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Decision Analysis Using Energy of Neutrosophic Hesitant Fuzzy Soft Sets

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Abstract. In this paper, we introduce a novel hybrid structure called the neutrosophic hesitant fuzzy soft set (NHFSS), which integrates the flexibility of hesitant fuzzy information with the expressive capacity of neutrosophic and soft set theories. Motivated by the need to handle multiple layers of uncertainty in decision-making problems, we propose a new framework for modeling such environments. Building upon this structure, we define two complementary measures: the pessimistic energy and the optimistic energy of an NHFSS, which capture lower-bound and upper-bound contributions of the hesitant and neutrosophic elements, respectively. These measures are then synthesized to derive the total energy of the NHFSS. We develop a decision-making algorithm based on this energy characterization and apply it to a real-world example to demonstrate its effectiveness. A comparative analysis is conducted to highlight the advantages of the proposed approach over existing models. Our findings suggest that the energy-based treatment of NHFSSs provides a robust and discriminative framework for multi-criteria decision-making under complex uncertainty.

AMS (MOS) Subject Classification Codes: 94D05; 90B50; 15A18
Key Words: neutrosophic hesitant fuzzy soft set, optimistic energy, pessimistic energy, decision-making, hybrid uncertainty model, multi-criteria analysis.

1. Introduction

In the era of artificial intelligence and data-driven decision-making, the need for frameworks capable of handling uncertainty, vagueness, and imprecision has become increasingly prominent. Classical models, based on binary logic and crisp data, often fall short

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in addressing real-world problems involving ambiguity. To overcome these challenges, researchers have introduced various mathematical structures tailored to better represent and process such data.

One of the earliest and most influential concepts in this area is the notion of fuzzy sets, introduced by Zadeh [36], which provides a flexible representation of partial membership. Recognizing the limitations of fuzzy sets in dealing with parametric uncertainty, Molodtsov [21] proposed the theory of soft sets as an alternative approach. This theory has since been further developed by numerous researchers, including Ali et al. [3], Sezgin and Atagun [25], who defined and analyzed operations on soft sets. A comprehensive overview of soft set theory, its historical development, and applications can be found in the review article by Alcantud et al. [2]. Soft sets provided the conceptual foundation for the development of hypersoft sets, which extend the original framework by allowing multi-parameter representations and higher structural flexibility; see, for example, studies [12, 13, 24].

Parallel to the development of soft sets, Atanassov [5] introduced intuitionistic fuzzy sets, which extend fuzzy sets by incorporating both membership and non-membership degrees. The combination of intuitionistic fuzzy sets and soft sets led to hybrid models such as fuzzy soft sets [16] and intuitionistic fuzzy soft sets [17], which further enhanced the ability to represent complex uncertain environments. These hybrid models have found practical applications in various fields, as shown by Alblowi et al. [1]. Recent studies also demonstrate that matrix-based soft extensions of these models play a crucial role in modern supervised learning frameworks, particularly in scenarios requiring efficient processing of large-scale uncertain information. For instance, picture fuzzy soft matrices have been successfully employed in classification via distance-based learning schemes [18], while intuitionistic fuzzy parameterized soft matrices have shown strong performance in data classification through refined similarity measures [19]. Moreover, adaptive machine-learning approaches built upon intuitionistic fuzzy soft matrix structures [20] further highlight the practical relevance and versatility of such hybrid models in real-world decision-making environments.

Despite these advancements, many decision-making problems require the explicit modeling of indeterminacy in addition to truth and falsity. This led to the introduction of neutrosophic sets by Smarandache [26], which are characterized by three membership functions: truth, indeterminacy, and falsity. Extensions of this concept, such as interval-valued neutrosophic sets [32] and single-valued neutrosophic sets [33], have been proposed to improve flexibility in modeling. Ye [34] further developed similarity measures and decision-making methods under simplified and hesitant neutrosophic environments.

The hybridization of neutrosophic sets and soft sets yielded the notion of neutrosophic soft sets, introduced by Maji [15]. Their extensions and applications in decision-making have been explored in works by Deli and Broumi [8], Dalkılıç [6], and others. In particular, Deli [7] developed a decision-making algorithm based on interval-valued neutrosophic soft sets, laying the groundwork for energy-based methods in this setting.

The notion of energy, originally introduced in graph theory by Gutman [10], has found broad applications in applied mathematics and optimization. Nikiforov [22, 23] extended the study of graph and matrix norms, while the nuclear norm, defined as the sum of singular values, has been used in matrix optimization problems [11, 14]. Inspired by these developments, energy-based models were introduced in the soft computing framework. Mudric

Staniskovski et al. proposed the concept of fuzzy soft set energy, Alcantud et al. defined the scored-energy of hesitant fuzzy soft sets, and Stojanovic et al. extended the energy framework to interval-valued hesitant fuzzy soft sets. Additional contributions in this direction include applications of energy-based models in decision-making [9], evaluation of cloud platforms, and the introduction of $Q[\varepsilon]$-fuzzy sets. Further developments include the formulation of decision-making algorithms grounded in the scored-energy of neutrosophic soft sets [27], as well as the analysis of complex decision-making environments through the energy of bipolar neutrosophic soft sets [28]. These results collectively illustrate the rapid expansion and versatility of energy-based methodologies within the broader soft computing paradigm.

Existing decision-making approaches developed for neutrosophic hesitant fuzzy environments, such as those proposed by Ye [35] and Wang and Li [31], provide important tools through aggregation operators and multi-criteria evaluation schemes. More recent contributions, including the partitioned Maclaurin symmetric mean operator introduced by Ali et al. [4], further demonstrate the versatility of neutrosophic hesitant fuzzy models. However, despite their usefulness, these methods often suffer from several limitations: they typically rely on aggregation procedures that do not fully exploit the structural richness of hesitant information, lack mechanisms for capturing global interactions among parameters, and may lead to ranking instability when alternatives exhibit similar evaluation patterns.

In contrast to problem-specific solutions, the objective of this work is to develop a mathematically grounded model for determining the optimal alternative in highly uncertain settings. Our goal is not to resolve a particular real-world decision problem, but to construct a new and more robust methodology capable of improving upon existing approaches. To this end, we introduce the neutrosophic hesitant fuzzy soft set (NHFSS), a hybrid structure that integrates hesitant fuzzy and neutrosophic data within a soft parameterization framework. This integration enables a richer and more coherent representation of uncertainty, capturing hesitation in truth, indeterminacy, and falsity values across multiple attributes.

To quantify the informational strength of an NHFSS, we draw motivation from the nuclear norm, a fundamental matrix norm defined as the sum of singular values and widely used in optimization and structural characterization. Prior research on energy-based measures in soft and fuzzy soft structures has demonstrated that singular value–based energy provides a meaningful way to capture global information patterns. Building on these ideas, we introduce two complementary numerical characteristics for NHFSSs: the *pessimistic energy*, derived from minimal hesitant evaluations, and the *optimistic energy*, derived from maximal hesitant evaluations. These measures reflect, respectively, the most conservative and the most favorable informational scenarios encoded in the data.

By synthesizing these two extremal measures, we define the total energy of an NHFSS, which serves as a comprehensive scalar indicator of its global informational content. The purpose of this construction is to establish a mathematically principled criterion for comparing alternatives and determining the optimal one. The resulting energy-based model thus provides a new methodological framework that overcomes several limitations of existing decision-making techniques and offers improved stability, interpretability, and sensitivity to structural variations in hesitant neutrosophic data.

The structure of the paper is as follows. Section 2 introduces the neutrosophic hesitant fuzzy soft set (NHFSS) by recalling the fundamental concepts and definitions on which it is based. In Section 3, we develop the notions of pessimistic and optimistic energy for NHFSSs, which jointly provide a complete energy-based characterization of the proposed structure. Section 4 describes a novel decision-making algorithm that incorporates these energy measures to evaluate and rank alternatives. In Section 5, the effectiveness of the proposed algorithm is demonstrated through its application to a practical problem. Section 6 offers a comparative performance analysis against existing approaches. The paper concludes in Section 7 with a summary of findings and suggestions for future research directions.

2. From Soft Sets to Neutrosophic Hesitant Fuzzy Soft Sets: Foundational Concepts

In this section, we present the essential concepts that support the construction of the hybrid model studied in this paper. Starting from soft sets, introduced as a parameterization tool for managing uncertainty, we connect this framework with fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets, each of which expands the ability to represent vagueness by incorporating additional degrees of information. These developments naturally led to more expressive models such as hesitant fuzzy sets (HFSs) and neutrosophic hesitant fuzzy sets (NHFSs), which allow multiple possible values to describe membership, indeterminacy, and non-membership in situations where experts may hesitate among several assessments.

While each of these structures provides useful mechanisms for dealing with uncertainty, combining them within a unified setting offers further expressive power. In this work, we focus on the integration of NHFSs with soft sets, resulting in the neutrosophic hesitant fuzzy soft set (NHFSS), a model capable of representing parameterized hesitation in all three membership dimensions. To prepare the ground for this construction, we briefly recall the essential properties of HFSs and NHFSs, which play a central role in understanding the proposed hybrid structure.

Definition 2.1. *(*[29, 30]*) Let X be a fixed universe. A* hesitant fuzzy set *(HFS) A on X is defined by a function $h_A : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ that assigns to each element $x \in X$ a finite, non-empty subset $h_A(x) \subseteq [0,1]$, representing the set of possible membership degrees of x to the set A. The hesitant fuzzy set A can be written as*

$$A = \{\langle x, h_A(x)\rangle \mid x \in X\},$$

where each $h_A(x)$ is called a hesitant fuzzy element *(HFE).*

This definition allows multiple membership degrees to coexist for a single element, thus providing a more flexible framework for representing uncertainty and hesitation in expert evaluations.

