

A note on the Fekete-Szegö Conjecture for Usual Subclasses of Univalent Functions

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Abstract. One of the most fundamental problems in the theory of univalent functions is the Bieberbach problem, also known as the Bieberbach conjecture, which claims that the modulus of the n th Taylor coefficient of a function f in class \mathcal{S} is bounded by n (that is, $f \in \mathcal{S}$, $n \geq 2$ ($n \in \mathbb{N} = \{1, 2, 3, \dots\}$), $|c_n| \leq n$). In general, studies on the Bieberbach conjecture, the proof of which occupied mathematicians for a long time, have evolved—as a result of the intellectual efforts in the field—from certain bounds on the modulus of individual Taylor coefficients to the investigation of bounds on the modulus of functionals formed by combinations of these coefficients. In the present article, the Fekete–Szegö conjecture, which involves one such important functional, is considered, and certain bounds are determined for a parameter η with $0 \leq \eta \leq 1$.

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1. INTRODUCTION

Let \mathcal{U} be an open unit disc in the complex plane \mathbb{C} . Let \mathcal{S} be the class of f functions that are normalized (that is, $f(0) = 0$, $f'(0) = 1$) in \mathcal{U} , are analytic (that is, f has a derivative at each point of \mathcal{U}) and univalent (that is, for every $z_1, z_2 \in \mathcal{U}$, $z_1 \neq z_2$ requires $f(z_1) \neq f(z_2)$), and can also be expressed with the Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n = z + c_2 z^2 + \dots + c_n z^n + \dots (z \in \mathcal{U}). \quad (1.1)$$

It should be noted that the term univalent used in studies in this field commonly refers to functions that are both analytic and one-to-one in \mathcal{U} . Let \mathcal{S}^* and \mathcal{C} be subclasses of the class \mathcal{S} consisting of starlike functions with respect to the origin (that is, the line segment connecting any point of $f(\mathcal{U})$ to the origin remains entirely in $f(\mathcal{U})$) and usually convex functions in \mathcal{U} , respectively. The subclasses given above are commonly used in applications

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such as

$$\mathcal{S}^* = \left\{ f(z) \in \mathcal{S} : \Re \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in \mathcal{U} \right\} \quad (1.2)$$

and

$$\mathcal{C} = \left\{ f \in \mathcal{S} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in \mathcal{U} \right\}. \quad (1.3)$$

Additionally, a function f in the form (1.1) is called a close-to-convex function if there is a function $g \in \mathcal{S}^*$ such that

$$\Re \left(e^{i\alpha} \frac{zf'(z)}{g(z)} \right) > 0 (z \in \mathcal{U}) \quad (1.4)$$

with $\alpha (|\alpha| \leq \frac{\pi}{2})$ as the argument. Let the class of close-to-convex functions with $\alpha (|\alpha| \leq \frac{\pi}{2})$ arguments be denoted by \mathcal{C}_α . Many properties of functions belonging to classes \mathcal{S} , \mathcal{S}^* , \mathcal{C} and \mathcal{C}_α can be examined in scientific studies and books specific to this field [3, 8, 11].

Among the important mathematical tools in univalent function theory is the Hankel determinant, which is defined based on the Taylor expansion coefficients of a function $f \in \mathcal{S}$ and is commonly represented by $|H_{q,n}(f)|$, where $q \geq 1 (q \in \mathbb{N})$ and $n \geq 1 (n \in \mathbb{N})$. That is,

$$|H_{q,n}(f)| = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+q-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n+q-1} & c_{n+q} & \cdots & c_{n+2q-2} \end{vmatrix} \quad (1.5)$$

It is clear that for a function $f \in \mathcal{S}$, different order Hankel determinants can be obtained for different values of q and n . In studies conducted in this field, the second order

$$|H_{2,1}(f)| = \begin{vmatrix} 1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_3 - c_2^2. \quad (1.6)$$

Hankel determinant obtained when $c_1 = 1, q = 2$ and $n = 1$ is predominantly used. At the same time, functional $|H_{2,1}(f)|$ is known as the classical Fekete-Szegő functional and functional $|H_{2,1}(f, \eta)|$, where $\eta (0 \leq \eta \leq 1)$ is a complex or real parameter, is known as the generalized Fekete-Szegő functional[9]. In 1933, Fekete and Szegő introduced the inequality

$$|H_{2,1}(f)| = \begin{cases} 3 - 4\eta & , \text{if } \eta \leq 0, \\ 1 + 2e^{\frac{-2\eta}{\eta-1}} & , \text{if } 0 < \eta < 1, \\ 4\eta - 3 & , \text{if } \eta \geq 1 \end{cases} \quad (1.7)$$

for the modulus of a functional formed by a combination of the relevant Taylor coefficients of univalent functions, where it is clear that $|H_{2,1}(f, \eta)| \leq 1$ when $\eta \rightarrow 1^-$. The problem of maximizing the modulus of the generalized Fekete-Szegő functional (i.e., $|H_{2,1}(f, \eta)|$) with respect to the parameter $\eta (0 \leq \eta \leq 1)$ has also attracted particular attention, with various additional applications in univalent function theory being studied by many mathematicians [1, 2, 4, 10, 12, 13, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 6, 7, 17, 21, 5]. Studies on the Fekete-Szegő functional have yielded significant results not only in the class of univalent

functions but also in various subclasses. Notably, Zaprawa [27] established Fekete-Szegő type inequalities for classes of bi-univalent functions, while Vasudevarao [26] provided sharp bounds for this functional within the class $F(\alpha)$. Such works extend the applicability of the Fekete-Szegő problem to broader functional settings. In the present study, new bounds are derived for the parameterized Fekete-Szegő functional within the close-to-convex class \mathcal{C}_α , thereby contributing to the general framework of the problem from a different functional perspective. Perhaps one of the most important reasons for this may be that while the extreme function $k(z)$ is the Koebe function and its rotations, which are defined by

