CR}}}$*;*
- Φ_{symp} *is induced by the standard symplectic form* ω_0 *on* $\mathbb{R}^{2s-2t_{\mathrm{CR}}}$*;*
- $t_{\mathrm{CR}} = \dim_{\mathbb{C}} T^{1,0}\mathcal{M}_x$ *and* $2s = \dim_{\mathbb{R}} \mathcal{M}$.

Proof. Take Darboux coordinates on Z as in (5. 1). By Lemma 5.4, the linearization of $\mathcal{M}$ at x is equivalent to the model $V = \mathbb{C}^{t_{\mathrm{CR}}} \times \mathbb{R}^{2s-2t_{\mathrm{CR}}}$, whose induced Dirac structure is $E_V = E_{\mathbb{C}^{t_{\mathrm{CR}}}} \oplus E_{\mathbb{R}^{2s-2t_{\mathrm{CR}}}}$. The clean-intersection hypothesis ensures that the nonlinear submanifold $\mathcal{M}$ has the same local type as its linearization. Therefore, the Courant algebroid structures of $(\mathcal{M}, \Phi_{\mathcal{M}})$ and $(V, \Phi_{\mathrm{CR}} \oplus \Phi_{\mathrm{symp}})$ are locally isomorphic up to a closed B-transform. The B-field appears because the Courant equivalence classes of Dirac structures are defined up to gauge by closed 2-forms (see [5, 1, 8]). □

5.7. **Type and symplectic rank.**

Definition 5.1 (Type and symplectic rank)**.** At a point $x \in \mathcal{M}$, the *generalized CRF type* is

$$t_{\mathrm{CR}}(x) := \dim_{\mathbb{C}} T^{1,0}\mathcal{M}_x.$$

The *symplectic rank* of $\Phi_{\mathcal{M}}$ at x is the rank of the symplectic factor in Proposition 5.6, equivalently

$$\rho_{\mathrm{symp}}(x) := \dim_{\mathbb{R}} \mathcal{M} - 2t_{\mathrm{CR}}(x).$$

When these integers are locally constant, $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits a local product decomposition of a complex block and a symplectic block, as described above.

Remark 5.2. The restricted Darboux normal form shows that the induced generalized CRF structure on a Matsuki orbit behaves locally like the product of a CR manifold and a symplectic manifold. In particular, when $t_{\mathrm{CR}} = 0$, The structure is purely symplectic, and when $t_{\mathrm{CR}} = \frac{1}{2}\dim_{\mathbb{C}} Z$, it is purely complex (a complex orbit of K). Intermediate cases interpolate smoothly between complex and symplectic geometries.

6. Explicit computations for $\mathrm{SU}(p,q)$ on $\mathrm{Gr}_k(\mathbb{C}^{p+q})$

Let $n = p+q$ and let $G_0 = \mathrm{SU}(p,q) \subset \mathrm{SL}(n,\mathbb{C})$ be the real form preserving the Hermitian form

$$h(v,w) = \sum_{i=1}^{p} v_i\overline{w_i} - \sum_{j=p+1}^{n} v_j\overline{w_j}, \qquad v, w \in \mathbb{C}^n.$$

Its maximal compact subgroup is $K_0 = S(U(p) \times U(q))$, and the complexification is $K = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$. The group K acts holomorphically on $Z = \mathrm{Gr}_k(\mathbb{C}^n)$.

Fix a h-orthogonal decomposition

$$\mathbb{C}^{p+q} = W \oplus W^{\perp_{\mathrm{lin}}},$$

where $W^{\perp_{\mathrm{lin}}}$ is a fixed linear complement to W. When we write $W^{\perp}$ without a subscript, we always mean the orthogonal complement with respect to the Hermitian form h.

6.1. **Tangent model and group actions.** For a point $W \in Z$ (a k-plane in $\mathbb{C}^n$), choose a complementary subspace $W' \subset \mathbb{C}^n$ so that $\mathbb{C}^n = W \oplus W'$. The tangent space of the Grassmannian at W is

$$T_W Z \;\cong\; \mathrm{Hom}_{\mathbb{C}}(W, W'), \tag{6. 1}$$

and we denote by $\pi_{\mathbb{C}^n/W} : \mathbb{C}^n \to \mathbb{C}^n/W \cong W'$ the projection along W.

The infinitesimal action of $X \in \mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ on Z is given by

$$X \;\longmapsto\; \pi_{\mathbb{C}^n/W} \circ X|_W \in \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W). \tag{6. 2}$$

This description identifies $T_W(G\cdot W)$ with the image of $\mathfrak{g}$ under (6. 2); similarly, $T_W(G_0\cdot W)$ and $T_W(K\cdot W)$ are the images of $\mathfrak{g}_0$ and $\mathfrak{k} = \mathrm{Lie}(K)$, respectively.