Definition 2.2. *Let U be a universal set of elements. A Neutrosophic Hesitant Fuzzy Set (NHFS) $\mathfrak{F}$ over U is defined as*

$$\mathfrak{F} = \{(u, T_{\mathfrak{F}}(u), I_{\mathfrak{F}}(u), F_{\mathfrak{F}}(u)) \mid u \in U\},$$

where $T_{\mathfrak{F}}(u)$, $I_{\mathfrak{F}}(u)$, and $F_{\mathfrak{F}}(u)$ denote the hesitant truth-membership, hesitant indeterminacy-membership, and hesitant falsity-membership functions, respectively. For each $u \in$

U, these functions assign a finite subset of $[0,1]$ representing possible degrees of membership, indeterminacy, and non-membership. The following condition holds for all $u \in U$:

$$0 \leq \sup T_{\mathfrak{F}}(u) + \sup I_{\mathfrak{F}}(u) + \sup F_{\mathfrak{F}}(u) \leq 3.$$

Each triplet $\Theta(u) = (T_{\mathfrak{F}}(u), I_{\mathfrak{F}}(u), F_{\mathfrak{F}}(u))$ is called a Neutrosophic Hesitant Fuzzy Element *(NHFE).*

In this subsection, we define the concept of the *neutrosophic hesitant fuzzy soft set* (NHFSS) as a hybrid structure obtained by combining the neutrosophic hesitant fuzzy set (NHFS) with the soft set (SS). Let U be the initial universal set, let $\mathcal{NHFS}(U)$ denote the collection of all neutrosophic hesitant fuzzy sets on U, and let E be the set of parameters with $A \subseteq E$.

Definition 2.3. *The ordered pair $(\mathfrak{F}, A)$ is called a* neutrosophic hesitant fuzzy soft set *(NHFSS) over U, where $\mathfrak{F} : E \to \mathcal{NHFS}(U)$ is a mapping such that for any $e \in E$,*

$$\mathfrak{F}(e) = \{(u, M(u), N(u), L(u)) \mid u \in U\} \in \mathcal{NHFS}(U),$$

where $M(u)$, $N(u)$, and $L(u)$ are finite subsets of $[0,1]$ representing the hesitant truth-membership, hesitant indeterminacy-membership, and hesitant falsity-membership degrees of the element $u \in U$ with respect to the parameter $e \in E$.

Instead of the notation $(\mathfrak{F}, A)$, we also write $\mathfrak{F}_A$, and the sets $M(u)$, $N(u)$, and $L(u)$ will often be denoted by $T_{\mathfrak{F}_A}(u)$, $I_{\mathfrak{F}_A}(u)$, and $F_{\mathfrak{F}_A}(u)$, respectively. The mapping $\mathfrak{F}$ is called the neutrosophic hesitant fuzzy approximating function *of the NHFSS $\mathfrak{F}_A$.*

$$\mathfrak{F}_A = \{(e, \mathfrak{F}(e)) \mid e \in A, \mathfrak{F}(e) \in \mathcal{NHFS}(U)\}.$$

The neutrosophic hesitant fuzzy soft set (NHFSS) provides a flexible and comprehensive framework for modeling decision-making scenarios involving uncertainty, hesitation, and indeterminacy. Each parameter in an NHFSS corresponds to a neutrosophic hesitant fuzzy evaluation over the universe of alternatives, making the structure particularly suitable for environments where traditional approaches fail to capture subtle informational ambiguities. To demonstrate its practical relevance, we now introduce a real-world decision-making example originally discussed by Ye ([35]), in which the attribute weights are not computed algorithmically but rather assigned subjectively based on expert judgment. This example illustrates how the NHFSS model accommodates real evaluations and expert-defined preferences within a unified decision-making framework.

Example 2.4. *Investment companies are frequently required to select the most suitable option among several potential investment alternatives, a problem that has been widely analyzed in the decision-making literature. In this context, identifying the* optimal *alternative becomes essential, especially when evaluations are uncertain, imprecise, or based on multiple expert opinions.*

Consider an investment company evaluating four possible choices: u_1 represents a car manufacturing company, u_2 a food industry company, u_3 a computer technology company, and u_4 an arms industry company. The decision depends on three key attributes: e_1 denotes the level of risk, e_2 reflects expected growth, and e_3 measures environmental impact.

To capture the uncertainty and hesitation inherent in expert assessments, these evaluations are represented using a neutrosophic hesitant fuzzy soft set $\mathfrak{F}_A$, presented in Table 1.

The goal is to determine which alternative provides the most favorable investment opportunity under this uncertain environment.

TABLE 1. Neutrosophic hesitant fuzzy soft set $\mathfrak{F}_A$ from Example 2.4

$\mathfrak{F}_A$	e_1	e_2	e_3
u_1	$(\{0.3,0.4,0.5\},\{0.1\},\{0.3,0.4\})$	$(\{0.5,0.6\},\{0.2,0.3\},\{0.3,0.4\})$	$(\{0.2,0.3\},\{0.1,0.2\},\{0.5,0.6\})$
u_2	$(\{0.6,0.7\},\{0.1,0.2\},\{0.2,0.3\})$	$(\{0.6,0.7\},\{0.1\},\{0.3\})$	$(\{0.6,0.7\},\{0.1,0.2\},\{0.1,0.2\})$
u_3	$(\{0.5,0.6\},\{0.4\},\{0.2,0.3\})$	$(\{0.6\},\{0.3\},\{0.4\})$	$(\{0.5,0.6\},\{0.1\},\{0.3\})$
u_4	$(\{0.7,0.8\},\{0.1\},\{0.1,0.2\})$	$(\{0.6,0.7\},\{0.1\},\{0.2\})$	$(\{0.3,0.5\},\{0.2\},\{0.1,0.2,0.3\})$

This example demonstrates how the NHFSS structure enables the simultaneous handling of hesitation, indeterminacy, and multiple membership assessments, providing a realistic foundation for determining the most suitable investment alternative.

3. ON THE NOTION OF ENERGY IN NHFSS THEORY

The main objective of this section is to present a method for representing a neutrosophic hesitant fuzzy soft set (NHFSS) in matrix form, thereby enabling the definition of quantitative characteristics associated with its structure. Specifically, we focus on the construction of certain matrices from which the singular values can be computed. The sum of the singular values of a matrix, plays a central role in this context. This concept has already been successfully applied in other branches of mathematical modeling, such as graph theory and optimization problems, which further motivates its application within the framework of NHFSSs.

To introduce a novel numerical measure that reflects the internal structure and informational content of a given NHFSS, we define the notion of energy of a neutrosophic hesitant fuzzy soft set. However, due to the inherent complexity of this structure - where each pair consisting of a parameter and an element from the universe is assigned hesitant values for truth, indeterminacy, and falsity - it is necessary to distinguish between two types of energy: optimistic and pessimistic. These two forms of energy arise from different strategies for interpreting hesitant information: the optimistic approach emphasizes the most favorable values, while the pessimistic approach highlights the least favorable ones.

Since an NHFSS is defined by three functions:

- T, which assigns a hesitant set of truth-membership values to each object-parameter pair,
- I, which assigns hesitant indeterminacy-membership values, and
- F, which assigns hesitant falsity-membership values,

it is possible to construct, from each of these functions, two matrices - one containing the maximum values from each hesitant set (optimistic interpretation) and the other containing the minimum values (pessimistic interpretation). In this way, a single NHFSS can be uniquely represented by a total of six rectangular matrices.

In Subsection 3.1, we formally define the concepts of optimistic and pessimistic energy, based on the matrices introduced above. Then, in Subsection 3.7, we present the central construction of this section - the general definition of the energy of a neutrosophic hesitant fuzzy soft set, expressed as a unified numerical measure derived from the spectral characteristics of all relevant matrices.

3.1. **Spectral Energy Analysis: Pessimistic and Optimistic Perspectives in NHFSSs.** In the context of neutrosophic hesitant fuzzy soft sets (NHFSSs), each of the functions T, I, and F assigns a discrete set of possible values to every pair consisting of an alternative and an attribute. This means that, instead of a single precise value, there may be multiple candidate degrees of truth, indeterminacy, and falsity, which reflects the presence of hesitation and uncertainty in the assessment process. While this structure allows for flexible modeling, it also requires additional procedures to extract numerical representations suitable for further analysis.

In this study, we focus on the minimum and maximum values within each hesitant set, which we consider representative indicators for constructing numerical models. Based on these values, we introduce six rectangular matrices for each NHFSS, two for each of the functions T, I, and F - one representing the optimistic case and the other the pessimistic case. These matrices serve as the foundation for defining the optimistic and pessimistic energies of a NHFSS, which are further used for the analysis and ranking of alternatives within a multi-criteria decision-making framework.

Definition 3.2. *Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universal set (the set of alternatives), and let $A = \{e_1, e_2, \ldots, e_m\} \subseteq E$, where E is the set of parameters (the set of attributes). Let $\mathfrak{F}_A$ be a NHFSS over (U, E).*

The matrices of the minimum and maximum values of the truth-membership function $T_{\mathfrak{F}_A}$ are defined as $n \times m$ matrices:

$$\Phi_{\mathfrak{F}_A}^{\min T} = \begin{bmatrix} \min T_{\mathfrak{F}_A}(e_1)(u_1) & \min T_{\mathfrak{F}_A}(e_2)(u_1) & \cdots & \min T_{\mathfrak{F}_A}(e_m)(u_1) \\ \min T_{\mathfrak{F}_A}(e_1)(u_2) & \min T_{\mathfrak{F}_A}(e_2)(u_2) & \cdots & \min T_{\mathfrak{F}_A}(e_m)(u_2) \\ \vdots & \vdots & \ddots & \vdots \\ \min T_{\mathfrak{F}_A}(e_1)(u_n) & \min T_{\mathfrak{F}_A}(e_2)(u_n) & \cdots & \min T_{\mathfrak{F}_A}(e_m)(u_n) \end{bmatrix},$$

and

$$\Phi_{\mathfrak{F}_A}^{\max T} = \begin{bmatrix} \max T_{\mathfrak{F}_A}(e_1)(u_1) & \max T_{\mathfrak{F}_A}(e_2)(u_1) & \cdots & \max T_{\mathfrak{F}_A}(e_m)(u_1) \\ \max T_{\mathfrak{F}_A}(e_1)(u_2) & \max T_{\mathfrak{F}_A}(e_2)(u_2) & \cdots & \max T_{\mathfrak{F}_A}(e_m)(u_2) \\ \vdots & \vdots & \ddots & \vdots \\ \max T_{\mathfrak{F}_A}(e_1)(u_n) & \max T_{\mathfrak{F}_A}(e_2)(u_n) & \cdots & \max T_{\mathfrak{F}_A}(e_m)(u_n) \end{bmatrix}.$$

Analogously, the matrices $\Phi_{\mathfrak{F}_A}^{\min I}$, $\Phi_{\mathfrak{F}_A}^{\max I}$ and $\Phi_{\mathfrak{F}_A}^{\min F}$, $\Phi_{\mathfrak{F}_A}^{\max F}$ are defined by replacing the function $T_{\mathfrak{F}_A}$ with $I_{\mathfrak{F}_A}$ and $F_{\mathfrak{F}_A}$, respectively, in the expressions above.