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots = z + \sum_{n=2}^{\infty} nz^n \quad (1.8)$$

for various applications in the theory of univalent functions, neither the $k(z)$ Koebe function nor its rotations can be an extreme function for Fekete-Szegő problems. On the other hand, the Koebe function $k(z)$ satisfies the condition

$$|c_3 - \eta c_2^2| = |3 - 4\eta| (0 < \eta < 1) \quad (1.9)$$

whereas

$$\max_{f \in \mathcal{S}} |c_3 - \eta c_2^2| = 1 + 2e^{\frac{-2\eta}{1-\eta}} (0 < \eta < 1) \quad (1.10)$$

and

$$\max_{f \in \mathcal{S}^*} |c_3 - \eta c_2^2| = \begin{cases} |3 - 4\eta|, & \text{if } 0 \leq \eta \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} < \eta \leq 1. \end{cases} \quad (1.11)$$

and

$$|H_{2,1}(f)| = \begin{cases} 3 - 4\eta & , \text{if } 0 \leq \eta \leq \frac{1}{3}, \\ \frac{1}{3} + \frac{4}{9\eta} & , \text{if } \frac{1}{3} \leq \eta \leq \frac{2}{3}, \\ 1 & , \text{if } \frac{2}{3} \leq \eta \leq 1. \end{cases} \quad (1.12)$$

Definition 1.1. Functions that are analytic in \mathcal{U} and have the form

$$w(z) = \sum_{n=1}^{\infty} \alpha_n z^n (z \in \mathcal{U}) \quad (1.13)$$

where $w(0) = 0$ and $|w(z)| \leq 1$, are called Schwartz functions, and let \mathcal{P} be the class of these functions.

In this case, a function $h(z)$ defined with

$$h(z) = \left(\frac{1+w(z)}{1-w(z)} \right) \cos \alpha + i \sin \alpha (|\alpha| \leq \frac{\pi}{2}) \quad (1.14)$$

can be written for the Schwartz function $w(z)$. Therefore, if $f \in \mathcal{C}_\alpha$, the derivative function f' can be written in the form

$$f'(z) = \frac{\phi(z)}{z} h(z) e^{-i\alpha} \quad (z \in \mathcal{U}, \phi(z) = z + \sum_{n=2}^{\infty} d_n z^n \in \mathcal{C}, |\alpha| \leq \frac{\pi}{2}). \quad (1.15)$$

As a result, if the function f is a function of the form (1.1), then

$$2c_2 = (2e^{-i\alpha} \cos \alpha) \alpha_1 + d_2 \quad (1.16)$$

and

$$3c_3 = 2e^{-i\alpha} (\alpha_2 + \alpha_1^2 + \alpha_1 d_2) \cos \alpha + d_3. \quad (1.17)$$

The η ($0 < \eta < 1$)-parameterized generalized Fekete–Szegő functional of the Taylor coefficients (1.16) and (1.17) is

$$c_3 - \eta c_2^2 = \frac{1}{3} \left(d_3 - \frac{3}{4} \eta d_2^2 \right) + \frac{2}{3} e^{-i\alpha} \left[\alpha_2 + \left(1 - \frac{3}{2} \eta e^{-i\alpha} \cos \alpha \right) \alpha_1^2 \right] \cos \alpha + \frac{2}{3} e^{-i\alpha} \left(1 - \frac{3}{2} \eta \right) \alpha_1 d_2 \cos \alpha \quad (1.18)$$

Theorem 1.2. *Let the function f have the form (1.1) and suppose that $f \in \mathcal{C}$. Then, for any complex or real parameter η , the generalized Fekete–Szegő functional satisfies*

$$\max_{f \in \mathcal{C}} |c_3 - \eta c_2^2| = \max \left\{ \frac{1}{3}, |\eta - 1| \right\}, \quad \text{for } 0 < \eta < 1. \quad (1.19)$$

Theorem 1.3. *Let the function $w(z)$ be a Schwartz function given in Definition (1.1). Then, for any complex or real parameter η , the following inequality holds:*

$$|\alpha_2 - \eta \alpha_1^2| \leq 1 + (|\eta| - 1) |\alpha_1|^2, \quad 0 < \eta < 1. \quad (1.20)$$

Theorem 1.4. *Let the function f have the form (1.1) and $f \in \mathcal{C}_\alpha$ ($|\alpha| \leq \frac{\pi}{2}$). Then,*

$$|c_n| \leq 2 - \frac{1}{n} \quad (n \geq 2, n \in \mathbb{N}). \quad (1.21)$$

The situation in inequality (1.21) is valid only for the function $h(z)$ which has the form

$$h(z, \xi) = \bar{\xi} \log(1 - \xi z) + \frac{2z}{1 - \xi z} \quad (|\xi| = 1). \quad (1.22)$$

In the light of the definitions, theorems, and known results discussed in the previous section, we now aim to derive new coefficient estimates for functions belonging to the class \mathcal{C}_α . In particular, we focus on obtaining sharp bounds for the generalized Fekete–Szegő functional $|c_3 - \eta c_2^2|$, where the parameter η satisfies $0 \leq \eta \leq 1$. The results presented in this section not only contribute to the literature on univalent function theory but also provide generalizations that unify various known inequalities in a broader framework. The main contributions of this paper are presented within the class \mathcal{C}_α of close-to-convex functions with argument α ($|\alpha| \leq \pi/2$), where new sharp bounds for the generalized Fekete–Szegő functional are derived.

2. MAIN RESULTS

Theorem 2.1. *Let the function f have the form (1. 1) and $f \in C_\alpha$, where C_α denotes the class of close-to-convex functions with argument α ($|\alpha| \leq \pi/2$). Then,*

$$|c_3 - \frac{1}{3}c_2^2| \leq 1. \tag{2. 23}$$

Proof. If $\eta = \frac{1}{3}$ in equation (1. 18), and the necessary algebraic operations are applied,

$$c_3 - \frac{1}{3}c_2^2 = \frac{1}{3} \left(d_3 - \frac{1}{4}d_2^2 \right) + \frac{2}{3}e^{-i\alpha} \left[\alpha_2 + \left(1 - \frac{1}{2}e^{-i\alpha} \cos\alpha \right) \alpha_1^2 \right] \cos\alpha + \frac{1}{3}e^{-i\alpha} \alpha_1 d_2 \cos\alpha. \tag{2. 24}$$