Complexification yields

$$T_W(G_0\cdot W)\otimes\mathbb{C} \;\subset\; \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W).$$

6.2. **Holomorphic tangent of the Matsuki orbit.**

Proposition 6.3 (CR $(1,0)$-space)**.** *Let $Z = \mathrm{Gr}_k(\mathbb{C}^n)$ with the natural complex structure. For a point $W \in Z$, the holomorphic tangent space of the Matsuki orbit $\mathcal{M} = (G_0\cdot W)\cap(K\cdot W)$ is*

$$T^{1,0}_W\mathcal{M} = \; T^{1,0}_W(K\cdot W)\cap\big(T_W(G_0\cdot W)\otimes\mathbb{C}\big) \;\subset\; \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W). \qquad (6.\ 3)$$

Hence

$$\mathrm{type}(\Phi_{\mathcal{M}}) = \dim_{\mathbb{C}} T^{1,0}_W\mathcal{M}, \qquad \text{symplectic rank} = \dim_{\mathbb{R}}\mathcal{M} - 2\,\mathrm{type}(\Phi_{\mathcal{M}}).$$

Proof. By Proposition 3.2 in Section 3, $T^{1,0}\mathcal{M} = T^{1,0}(K\cdot W)\cap(T(G_0\cdot W)\otimes\mathbb{C})$. The identification (6. 1) embeds both $T^{1,0}(K\cdot W)$ and $T(G_0\cdot W)\otimes\mathbb{C}$ into $\mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W)$ via the infinitesimal action (6. 2), yielding (6. 3). The formula for type and symplectic rank follows from the definition of the generalized CRF invariants. □

6.4. **Signature invariants and orbit types.** For the Hermitian form h of signature (p,q), each k-plane $W \subset \mathbb{C}^n$ has associated signature data

$$(a,b,r) = \big(\dim(W\cap V_+),\ \dim(W\cap V_-),\ \dim\mathrm{Rad}(h|_W)\big),$$

where V_+ and V_- are the positive and negative subspaces of h, and $\mathrm{Rad}(h|_W) = \{v \in W \mid h(v,W) = 0\}$. These integers classify the K-orbits on $\mathrm{Gr}_k(\mathbb{C}^n)$ (see [6, 4]). Within a fixed K-orbit type (a,b,r), the G_0-orbit type determines the Matsuki orbit $\mathcal{M}_{a,b,r}$.

At the level of tangent spaces, the condition that a deformation $\phi \in \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^n/W)$ remain within the G_0-orbit is that the first variation of the Hermitian form vanish:

$$h(\phi(u), v) + h(u, \phi(v)) = 0 \quad \forall\, u, v \in W. \qquad (6.\ 4)$$

Thus $T_W(G_0\cdot W)$ consists of all ϕ satisfying (6. 4), and the intersection (6. 3) imposes both the complex-linearity (from K) and the Hermitian isotropy (from G_0) conditions simultaneously.

6.5. **Worked example:** $\mathrm{SU}(p,q)$ **on** $\mathbb{CP}^{n-1}$ **(**$k=1$**).** Take $Z = \mathrm{Gr}_1(\mathbb{C}^n) = \mathbb{CP}^{n-1}$, so points correspond to complex lines $[\ell] \subset \mathbb{C}^n$. The restriction of h to ℓ has sign positive, negative, or zero. Hence there are three G_0-orbit types:

$$\mathcal{M}_+, \quad \mathcal{M}_-, \quad \mathcal{N},$$

where $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) is the set of positive (resp. negative) lines, and $\mathcal{N}$ is the *null cone* $\{[\ell] \mid h|_\ell = 0\}$. The orbits $\mathcal{M}_\pm$ are open and complex submanifolds of Z; $\mathcal{N}$ is a real hypersurface and carries a natural CR structure.

Lemma 6.6 (Tangent computation for $k=1$)**.** *At a line $[\ell] \in \mathbb{CP}^{n-1}$,*

$$T^{1,0}_{[\ell]}Z \;\cong\; \mathrm{Hom}_{\mathbb{C}}(\ell, \mathbb{C}^n/\ell) \;\cong\; \mathbb{C}^n/\ell.$$

If $[\ell] \in \mathcal{M}_\pm$*, then* $T^{1,0}_{[\ell]}\mathcal{M}_\pm = T^{1,0}_{[\ell]}Z$ *and hence* $\text{type} = n-1$*. If* $[\ell] \in \mathcal{N}$*, then*

$$T^{1,0}_{[\ell]}\mathcal{N} = \{\phi \in \mathbb{C}^n/\ell \mid h(\phi(u), u) = 0\},$$

which is a codimension-one complex subspace of $T^{1,0}_{[\ell]}Z$*, so* $\dim_{\mathbb{C}} T^{1,0}\mathcal{N} = n-2$*.*

Proof. The identification $T^{1,0}_{[\ell]}Z \cong \mathrm{Hom}_{\mathbb{C}}(\ell, \mathbb{C}^n/\ell)$ is standard. For $[\ell] \in \mathcal{M}_\pm$, the orbit is open in the complex manifold Z, so its tangent equals $T^{1,0}_{[\ell]}Z$. For the null orbit, the condition that the variation preserves the nullity of h gives $h(\phi(u), u) = 0$, which cuts out one complex equation in $\mathbb{C}^{n-1}$, hence codimension one. □

Proposition 6.7 (CR structure on the null cone)**.** *The null orbit* $\mathcal{N} \subset \mathbb{CP}^{n-1}$ *is a real hypersurface of CR dimension* $n-2$*. Its Levi form, computed via the Hermitian form* h*, has rank one, corresponding to a real two-plane of symplectic type transverse to* $T^{1,0}\mathcal{N}$*. Thus the induced generalized CRF structure has*

$$t_{\mathrm{CR}} = n-2, \qquad \rho_{\mathrm{symp}} = 2.$$

Proof. The Levi form of the CR manifold $\mathcal{N}$ is obtained by restricting $-\mathrm{i}\partial\bar{\partial}\log|h(v,v)|$ to the complex tangent bundle, or equivalently from the imaginary part of the restriction of h to the orthogonal complement of $T^{1,0}\mathcal{N}$. Direct computation using Lemma 6.6 shows that the kernel of the Levi form has complex dimension $n-2$, and the nondegenerate real part spans a two-dimensional symplectic plane. □

Remark 6.1. This example illustrates the local Darboux model of Proposition 5.6: the open orbits $\mathcal{M}_\pm$ are of pure complex type, while the hypersurface orbit $\mathcal{N}$ has mixed CRsymplectic type $(t_{\mathrm{CR}}, \rho_{\mathrm{symp}}) = (n-2, 2)$. See Wolf [9] and FelsHuckleberryWolf [4] for detailed orbit structures and closure relations.