Based on the previous definition, every NHFSS can be precisely represented using six rectangular matrices that contain the minimum and maximum values of the truth, indeterminacy, and falsity functions. These matrices enable the transformation of discrete information from hesitant values into deterministic forms suitable for further numerical analysis. For each of the introduced matrices, the singular values can be computed using standard procedures from linear algebra. The sum of all singular values of a given matrix is known as the nuclear norm, which is widely used in optimization problems, especially in low-rank matrix approximation. In graph theory, a similar construction is referred to as graph energy and is used as a numerical indicator of the structural complexity of a graph. Building on these ideas, the main objective of this paper is to formally introduce a new numerical measure called the energy of a NHFSS. This measure is based on the singular values (SV) of the matrices that describe the structural characteristics of an NHFSS and provides additional insight into the distribution of information within the set. Due to the nature of hesitant structures and the presence of multiple values per component, two perspectives are especially considered: optimistic and pessimistic. These perspectives allow us to distinguish between the best-case and worst-case scenarios in interpreting the data.

Definition 3.3. *Let $\mathfrak{F}_A$ be a NHFSS. The* optimistic energy *of $\mathfrak{F}_A$, denoted by $\mathbb{E}^+(\mathfrak{F}_A)$, is defined as*

$$\mathbb{E}^+(\mathfrak{F}_A) = \sum_{i=1}^{n} \left(\sigma_i^{\max T} + \sigma_i^{\max I} - \sigma_i^{\min F}\right),$$

where:

- $\sigma_1^{\max T} \geqslant \sigma_2^{\max T} \geqslant \cdots \geqslant \sigma_n^{\max T} \geqslant 0$ *are the SV of the matrix* $\Phi_{\mathfrak{F}_A}^{\max T}$,
- $\sigma_1^{\max I} \geqslant \sigma_2^{\max I} \geqslant \cdots \geqslant \sigma_n^{\max I} \geqslant 0$ *are the SV of the matrix* $\Phi_{\mathfrak{F}_A}^{\max I}$,
- $\sigma_1^{\min F} \geqslant \sigma_2^{\min F} \geqslant \cdots \geqslant \sigma_n^{\min F} \geqslant 0$ *are the SV of the matrix* $\Phi_{\mathfrak{F}_A}^{\min F}$.

Definition 3.4. *Let $\mathfrak{F}_A$ be a NHFSS. The* pessimistic energy *of $\mathfrak{F}_A$, denoted by $\mathbb{E}^-(\mathfrak{F}_A)$, is defined as*

$$\mathbb{E}^-(\mathfrak{F}_A) = \sum_{i=1}^{n} \left(\sigma_i^{\min T} + \sigma_i^{\min I} - \sigma_i^{\max F}\right),$$

where:

- $\sigma_1^{\min T} \geqslant \sigma_2^{\min T} \geqslant \cdots \geqslant \sigma_n^{\min T} \geqslant 0$ *are the SV of the matrix* $\Phi_{\mathfrak{F}_A}^{\min T}$,
- $\sigma_1^{\min I} \geqslant \sigma_2^{\min I} \geqslant \cdots \geqslant \sigma_n^{\min I} \geqslant 0$ *are the SV of the matrix* $\Phi_{\mathfrak{F}_A}^{\min I}$,
- $\sigma_1^{\max F} \geqslant \sigma_2^{\max F} \geqslant \cdots \geqslant \sigma_n^{\max F} \geqslant 0$ *are the SV of the matrix* $\Phi_{\mathfrak{F}_A}^{\max F}$.

We now utilize Example 2.4, previously introduced in Section 2, as the basis for illustrating the procedure for calculating the energy of a NHFSS. Using the specific data presented in that example, Example 3.5 will demonstrate in detail how to compute the optimistic and pessimistic energy values. These results will then be used to determine the overall energy of the same set in Subsection 3.7.

Example 3.5. *By applying Definition 3.2, this NHFSS can be uniquely represented using six rectangular matrices:*

$$\Phi_{\mathfrak{F}_A}^{\min T} = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.7 & 0.6 & 0.3 \end{bmatrix}, \quad \Phi_{\mathfrak{F}_A}^{\max T} = \begin{bmatrix} 0.5 & 0.6 & 0.3 \\ 0.7 & 0.7 & 0.7 \\ 0.6 & 0.6 & 0.6 \\ 0.8 & 0.7 & 0.5 \end{bmatrix},$$

$$\Phi_{\mathfrak{F}_A}^{\min I} = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.4 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}, \quad \Phi_{\mathfrak{F}_A}^{\max I} = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0.4 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix},$$

$$\Phi_{\mathfrak{F}_A}^{\min F} = \begin{bmatrix} 0.3 & 0.3 & 0.5 \\ 0.2 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \quad \Phi_{\mathfrak{F}_A}^{\max F} = \begin{bmatrix} 0.4 & 0.4 & 0.6 \\ 0.3 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.3 \end{bmatrix}.$$

Furthermore, to determine the singular values of these matrices, we multiply each of these matrices by their respective transpose, so we form the matrices

$$\begin{gathered} \Phi_{\mathfrak{F}_A}^{\min T} \cdot (\Phi_{\mathfrak{F}_A}^{\min T})^{\mathbf{T}}, \;\; \Phi_{\mathfrak{F}_A}^{\max T} \cdot (\Phi_{\mathfrak{F}_A}^{\max T})^{\mathbf{T}}, \\ \Phi_{\mathfrak{F}_A}^{\min I} \cdot (\Phi_{\mathfrak{F}_A}^{\min I})^{\mathbf{T}}, \;\; \Phi_{\mathfrak{F}_A}^{\max I} \cdot (\Phi_{\mathfrak{F}_A}^{\max I})^{\mathbf{T}}, \\ \Phi_{\mathfrak{F}_A}^{\min F} \cdot (\Phi_{\mathfrak{F}_A}^{\min F})^{\mathbf{T}}, \;\; \Phi_{\mathfrak{F}_A}^{\max F} \cdot (\Phi_{\mathfrak{F}_A}^{\max F})^{\mathbf{T}}. \end{gathered}$$

The singular values are obtained as the square roots of the eigenvalues of the resulting matrices, yielding

$$\begin{gathered} \sigma_1^{\min T} = 1.7814, \;\; \sigma_2^{\min T} = 0.24391, \;\; \sigma_3^{\min T} = 0.16481, \\ \sigma_1^{\max T} = 2.1384, \;\; \sigma_2^{\max T} = 0.21839, \;\; \sigma_3^{\max T} = 0.09871, \\ \sigma_1^{\min I} = 0.61099, \;\; \sigma_2^{\min I} = 0.17321, \;\; \sigma_3^{\min I} = 0.08183, \\ \sigma_1^{\max I} = 0.69246, \;\; \sigma_2^{\max I} = 0.2175, \;\; \sigma_3^{\max I} = 0.15228, \\ \sigma_1^{\min F} = 0.93026, \;\; \sigma_2^{\min F} = 0.22336, \;\; \sigma_3^{\min F} = 0.06874, \\ \sigma_1^{\max F} = 1.1716, \;\; \sigma_2^{\max F} = 0.18644, \;\; \sigma_3^{\max F} = 0.05118. \end{gathered}$$

Using Definition 3.3, we find that the optimistic energy of the observed NHFSS $\mathfrak{F}_A$ is

$$\mathbb{E}^+(\mathfrak{F}_A) = \sum_{i=1}^{3} (\sigma_i^{\max T} + \sigma_i^{\max I} - \sigma_i^{\min F}) = 2.29538,$$

and using Definition 3.4, the pessimistic energy is

$$\mathbb{E}^-(\mathfrak{F}_A) = \sum_{i=1}^{3} (\sigma_i^{\min T} + \sigma_i^{\min I} - \sigma_i^{\max F}) = 1.64693.$$

In the previous example, it was shown that the obtained values of optimistic and pessimistic energy are finite real numbers. This observation is not limited to the specific case but can be generalized. In the following, we present a theorem that defines the general bounds for the energy values within the NHFSS model.

Theorem 3.6. *Let $\mathfrak{F}_A$ be a neutrosophic hesitant fuzzy soft set, where $U = \{u_1, u_2, \ldots, u_n\}$ is the universe, $E = \{e_1, e_2, \ldots, e_m\}$ is the set of parameters, and $A \subseteq E$ is the set of considered parameters. Then, the optimistic and pessimistic energies of $\mathfrak{F}_A$ satisfy:*

$$-n\sqrt{m} \leqslant \mathbb{E}^+(\mathfrak{F}_A) \leqslant 2n\sqrt{m},$$

and

$$-n\sqrt{m} \leqslant \mathbb{E}^-(\mathfrak{F}_A) \leqslant 2n\sqrt{m}.$$

Proof. We first establish the bound

$$-n\sqrt{m} \leqslant \mathbb{E}^{+}(\mathfrak{F}_A) \leqslant 2n\sqrt{m}.$$

The corresponding inequality for $\mathbb{E}^{-}(\mathfrak{F}_A)$ can be proven in an analogous way.

Applying the inequality A-G means to the values $\sigma_1^{\max T}, \sigma_2^{\max T}, \dots, \sigma_n^{\max T}$, we obtain

$$\sum_{i=1}^{n} \sigma_i^{\max T} \leqslant \sqrt{n \sum_{i=1}^{n} \left(\sigma_i^{\max T}\right)^2}.$$

Using the basic properties of matrices, their eigenvalues and SV, together with the fact that $T_{\mathfrak{F}_A}(e_j)(u_i) \subseteq [0,1]$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$, it follows that

$$\sum_{i=1}^{n} \left(\sigma_i^{\max T}\right)^2 = \operatorname{tr}\left(\Phi_{\mathfrak{F}_A}^{\max T} \cdot \left(\Phi_{\mathfrak{F}_A}^{\max T}\right)^{\mathbf{T}}\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\sup T_{\mathfrak{F}_A}(e_j)(u_i)\right)^2 \leqslant mn.$$

Therefore,

$$\sum_{i=1}^{n} \sigma_i^{\max T} \leqslant \sqrt{n^2 m} = n\sqrt{m}.$$

By the same reasoning, we obtain $\sum_{i=1}^{n} \sigma_i^{\max I} \leqslant n\sqrt{m}$.

