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# Novel Solitary Wave Solutions for Nonlinear Fractional BBM and (2+1)-Dimensional Extended KP Equations via $e^{-\psi(\zeta)}$ Expansion Approach

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**Abstract.** In this article nonlinear space-time fractional Benjamin Bona Mahony (BBM) and (2+1) dimensional extended Kadomtsev Petviashvili (KP) equations are analyzed, which represents many physical phenomena, such as shallow water wave propagation, plasma physics, nonlinear optics, and fluid dynamics. The Generalized closed-form soliton solutions are obtained using the well-known $e^{-\psi(\zeta)}$ expansion approach. Through the suitable traveling wave transformation, the nonlinear fractional partial differential equations are converted into nonlinear ordinary differential equations, which are subsequently reduced to a system of nonlinear algebraic equations. The resulting system of algebraic equations is then solved for the required parameters with the assistance of MATLAB. The closed-form soliton solutions obtained are expressed in terms of transcendental functions, and graphs are drawn to illustrate the physical behavior of the derived solutions, which represent bright, dark, and kink solutions. It is important to mention that the $e^{-\psi(\zeta)}$ expansion approach is an excellent mathematical tool to derive soliton solutions of nonlinear fractional differential equations and does not require any type of linearization, perturbation or unnecessarily assumption, which may change the physical behavior of the problem. Furthermore, the obtained generalized soliton

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solutions are not reported in the literature and will be helpful in understanding the complex dynamics of the equations considered.

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**Keywords:** Space-Time Fractional Partial Differential Equation, Soliton solutions, closed-form wave solution, Riemann-Liouville derivative

## 1. INTRODUCTION

Fractional PDEs are useful across a vast range of scientific as well as engineering fields. Researchers study fractional partial differential equations and related integral equations because they appear again and again in real-world applications, the model of complex phenomena in areas as diverse as processing signals [23], understanding biological systems like the DNA model [2], the gas dynamic model [30], optical fiber[34], fluid dynamics [13], characterizing materials like viscoelastic polymers [28], conservation laws, and lie symmetry analysis of nonlinear PDEs [27], and describing diffusion processes that don't follow the usual rules. This broad relevance has naturally captured the intense interest of scientists and engineers world wide. In the last few years, more works have been done to obtain analytic and numerical solutions of ODE and FPDE [24]. Recently, a lot of high-powered approaches have been given to achieve the exact solution and numerical solution of (FPDE's) [5] like Homotopy-perturbation approach [3], Variational iteration method [21], The standard tanh and extended tanh methods [6],The exp function method [26], Hirota bilinear methods [14], the F-expansion [1] and so on. A wide range of closed-form wave solutions, including as solitons, breathers, lump waves, and rogue waves (RWs), are admitted by nonlinear evolution equations (NLEEs) [16, 29, 17, 15]. Localized waves that fluctuate on a regular basis are called breather solutions. Line breathers and the Akhmediev [33] breather are typical examples. Ma's [18] algebraic method for integrable systems advanced the study of lump waves, which Petviashvili had initially found numerically. Localized in all spatial directions, lump solutions are special rational function solutions of NLEEs. Peregrine [25] was the first to mathematically study rogue waves using the Schrödinger equation. In addition to being localized in space and time, these waves are rational solutions of NLEEs. The surrounding background waves height can be tripled by their amplitude. Higher-dimensional models are typically home to breather, lump, and rogue wave solutions, which help to create nonlinear dynamics in more intricate systems. In this article, new and more general closed-form soliton solutions are determined for two nonlinear space-time fractional partial differential equations, namely the BBM and (2+1) EKP equations in the sense of accordant fractional derivative. The application of the $e^{-\psi(\zeta)}$ expansion approach to these fractional models, which had not been previously reported in the literature, was what made the current work novel. The first model considered is BBM, a fundamental nonlinear equation that emerges in the study of relativistic wave equations in quantum physics. This equation finds widespread use in both classical and quantum field theory, serving as a crucial tool in various branches of physics [11]. This model is given by

$$D_t^{\beta} w + D_x^{\alpha} w + w D_x^{\alpha} w - D_x^{2\alpha} D_t^{\beta} w = 0; \quad 0 < \alpha, \beta \leq 1,\, t > 0 \tag{1. 1}$$

The second model studied is the $(2+1)$ EKP equation. It is a generalized form of the KP equation [35], which itself is a mathematical equation describing the evolution of weakly nonlinear, dispersive waves in a two-dimensional medium. The extended KP equation has many applications.

$$D_x^{\alpha}(D_t^{\gamma}w+6wD_x^{\alpha}w+D_x^{3\alpha}w)-D_y^{2\beta}w+\lambda D_t^{2\gamma}w+\mu D_t^{\gamma}D_y^{\beta}w=0; \quad 0<\alpha,\beta,\gamma\leq 1,\, t>0 \tag{1. 2}$$

In these models, $\lambda$ and $\mu$ are random parameters, and fractional parameters are represented by $\alpha, \beta$, and $\gamma$.