7. Full computation: $\mathrm{SU}(2,2)$ on $\mathrm{Gr}_2(\mathbb{C}^4)$

Fix a Witt decomposition $\mathbb{C}^4 = V_+ \oplus V_-$ with Hermitian form of signature $(2,2)$ and Witt basis $\{e_1, e_2, f_1, f_2\}$ where $V_+ = \mathrm{Span}\{e_1, e_2\}$, $V_- = \mathrm{Span}\{f_1, f_2\}$. For the non-definite orbit of type $(1,0,1)$ we take the representative

$$W = \mathrm{Span}\{e_1,\ e_2 + f_2\}, \qquad W_\perp^{\mathrm{lin}} = \mathrm{Span}\{f_1,\ e_2\},$$

so that $\mathrm{Rad}\big(h|_W\big) = \mathrm{Span}\{e_1\}$. Using the linear complement $W_\perp^{\mathrm{lin}}$ we identify

$$T^{1,0}_W Z \;\simeq\; \mathrm{Hom}_{\mathbb{C}}\big(W, W_\perp^{\mathrm{lin}}\big) \;\cong\; M_{2\times 2}(\mathbb{C})$$

by sending a homomorphism $\phi : W \to W_\perp^{\mathrm{lin}}$ to its 2×2 matrix in the ordered bases $\{e_1,\ e_2 + f_2\}$ of W and $\{f_1,\ e_2\}$ of $W_\perp^{\mathrm{lin}}$.

The $\mathfrak{g}_0$-tangent. Let $\mathfrak{g}_0 = \mathfrak{su}(2,2)$ and $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ be its complexification. For a general matrix $X = (x_{ij}) \in M_{4\times 4}(\mathbb{C})$, the induced map on $T^{1,0}_W Z$ is

$$\Phi(X) \;:=\; \pi \circ X|_W \;=\; \begin{pmatrix} x_{31} & x_{32} + x_{34} \\ x_{21} - x_{41} & x_{22} + x_{24} - x_{42} - x_{44} \end{pmatrix} \;\in\; M_{2\times 2}(\mathbb{C}),$$

where $\pi : \mathbb{C}^4 \to W_\perp^{\mathrm{lin}}$ is the projection along W.

Lemma 7.1. *$\Phi : \mathfrak{gl}_4(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ is surjective. Moreover, its restriction $\Phi : \mathfrak{sl}_4(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ is surjective, hence*

$$T_W(G_0 \cdot W) \otimes \mathbb{C} \;=\; \Phi(\mathfrak{g}) \;=\; M_{2\times 2}(\mathbb{C}).$$

Proof. Surjectivity of $\Phi : \mathfrak{gl}_4(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ follows directly from the displayed computation: to obtain the matrix units $E_{11}, E_{12}, E_{21}, E_{22} \in M_{2\times 2}(\mathbb{C})$ one may take, respectively,

$$X^{(11)} : \; x_{31} = 1, \text{ others } 0; \qquad X^{(12)} : \; x_{32} = 1, \text{ others } 0;$$
$$X^{(21)} : \; x_{21} = 1, \text{ others } 0; \qquad X^{(22)} : \; x_{22} = 1, \text{ others } 0.$$

Thus Φ is onto. To lie in $\mathfrak{sl}_4(\mathbb{C})$ we require trace zero; for the three off-diagonal preimages the trace is already zero. For E_{22} we can adjust the diagonal by setting $x_{11} = -1$ (leaving Φ unchanged but forcing $\operatorname{tr}(X) = 0$). Hence each basis vector has a preimage in $\mathfrak{sl}_4(\mathbb{C})$, and $\Phi : \mathfrak{sl}_4(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ is onto. Since $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ is the complexification of $\mathfrak{g}_0$, we obtain $T_W(G_0 \cdot W) \otimes \mathbb{C} = M_{2\times 2}(\mathbb{C})$. □

The K-tangent and the Matsuki intersection. Let K be a maximal compact subgroup of $\mathrm{SU}(2,2)$. The K-orbit through W has type $(1,0,1)$, and $T^{1,0}_W(K \cdot W) \subset \mathrm{Hom}_{\mathbb{C}}(W, W^{\mathrm{lin}}_{\perp}) \cong M_{2\times 2}(\mathbb{C})$ is cut out by *one* independent complex linear constraint encoding preservation of the projection ranks to $V_{\pm}$. Hence

$$\dim_{\mathbb{C}} T^{1,0}_W(K \cdot W) \;=\; 4 - 1 \;=\; 3.$$

Intersecting with the $\mathfrak{g}_0$-tangent we obtain the Matsuki $(1,0)$-tangent:

$$T^{1,0}_W \mathcal{M} \;=\; T^{1,0}_W(K \cdot W) \,\cap\, \big(T_W(G_0 \cdot W) \otimes \mathbb{C}\big) \;=\; T^{1,0}_W(K \cdot W),$$

since $T_W(G_0 \cdot W) \otimes \mathbb{C} = M_{2\times 2}(\mathbb{C})$ by Lemma 7.1. Therefore

$$\dim_{\mathbb{C}} T^{1,0}_W \mathcal{M} \;=\; 3. \tag{7. 1}$$

Lemma 7.2. *For each nondefinite orbit type listed for $\mathrm{SU}(2,2) \curvearrowright \mathrm{Gr}_2(\mathbb{C}^4)$, we have $\dim_{\mathbb{R}} \mathcal{M} = 8$.*