Since $\sum_{i=1}^{n} \sigma_i^{\max I} \geqslant 0$ and $\sum_{i=1}^{n} \sigma_i^{\min F} \geqslant 0$, we conclude that

$$\mathbb{E}^{+}(\mathfrak{F}_A) = \sum_{i=1}^{n} \left(\sigma_i^{\max T} + \sigma_i^{\max I} - \sigma_i^{\min F}\right) \leqslant 2n\sqrt{m}.$$

A similar argument shows that

$$-n\sqrt{m} \leqslant \mathbb{E}^{+}(\mathfrak{F}_A),$$

which completes the proof. □

3.7. **Energy Analysis of NHFSS.** In this paper, we introduce a new numerical measure called the energy of a NHFSS, defined as the arithmetic mean of its optimistic and pessimistic energies. In this way, a balanced value is obtained that takes into account both the most favorable and the least favorable estimates, allowing for a more comprehensive assessment of the information contained in the structure of the considered set.

Definition 3.8. *The energy of the NHFSS $\mathfrak{F}_A$, denoted by $\mathbb{E}(\mathfrak{F}_A)$, is defined as*

$$\mathbb{E}(\mathfrak{F}_A) = \frac{\mathbb{E}^{+}(\mathfrak{F}_A) + \mathbb{E}^{-}(\mathfrak{F}_A)}{2},$$

where $\mathbb{E}^{+}(\mathfrak{F}_A)$ and $\mathbb{E}^{-}(\mathfrak{F}_A)$ are the optimistic and pessimistic energies of the NHFSS $\mathfrak{F}_A$.

The definition of the total energy as the arithmetic mean of the optimistic and pessimistic energies is mathematically justified by the complementary nature of these two extremal measures. While the pessimistic energy captures the most conservative interpretation of the available hesitant neutrosophic information, the optimistic energy reflects its most favorable realization. Taking their average provides a balanced scalar characterization that lies between these two bounds and preserves monotonicity with respect to both components. Moreover, this aggregation is consistent with classical energy-based constructions motivated by the nuclear norm, where the sum of singular values yields a stable and representative global measure of structural information. As a result, the total energy offers a robust and interpretable criterion for comparing alternatives in the proposed decision-making framework.

Example 3.9. *In Example 3.5, we obtained that the optimistic energy of the observed neutrosophic hesitant fuzzy soft set is* $\mathbb{E}^{+}(\mathfrak{F}_A) = 2.29538$*, while the pessimistic energy of the same set is* $\mathbb{E}^{-}(\mathfrak{F}_A) = 1.64693$*. These values represent the boundary cases in the evaluation of the set* $\mathfrak{F}_A$ *- the optimistic energy corresponds to the scenario in which the degrees of truth-membership are maximal, the degrees of indeterminacy-membership are minimal, and the degrees of falsity-membership are as low as possible, whereas the pessimistic energy reflects the opposite configuration.*

Using the previous Definition 3.8, which defines the energy of the set as the arithmetic mean of these two extreme values, we find that the overall energy of the observed NHFSS is

$$\mathbb{E}(\mathfrak{F}_A) = \frac{\mathbb{E}^{+}(\mathfrak{F}_A) + \mathbb{E}^{-}(\mathfrak{F}_A)}{2} = \frac{2.29538 + 1.64693}{2} = 1.971155.$$

This value can be interpreted as a moderate estimate of the overall energy potential of the set $\mathfrak{F}_A$*, taking into account all aspects of uncertainty and the multiple membership values that characterize its internal structure.*

Given the way in which the optimistic and pessimistic energies are constructed, one might naturally expect the inequality

$$\mathbb{E}^{-}(\mathfrak{F}_A) \leqslant \mathbb{E}(\mathfrak{F}_A) \leqslant \mathbb{E}^{+}(\mathfrak{F}_A)$$

to always hold. However, it turns out that the pessimistic energy does not necessarily have to be smaller than the optimistic energy. The order of these values depends heavily on the specific internal structure of the data within the neutrosophic hesitant fuzzy soft set. To illustrate this, consider the following example.

Example 3.10. *Consider the NHFSS* $\mathfrak{F}_A$ *given in Table 2. This example illustrates a situation in which the usual expectation regarding the ordering of pessimistic, average, and optimistic energies does not hold.*

TABLE 2. The NHFSS $\mathfrak{F}_A$ used in Example 3.10

$\mathfrak{F}_A$	e_1	e_2	e_3
u_1	$(\{0.2\},\{0.2\},\{0.1\})$	$(\{0.6\},\{0.6\},\{0.6\})$	$(\{0.1\},\{0.2\},\{0.2\})$
u_2	$(\{0.4\},\{0.4\},\{0.2\})$	$(\{0.7\},\{0.7\},\{0.7\})$	$(\{0.9\},\{0.9\},\{0.4\})$
u_3	$(\{0.8\},\{0.8\},\{0.3\})$	$(\{0.5\},\{0.5\},\{0.6\})$	$(\{0.2,0.4\},\{0.4,0.6\},\{0.4\})$
u_4	$(\{0.8\},\{0.8\},\{0.8\})$	$(\{0.2\},\{0.2\},\{0.3\})$	$(\{0.5\},\{0.5\},\{0.6\})$

By applying Definition 3.3 and Definition 3.4, we calculate the optimistic energy of the set $\mathfrak{F}_A$ *as* $\mathbb{E}^{+}(\mathfrak{F}_A) = 3.3938$*, and the pessimistic energy as* $\mathbb{E}^{-}(\mathfrak{F}_A) = 3.42739$*.*

Then, using Definition 3.8, the total energy of $\mathfrak{F}_A$ *is given by:*

$$\mathbb{E}(\mathfrak{F}_A) = \frac{\mathbb{E}^{+}(\mathfrak{F}_A) + \mathbb{E}^{-}(\mathfrak{F}_A)}{2} = \frac{3.3938 + 3.42739}{2} = 3.410595.$$

Therefore, in this particular case, we observe that the energy of the NHFSS lies between the optimistic and pessimistic energies, but in reverse order:

$$\mathbb{E}^+(\mathfrak{F}_A) < \mathbb{E}(\mathfrak{F}_A) < \mathbb{E}^-(\mathfrak{F}_A).$$

This confirms that the usual assumption regarding the ordering does not always hold, and that the structure of the data significantly influences the outcome.

4. Decision Making Procedure Based on NHFSS

In real-world situations, we are often confronted with the challenge of selecting the least or most suitable alternative among a set of options. Making a rational and well-informed decision in such cases requires identifying the appropriate alternative. However, determining the optimal option is not a straightforward task; rather, it requires the application of a clearly defined and efficient algorithm. An algorithm that successfully identifies such alternatives can be considered both useful and effective in practice. The aim of this section is to formulate such an algorithm, specifically adapted for processing data represented in the form of a neutrosophic hesitant fuzzy soft set (NHFSS). The algorithm is based on the numerical measures introduced in the previous section, including optimistic, pessimistic, and average energy. It consists of eight consecutive steps, which will be thoroughly explained and discussed in the following.

Algorithm Based on the Energy of NHFSS:

Input: A NHFSS $\mathfrak{F}_A$ defined on the universe of alternatives $U = \{u_1, u_2, \ldots, u_n\}$ and the attribute set $E = \{e_1, e_2, \ldots, e_m\}$, along with a weight vector $w = (w_1, w_2, \ldots, w_m)$.

Output: The optimal alternative u_i that contributes the most to the system composed of all alternatives in U.

Step 1: Input the NHFSS $\mathfrak{F}_A$ and the weight vector w.

Step 2: Construct the weighted NHFSS $\overline{\mathfrak{F}}_A$ by multiplying each attribute value by the corresponding weight:

$$\overline{\mathfrak{F}}_A(u_i, e_j) = w_j \cdot \mathfrak{F}_A(u_i, e_j)$$

Step 3: For each $u_i \in U$, construct the reduced NHFSS $\overline{\mathfrak{F}}_A^{(i)}$ over $U \setminus \{u_i\}$.

Step 4: For each reduced NHFSS $\overline{\mathfrak{F}}_A^{(i)}$, form the six corresponding matrices:

$$\Phi^{\min T}_{\overline{\mathfrak{F}}_A^{(i)}}, \Phi^{\max T}_{\overline{\mathfrak{F}}_A^{(i)}}, \Phi^{\min I}_{\overline{\mathfrak{F}}_A^{(i)}}, \Phi^{\max I}_{\overline{\mathfrak{F}}_A^{(i)}}, \Phi^{\min F}_{\overline{\mathfrak{F}}_A^{(i)}}, \Phi^{\max F}_{\overline{\mathfrak{F}}_A^{(i)}}$$

Step 5: For each matrix obtained in Step 4, calculate its singular values.

Step 6: Compute the optimistic and pessimistic energies for each $\overline{\mathfrak{F}}_A^{(i)}$ using: $\mathbb{E}^+(\overline{\mathfrak{F}}_A^{(i)}) = \sum_{j=1}^{n-1} \left(\sigma_j^{\max T} + \sigma_j^{\max I} - \sigma_j^{\min F}\right)$,
$\mathbb{E}^-(\overline{\mathfrak{F}}_A^{(i)}) = \sum_{j=1}^{n-1} \left(\sigma_j^{\min T} + \sigma_j^{\min I} - \sigma_j^{\max F}\right)$

Step 7: Calculate the energy of each reduced NHFSS $\overline{\mathfrak{F}}_A^{(i)}$:

$$\mathbb{E}(\overline{\mathfrak{F}}_A^{(i)}) = \frac{\mathbb{E}^+(\overline{\mathfrak{F}}_A^{(i)}) + \mathbb{E}^-(\overline{\mathfrak{F}}_A^{(i)})}{2}$$

Step 8: Identify the alternative u_i for which the energy $\mathbb{E}(\overline{\mathfrak{F}}_A^{(i)})$ is minimized:

$$\min_{1 \le i \le n} \mathbb{E}(\overline{\mathfrak{F}}_A^{(i)})$$

The optimal alternative is the one whose exclusion leads to the greatest decrease in system energy, indicating its maximal contribution to the full system.

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universal set (the set of alternatives), and let $A = \{e_1, e_2, \ldots, e_m\} \subseteq E$, where E is the set of parameters (attributes). Let $\mathfrak{F}_A$ be a neutrosophic hesitant fuzzy soft set defined over (U, E), and let $w = (w_1, w_2, \ldots, w_m)$ be the weight vector associated with the attribute set. Under the assumption that $\sum_{k=1}^{m} w_k = 1$ and $w_k \in [0, 1]$ for all k, the vector w represents the relative importance of each attribute in the decision-making process. The algorithm is based on the numerical characteristics defined in the previous section and consists of eight steps.