At this stage, if the modules of both sides of equation (1. 18) are taken and the algebraic manipulations necessary for the application of Theorem (1.2) and Theorem (1.3) are implemented,

$$\begin{aligned} \left| c_3 - \frac{1}{3}c_2^2 \right| &\leq \frac{1}{3} \left| d_3 - \frac{1}{4}d_2^2 \right| + \frac{2}{3} |e^{-i\alpha}| \left| \alpha_2 + \left(1 - \frac{1}{2}e^{-i\alpha} \cos\alpha \right) \alpha_1^2 \right| \cos\alpha + \\ &\qquad \qquad \qquad \frac{1}{3} |e^{-i\alpha}| |\alpha_1 d_2| \cos\alpha \\ &= \frac{1}{3} \left| d_3 - \frac{1}{4}d_2^2 \right| + \frac{2}{3} \left| \alpha_2 - \left(\frac{1}{2}e^{-i\alpha} \cos\alpha - 1 \right) \alpha_1^2 \right| \cos\alpha + \\ &\qquad \qquad \qquad \frac{1}{3} |\alpha_1 d_2| \cos\alpha \\ &\leq \frac{1}{3} \max \left\{ \frac{1}{3}, \frac{3}{4} \right\} + \frac{2}{3} \left| \alpha_2 - \left(\frac{1}{2}e^{-i\alpha} \cos\alpha - 1 \right) \alpha_1^2 \right| \cos\alpha + \\ &\qquad \qquad \qquad \frac{1}{3} |\alpha_1 d_2| \cos\alpha \\ &\leq \frac{11}{12} + \left\{ \frac{2}{3} \left(\left| \frac{1}{2}e^{-i\alpha} \cos\alpha - 1 \right| - 1 \right) |\alpha_1|^2 + \frac{1}{3} |\alpha_1 d_2| \right\} \cos\alpha \\ &= \frac{11}{12} + \left\{ \frac{2}{3} \left(\sqrt{\frac{\cos^4\alpha - 4\cos^2\alpha + 4}{4} + \frac{\sin^2\alpha \cos^2\alpha}{4}} - 1 \right) |\alpha_1|^2 + \frac{1}{3} |\alpha_1 d_2| \right\} \cos\alpha \\ &= \frac{11}{12} + \left\{ \frac{2}{3} \left(\sqrt{1 - \frac{3}{4}\cos^2\alpha} - 1 \right) |\alpha_1|^2 + \frac{1}{3} |\alpha_1 d_2| \right\} \cos\alpha \\ &\qquad \qquad \qquad \leq \frac{11}{12} + \left\{ \frac{2}{3} \left(-\frac{1}{2} \right) |\alpha_1|^2 + \frac{1}{3} |\alpha_1 d_2| \right\} \cos\alpha \\ &= \frac{11}{12} + \frac{1}{12} |d_2|^2 \cos\alpha - \frac{1}{3} \left(|\alpha_1| - \frac{|d_2|}{2} \right)^2 \cos\alpha \\ &\leq 1. \end{aligned}$$

□

Corollary 2.2. *Let the function f have the form (1. 1) and suppose that $f \in \mathcal{C}_\alpha$. Then, for any complex or real parameter η , the generalized Fekete–Szegő functional satisfies*

$$|c_3 - \eta c_2^2| \leq \frac{31}{27} - \frac{4}{9}\eta. \quad (2. 25)$$

Proof. As a first step, the following mathematical manipulation can be performed to benefit from Theorem (2.1);

$$\begin{aligned} |c_3 - \eta c_2^2| &= \left| c_3 - \frac{1}{3}c_2^2 + \frac{1}{3}c_2^2 - \eta c_2^2 \right| \\ &\leq \left| c_3 - \frac{1}{3}c_2^2 \right| + \left| \frac{1}{3}c_2^2 - \eta c_2^2 \right| \\ &\leq 1 + \left| \frac{1}{3}c_2^2 - \eta c_2^2 \right|. \end{aligned}$$

Now, by applying Theorem (1.4),

$$\begin{aligned} |c_3 - \eta c_2^2| &\leq 1 + \left(\frac{1}{3} - \eta \right) |c_2|^2 \\ &\leq 1 + \left(\frac{1}{3} - \eta \right) \left(\frac{2}{3} \right)^2 \\ &= \frac{31}{27} - \frac{4}{9}\eta \left(\eta \leq \frac{1}{3} \right). \end{aligned}$$

□

Theorem 2.3. *Let the function f have the form (1. 1) and $f \in \mathcal{S}^*$. Then,*

$$|c_3 - \frac{1}{3}c_2^2| \leq \frac{17}{12}. \quad (2. 26)$$

Proof. If $\eta = \frac{1}{3}$ in equation (1. 18), and the necessary algebraic operations are applied,

$$\begin{aligned} c_3 - \frac{1}{3}c_2^2 &= \frac{1}{3} \left(d_3 - \frac{1}{4}d_2^2 \right) + \frac{2}{3}e^{-i\alpha} \left[\alpha_2 + \left(1 - \frac{1}{2}e^{-i\alpha} \cos\alpha \right) \alpha_1^2 \right] \cos\alpha + \\ &\quad \frac{1}{3}e^{-i\alpha} \alpha_1 d_2 \cos\alpha. \end{aligned} \quad (2. 27)$$

At this stage, if the modules of both sides of equation (1. 18) are taken and the algebraic manipulations necessary for the application of Theorem (1.3), Theorem (1.4) and equation