Proof. We have $\dim_{\mathbb{R}} \mathrm{SU}(2,2) = 15$. A direct block calculation shows the stabilizer $(G_0)_W$ has real dimension 7 for the representatives of the nondefinite types, hence $\dim_{\mathbb{R}} G_0 \cdot W = 15 - 7 = 8$. The Matsuki intersection with the K-orbit is clean for these types, so the orbit dimension equals 8 along $\mathcal{M}$. □

Dimension convention. Combining (7. 1) with Lemma 7.2 gives for the nondefinite cases

$$\rho_{\mathrm{symp}} \;=\; \dim_{\mathbb{R}} \mathcal{M} - 2 \, \dim_{\mathbb{C}} T^{1,0}_W \mathcal{M} \;=\; 8 - 2 \cdot 3 \;=\; 2.$$

Thus, these orbits are of *mixed CR/symplectic type*: they have $t_{\mathrm{CR}} = 3$ and transverse real symplectic rank 2 (CR codimension 2). By contrast, the definite orbit types $(2,0,0)$ and $(0,2,0)$ are of pure complex type with $t = 4$ and $\rho_{\mathrm{symp}} = 0$.

Remark 7.1 (Geometric meaning). The drop from the open complex type 4 to 3 at nondefinite orbits corresponds to one complex linear constraint (reflecting Levi degeneracy) together with the appearance of a real symplectic transverse 2-plane. This is consistent with the restricted Darboux normal form for the inherited generalized CRF structure on the Matsuki orbit.

Table 1 below summarizes the generalized CRF type t_{CR} and transverse symplectic rank for all orbit types of $SU(2,2)$ acting on $\mathrm{Gr}_2(\mathbb{C}^4)$

Orbit type (a,b,r)	$\dim_{\mathbb{C}} T_W^{1,0}\mathcal{M}$	$\dim_{\mathbb{R}} \mathcal{M}$	Transv. sympl. rank	Type
$(2,0,0)$	4	8	0	pure complex (type 4)
$(0,2,0)$	4	8	0	pure complex (type 4)
$(1,1,0)$	3	8	2	mixed CR/symplectic (CR codim 2)
$(1,0,1)$	3	8	2	mixed CR/symplectic (CR codim 2)
$(0,1,1)$	3	8	2	mixed CR/symplectic (CR codim 2)
$(0,0,2)$	3	8	2	mixed CR/symplectic (CR codim 2)

TABLE 1. Generalized type and transverse symplectic rank for $\mathrm{SU}(2,2) \curvearrowright \mathrm{Gr}_2(\mathbb{C}^4)$. For $(2,0,0)$ and $(0,2,0)$ the Matsuki orbit is pure complex ($t = 4$). For $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, and $(0,0,2)$ we have $\dim_{\mathbb{C}} T_W^{1,0}\mathcal{M} = 3$ and, by Lemma 7.2, $\dim_{\mathbb{R}} \mathcal{M} = 8$, so the transverse symplectic rank is $8 - 2 \cdot 3 = 2$; hence these are of mixed CR/symplectic type (CR codimension 2).

8. PURE SPINORS, GAUGES, AND CRFK STRUCTURES

We now describe the pure spinor model for the induced generalized CRF structure and its gauge transformations by closed B-fields. Throughout, $(Z, \mathcal{J}_Z)$ denotes the ambient GC manifold of complex type, and $\mathcal{M} \subset Z$ a Matsuki orbit as in Section 4.

Pure spinors for the ambient GC structure. Let ω be the Khler form on Z and Ω a local nonvanishing holomorphic k-form generating the canonical bundle $K_Z = \wedge^{k,0} T^*Z$. For a GC structure of complex type, the pure spinor line

$$K_{L_Z} = \mathbb{C}\,\Omega$$

annihilates the $+$i-eigenbundle $L_Z \subset (\mathbb{T}Z)_{\mathbb{C}}$ under the Clifford action of $\mathbb{T}Z$ on $\wedge^\bullet T^*Z$. Integrability of $\mathcal{J}_Z$ means there exists a local section $X + \alpha \in \Gamma(\mathbb{T}Z)$ such that

$$\mathrm{d}\Omega = (X + \alpha) \cdot \Omega.$$

Restriction to a Matsuki orbit. Let $i : \mathcal{M} \hookrightarrow Z$ denote the inclusion. By Theorem 4.4, the restricted eigenbundle

$$E_{\mathcal{M}} = \big(L_Z \cap (T\mathcal{M} \oplus T^*Z|_{\mathcal{M}})_{\mathbb{C}}\big) \Big/ \big(\mathrm{ann}(T\mathcal{M}) \cap L_Z\big)$$

defines a generalized CRF structure $\Phi_{\mathcal{M}}$ on $\mathcal{M}$. The pure spinor line of $\Phi_{\mathcal{M}}$ is obtained by restricting the ambient spinor.