In Step 1, we construct the initial neutrosophic hesitant fuzzy soft set $\mathfrak{F}_A$ from the input data. In Step 2, we form a new NHFSS $\overline{\mathfrak{F}}_A$ by applying the weight vector. Specifically, for each membership, indeterminacy, and non-membership value related to attribute e_k, we multiply the values by the corresponding weight w_k, thereby obtaining a weighted version of the original NHFSS. In Step 3, we generate n new NHFSSs from $\overline{\mathfrak{F}}_A$, where each is defined over the universal set $U \setminus \{u_i\}$. That is, each new set excludes one particular alternative, resulting in n sets, each having $n-1$ alternatives. In Step 4, for each of these n NHFSSs, we form six rectangular matrices: $\Phi^{\min T}_{\overline{\mathfrak{F}}^{(i)}_A}, \Phi^{\max T}_{\overline{\mathfrak{F}}^{(i)}_A}, \Phi^{\min I}_{\overline{\mathfrak{F}}^{(i)}_A}, \Phi^{\max I}_{\overline{\mathfrak{F}}^{(i)}_A}, \Phi^{\min F}_{\overline{\mathfrak{F}}^{(i)}_A}, \Phi^{\max F}_{\overline{\mathfrak{F}}^{(i)}_A}$. These matrices correspond to the minimum and maximum values for the truth, indeterminacy, and falsity membership degrees, respectively.

In Step 5, we determine the singular values of all the matrices constructed in the previous step using tools from linear algebra. Step 6 involves calculating both the optimistic and pessimistic energies for each NHFSS, following the definitions given earlier. In Step 7, the total energy of each NHFSS is obtained by computing the arithmetic mean of its optimistic and pessimistic energies. This results in n energy values, each corresponding to the system without one of the alternatives. Finally, in Step 8, we identify the minimum energy among these values. It is important to emphasize that, within the proposed framework, the alternative corresponding to the *minimum* total energy is interpreted as the optimal one. This criterion is based on the idea that the energy of an NHFSS quantifies the global informational contribution of the remaining alternatives through singular value–based characteristics. When a particular alternative u_i is removed and the resulting reduced NHFSS exhibits a significantly lower energy, this indicates that u_i had a strong structural influence on the original system. Consequently, the exclusion of the most influential alternative produces the greatest decrease in total energy, which justifies selecting the alternative associated with the minimum value of $\mathbb{E}(\overline{\mathfrak{F}}^{(i)}_A)$ as the optimal one. This interpretation is consistent with energy-based principles motivated by the nuclear norm, where lower energy reflects a loss of dominant structural components.

4.1. **Computational Complexity of the Proposed Algorithm.** The efficiency of the decision-making procedure described in Algorithm is strongly influenced by the matrix operations and singular value decompositions (SVDs) performed during the evaluation of each reduced NHFSS. In this subsection, we analyze the computational complexity of the algorithm in terms of the number of alternatives n and the number of attributes m.

The construction of the weighted NHFSS $\overline{\mathfrak{F}}_A$ in Step 2 requires processing all hesitant truth, indeterminacy, and falsity values. Since each parameter evaluation appears exactly once for each alternative, this step runs in time $O(nm)$. Step 3 generates n reduced NHF-SSs, each obtained by deleting a single alternative. This operation also has linear cost with respect to the size of the dataset, amounting to $O(nm)$ for the entire step.

The dominant part of the computation arises in Step 4 and Step 5, where six matrices are constructed for each reduced NHFSS and their singular values are computed. The matrices are of size $(n-1)\times m$, and a full SVD of an $a \times b$ matrix requires

$$O\big(\min(ab^2,\, a^2b)\big)$$

floating-point operations. Consequently, computing the SVD for all six matrices of a single reduced NHFSS requires

$$6\cdot O\big(\min((n-1)m^2,\, (n-1)^2m)\big) = O\big(\min(nm^2,\, n^2m)\big),$$

since constant factors do not affect asymptotic growth.

As Algorithm constructs such SVD-based evaluations for each of the n reduced NHF-SSs, the total time required for Steps 4 and 5 becomes

$$n\cdot O\big(\min(nm^2,\, n^2m)\big) = O\big(\min(n^2m^2,\, n^3m)\big).$$

The subsequent computation of optimistic, pessimistic, and total energy values in Steps 6 and 7 involves summations over at most $n-1$ singular values, leading to a cost of $O(n)$ per reduced NHFSS and $O(n^2)$ overall, which is negligible in comparison with the SVD computations. Step 8 requires a simple comparison across the n obtained energy values, contributing an additional $O(n)$ time.

Combining all steps, the overall computational complexity of the proposed decision-making procedure is dominated by the SVD computations, yielding

$$O\big(\min(n^2m^2,\, n^3m)\big).$$

This shows that the algorithm is computationally feasible for moderate values of n and m, while for large-scale applications one may rely on truncated SVD techniques, randomized matrix decompositions, or parallel implementations to significantly reduce the computational burden. Such optimizations are natural directions for future research aimed at improving scalability.

5. Solving a practical problem using the developed NHFSS algorithm

In this section, we demonstrate the application of the algorithm presented in Section 4 to a decision-making example with the goal of identifying the most optimal alternative among the considered options. We employ the same real-world motivated case study examined by Ye [35] and Wang and Li [31], in which the attribute weights are provided subjectively (based on expert preference) and are not derived through any additional weighting procedure. Hence, in our setting, the weights are taken as given input parameters rather than quantities to be computed.

It is also worth noting that the proposed algorithm remains applicable when all attributes are considered equally important. In such a case, one may simply adopt a uniform weight distribution, that is, $w_j = 1/m$ for all $j = 1,\dots,m$. Moreover, even without explicitly enforcing normalization through equal weights, the computation of singular values in Step 5 depends on the constructed matrices themselves and not on any particular normalization scheme; therefore, the use of uniform weights is a convenient and natural choice in the absence of preference information, but it is not a restrictive requirement for applying the method. The considered example thus serves to explain each step of the algorithm in detail and to show how the numerical characteristics of an NHFSS can support the selection of an optimal decision.

Example 5.1. *Let us now revisit Example 2.4, originally described in the aforementioned study, and apply the decision-making algorithm proposed in this paper. The attribute weight vector is given as* $w = (0.35, 0.25, 0.4)$, *representing the relative importance of the three criteria under consideration.*

Step 1: *In this step, we begin with the neutrosophic hesitant fuzzy soft set* $\mathfrak{F}_A$ *defined in Table 1, which serves as the initial representation of the alternatives and their evaluations with respect to the selected attributes.*

Step 2: *Using the given weight vector, we construct a new NHFSS denoted by* $\overline{\overline{\mathfrak{F}}}_A$, *in which the influence of each attribute has been adjusted according to its respective weight. The resulting structure is shown in Table 3.*

TABLE 3. The weighted NHFSS $\overline{\overline{\mathfrak{F}}}_A$ from Example 5.1

$\overline{\overline{\mathfrak{F}}}_A$	e_1	e_2
u_1	$(\{0.105, 0.14, 0.175\}, \{0.035\}, \{0.105, 0.14\})$	$(\{0.125, 0.15\}, \{0.05, 0.075\}, \{0.075, 0.1\})$
u_2	$(\{0.21, 0.245\}, \{0.035, 0.07\}, \{0.07, 0.105\})$	$(\{0.15, 0.175\}, \{0.025\}, \{0.075\})$
u_3	$(\{0.175, 0.21\}, \{0.14\}, \{0.07, 0.105\})$	$(\{0.15\}, \{0.075\}, \{0.1\})$
u_4	$(\{0.245, 0.28\}, \{0.035\}, \{0.035, 0.07\})$	$(\{0.15, 0.175\}, \{0.025\}, \{0.05\})$
	e_3	
u_1	$(\{0.08, 0.12\}, \{0.04, 0.08\}, \{0.2, 0.24\})$	
u_2	$(\{0.24, 0.28\}, \{0.04, 0.08\}, \{0.04, 0.08\})$	
u_3	$(\{0.2, 0.24\}, \{0.04\}, \{0.12\})$	
u_4	$(\{0.12, 0.2\}, \{0.08\}, \{0.04, 0.08, 0.24\})$	

By applying Definition 3.2, this NHFSS can be uniquely represented using six rectangular matrices:

$$\Phi^{\min T}_{\overline{\overline{\mathfrak{F}}}_A} = \begin{bmatrix} 0.105 & 0.125 & 0.08 \\ 0.21 & 0.15 & 0.24 \\ 0.175 & 0.15 & 0.2 \\ 0.245 & 0.15 & 0.12 \end{bmatrix}, \quad \Phi^{\max T}_{\overline{\overline{\mathfrak{F}}}_A} = \begin{bmatrix} 0.175 & 0.15 & 0.12 \\ 0.245 & 0.175 & 0.28 \\ 0.21 & 0.15 & 0.24 \\ 0.28 & 0.175 & 0.2 \end{bmatrix},$$

$$\Phi^{\min I}_{\overline{\overline{\mathfrak{F}}}_A} = \begin{bmatrix} 0.035 & 0.05 & 0.04 \\ 0.035 & 0.025 & 0.04 \\ 0.14 & 0.075 & 0.04 \\ 0.035 & 0.025 & 0.08 \end{bmatrix}, \quad \Phi^{\max I}_{\overline{\overline{\mathfrak{F}}}_A} = \begin{bmatrix} 0.035 & 0.075 & 0.08 \\ 0.07 & 0.025 & 0.08 \\ 0.14 & 0.075 & 0.04 \\ 0.035 & 0.025 & 0.08 \end{bmatrix},$$

$$\Phi_{\overline{\mathfrak{F}}_A}^{\min F} = \begin{bmatrix} 0.105 & 0.075 & 0.2 \\ 0.07 & 0.075 & 0.04 \\ 0.07 & 0.1 & 0.12 \\ 0.035 & 0.05 & 0.04 \end{bmatrix}, \quad \Phi_{\overline{\mathfrak{F}}_A}^{\max F} = \begin{bmatrix} 0.14 & 0.1 & 0.24 \\ 0.105 & 0.075 & 0.08 \\ 0.105 & 0.1 & 0.12 \\ 0.07 & 0.05 & 0.12 \end{bmatrix}.$$

Step 3: *Next, for each alternative $u_i \in U$, we construct the corresponding NHFSS over the reduced universe $U \setminus \{u_i\}$, denoted as $\overline{\mathfrak{F}}_A^{(i)}$. In what follows, we explicitly derive the set $\overline{\mathfrak{F}}_A^{(1)}$, while the remaining sets $\overline{\mathfrak{F}}_A^{(2)}, \ldots, \overline{\mathfrak{F}}_A^{(3)}$ can be obtained in the same manner. The resulting neutrosophic hesitant fuzzy soft set $\overline{\mathfrak{F}}_A^{(1)}$ is displayed in Table 4.*