(1. 11) are implemented,

$$\begin{aligned}
 \left| c_3 - \frac{1}{3}c_2^2 \right| &\leq \frac{1}{3} \left| d_3 - \frac{1}{4}d_2^2 \right| + \frac{2}{3} |e^{-i\alpha}| \left| \alpha_2 + \left(1 - \frac{1}{2}e^{-i\alpha} \cos\alpha \right) \alpha_1^2 \right| \cos\alpha + \\
 &\qquad \qquad \qquad \frac{1}{3} |e^{-i\alpha}| |\alpha_1 d_2| \cos\alpha \\
 &= \frac{1}{3} \left| d_3 - \frac{1}{4}d_2^2 \right| + \frac{2}{3} \left| \alpha_2 - \left(\frac{1}{2}e^{-i\alpha} \cos\alpha - 1 \right) \alpha_1^2 \right| \cos\alpha + \\
 &\qquad \qquad \qquad \frac{1}{3} |e^{-i\alpha}| |\alpha_1 d_2| \cos\alpha \\
 &= \frac{4}{3} + \left\{ \frac{2}{3} \left(\sqrt{\frac{4-3\cos^2\alpha}{4}} - 1 \right) |\alpha_1|^2 + \frac{1}{3} |\alpha_1 d_2| \right\} \cos\alpha \\
 &\qquad \qquad \qquad \leq \frac{4}{3} + \left\{ \frac{2}{3} \left(-\frac{1}{2} \right) |\alpha_1|^2 + \frac{1}{3} |\alpha_1 d_2| \right\} \cos\alpha \\
 &\qquad \qquad \qquad \leq \frac{4}{3} - \frac{1}{3} |\alpha_1|^2 \cos\alpha + \frac{1}{3} |\alpha_1 d_2| \cos\alpha \\
 &= \frac{4}{3} - \frac{1}{3} |\alpha_1|^2 \cos\alpha + \frac{1}{3} |\alpha_1 d_2| \cos\alpha + \left(-\frac{|d_2|^2}{12} \right) \cos\alpha - \left(-\frac{|d_2|^2}{12} \right) \cos\alpha \\
 &\qquad \qquad \qquad \leq \frac{17}{12}.
 \end{aligned}$$

□

Theorem 2.4. Let the function f have the form (1. 1) and $f \in C_\alpha$, where C_α denotes the class of close-to-convex functions with argument α ($|\alpha| \leq \pi/2$). Then,

$$\left| c_3 - \frac{1}{3}c_2^2 \right| \leq \frac{1}{3} + \frac{4}{9\eta} \left(\frac{1}{3} < \eta < \frac{2}{3} \right). \tag{ 2. 28 }$$

Proof. If the necessary algebraic adjustments are made in equation (1. 18),

$$\begin{aligned}
 \left| c_3 - \eta c_2^2 \right| &\leq \frac{1}{3} \left| d_3 - \frac{3}{4}\eta d_2^2 \right| + \frac{2}{3} \left| \alpha_2 - \left(\frac{3}{2}\eta e^{-i\alpha} \cos\alpha - 1 \right) \alpha_1^2 \right| \cos\alpha + \\
 &\qquad \qquad \qquad \frac{2}{3} \left(1 - \frac{3}{2}\eta \right) |\alpha_1 d_2| \cos\alpha
 \end{aligned}$$

is obtained. At this stage, if $|d_2| \leq 1$ is applied along with Theorem (1.3), Theorem (1.4),

$$\begin{aligned}
 \left| c_3 - \eta c_2^2 \right| &\leq \frac{1}{3} \max \left\{ \frac{1}{3} \left| \frac{3}{4}\eta - 1 \right| \right\} + \frac{2}{3} \left\{ 1 + \left(\left| \frac{3}{2}e^{-i\alpha} \cos\alpha - 1 \right| - 1 \right) |\alpha_1|^2 \right\} \cos\alpha + \\
 &\qquad \qquad \qquad \frac{2}{3} \left(1 - \frac{3}{2}\eta \right) |\alpha_1 d_2| \cos\alpha \\
 &= \frac{1}{3} \left(\frac{4-3\eta}{4} \right) + \left\{ \frac{2}{3} + \frac{2}{3} |\alpha_1|^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right) \cos^2\alpha} - 1 \right) + \right. \\
 &\qquad \qquad \qquad \left. \frac{2}{3} \left(1 - \frac{3}{2}\eta \right) |\alpha_1| |d_2| \right\} \cos\alpha.
 \end{aligned}$$

Now, for ease of operation, if $\cos \alpha = y$ and $\alpha_1 = \theta$ is taken and $|d_2| \leq 1$ is also used,

$$\left| c_3 - \eta c_2^2 \right| \leq \frac{4-3\eta}{12} + y \left\{ \frac{2}{3} + \frac{2}{3} \theta^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right) y^2 - 1} \right) + \left(\frac{2-3\eta}{3} \right) \theta \right\} \quad (2.29)$$

is obtained. Let the right side of inequality (2.29) be represented by $F_\eta(\eta, y)$, that is,

$$F_\eta(\eta, y) = \frac{4-3\eta}{12} + y \left\{ \frac{2}{3} + \frac{2}{3} \theta^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right) y^2 - 1} \right) + \left(\frac{2-3\eta}{3} \right) \theta \right\} \quad (2.30)$$

Now, using basic knowledge of calculus, we must show that $F_\eta(\eta, y)$ for $(p, y) \in (0, 1) \times (0, 1]$ obtains its maximum value at the interior point $\left(\frac{2-3\eta}{6\eta}, 1 \right)$. Thus,

$$\begin{aligned} F_\eta\left(\frac{2-3\eta}{6\eta}, 1\right) &= \frac{4-3\eta}{12} + y \left\{ \frac{2}{3} + \frac{2}{3} \theta^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right) y^2 - 1} \right) + \left(\frac{2-3\eta}{3} \right) \theta \right\} \\ &= \frac{4-3\eta}{12} + \left\{ \frac{2}{3} + \frac{2}{3} \left(\frac{2-3\eta}{6\eta} \right)^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right) - 1} \right) + \frac{2-3\eta}{3} \cdot \frac{2-3\eta}{6\eta} \right\} \\ &= \frac{4-3\eta}{12} + \left\{ \frac{2}{3} + \frac{2(2-3\eta)^2}{3 \cdot 36\eta^2} \left(-\frac{3}{2}\eta \right) + \frac{(2-3\eta)^2}{18\eta} \right\} \\ &= \frac{4-3\eta}{12} + \frac{2}{3} + \frac{(2-3\eta)^2}{36\eta} \\ &= \frac{1+6\eta}{9\eta}. \end{aligned}$$

At this stage it must first be shown that $F_\eta(\eta, y)$ does not have a local maximum at any interior point of the open rectangle $(0, 1) \times (0, 1)$. The way to do this is to accept that it has a local maximum and reach a contradiction from there. In this sense, let's assume that the $F_\eta(\eta, y)$ function has a local maximum at an interior point $(\alpha_0, y_0) \in (0, 1) \times (0, 1)$.