1.1. **Literature review.** Several scholars have thoroughly examined Eq ( 1. 1 ) through a variety of studies. Numerous soliton formations, such as bright and dark waveforms, were identified by Beenish et al. [10] using the generalized arnous method, for various parameter combinations of Eq ( 1. 1 ) , Kaihua Shao et al [32] predicts multiple typical soliton solutions, including bright and dark solitons for modified version of Eq ( 1. 1 ) using physics-informed neural networks approach. mixed periodic, singular, shock-singular, complex solitary-shock, and planewave solutions are observed in Dean Chou et al [12] for Eq ( 1. 1 ) via the new extended direct algebraic approach. Weaam Alhejaili et al [4] explore a class of lump solutions for Eq ( 1. 2 ) using the simplified Hirota method. Asadullah, M et al [7] explore bright envelope soliton solutions, dark envelope soliton solutions, periodic solutions for Eq ( 1. 2 ) using the simplified Hirota method and Şenol, M et al [31] shows the family of admissible solitary and lump-type wave structures for Eq ( 1. 2 ) with Boussinesq model using the $exp(-\psi(\xi))$-expansion method. According to comparative analysis from the literature [20, 19, 8, 22], the proposed approach is a more comprehensive class of novel solitary wave solutions than the previous research. This work was novel because it was the first time that the fractional-order BBM and (2+1)-dimensional Extended KP equations via $e^{-\psi(\zeta)}$ expansion approach, defined using the accordant fractional derivative, have been studied. As a result, new closed-form soliton solutions are obtained and their profiles are different from literature.

Section 2 provides the definition of the modified Riemann-Liouville derivative operator as well as some of its typical characteristics. Section 3 presents a detailed methodology. Section 4 presents the BBM soliton solutions and EKPE are calculated, after which Sections 5 and 6 offer the numerical results, respectively.

## 2. Modified Riemann-Liouville Derivative and its Properties

In this article the modified Riemann-Liouville fractional operator has been used in Eq.( 1. 1 ) and Eq.( 1. 2 ) and is defined as: [9].

$$D_t^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\eta)^{-\alpha-1}[f(\eta)-f(0)]d\eta, & \alpha<0\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\eta)^{-\alpha}[f(\eta)-f(0)]d\eta, & 0<\alpha<1\\ [f^{(\alpha-m)}(t)]^{(m)},\ m\leq\alpha<m+1,\, m\geq 1\end{cases} \tag{2. 3}$$

$$\Gamma(\alpha)=\lim_{m\to\infty}\frac{m^{\alpha}\,\Gamma(m)}{(\alpha+1)(\alpha+2)\cdots(\alpha+m)}$$

Furthermore, some well-known properties are given as follows:

- $D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(r-\alpha+1)} t^{r-\alpha}$
- $D_t^{\alpha}[f(t)g(t)] = g(t)D_t^{\alpha}f(t) + f(t)D_t^{\alpha}g(t)$
- $D_t^{\alpha} f[g(t)] = \frac{df(g)}{dg} D_t^{\alpha} g(t)$

## 3. Description of $e^{-\psi(\zeta)}$-expansion approach

To understand the $e^{-\psi(\zeta)}$expansion approach, consider the following nonlinear STFPDE using $x$ and $t$ as two independent parameters:

$$P\big(w, w_x, w_t, w_{xx}, \ldots, D_t^{\beta} w, D_x^{\alpha} w, \ldots\big) = 0,\ 0 < \alpha, \beta \leq 1. \tag{3. 4}$$

Here, $D_t^{\beta} w$ and $D_x^{\alpha} w$ denote fractional derivative operators, $w(x,t)$ is an unknown function and $P$ represents a polynomial in $w$ and its various fractional derivatives, including both linear and nonlinear terms.

Step 1. Assume the transformation

$$w(x,t) = w(\zeta), \qquad \zeta = \frac{x^{\alpha}}{\Gamma(1+\alpha)} - c\frac{t^{\beta}}{\Gamma(1+\beta)}, \qquad c \neq 0, \tag{3. 5}$$

where $c$ a wave speed parameter is to be determined later. Using the wave variable $\zeta$, Eq. ( 3. 4 ) can be reduced to a nonlinear ODE of the form

$$P\big(w, w', -cw', w'', \ldots\big) = 0, \tag{3. 6}$$

where the prime denotes differentiation w.r.t $\zeta$.

Step 2. Integrate Eq.( 3. 6 ) if it contains derivatives in each term, and consider the integration constant is zero.

Step 3. Assume that the following is an expression for the nonlinear ODE Eq.( 3. 6 ) solution:

$$w(\zeta) = \sum_{i=0}^{m} c_i \big(e^{-\psi(\zeta)}\big)^i, \qquad c_m \neq 0, \tag{3. 7}$$

where $c_i$ $(i = 0, 1, 2, \ldots, m)$ are unknown coefficient constants. The auxiliary ODE is satisfied by the function.

$$\psi'(\zeta) = e^{-\psi(\zeta)} + \mu e^{\psi(\zeta)} + \lambda, \tag{3. 8}$$

where $\psi' = d\psi/d\zeta$, and $\lambda$ and $\mu$ are free parameters.