TABLE 4. The table for NHFSS $\overline{\mathfrak{F}}_A^{(1)}$ from Example 5.1

$\overline{\mathfrak{F}}_A^{(1)}$	e_1	e_2
u_2	$(\{0.21, 0.245\}, \{0.035, 0.07\}, \{0.07, 0.105\})$	$(\{0.15, 0.175\}, \{0.025\}, \{0.075\})$
u_3	$(\{0.175, 0.21\}, \{0.14\}, \{0.07, 0.105\})$	$(\{0.15\}, \{0.075\}, \{0.1\})$
u_4	$(\{0.245, 0.28\}, \{0.035\}, \{0.035, 0.07\})$	$(\{0.15, 0.175\}, \{0.025\}, \{0.05\})$
	e_3	
u_2	$(\{0.24, 0.28\}, \{0.04, 0.08\}, \{0.04, 0.08\})$	
u_3	$(\{0.2, 0.24\}, \{0.04\}, \{0.12\})$	
u_4	$(\{0.12, 0.2\}, \{0.08\}, \{0.04, 0.08, 0.24\})$	

Step 4: *In this step, for each neutrosophic hesitant fuzzy soft set $\overline{\mathfrak{F}}_A^{(i)}$ obtained in the previous phase, we construct the associated six matrices. These matrices are generated by omitting the i-th row from the global matrices $\Phi_{\overline{\mathfrak{F}}_A}^{\min T}, \Phi_{\overline{\mathfrak{F}}_A}^{\max T}, \Phi_{\overline{\mathfrak{F}}_A}^{\min I}, \Phi_{\overline{\mathfrak{F}}_A}^{\max I}, \Phi_{\overline{\mathfrak{F}}_A}^{\min F}$ and $\Phi_{\overline{\mathfrak{F}}_A}^{\max F}$ defined in Step 2.*

Step 5: *For every matrix obtained in Step 4, we compute the singular values using standard techniques from linear algebra. These values capture essential structural information from the matrices and will be used in subsequent computations.*

Step 6: *Based on the singular values determined in Step 5, and using the relevant definitions, we evaluate the optimistic and pessimistic energies for each of the NHFSSs. For the set $\overline{\mathfrak{F}}_A^{(1)}$, we obtain:*

$$\mathbb{E}^+(\overline{\mathfrak{F}}_A^{(1)}) = 0.76063, \quad \mathbb{E}^-(\overline{\mathfrak{F}}_A^{(1)}) = 0.57678.$$

Applying the same procedure to the remaining sets yields the following results:

$$\mathbb{E}^+(\overline{\mathfrak{F}}_A^{(2)}) = 0.6304, \quad \mathbb{E}^-(\overline{\mathfrak{F}}_A^{(2)}) = 0.43442,$$
$$\mathbb{E}^+(\overline{\mathfrak{F}}_A^{(3)}) = 0.61303, \quad \mathbb{E}^-(\overline{\mathfrak{F}}_A^{(3)}) = 0.37794,$$
$$\mathbb{E}^+(\overline{\mathfrak{F}}_A^{(4)}) = 0.406, \quad \mathbb{E}^-(\overline{\mathfrak{F}}_A^{(4)}) = 0.34772.$$

Step 7: *We calculate the energy* $\mathbb{E}(\overline{\mathfrak{F}}_A^{(1)})$ *as the arithmetic mean of the optimistic energy* $\mathbb{E}^+(\overline{\mathfrak{F}}_A^{(1)})$ *and the pessimistic energy* $\mathbb{E}^-(\overline{\mathfrak{F}}_A^{(1)})$*, so we obtain* $\mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}) = 0.668705$. *Similarly, we find:* $\mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}) = 0.53241, \mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}) = 0.495485, \mathbb{E}(\overline{\mathfrak{F}}_A^{(4)}) = 0.37686.$

Step 8: *Using the values of the energies computed in the previous step, we now rank the alternatives according to their contribution to the overall system. Since the energy reflects the amount of information lost when an alternative is removed, the optimal choice corresponds to the alternative whose removal causes the greatest loss, i.e., the one associated with the lowest energy value. Based on the computed values, we observe the following strict inequality:*

$$\mathbb{E}(\overline{\mathfrak{F}}_A^{(4)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}).$$

This implies that the most significant contributor to the system is the alternative u_4*, representing the arms company, since its absence results in the smallest overall system energy. Conversely, the least impactful alternative is* u_1*, corresponding to the car company, as its removal results in the highest system energy. Accordingly, the final ranking of the alternatives in descending order of their contribution is given by:*

$$u_4 \succ u_3 \succ u_2 \succ u_1. \tag{5. 1}$$

In the next section, we will further analyze the obtained results to gain deeper insights into the implications of the proposed decision-making framework. The numerical example considered above clearly demonstrates the applicability and effectiveness of the algorithm based on the energy of neutrosophic hesitant fuzzy soft sets. Through a systematic and structured approach involving the elimination of each alternative and evaluating its impact on the total system energy, we were able to determine a complete ranking of the available options.

The fact that the alternative u_4 exhibits the lowest energy confirms its dominant role in the decision system, indicating that its removal would result in the most significant loss of information. On the other hand, the highest energy associated with u_1 suggests that this alternative has the least influence on the overall system, and thus represents the least preferable choice.

This kind of analysis not only enables the identification of the most optimal alternative but also provides a deeper understanding of the relative importance and influence of each option in the context of all others. Such insights are crucial in real-world decision-making scenarios, where multiple criteria and uncertainty factors must be simultaneously considered. Therefore, the results obtained from this example validate the practical value and robustness of the proposed energy-based algorithm, setting a strong foundation for further investigations and applications in the next section.

6. Evaluation of the Proposed Method Against Existing NHFSS Techniques

The development of a novel decision-making algorithm requires a careful comparison with existing methods in order to confirm its importance and contribution. Although previous algorithms such as those described in [35], [31], and [4] have proven to be effective and broadly applicable in neutrosophic hesitant fuzzy environments, they still exhibit certain shortcomings. In particular, they may fail to provide consistent optimality or to distinguish clearly among close alternatives in complex decision problems. As will be illustrated in Subsections 6.1–6.3, these methods are often capable of separating clearly inferior alternatives from better ones, but in some situations they do not yield a unique optimal solution, which is a serious drawback in practical applications where a clear and definitive choice is required. To justify the relevance of the proposed energy-based approach, we analyze the performance of these methods and point out specific cases where they do not produce satisfactory or interpretable results. In each of the examples considered in Subsections 6.1–6.3, the energy-based method succeeds in providing a complete (linear) ranking of the alternatives and in selecting a single best option. In contrast, the algorithm we introduce in Section 4 makes use of both pessimistic and optimistic energy values to establish a stable and understandable ranking of alternatives. Additionally, it provides insight into the extent to which each alternative contributes to the overall decision context. Finally, in Subsection 6.4, we apply all the considered methods to the same numerical example, analyze the obtained rankings in detail, and complement the comparison with a graphical illustration of the results, thereby highlighting more clearly the specific advantages of the proposed energy-based procedure.

6.1. **On the Limitations of the Neutrosophic Hesitant Fuzzy Method Proposed by Ye [35].** In this subsection, we analyze the decision-making algorithm proposed by Jun Ye in [35], which employs the single-valued neutrosophic hesitant fuzzy weighted average operator (SVNHFWA) or single-valued neutrosophic hesitant fuzzy weighted geometric operator (SVNHFWG). The aim is to explore potential limitations of this method and to contrast its performance with the advantages offered by an energy-based decision-making framework. As demonstrated in the following example, there are cases in which Ye's method does not produce the most suitable ranking of alternatives, highlighting the relevance of the proposed approach.

Example 6.2. *Let* $U = \{u_1, u_2, u_3\}$ *be the universal set of alternatives,* $A = \{e_1, e_2, e_3\}$ *the set of parameters, and* $w = (0.4, 0.3, 0.3)$ *the weight vector. The tabular representation of the obtained NHFSS* $\mathfrak{F}_A$ *is given in Table 5.*

TABLE 5. NHFSS $\mathfrak{F}_A$ from Example 6.2

$\mathfrak{F}_A$	e_1	e_2	e_3
u_1	$(\{0.5, 0.6\}, \{0.1\}, \{0.2, 0.3\})$	$(\{0.6\}, \{0.2, 0.3\}, \{0.2\})$	$(\{0.7\}, \{0.2\}, \{0.1\})$
u_2	$(\{0.5, 0.6\}, \{0.2, 0.3\}, \{0.1\})$	$(\{0.6\}, \{0.2\}, \{0.2, 0.3\})$	$(\{0.7\}, \{0.1\}, \{0.2\})$
u_3	$(\{0.5, 0.6\}, \{0.3\}, \{0.5\})$	$(\{0.7\}, \{0.3\}, \{0.5\})$	$(\{0.6\}, \{0.4\}, \{0.6\})$

We first apply the algorithm proposed in [35]*, which uses the SVNHFWA operator. For alternative u_1, we obtain:*

$$
\begin{aligned}
n_1 &= \textit{SVNHFWA}(n_{11}, n_{12}, n_{13}) \\
&= \big\{\{1-(1-0.5)^{0.4}(1-0.6)^{0.3}(1-0.7)^{0.3},\ 1-(1-0.6)^{0.4}(1-0.6)^{0.3}(1-0.7)^{0.3}\}, \\
&\quad \{0.1^{0.4}\cdot 0.2^{0.3}\cdot 0.2^{0.3},\ 0.1^{0.4}\cdot 0.3^{0.3}\cdot 0.2^{0.3}\}, \\
&\quad \{0.2^{0.4}\cdot 0.2^{0.3}\cdot 0.1^{0.3},\ 0.3^{0.4}\cdot 0.2^{0.3}\cdot 0.1^{0.3}\}\big\} \\
&= \big\{\{0.59882, 0.63307\}, \{0.15157, 0.17118\}, \{0.16245, 0.19106\}\big\}.
\end{aligned}
$$

Similarly, we get:

$$n_2 = \big\{\{0.59882, 0.63307\}, \{0.16245, 0.19106\}, \{0.15157, 0.17118\}\big\},$$
$$n_3 = \big\{\{0.59882, 0.63307\}, \{0.32704\}, \{0.52811\}\big\}.$$

The cosine measures with respect to the ideal element n^* are:

$$\cos(n_1, n^*) = 0.93211, \quad \cos(n_2, n^*) = 0.93211, \quad \cos(n_3, n^*) = 0.70411.$$

Thus, u_1 and u_2 outperform u_3, but the method cannot distinguish between u_1 and u_2.