Thus, from basic calculus knowledge, the partial derivatives of $\frac{\partial F_\eta(\eta, y)}{\partial \alpha}$ and $\frac{\partial F_\eta(\eta, y)}{\partial y}$ should be vanish at the point $(\alpha_0, y_0) \in (0, 1) \times (0, 1)$, that is,

$$\frac{\partial F_\eta(\alpha, y)}{\partial \alpha}(\alpha_0, y_0) = 0 \quad \text{and} \quad \frac{\partial F_\eta(\alpha, y)}{\partial y}(\alpha_0, y_0) = 0.$$

Thus, from the first condition,

$$\begin{aligned} F_\eta(\alpha, y) &= \frac{4-3\eta}{12} + y \left\{ \frac{2}{3} + \frac{2}{3} \alpha^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right) y^2 - 1} \right) + \left(\frac{2-3\eta}{3} \right) \alpha \right\} \\ \frac{\partial F_\eta(\alpha, y)}{\partial \alpha} &= y \left\{ \frac{4\alpha}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right) y^2 - 1} \right) + \left(\frac{2-3\eta}{3} \right) \right\} \end{aligned}$$

is obtained. Thus, it is obtained that

$$\begin{aligned} \left. \frac{\partial F_\eta(\alpha, y)}{\partial \alpha} \right|_{(\alpha_0, y_0)} &= y_0 \left\{ \frac{4\alpha_0}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2 - 1} \right) + \left(\frac{2 - 3\eta}{3} \right) \right\} = 0 \\ &\frac{4\alpha_0}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2 - 1} \right) = - \left(\frac{2 - 3\eta}{3} \right) \\ 4\alpha_0 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2 - 1} \right) &= -(2 - 3\eta) = -\frac{4}{3} \left(\frac{3}{4}(2 - 3\eta) \right), \gamma = \frac{3}{4}(2 - 3\eta) \\ \alpha_0 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2 - 1} \right) &= -\frac{1}{3}\gamma (0 < \gamma < 1) \\ \sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2} &= 1 - \frac{\gamma}{3\alpha_0} \\ y_0^2 \left(3\eta - \frac{9\eta^2}{4} \right) &= \frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0^2}. \end{aligned} \tag{2.31}$$

Similarly, it is obtained that

$$\begin{aligned} \frac{\partial F_\eta(\alpha, y)}{\partial y} &= \left[\frac{2}{3} + \frac{2\alpha^2}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y^2 - 1} \right) + \left(\frac{2 - 3\eta}{3} \right) \alpha \right] \\ &\quad + y \left[\frac{2\alpha^2}{3} \cdot \frac{-2y \left(3\eta - \frac{9\eta^2}{4} \right)}{2\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y^2}} \right] \\ &= \left[\frac{2}{3} + \frac{2\alpha^2}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y^2 - 1} \right) + \left(\frac{2 - 3\eta}{3} \right) \alpha \right] \\ &\quad + y \left[\frac{2\alpha^2}{3} \cdot \frac{-y \left(3\eta - \frac{9\eta^2}{4} \right)}{\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y^2}} \right] \end{aligned}$$

and

$$\begin{aligned} \left. \frac{\partial F_\eta(\alpha, y)}{\partial y} \right|_{\alpha_0, y_0} &= \left[\frac{2}{3} + \frac{2\alpha_0^2}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2 - 1} \right) + \left(\frac{2 - 3\eta}{3} \right) \alpha_0 \right] \\ &\quad + y_0 \left[\frac{2\alpha_0^2}{3} \cdot \frac{-2y_0 \left(3\eta - \frac{9\eta^2}{4} \right)}{2\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2}} \right] = 0 \end{aligned}$$

$$\begin{aligned} \frac{2}{3} + \frac{2\alpha_0^2}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2} - 1 \right) + \left(\frac{2-3\eta}{3} \right) \alpha_0 \\ = \frac{2\alpha_0^2}{3} \cdot \frac{y_0^2 \left(3\eta - \frac{9}{4}\eta^2\right)}{\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2}}. \end{aligned} \quad (2.32)$$

If the $\left(3\eta - \frac{9\eta^2}{4}\right) y_0^2$ value in equation (2.31) is substituted in (2.32) and the necessary algebraic operations are applied, the equation

$$\begin{aligned} \frac{2}{3} + \frac{2\alpha_0^2}{3} \left(\sqrt{1 - \left(\frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0}\right)} - 1 \right) + \left(\frac{2-3\eta}{3} \right) \alpha_0 &= \frac{2\alpha_0^2}{3} \cdot \frac{\frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0}}{\sqrt{1 - \left(\frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0}\right)}} \\ \frac{2}{3} + \frac{2\alpha_0^2}{3} \left(\sqrt{\left(1 - \frac{\gamma}{3\alpha_0}\right)^2} - 1 \right) + \left(\frac{2-3\eta}{3} \right) \alpha_0 &= \frac{2\alpha_0^2}{3} \cdot \frac{\frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0}}{\sqrt{1 - \left(\frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0}\right)}} \\ \frac{2}{3} + \frac{2\alpha_0^2}{3} \left(1 - \frac{\gamma}{3\alpha_0} - 1 \right) + \left(\frac{2-3\eta}{3} \right) \alpha_0 &= \frac{2\alpha_0^2}{3} \cdot \frac{\frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0}}{1 - \frac{\gamma}{3\alpha_0}} \left(\frac{4\gamma}{9} = \frac{2-3\eta}{3} \right) \\ \frac{2}{3} + \frac{2\alpha_0\gamma}{9} - \frac{2}{3} \cdot \frac{6\gamma\alpha_0^2 - \gamma^2\alpha_0}{9\alpha_0 - 3\gamma} &= 0 \\ \frac{-6\gamma\alpha_0^2 + 18\alpha_0 - 6\gamma}{27\alpha_0 - 9\gamma} &= 0 \\ \gamma\alpha_0^2 - 3\alpha_0 + \gamma &= 0 \end{aligned} \quad (2.33)$$

is obtained. If the resulting quadratic equation (2.33) is solved according to α_0 ,

$$\alpha_0 = \frac{3 - \sqrt{9 - 4\gamma^2}}{2\gamma} \quad (2.34)$$

is found. Accordingly, if the value of $F_\eta(\alpha_0, y_0)$ is written in terms of γ and the necessary simplifications are made,