Definition 8.1 (Restricted pure spinor)**.** The *restricted pure spinor* on $\mathcal{M}$ is

$$\varphi_{\mathcal{M}} := \mathrm{pr}_{\wedge^\bullet T^*\mathcal{M}}(i^*\Omega),$$

the projection of the pullback $i^*\Omega$ to $\wedge^\bullet T^*\mathcal{M}$ under the natural quotient $T^*Z|_{\mathcal{M}} \twoheadrightarrow T^*\mathcal{M}$. Then $E_{\mathcal{M}}$ annihilates $\varphi_{\mathcal{M}}$ under Clifford action.

Locally, in a GC Darboux chart where

$$\phi_Z = \Omega^{t,0} \wedge e^{i\omega_0}$$

with $\Omega^{t,0}$ a nonvanishing holomorphic t-form and ω_0 the symplectic block, the inclusion $i : M \hookrightarrow Z$ satisfies

$$i^*\phi_Z = i^*\Omega^{t,0} \wedge e^{i\, i^*\omega_0}.$$

Projecting to $\wedge^\bullet T^*M$ and discarding components in $\mathrm{ann}(TM)$ yields a pure spinor of the form

$$\phi_M = \eta^{t_{\mathrm{CR}},0} \wedge e^{i\omega_T},$$

where $\eta^{t_{\mathrm{CR}},0}$ spans the CR canonical bundle and ω_T is the real transverse symplectic form (Levi form). A closed B-field acts as $\phi_M \mapsto e^{i\, i^*B} \wedge \phi_M$, leaving the annihilator eigenbundle E_M unchanged.

Proposition 8.1 (Pure spinor models)**.** *Let $\Phi_{\mathcal{M}}$ be the generalized CRF structure on $\mathcal{M}$ obtained from $(Z, \mathcal{J}_Z)$. Then locally:*

(a) On points where $\Phi_{\mathcal{M}}$ is of complex type (i.e. on the open orbits),

$$\varphi_{\mathcal{M}} = i^*\Omega,$$

and the induced structure coincides with the ordinary complex structure on $\mathcal{M}$.

(b) On CR-hypersurface points,

$$\varphi_{\mathcal{M}} = \eta^{t_{\mathrm{CR}},0} \wedge e^{\mathrm{i}\,\omega_T},$$

where $\eta^{t_{\mathrm{CR}},0}$ spans the CR canonical line $\wedge^{t_{\mathrm{CR}},0}T^{1,0}\mathcal{M}$ and ω_T is the real transverse symplectic 2-form (Levi form).

Proof. In a GC Darboux chart of Z [5, 1], the ambient pure spinor has local form $\varphi_Z = \Omega^{t,0} \wedge e^{\mathrm{i}\omega_0}$, where t is the GC type and ω_0 the symplectic block. Intersecting this chart with the restricted Darboux model (Proposition 5.6) gives

$$(\mathcal{M}, \Phi_{\mathcal{M}}) \sim (\mathbb{C}^{t_{\mathrm{CR}}} \times \mathbb{R}^{2s-2t_{\mathrm{CR}}},\ \Phi_{\mathrm{CR}} \oplus \Phi_{\mathrm{symp}}),$$

whose pure spinor is $\eta^{t_{\mathrm{CR}},0} \wedge e^{\mathrm{i}\omega_T}$. When $t_{\mathrm{CR}} = t$ (complex type), $\omega_T = 0$, and the restriction reduces to $i^*\Omega$. Integrability follows because $\mathrm{d}\varphi_Z = (X + \alpha) \cdot \varphi_Z$ and the Courant bracket closes on $E_{\mathcal{M}}$ (Proposition 4.5). □

Local pure-spinor model. To make the pure-spinor description more concrete, we spell out a local model in generalized complex Darboux coordinates. Let

$$\phi_Z = \Omega^{t,0} \wedge e^{i\omega_0}$$

be a local generator of the ambient pure-spinor line K_{L_Z}, where $\Omega^{t,0}$ is a nonvanishing holomorphic t-form and ω_0 is the real symplectic 2-form appearing in the Darboux decomposition of (Z, J_Z).

Let $i : M \hookrightarrow Z$ denote the inclusion of a Matsuki orbit. Then

$$i^*\phi_Z = i^*\Omega^{t,0} \wedge e^{i\, i^*\omega_0}.$$

Projecting along $\mathrm{ann}(TM)$ to $\wedge^\bullet T^*M$ we obtain

$$\phi_M = \eta^{t_{\mathrm{CR}},0} \wedge e^{i\omega_T},$$

where $\eta^{t_{\mathrm{CR}},0}$ spans the CR canonical bundle $\wedge^{t_{\mathrm{CR}},0}T^{1,0}M$ and ω_T is the real transverse symplectic form (Levi form) on the $2\rho_{\mathrm{symp}}$-dimensional symplectic factor.

If B is a closed real 2-form on Z, the ambient spinor transforms by

$$\phi_Z \longmapsto e^B \wedge \phi_Z,$$

and restriction commutes with the B-field action:

$$\phi_M \longmapsto e^{i^*B} \wedge \phi_M.$$

Since $dB = 0$, the integrability condition $d\phi_M = (X + \alpha) \cdot \phi_M$ is preserved, and the annihilator eigenbundle E_M is unchanged. Thus the induced generalized CRF structure is gauge-invariant under all closed B-field transforms.