Applying the SVNHFWG operator yields:

$$
\begin{aligned}
n_1 &= \big\{\{0.5842, 0.6284\}, \{0.16141, 0.19434\}, \{0.17123, 0.21433\}\big\}, \\
n_2 &= \big\{\{0.5842, 0.6284\}, \{0.17123, 0.21433\}, \{0.16141, 0.19434\}\big\}, \\
n_3 &= \big\{\{0.5842, 0.6284\}, \{0.33163\}, \{0.53238\}\big\}.
\end{aligned}
$$

The cosine measures are:

$$\cos(n_1, n^*) = \cos(n_2, n^*) = 0.91779, \quad \cos(n_3, n^*) = 0.69501,$$

again failing to select a unique optimal alternative.

In contrast, applying the energy-based algorithm from Section 5 gives:

$$\mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}) = 0.437922, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}) = 0.406523, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}) = 0.500295.$$

This yields the ranking:

$$\mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}),$$

i.e.,

$$u_2 \succ u_1 \succ u_3,$$

and the optimal solution is u_2. The obtained rankings and optimal solutions for the considered algorithms are summarized in Table 6.

TABLE 6. Obtained rankings and optimal solutions for the considered algorithms

Procedure	*Ranking of alternatives*	*Optimal solution*
Jun Ye SVNHFWA operator [35]	$u_1 = u_2 \succ u_3$	×
Jun Ye SVNHFWG operator [35]	$u_1 = u_2 \succ u_3$	×
Method based on energy (this paper)	$u_2 \succ u_1 \succ u_3$	u_2

Based on Example 6.2, it becomes evident that the main drawback of the decision-making algorithm proposed in [35] lies in its inability to consistently yield a unique optimal solution. While the method is capable of distinguishing less favorable alternatives from better ones, it may produce ties among top-ranked options, leaving the decision-maker without clear guidance for final selection. This limitation is particularly critical in practical scenarios where a definitive choice is required. Furthermore, as shown in the example, both the SVNHFWA and SVNHFWG operators fail to discriminate between the best-performing alternatives, even when subtle differences exist in their underlying neutrosophic hesitant fuzzy evaluations.

In contrast, the energy-based approach developed in this paper overcomes this shortcoming by incorporating both optimistic and pessimistic perspectives into a unified numerical measure of performance. This allows for a complete and unambiguous ranking of alternatives, ensuring that the optimal choice can always be identified. The ability of the energy-based method to differentiate between alternatives with similar aggregated values demonstrates its potential as a more reliable and informative decision-making tool in complex neutrosophic hesitant fuzzy environments.

6.3. **Evaluation of the Ali et al.** [4] **Decision-Making Method.** In this subsection, we analyze the decision-making algorithm proposed by Ali et al. [4], which introduces an innovative multiple-criteria decision-making (MCDM) approach based on the NHFWPMSM operator. This method is designed to determine the most suitable alternative from a given set and demonstrates strong performance when compared with other established algorithms. Nevertheless, it should be emphasized that the method may not consistently produce the optimal solution in every scenario. To demonstrate this limitation, we present the following example.

Example 6.4. *Let* $U = \{u_1, u_2, u_3, u_4\}$ *be the universal set of alternatives,* $A = \{e_1, e_2\}$ *the set of parameters, and* $w = (0.5, 0.5)$ *the weight vector. Both attributes are categorized into the same group, and let* $\rho = 2$*. The tabular representation of the obtained NHFSS* $\mathfrak{F}_A$ *is given in Table 7.*

TABLE 7. NHFSS $\mathfrak{F}_A$ from Example 6.4

$\mathfrak{F}_A$	e_1	e_2
u_1	$(\{0.9\}, \{0.1\}, \{0.1\})$	$(\{0.5\}, \{0.3\}, \{0.4\})$
u_2	$(\{0.8\}, \{0.2\}, \{0.2\})$	$(\{0.6\}, \{0.2\}, \{0.3\})$
u_3	$(\{0.6\}, \{0.3\}, \{0.3\})$	$(\{0.8\}, \{0.2\}, \{0.2\})$
u_4	$(\{0.5\}, \{0.3\}, \{0.4\})$	$(\{0.9\}, \{0.1\}, \{0.1\})$

Using the method from the aforementioned study, the Score values for the alternatives u_i *are calculated as:*

$$s(u_1) = 0.50165, \quad s(u_2) = 0.50118, \quad s(u_3) = 0.48359, \quad s(u_4) = 0.50165.$$

From these results, we see that u_1 and u_4 outperform the remaining two, while u_2 is better than u_3. However, since u_1 and u_4 have identical Score values, the Accuracy function must be applied to break the tie. It turns out that the Accuracy values for u_1 and u_4 are also equal, meaning that this method fails to produce a unique optimal solution.

If we instead apply the energy-based algorithm, we obtain:

$$\mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}) = 0.92575, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}) = 0.97799, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}) = 0.95183, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(4)}) = 0.94763.$$

This leads to the following linear order of energies:

$$\mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(4)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}) < \mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}),$$

from which we conclude that the optimal solution is alternative u_1. The obtained results are summarized in Table 8.

TABLE 8. Obtained rankings and optimal solutions for the considered algorithms

Procedure	*Ranking of alternatives*	*Optimal solution*
Ali et al. method [4]	$u_1 = u_4 \succ u_2 \succ u_3$	$\times$
Method based on energy (this paper)	$u_1 \succ u_4 \succ u_3 \succ u_2$	u_1

Based on Example 6.4, it is evident that the method proposed in [4], although effective in many decision-making scenarios, may fail to produce a unique optimal solution when multiple alternatives yield identical Score and Accuracy values. In such cases, the decision-making process remains inconclusive, which can be a significant drawback in applications where a single best choice is required.

In contrast, the proposed energy-based approach successfully distinguishes between all alternatives by providing a complete and strict ranking according to their calculated energy values. This not only resolves situations involving ties but also offers an additional interpretative advantage, as the energy measure reflects the overall contribution of each alternative to the system. Therefore, the energy-based method represents a more robust and reliable tool for decision-making in neutrosophic hesitant fuzzy soft set environments.

6.5. **Analysis of the Wang and Li Method** [31] **with Identified Limitations.** The focus of this subsection is the decision-making algorithm proposed by Wang and Li in [31], which employs two types of aggregation operators: the generalized single-valued neutrosophic hesitant fuzzy prioritized weighted average (GSVNHFPWA) and generalized single-valued neutrosophic hesitant fuzzy prioritized weighted geometric (GSVNHFPWG) operator. While the method offers a systematic approach to handling complex multi-criteria decision-making problems, it also presents certain limitations that may affect its ability to consistently produce an optimal solution. Our objective here is to identify and analyze these limitations, while at the same time emphasizing the strengths of the proposed energy-based decision-making algorithm. The following example serves to illustrate these observations.

Example 6.6. *Let $U = \{u_1, u_2, u_3, u_4\}$ be the universal set of alternatives, $A = \{e_1, e_2, e_3, e_4\}$ the set of parameters, and $w = (0.4, 0.3, 0.2, 0.1)$ the weight vector. The tabular representation of the NHFSS $\mathfrak{F}_A$ is given in Table 9.*

TABLE 9. NHFSS $\mathfrak{F}_A$ from Example 6.6

$\mathfrak{F}_A$	e_1	e_2	e_3	e_4
u_1	$(\{0.7\},\{0.3\},\{0.2\})$	$(\{0.8\},\{0.3\},\{0.3\})$	$(\{0.6\},\{0.4\},\{0.3\})$	$(\{0.7\},\{0.3\},\{0.2\})$
u_2	$(\{0.6\},\{0.3\},\{0.3\})$	$(\{0.8\},\{0.4\},\{0.2\})$	$(\{0.7\},\{0.3\},\{0.3\})$	$(\{0.7\},\{0.3\},\{0.2\})$
u_3	$(\{0.7\},\{0.2\},\{0.3\})$	$(\{0.8\},\{0.3\},\{0.3\})$	$(\{0.6\},\{0.3\},\{0.4\})$	$(\{0.7\},\{0.2\},\{0.3\})$
u_4	$(\{0.5\},\{0.3\},\{0.4\})$	$(\{0.6\},\{0.3\},\{0.5\})$	$(\{0.5\},\{0.4\},\{0.4\})$	$(\{0.6\},\{0.4\},\{0.5\})$

Applying the procedure described in [31]*, we first compute the elements of the matrix* T_{ij} *as:*

$$T_{ij} = \begin{bmatrix} 1 & 0.73333 & 0.5377 & 0.3405 \\ 1 & 0.66667 & 0.4889 & 0.3424 \\ 1 & 0.73333 & 0.5377 & 0.3405 \\ 1 & 0.6 & 0.36 & 0.204 \end{bmatrix}.$$

Using the GSVNHFPWA operator with $\lambda = 1$*, the SVNHFE values corresponding to the alternatives are obtained as:*

$$\tilde{n}_1 = \{\{0.7155\},\{0.3187\},\{0.2440\}\}, \quad \tilde{n}_2 = \{\{0.6983\},\{0.3242\},\{0.2551\}\},$$
$$\tilde{n}_3 = \{\{0.7155\},\{0.2440\},\{0.3187\}\}, \quad \tilde{n}_4 = \{\{0.5396\},\{0.3233\},\{0.4344\}\}.$$

The corresponding score function values are:

$$s(\tilde{n}_1) = 0.7176, \quad s(\tilde{n}_2) = 0.7063, \quad s(\tilde{n}_3) = 0.7176, \quad s(\tilde{n}_4) = 0.5940.$$

This results in the ranking $u_1 = u_3 \succ u_2 \succ u_4$*, indicating that the method cannot distinguish between* u_1 *and* u_3 *as the optimal choice. Applying the GSVNHFPWG operator with* $\lambda = 1$ *yields the same ranking.*

In contrast, applying the proposed energy-based algorithm gives:

$$\mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}) = 0.59846, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}) = 0.58761, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}) = 0.62727, \quad \mathbb{E}(\overline{\mathfrak{F}}_A^{(4)}) = 0.72177.$$

This leads to the ranking $u_2 \succ u_3 \succ u_1 \succ u_4$*, clearly identifying* u_2 *as the unique optimal alternative. The results are summarized in Table 10.*

TABLE 10. Obtained rankings and optimal solutions for the considered algorithms

Procedure	*Ranking of alternatives*	*Optimal solution*
Wang and Li GSVNHFPWA operator [31]	$u_1 = u_3 \succ u_2 \succ u_4$	$\times$
Wang and Li GSVNHFPWG operator [31]	$u_1 = u_3 \succ u_2 \succ u_4$	$\times$
Method based on energy (this paper)	$u_2 \succ u_3 \succ u_1 \succ u_4$	u_2

Based on Example 6.6, it can be concluded that the algorithm proposed in [31] is not always capable of identifying a unique optimal solution, as it may assign the same ranking to different alternatives. This limitation reduces its effectiveness in situations where a clear and decisive choice is required. In contrast, the proposed energy-based approach not only differentiates between all alternatives but also provides a consistent linear ranking, thereby offering a more reliable and precise decision-making outcome.