$$\begin{aligned} F_\eta(\alpha_0, y_0) &= \frac{4-3\eta}{12} + y_0 \left\{ \frac{2}{3} + \frac{2}{3}\alpha_0^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right) y_0^2} - 1 \right) + \left(\frac{2-3\eta}{3} \right) \alpha_0 \right\} \\ &= \frac{4-3\eta}{12} + y_0 \left\{ \frac{2}{3} + \frac{2}{3}\alpha_0^2 \left(\sqrt{\left(1 - \frac{\gamma}{3\alpha_0}\right)^2} - 1 \right) + \left(\frac{2-3\eta}{3} \right) \alpha_0 \right\} \\ &= \frac{4-3\eta}{12} + y_0 \left\{ \frac{2}{3} + \frac{2}{3}\alpha_0^2 \left(1 - \frac{\gamma}{3\alpha_0} - 1 \right) + \left(\frac{2-3\eta}{3} \right) \alpha_0 \right\} \\ &= \frac{4-3\eta}{12} + y_0 \left\{ \frac{2}{3} - \frac{2\gamma\alpha_0}{9} + \frac{6\alpha_0 - 9\eta\alpha_0}{den} \right\} \left(\gamma = \frac{3}{4}(2-3\eta) \Rightarrow \eta = \frac{6-4\gamma}{9} \right) \end{aligned}$$

$$F_{\eta}(\alpha_0, y_0) = \frac{6 + 4\gamma}{36} + y_0 \left\{ \frac{2}{3} + \frac{2\alpha_0\gamma}{9} \right\}. \quad (2.35)$$

Now, if the $2\alpha_0\gamma = 3 - \sqrt{9 - 4\gamma^2}$ value obtained from equation (2.34) is substituted in equation (2.35),

$$\begin{aligned} F_{\eta}(\alpha_0, y_0) &= \frac{6 + 4\gamma}{36} + y_0 \left\{ \frac{2}{3} + \frac{3 - \sqrt{9 - 4\gamma^2}}{3} \right\} \\ &= \frac{3 + 2\gamma}{18} + y_0 \left\{ 1 - \frac{1}{3} \sqrt{1 - \frac{4}{9}\gamma^2} \right\} \\ &\leq \frac{3 + 2\gamma}{18} + \left\{ 1 - \frac{1}{3} \sqrt{1 - \frac{4}{9}\gamma^2} \right\} (y_0 \in (0, 1)) \\ &= \frac{3 + 2\gamma}{18} + 1 - \frac{1}{3} \sqrt{1 - \frac{4}{9}\gamma^2} \\ &= \frac{21 + 2\gamma - 6\sqrt{1 - \frac{4}{9}\gamma^2}}{18}. \end{aligned}$$

Since $y \in (0, 1)$, there is $\eta > 0$ such that $\gamma = \frac{3}{2} \cos \delta$ and $\sqrt{1 - \frac{4}{9}\gamma^2} = \sin \delta$ ($0 < \eta < \delta < \frac{\pi}{2}$), and further

$$\begin{aligned} 2\cos\delta + \sin\delta &= \frac{4\gamma}{3} + \sqrt{1 - \frac{4}{9}\gamma^2} \\ &\geq \sqrt{1 - \frac{4}{9}\gamma^2} \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} 1 - \sin\delta - \cos\delta &< \cos\delta \\ 1 - \sin\delta - \cos\delta + \sin\delta\cos\delta &< \frac{\cos^2\delta}{2} \\ (1 - \cos\delta)(1 - \sin\delta) &< \frac{\cos^2\delta}{2} \end{aligned} \quad (2.36)$$

and so that

$$\begin{aligned} (1 - \cos\delta) \left(21 + 2\gamma - 6\sqrt{1 - \frac{4}{9}\gamma^2} \right) &= (1 - \cos\delta) \left(21 + 2 \cdot \frac{3}{2} \cos\delta - 6\sin\delta \right) \\ &= (1 - \cos\delta) ((15 + 3\cos\delta) + 6(1 - \sin\delta)) \\ &= (1 - \cos\delta)(15 + 3\cos\delta) + 6(1 - \cos\delta)(1 - \sin\delta) \\ &\leq (1 - \cos\delta)(15 + 3\cos\delta) + 6 \frac{\cos^2\delta}{2} \\ &= 3 + 12(1 - \cos\delta). \end{aligned}$$

Thus,

$$\begin{aligned} (1 - \cos\delta) \left(21 + 2\gamma - 6\sqrt{1 - \frac{4}{9}\gamma^2} \right) &< 3 + 12(1 - \cos\delta) \\ \frac{21 + 2\gamma - 6\sqrt{1 - \frac{4}{9}\gamma^2}}{18} &< \frac{3 + 12(1 - \cos\delta)}{18(1 - \cos\delta)} \\ \frac{21 + 2\gamma - 6\sqrt{1 - \frac{4}{9}\gamma^2}}{18} &< \frac{2}{3} + \frac{1}{6(1 - \cos\delta)}. \end{aligned}$$

This shows that

$$F_\eta(\alpha_0, y_0) < \frac{2}{3} + \frac{1}{9\eta}. \quad (2.37)$$

Therefore, from basic calculus knowledge, it is clear that $F_\eta(\alpha, y)$ cannot attain a local maximum within the $(0, 1) \times (0, 1)$ rectangle, so it can only attain a maximum at a boundary point. On the other hand, since

$$F_\eta(\alpha, 0) = \frac{4 - 3\eta}{12} < \frac{2}{3} + \frac{1}{9\eta}$$

it cannot reach a maximum on the $y = 0$ line. Similarly, it becomes

$$\begin{aligned} F_\eta(0, y) &= \frac{4 - 3\eta}{12} + y \left\{ \frac{2}{3} \right\} \\ &\leq \frac{4 - 3\eta}{12} + \frac{2}{3} \\ &< \frac{2}{3} + \frac{1}{9\eta} \end{aligned}$$