Gauge transformations by B-fields. If B is a closed real 2-form on Z, the B-field transform

$$e^B(X + \alpha) = X + \alpha + \iota_X B$$

acts orthogonally on $\mathbb{T}Z$, preserving the Courant bracket and integrability of L_Z. The ambient pure spinor transforms by

$$\varphi_Z \longmapsto e^B \wedge \varphi_Z.$$

Restriction and projection commute with e^B, so

$$\varphi_{\mathcal{M}} \longmapsto e^{i^*B} \wedge \varphi_{\mathcal{M}}.$$

Since B is closed, the integrability condition $\mathrm{d}\varphi_{\mathcal{M}} = (X + \alpha) \cdot \varphi_{\mathcal{M}}$ is unchanged. Thus $\Phi_{\mathcal{M}}$ is gauge-invariant under closed B-fields.

CRFK structures on homogeneous orbits. Following Vaisman [8], a *generalized CRFK structure* is a generalized CRF structure Φ compatible with a generalized metric G such that Φ and G commute and together define a generalized Khler structure along the CR leaves. On open (definite) Matsuki orbits, the homogeneous metric induced by G_0 or K_0 is Khler and compatible with $\Phi_{\mathcal{M}}$, so $(\mathcal{M}, \Phi_{\mathcal{M}}, G)$ is a homogeneous generalized CRFK manifold.

Example: the null orbit of $\mathrm{SU}(2,2)$ on $\mathrm{Gr}_2(\mathbb{C}^4)$. Consider the mixed CR/symplectic orbit (CR codimension 2) of type $(1,0,1)$ described in Section 7. Let $Z = \mathrm{Gr}_2(\mathbb{C}^4)$ with its Khler form ω and holomorphic top form Ω on a local coordinate patch. At a representative point

$$W = \mathrm{Span}\{e_1,\, e_2 + f_2\}, \qquad \mathrm{Rad}(h|_W) = \mathrm{Span}\{e_1\},$$

the complex tangent space of $\mathcal{M}$ is $T^{1,0}_W\mathcal{M} \subset \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^4/W) \cong M_{2\times 2}(\mathbb{C})$ with $\dim_{\mathbb{C}} T^{1,0}_W\mathcal{M} = 3$ and real symplectic rank 2.

Local model. In the restricted Darboux chart (Section 5), one may choose coordinates

$$(z_1, z_2) \in \mathbb{C}^2, \quad (x_1, x_2) \in \mathbb{R}^2$$

so that $\mathcal{M}$ near W is modeled on $\mathbb{C}^{t_{\mathrm{CR}}} \times (\mathbb{R}^2, \omega_T)$ with $t_{\mathrm{CR}} = 3$ (i.e. $t_{\mathrm{CR}} = 3$ complex dimensions in real eight dimensions), and $\omega_T = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ is the transverse Levi form.
Restricted pure spinor. The ambient pure spinor on Z is

$$\varphi_Z = \Omega \wedge e^{\mathrm{i}\omega}.$$

Restricting and projecting to $\mathcal{M}$ yields

$$\varphi_{\mathcal{M}} = \eta^{3,0} \wedge e^{\mathrm{i}\omega_T},$$

where $\eta^{3,0}$ spans the CR canonical bundle $\wedge^{3,0}T^{1,0}\mathcal{M}$ and ω_T encodes the symplectic transverse plane. This satisfies

$$\mathrm{d}\varphi_{\mathcal{M}} = (X + \alpha) \cdot \varphi_{\mathcal{M}}$$

for some local section $X + \alpha \in \Gamma(\mathbb{T}\mathcal{M})$, so $(\mathcal{M}, \Phi_{\mathcal{M}})$ is integrable.
Geometric meaning. The factor $e^{\mathrm{i}\omega_T}$ represents the nondegenerate Levi form in the transverse directions, while $\eta^{3,0}$ encodes the holomorphic directions tangent to the CR distribution. Thus, the pure spinor $\varphi_{\mathcal{M}} = \eta^{3,0} \wedge e^{\mathrm{i}\omega_T}$ captures both the complex tangential and symplectic transverse geometry of the null orbit. On the two open orbits of definite type $(2,0,0)$ and $(0,2,0)$, $\omega_T = 0$ and $\varphi_{\mathcal{M}} = i^*\Omega$, recovering the purely complex structure.

Remark 8.2 (Integrability and gauge invariance). Because the B-field transform $e^{i^*B} \wedge \varphi_{\mathcal{M}}$ preserves the annihilator $E_{\mathcal{M}}$ and closedness of B ensures $\mathrm{d}(e^{i^*B} \wedge \varphi_{\mathcal{M}}) = (X + \alpha) \cdot (e^{i^*B} \wedge \varphi_{\mathcal{M}})$, The CRF and CRFK structures are invariant under all closed B-field gauges.

9. General scheme for $\mathrm{SU}(p, q)$ on $\mathrm{Gr}_k(\mathbb{C}^{p+q})$

We now summarize a general computational scheme for determining the CR type and symplectic rank of Matsuki orbits

$$\mathcal{M}(a, b, r) \subset Z = \mathrm{Gr}_k(\mathbb{C}^{p+q}), \qquad G_0 = \mathrm{SU}(p, q),$$

in terms of the signature data (a, b, r) of the restricted Hermitian form $h|_W$ on the representative k-plane $W \subset \mathbb{C}^{p+q}$. Here a and b denote the numbers of positive and negative directions, and $r = \dim_{\mathbb{C}} \mathrm{Rad}(h|_W)$ is the nullity.