6.7. **Comparative evaluation of the proposed method and existing approaches.** In this subsection, we conduct a comparative evaluation of the proposed energy-based decision-making algorithm against several well-known approaches from the literature. To ensure a fair and consistent assessment, all methods are applied to the same numerical example, enabling a direct comparison of their decision outputs. This approach allows us to identify not only the similarities and differences in the obtained rankings, but also to highlight specific situations in which the proposed method demonstrates advantages in terms of solution uniqueness, interpretability, and the ability to capture subtle distinctions between alternatives.

Example 6.8. *Let us revisit Example 5.1, which has also been examined in* [35] *and* [31]. *For consistency, we apply all considered algorithms to the same dataset and record the values on which each procedure bases its decision. These values are summarized in Table 11, while the resulting rankings and identified optimal solutions are given in Table 12. As established earlier, the proposed energy-based algorithm produces the ranking given in (5. 1), with u_4 emerging as the optimal alternative.*

TABLE 11. Decision values obtained using the considered algorithms

Procedure	***Decision values***
Jun Ye SVNHFWA operator [35]	$\cos(n_1, n^*) = 0.6636,\ \cos(n_2, n^*) = 0.9350,$ $\cos(n_3, n^*) = 0.8353,\ \cos(n_4, n^*) = 0.9426$
Jun Ye SVNHFWG operator [35]	$\cos(n_1, n^*) = 0.6040,\ \cos(n_2, n^*) = 0.9259,$ $\cos(n_3, n^*) = 0.8080,\ \cos(n_4, n^*) = 0.9232$
Ali et al. method [4]	$\mathbb{S}(u_1) = 0.2741,\ \mathbb{S}(u_2) = 0.3845,$ $\mathbb{S}(u_3) = 0.3001,\ \mathbb{S}(u_4) = 0.3889$
Wang and Li GSVNHFPWA operator [31]	$s(\tilde{n}_1) = 0.5902,\ s(\tilde{n}_2) = 0.7711,$ $s(\tilde{n}_3) = 0.6882,\ s(\tilde{n}_4) = 0.7623$
Wang and Li GSVNHFPWG operator [31]	$s(\tilde{n}_1) = 0.5669,\ s(\tilde{n}_2) = 0.7622,$ $s(\tilde{n}_3) = 0.6663,\ s(\tilde{n}_4) = 0.7358$
Method based on energy (this paper)	$\mathbb{E}(\overline{\mathfrak{F}}_A^{(1)}) = 0.668705,\ \mathbb{E}(\overline{\mathfrak{F}}_A^{(2)}) = 0.53241,$ $\mathbb{E}(\overline{\mathfrak{F}}_A^{(3)}) = 0.495485,\ \mathbb{E}(\overline{\mathfrak{F}}_A^{(4)}) = 0.37686$

TABLE 12. Obtained rankings and optimal solutions for the considered algorithms

Procedure	*Ranking of alternatives*	*Optimal solution*
Jun Ye SVNHFWA operator [35]	$u_4 \succ u_2 \succ u_3 \succ u_1$	u_4
Jun Ye SVNHFWG operator [35]	$u_2 \succ u_4 \succ u_3 \succ u_1$	u_2
Ali et al. method [4]	$u_4 \succ u_2 \succ u_3 \succ u_1$	u_4
Wang and Li GSVNHFPWA operator [31]	$u_2 \succ u_4 \succ u_3 \succ u_1$	u_2
Wang and Li GSVNHFPWG operator [31]	$u_2 \succ u_4 \succ u_3 \succ u_1$	u_2
Method based on energy (this paper)	$u_4 \succ u_3 \succ u_2 \succ u_1$	u_4

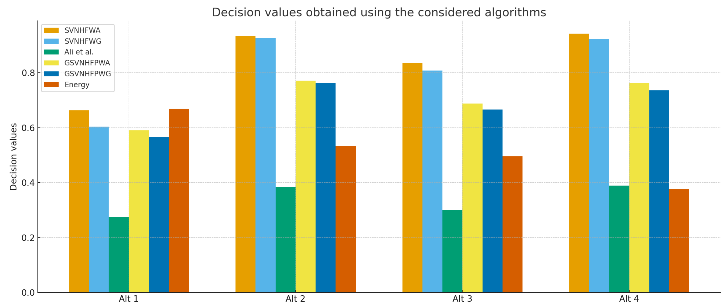

FIGURE 1. Comparison of decision values obtained by the considered algorithms in Example 6.8

Figures 1 and 2 provide a graphical illustration of the numerical results reported in Tables 11 and 12, further highlighting the differences among the considered decision-making approaches.

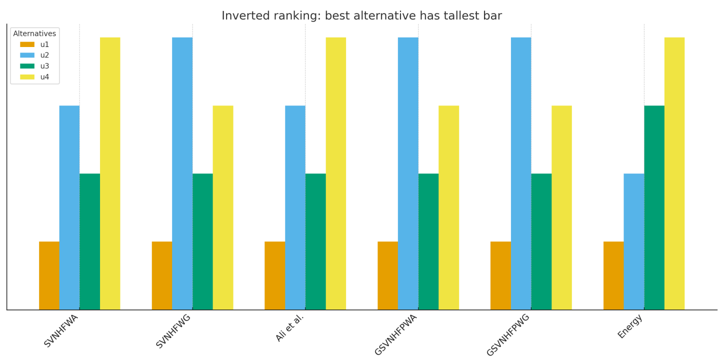

FIGURE 2. Comparison of rankings and identified optimal alternatives for the considered methods in Example 6.8

A comparative examination of the results presented in Tables 11 and 12 reveals that the outcomes generated by the proposed energy-based algorithm are not always identical to those obtained using other existing approaches. In several cases, the algorithm identifies a different optimal alternative, which demonstrates its potential to offer new perspectives and insights in the decision-making process. This adaptability allows for decisions that may be more closely aligned with the specific characteristics of the problem at hand.

One of the notable strengths of the proposed method, as discussed in Subsections 6.1–6.5, lies in its ability to consistently produce an optimal solution across all tested scenarios. This is in contrast to certain existing algorithms which, while effective in specific situations, do not guarantee an optimal result under all conditions. Consequently, such algorithms may yield ambiguous or incomplete recommendations.

Although the energy measure is a newly introduced characteristic in the context of neutrosophic hesitant fuzzy soft sets, it is not an unfamiliar notion in mathematics, particularly in graph theory. Its computation is based on established concepts of linear algebra, ensuring that the algorithm remains both straightforward to implement and computationally reliable.

Furthermore, in the case study given in Example 5.1, the SVNHFWA-based approach proposed in [35], the method developed in [4], and our energy-based method converge to the same optimal solution, even though the rankings of the alternatives differ. This demonstrates that the proposed approach is not only capable of validating existing results but can also refine the decision-making process by providing additional differentiation among alternatives.

The most significant comparative observations can be summarized as follows:

- The proposed energy-based algorithm is capable of producing a different optimal alternative from other methods, thereby offering alternative viewpoints and potentially better problem-specific solutions.
- It guarantees the identification of an optimal solution in all tested cases, unlike several existing algorithms that may fail to do so under certain conditions.
- The method is grounded in well-established mathematical principles, making it easy to implement while maintaining reliability and computational efficiency.
- In some scenarios (e.g., Example 5.1), it confirms the optimal solution obtained by other methods, but with a different ranking structure, thereby adding granularity to the decision-making process.

7. Summary and Conclusion

Advances in the field of uncertainty modeling and multi-criteria decision-making increasingly depend on the development of methods capable of integrating various types of information while providing a clear basis for ranking alternatives. Among the numerous extensions of neutrosophic soft sets, the introduction of the neutrosophic hesitant fuzzy soft set (NHFSS) offers additional flexibility in modeling situations where both hesitation and multiple parameter values occur simultaneously.

In this paper, we have defined the pessimistic and optimistic energy of an NHFSS as two complementary numerical characteristics that enable a deeper understanding of the contribution of each alternative within a decision-making system. Based on these measures, we have constructed a novel energy-based decision-making algorithm, which allows an intuitive yet mathematically grounded ranking of alternatives and successfully resolves situations in which several existing methods produce tied or ambiguous results.

Through a detailed comparison with the algorithms proposed in [35], [31], and [4], we have shown that the proposed approach consistently produces a unique optimal solution, even in scenarios where other decision-making procedures fail to distinguish between competing alternatives. These findings confirm that energy, when defined via singular value–based characteristics, represents a robust and informative aggregate measure for neutrosophic hesitant fuzzy soft sets.

Overall, the results indicate that energy-based modeling within the NHFSS framework has strong potential for further applications in multi-criteria decision-making problems characterized by multiple sources of uncertainty, hesitation, and incomplete information.

Possible directions for future research include:

- **Analytical investigation of NHFSS energy bounds:** A systematic study of lower and upper bounds for pessimistic and optimistic energy values, as well as their dependence on the distribution of truth, indeterminacy, and falsity memberships, may provide deeper theoretical insight and support the reconstruction of incomplete or partially known NHFSSs.
- **Extension of energy concepts to other fuzzy soft frameworks:** Motivated by the effectiveness of the proposed approach, a natural continuation of this work is the definition and analysis of energy measures for *intuitionistic fuzzy soft sets* and *Pythagorean fuzzy soft sets*. Such extensions would allow direct comparison between different uncertainty models within a unified energy-based decision-making paradigm.
- **Algorithm optimization for large-scale problems:** For decision-making scenarios involving a large number of alternatives or attributes, it is important to develop optimized implementations that reduce computational complexity, for example through truncated singular value decompositions or parallel computation, while preserving the discriminative power of the energy measure.

Credit Authorship Contribution's Statement

Milica Dabic: Conceptualization, Methodology, Formal Analysis, Writing – Original Draft, Investigation.
Nenad Stojanovic: Supervision, Validation, Resources, Writing – Review & Editing, Project Administration.

DECLARATIONS

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