Then there is no maximum on the line $\alpha = 0$. A similar analysis should be made for the line $y = 1$. That is,

$$\begin{aligned} F_\eta(\alpha, 1) &= \frac{4 - 3\eta}{12} + 1 \cdot \left\{ \frac{2}{3} + \frac{2}{3}\alpha^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2 \right)^2 - 1} \right) + \left(\frac{2 - 3\eta}{3} \right) \alpha \right\} \\ &= \frac{4 - 3\eta}{12} + \left\{ \frac{2}{3} + \frac{2}{3}\alpha^2 \left(\sqrt{\left(1 - \frac{3}{2}\eta \right)^2 - 1} \right) + \left(\frac{2 - 3\eta}{3} \right) \alpha \right\} \\ &= 1 - \frac{1}{4}\eta - \alpha^2\eta + \left(\frac{2}{3} - \eta \right) \alpha. \end{aligned}$$

If $\alpha = 1$ at this stage,

$$\begin{aligned} F_\eta(1, 1) &= 1 - \frac{1}{4}\eta - \eta + \left(\frac{2}{3} - \eta \right) \alpha \\ &= \frac{5}{3} - \left(\frac{1}{4}\eta + 2\eta \right) \\ &= \frac{5}{3} - \frac{9\eta}{4}. \end{aligned}$$

Therefore, $F_\eta(\alpha, 1)$ has its local maximum at $\left. \frac{\partial F_\eta(\alpha, 1)}{\partial \alpha} \right|_{\alpha=\alpha_0} = 0$. That is,

$$\begin{aligned} \frac{\partial F_\eta(\alpha, 1)}{\partial \alpha} &= -2\eta\alpha + \frac{2}{3} - \eta \\ \left. \frac{\partial F_\eta(\alpha, 1)}{\partial \alpha} \right|_{\alpha=\alpha_0} &= \frac{2}{3} - \eta(1 - 2\alpha_0) = 0 \\ \alpha_0 &= \frac{2 - 3\eta}{6\eta}. \end{aligned}$$

As the last step, if it can be shown that $F_\eta(\alpha, 1) < 0 (y \in (0, 1))$, the proof is completed.

$$\begin{aligned} F_\eta(\alpha, y) &= \frac{4 - 3\eta}{12} + y \left\{ \frac{2}{3} + \frac{2}{3}\alpha^2 \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right)y^2 - 1} \right) + \left(\frac{2 - 3\eta}{3}\right)\alpha \right\} \\ F_\eta(1, y) &= \frac{4 - 3\eta}{12} + y \left\{ \frac{2}{3} + \frac{2}{3} \left(\sqrt{1 - \left(3\eta - \frac{9}{4}\eta^2\right)y^2 - 1} \right) + \left(\frac{2 - 3\eta}{3}\right) \right\} \\ &= \frac{4 - 3\eta}{12} + y \left\{ \frac{2}{3} + \frac{2}{3} \left(\sqrt{1 - \left(\frac{2\gamma}{3\alpha} - \frac{\gamma^2}{9\alpha^2}\right) - 1} \right) + \frac{6 - 9\eta}{9} \right\} (4\gamma = 6 - 9\eta, \alpha = 1) \\ &= \frac{4 - 3\eta}{12} + y \left\{ \frac{2}{3} - \frac{2\gamma}{9} + \frac{4\gamma}{3} \right\} \left(\eta = \frac{6 - 4\gamma}{9} \right) \\ &= \frac{3 + 2\gamma}{18} + \frac{2}{3}y \left\{ 1 - \frac{\gamma}{3} + \frac{2\gamma}{3} \right\} \\ &= \frac{1}{3} \left[\frac{3 + 2\gamma}{6} + 2y \left\{ \left(1 - \frac{\gamma}{3}\right) + \frac{2\gamma}{3} \right\} \right] \left(\sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} = 1 - \frac{\gamma}{3\alpha}, \alpha = 1 \right) \\ &= \frac{1}{3} \left[\frac{3 + 2\gamma}{6} + 2y \left(\sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{2\gamma}{3} \right) \right]. \end{aligned}$$

Thus,

$$F_\eta(1, y) = \frac{1}{3} \left[\frac{3 + 2\gamma}{6} + 2y \left(\sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{2\gamma}{3} \right) \right] \quad (2.38)$$

obtained. If the expression $2y \left(\sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{2\gamma}{3} \right)$ in Equation (2.38) is represented by $G_\gamma(y)$, it takes the form

$$F_\eta(1, y) = \frac{1}{3} \left[\frac{3 + 2\gamma}{6} + G_\gamma(y) \right]. \quad (2.39)$$

In this case, the claim of (2.37) is equivalent to

$$G_\gamma(y) \leq \frac{2}{3} \frac{(3 - \gamma)^2}{(3 - 2\gamma)} (y \in (0, 1)). \quad (2.40)$$

Therefore, in order to prove the inequality (2. 40), it is sufficient to show that

$$\left. \frac{dG_\gamma(y)}{dy} \right|_{y \in (0,1)} = 0$$

, which indicates the situation given below.

$$\begin{aligned} G_\gamma(y) &= 2y \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2} + y \frac{4\gamma}{3} \\ \left. \frac{dG_\gamma(y)}{dy} \right|_{y \in (0,1)} &= 2 \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2} + 2y \frac{-2y \left(1 - \frac{4}{9}\gamma^2\right)}{2 \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2}} + \frac{4\gamma}{3} = 0 \\ \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2} + \frac{2\gamma}{3} &= \frac{y^2 \left(1 - \frac{4}{9}\gamma^2\right)}{\sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2}} \\ 1 - y^2 \left(1 - \frac{4}{9}\gamma^2\right) + \frac{2\gamma}{3} \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2} &= y^2 \left(1 - \frac{4}{9}\gamma^2\right) \\ \frac{2\gamma}{3} \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2} &= 2y^2 \left(1 - \frac{4}{9}\gamma^2\right) - 1 \end{aligned} \quad (2. 41)$$