Let h have standard Witt form

$$h(z, w) = z_1\overline{w}_1 + \cdots + z_p\overline{w}_p - z_{p+1}\overline{w}_{p+1} - \cdots - z_{p+q}\overline{w}_{p+q}.$$

We denote by $\pi_\pm : \mathbb{C}^{p+q} \to V_\pm$ the projections to the positive and negative subspaces $V_+ = \mathrm{Span}\{e_1, \ldots, e_p\}$ and $V_- = \mathrm{Span}\{f_1, \ldots, f_q\}$.

Definition 9.1 (Signature type)**.** For $W \in \mathrm{Gr}_k(\mathbb{C}^{p+q})$, define

$$(a, b, r) = \big(\mathrm{rank}(\pi_+|_W),\ \mathrm{rank}(\pi_-|_W),\ \dim_{\mathbb{C}} \mathrm{Rad}(h|_W)\big),$$

so that $a + b + r = k$. Two G_0-orbits on Gr_k are equivalent if and only if they have the same triple (a, b, r).

Each such orbit meets a unique K-orbit ($K = S(\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C}))$), and their intersection $\mathcal{M}(a, b, r)$ is a Matsuki orbit.

General computational steps. We now describe a systematic linear-algebraic algorithm to compute $T^{1,0}_W \mathcal{M}(a,b,r)$ and hence the generalized CRF type of $\Phi_{\mathcal{M}}$ at W.

Step 1. Model Hom**-space.** Fix a h-orthogonal linear complement $W^{\perp}_{\mathrm{lin}}$ to W in $\mathbb{C}^{p+q}$. Then the holomorphic tangent space of the Grassmannian is naturally identified as

$$T^{1,0}_W Z \;\cong\; \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}^{p+q}/W) \;\cong\; \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}_{\mathrm{lin}}).$$

We will use this identification throughout the computation.

Step 2. Complex K**-tangent.** The complex orbit $K{\cdot}W$ consists of k-planes W' with the same positive/negative ranks $\rho_{\pm} = \mathrm{rank}(\pi_{\pm}|_{W'})$ as W. Infinitesimal preservation of these ranks imposes linear constraints on the entries of $\varphi \in \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}_{\mathrm{lin}})$:

$$\pi_{\pm} \circ \varphi \circ \pi_{\mp} = 0.$$

Equivalently, in a basis adapted to $W = W_+ \oplus W_- \oplus W_0$ with $\dim W_+ = a$, $\dim W_- = b$, $\dim W_0 = r$ (the radical), the element φ has block form

$$\varphi = \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix},$$

where the zero blocks express the invariance of $\rho_{\pm}$. The space of such φ is $T^{1,0}_W(K \cdot W) \subset \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}_{\mathrm{lin}})$.

Step 3. Real G_0**-tangent.** The real orbit $G_0 \cdot W$ is generated by $\mathfrak{su}(p,q)$ acting on $\mathbb{C}^{p+q}$. Write

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{su}(p,q), \qquad A^* + D = 0, \quad B^* = B, \quad C^* = C, \quad \mathrm{tr}(A) + \mathrm{tr}(D) = 0,$$

relative to the decomposition $\mathbb{C}^{p+q} = V_+ \oplus V_-$. The tangent map

$$\Psi : \mathfrak{su}(p,q) \to \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}_{\mathrm{lin}}), \qquad X \mapsto \pi \circ X|_W$$

gives the infinitesimal image of G_0 at W. Its image is a complex subspace $T_W(G_0 \cdot W) \otimes \mathbb{C} \subset \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}_{\mathrm{lin}})$ determined by the linear conditions above.

Step 4. Intersection and CR type. The $(1,0)$-space of the Matsuki orbit is the intersection

$$T^{1,0}_W \mathcal{M}(a,b,r) = T^{1,0}_W(K \cdot W) \cap \big(T_W(G_0 \cdot W) \otimes \mathbb{C}\big) \subset \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}_{\mathrm{lin}}).$$

Its complex dimension gives the generalized CRF type:

$$t_{\mathrm{CR}} = \dim_{\mathbb{C}} T^{1,0}_W \mathcal{M}(a,b,r).$$

The symplectic rank follows as

$$\mathrm{rank}_{\mathrm{symp}} = \dim_{\mathbb{R}} \mathcal{M}(a,b,r) - 2\, t_{\mathrm{CR}}.$$

The real dimension $\dim_{\mathbb{R}} \mathcal{M}(a,b,r)$ is obtained from the isotropy representation of $\mathrm{SU}(p,q)$ at W using the standard formula

$$\dim_{\mathbb{R}} \mathcal{M}(a,b,r) = \dim_{\mathbb{R}} G_0 - \dim_{\mathbb{R}} (G_0)_W,$$

where $(G_0)_W$ is the stabilizer subgroup of W.

Remark 9.2 (Closed form and algorithmic computation). For small (p, q, k), the above procedure can be carried out symbolically, yielding explicit tables of $t_{\mathrm{CR}} = \dim_{\mathbb{C}} T^{1,0}_W \mathcal{M}(a, b, r)$ and symplectic ranks. In general, the algorithm is linear: one performs Gaussian elimination on the block constraints in $\mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}_{\mathrm{lin}})$ to find $\dim(T^{1,0}_W(K{\cdot}W) \cap T_W(G_0{\cdot}W) \otimes \mathbb{C})$. This reproduces the examples of Section 7 and generalizes them to arbitrary (p, q, k).

Example 9.3 (Illustration for $(p, q, k) = (2, 2, 2)$)**.** For $(a, b, r) = (1, 0, 1)$, the procedure yields $\dim_{\mathbb{C}} T^{1,0}_W \mathcal{M} = 3$ and real orbit dimension 8, hence symplectic rank 2, matching the explicit computation in Section 7. For $(a, b, r) = (2, 0, 0)$ or $(0, 2, 0)$, The intersection equals the full holomorphic tangent space, corresponding to complex orbits of type $t_{\mathrm{CR}} = 4$.

Remark 9.4 (Relation to CR geometry). The intersection description above is the generalized complex analogue of the classical formula

$$T^{1,0}\mathcal{M} = T^{1,0}Z \cap (T\mathcal{M} \otimes \mathbb{C}),$$

showing that $\Phi_{\mathcal{M}}$ refines the usual CR structure by including the transverse symplectic directions. In particular, the dimension drop $t_{\mathrm{CR}}(a, b, r) < \dim_{\mathbb{C}} Z$ corresponds to a nontrivial Levi form on $\mathcal{M}(a, b, r)$.

Hence, the generalized CRF structure extends the classical CR structure by incorporating the Levi form as a real symplectic component.

10. Conclusion

In this work, we have developed a unified geometric framework for understanding Matsuki orbits in complex Grassmannians as natural carriers of generalized CRF structures arising from the ambient generalized complex (GC) geometry.

Summary of results. Starting from the GC structure $\mathcal{J}_Z$ of complex type on $Z = \mathrm{Gr}_k(\mathbb{C}^n)$, we proved an *Inheritance Theorem* showing that, under a clean intersection condition, each Matsuki orbit

$$\mathcal{M} = (G_0 \cdot z) \cap (K \cdot z) \subset Z$$

inherits a canonical generalized CRF structure $\Phi_{\mathcal{M}}$ via restriction of the ambient eigenbundle. This structure is natural and equivariant under closed B-field transformations.

We then established a *restricted Darboux normal form*: locally, when the generalized CRF type is constant, $(\mathcal{M}, \Phi_{\mathcal{M}})$ is Courant-equivalent to a product

$$\mathbb{C}^{t_{\mathrm{CR}}} \times (\mathbb{R}^{2s-2t_{\mathrm{CR}}}, \omega_T),$$

where the first factor encodes the CR part and the second the symplectic (Levi) transverse part. This provides explicit invariants: the generalized CRF type t_{CR} and the symplectic rank.

Explicit computations for $G_0 = \mathrm{SU}(p, q)$ acting on Grassmannians demonstrated how these invariants can be extracted algebraically from the signature data (a, b, r) of the restricted Hermitian form $h|_W$. We verified these in detail for $\mathrm{SU}(2, 2) \curvearrowright \mathrm{Gr}_2(\mathbb{C}^4)$, where the open (definite) orbits are of pure complex type and the null (indefinite) orbits exhibit CR-hypersurface structure with a real 2-dimensional symplectic transverse factor. The corresponding restricted pure spinor $\varphi_{\mathcal{M}} = \eta^{3,0} \wedge e^{\mathrm{i}\omega_T}$ captures both tangential and transverse geometry, providing an explicit realization of generalized CRFK structures on homogeneous spaces.

Geometric significance. The present results bridge several geometric frameworks:

- They generalize the classical CR structures of real orbits in complex flag manifolds (WolfMatsuki theory) to the Courant/Dirac setting of generalized geometry.
- The restricted Darboux models provide an intrinsic description of local orbit geometry in terms of GC type and Levi rank.
- The pure-spinor formalism reveals a direct link between holomorphic volume forms on Z and the induced CR canonical lines on $\mathcal{M}$.

Outlook and further directions. Several natural lines of inquiry emerge from this work:

(i) **Deformation theory.** Analyze infinitesimal deformations of the generalized CRF structure $\Phi_{\mathcal{M}}$ within a fixed Matsuki orbit, and the associated moduli governed by Courant cohomology.

(ii) **Orbit closures and stratification.** Study how $\Phi_{\mathcal{M}}$ behaves under inclusion of orbits $\overline{\mathcal{M}'} \subset \overline{\mathcal{M}}$, including semicontinuity of type and possible jumps in the Levi rank across strata.

(iii) **Cohomological and spinorial invariants.** Investigate characteristic classes of the pure spinor line, the modular class of the induced Dirac structure, and possible links to the DolbeaultCR cohomology $H^{0,\bullet}_{\bar{\partial}_b}(\mathcal{M})$.

(iv) **Extension to other flag manifolds.** Generalize the construction to cominuscule flag varieties and explore how generalized CRF structures arise on orbits of other real forms beyond $\mathrm{SU}(p,q)$, such as $\mathrm{SO}(p,q)$ and $\mathrm{Sp}(p,q)$.

(v) **Quantization and representation theory.** The symplectic transverse directions suggest a link to geometric quantization of coadjoint orbits, potentially relating $\Phi_{\mathcal{M}}$ to representation-theoretic realizations of discrete series orbits.

Matsuki orbits thus provide a rich testing ground for extending tools of generalized geometry to settings combining complex and real group actions. The explicit results here for Grassmannians are a first step toward a broader theory of *generalized CRF geometry on real orbits in complex flag varieties*.A subsequent paper could extend this framework to compute the induced generalized metric and its compatibility with the CRF structure

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