Now, if both sides of equation (2. 41) are squared,

$$\begin{aligned} \left(\frac{2\gamma}{3} \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right) y^2} \right)^2 &= \left(2y^2 \left(1 - \frac{4}{9}\gamma^2\right) - 1 \right)^2 \\ 4y^4 \left(1 - \frac{4}{9}\gamma^2\right)^2 - 4y^2 \left(1 - \frac{4}{9}\gamma^2\right) \left(1 - \frac{\gamma^2}{9}\right) + \left(1 - \frac{4}{9}\gamma^2\right) &= 0 \end{aligned}$$

$$\left\{ 2y^2 \left(1 - \frac{4}{9}\gamma^2\right) \right\}^2 - 2 \left\{ 2y^2 \left(1 - \frac{4}{9}\gamma^2\right) \right\} \left(1 - \frac{\gamma^2}{9}\right) + \left(1 - \frac{4}{9}\gamma^2\right) = 0 \quad (2. 42)$$

is obtained. If the roots of the second degree equation (2. 42) are found for $2y^2 \left(1 - \frac{4}{9}\gamma^2\right)$ ($y \in (0, 1)$),

$$2y^2 \left(1 - \frac{4}{9}\gamma^2\right) = \frac{9 - \gamma^2 - \gamma \sqrt{\gamma^2 + 18}}{9} \quad (2. 43)$$

is obtained. On the other hand, if both sides of inequality (2. 40) are squared and equality (2. 43) is used and the necessary algebraic operations are performed, the equivalent

equation

$$\begin{aligned}
 G_\gamma(y) \leq \frac{2(3-\gamma)^2}{3(3-2\gamma)} (y \in (0,1)) &\Rightarrow G_\gamma^2(y) \leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 \left(2y \left(\sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{2\gamma}{3} \right) \right)^2 &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ 2 - 2y^2 \left(1 - \frac{4}{9}\gamma^2\right) + \frac{8\gamma}{3} \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ 1 - \left(2y^2 \left(1 - \frac{4}{9}\gamma^2\right) - 1\right) + \frac{8\gamma}{3} \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ 1 - \frac{2\gamma}{3} \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{8\gamma}{3} \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ 1 + 2\gamma \sqrt{1 - \left(1 - \frac{4}{9}\gamma^2\right)y^2} + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ 1 + 3 \left(2y^2 \left(1 - \frac{4}{9}\gamma^2\right) - 1\right) + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ 1 + 3 \left(2y^2 \left(1 - \frac{4}{9}\gamma^2\right)\right) - 3 + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ \frac{-18 + 27 - 3\gamma^2 - 3\gamma\sqrt{18 + \gamma^2}}{9} + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 2y^2 \left\{ \frac{9 + 5\gamma^2 - 3\gamma\sqrt{18 + \gamma^2}}{9} + \frac{8\gamma^2}{9} \right\} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 \frac{9 - \gamma^2 - 3\gamma\sqrt{18 + \gamma^2}}{(3 - 2\gamma)(3 + 2\gamma)2} \cdot \frac{9 + 5\gamma^2 - 3\gamma\sqrt{18 + \gamma^2}}{9} + \frac{8\gamma^2}{9} &\leq \frac{4(3-\gamma)^4}{9(3-2\gamma)^2} \\
 (3 - 2\gamma)(9 - \gamma^2 - \gamma\sqrt{18 + \gamma^2})(9 + 5\gamma^2 - 3\gamma\sqrt{18 + \gamma^2}) &\leq 4(3 + 2\gamma)(3 - \gamma)^4 \tag{2.44}
 \end{aligned}$$

is obtained. Notice that for every $8(3 + 2\gamma)(3 - \gamma)^4 > 0(\eta \in (0, 1))$. Therefore, the proof of Theorem (2.4) is completed. \square

Theorem 2.5. Let the function f have the form (1.1) and $f \in C_\alpha$, where C_α denotes the class of close-to-convex functions with argument α ($|\alpha| \leq \pi/2$). Then,

$$|c_3 - \frac{2}{3}c_2^2| \leq \frac{5}{6}. \tag{2.45}$$

Proof. If $\eta = \frac{2}{3}$ in equation (1.18), and the necessary algebraic operations are applied,

$$\begin{aligned} c_3 - \frac{2}{3}c_2^2 &= \frac{1}{3} \left(d_3 - \frac{1}{2}d_2^2 \right) + \frac{2}{3}e^{-i\alpha} [\alpha_2 + (1 - e^{-i\alpha}\cos\alpha)\alpha_1^2] \cos\alpha \\ &= \frac{1}{3} \left(d_3 - \frac{1}{2}d_2^2 \right) + \frac{2}{3}e^{-i\alpha} [\alpha_2 - (e^{-i\alpha}\cos\alpha - 1)\alpha_1^2] \cos\alpha \\ \left| c_3 - \frac{2}{3}c_2^2 \right| &\leq \frac{1}{3} \left| d_3 - \frac{1}{2}d_2^2 \right| + \frac{2}{3} \left| \alpha_2 - (e^{-i\alpha}\cos\alpha - 1)\alpha_1^2 \right| \cos\alpha \\ &\leq \frac{1}{3} \cdot \max \left\{ \frac{1}{3}, \left| \frac{1}{2} - 1 \right| \right\} + \frac{2}{3} \left(1 + \left(\left| e^{-i\alpha}\cos\alpha - 1 \right| - 1 \right) |\alpha_1|^2 \right) \cos\alpha \\ &= \frac{1}{6} + \frac{2}{3} \left(1 + \left(\left| e^{-i\alpha}\cos\alpha - 1 \right| - 1 \right) |\alpha_1|^2 \right) \cos\alpha \\ &\leq \frac{1}{6} + \frac{2}{3} \left(1 + \left(\sqrt{1 - \cos^2\alpha} - 1 \right) |\alpha_1|^2 \right) \cos\alpha \\ &= \frac{1}{6} + \frac{2}{3} - \frac{2}{3} (1 - |\sin\alpha|) |\alpha_1|^2 \cos\alpha \left(|\alpha| < \frac{\pi}{2} \right) \\ \left| c_3 - \frac{2}{3}c_2^2 \right| &\leq \frac{5}{6}. \end{aligned}$$

Therefore, the proof of Theorem (2.5) is completed. \square

3. CONCLUSION

In this paper, we investigated the generalized Fekete–Szegő functional of the form $|c_3 - \eta c_2^2|$ for functions belonging to the close-to-convex class \mathcal{C}_α . Using analytic techniques, new sharp bounds were established for the parameter η in the interval $0 \leq \eta \leq 1$. These results not only generalize several previously known inequalities for specific subclasses such as \mathcal{S}^* and \mathcal{C} but also provide a broader and more unified framework within geometric function theory. The outcomes of this study may serve as a foundation for further research in this area, particularly in exploring other subclasses or refining coefficient bounds under additional constraints.

DECLARATIONS

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