The non-negative integer $m$ in Eq.( 3. 7 ) can be found using the greatest order and the homogeneity balance between nonlinear terms present in Eq.( 3. 6 ) and by using the following results:

$$\deg\left(\frac{d^p w}{d\zeta^p}\right) = m + p, \tag{3. 9}$$

and

$$\deg\left(w^s \left(\frac{d^p w}{d\zeta^p}\right)^q\right) = ms + q(m+p). \tag{3. 10}$$

By equating Eq. ( 3. 9 ) to Eq. ( 3. 10 ) will provide us with a non-negative integer $m$. Furthermore, the generalized and possible solutions of Eq. ( 3. 8 ) are presented as follows:

Case 1. For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$

$$\psi(\zeta) = \log\left(\frac{-\sqrt{\lambda^2 - 4\mu}\,\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\zeta + k)\right) - \lambda}{2\mu}\right). \quad (3.\ 11)$$

and

$$\psi(\zeta) = \log\left(\frac{-\sqrt{\lambda^2 - 4\mu}\,\coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\zeta + k)\right) - \lambda}{2\mu}\right). \quad (3.\ 12)$$

Case 2. For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$

$$\psi(\zeta) = \log\left(\frac{\sqrt{4\mu - \lambda^2}\,\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\zeta + k)\right) - \lambda}{2\mu}\right). \quad (3.\ 13)$$

and

$$\psi(\zeta) = \log\left(\frac{\sqrt{4\mu - \lambda^2}\,\cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\zeta + k)\right) - \lambda}{2\mu}\right). \quad (3.\ 14)$$

Case 3. For $\lambda^2 - 4\mu > 0$ and $\mu = 0$, and $\lambda \neq 0$

$$\psi(\zeta) = -\log\left(\frac{\lambda}{e^{\lambda(\zeta+k)} - 1}\right). \quad (3.\ 15)$$

Case 4. For $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, and $\lambda \neq 0$

$$\psi(\zeta) = \log\left(-\frac{2\big(\lambda(\zeta + k) + 2\big)}{\lambda^2(\zeta + k)}\right). \quad (3.\ 16)$$

Case 5. For $\lambda^2 - 4\mu = 0$ and $\mu = 0$, and $\lambda = 0$

$$\psi(\zeta) = \log(\zeta + k). \quad (3.\ 17)$$

Step 4. By plugging Eq.( 3. 7 ) and Eq.( 3. 8 ) into Eq.( 3. 6 ) will produce polynomials in powers of $e^{-\psi(\zeta)}$. By arranging the coefficients of like powers of $e^{-\psi(\zeta)}$ to zero, a nonlinear set of algebraic equations will be obtained in terms of unknown constants $c_i$ $(i = 0, 1, 2, \ldots, m)$ and parameters $\lambda$, $\mu$, $k$, and $c$.
Step 5. The resulting algebraic set obtained in Step 4 will be solved by using MATLAB. The solutions of the nonlinear ODE Eq. ( 3. 8 ) are already known, substituting the obtained values of $c_i$, $k$, and $c$ into the solutions of Eq.( 3. 8 ), and then into Eq. ( 3. 7 ), yields a family of closed soliton solutions of the nonlinear evolution equation ( 3. 8 ).

## 4. Applications of $e^{-\psi(\zeta)}$-expansion approach

The aim of this section is to achieve the closed soliton solutions of nonlinear STFDE's given in Eq.( 1. 1 ) and Eq.( 1. 2 ) with the application of $e^{-\psi(\zeta)}$-expansion approach.

### 4.1. **Space-Time Fractional Benjamin Bona Mahony Equation (STFBBME).**

$$D_t^{\beta} w + D_x^{\alpha} w + w D_x^{\alpha} w - D_x^{2\alpha}\left(D_t^{\beta} w\right) = 0, \qquad 0 < \alpha, \beta \leq 1,\ t > 0. \tag{4. 18}$$

By plugging the traveling wave transformation into Eq.( 3. 5 ) and its required derivatives with the help of the chain rule into Eq.( 4. 18 ) it yields the following nonlinear ordinary differential equation Eq.( 4. 19 ) where $c \neq 0$ is a constant to be found later and the fractional parameters $\alpha, \beta$ lie in $0 < \alpha, \beta \leq 1$. By plugging Eq. ( 3. 5 ) into Eq.( 1. 1 ) it yields

$$2cw'' + 2(1-c)w + w^2 = 0. \tag{4. 19}$$

Considering the solution of nonlinear ODE Eq. ( 4. 19 ) is presented by Eq. ( 3. 7 ).

Furthermore, the non negetive integer $m$ can be obtained using Eq. ( 3. 9 ) and Eq. ( 3. 10 ) and is given as follows:

$$2 + m = 2m \quad \Rightarrow \quad m = 2.$$

By inserting in the value of $m = 2$ Eq.( 3. 7 ), we have the following results: $w(\zeta)$, $w^2(\zeta)$, and , $w''(\zeta)$ respectively, as follows:

$$w(\zeta) = c_0 + c_1 e^{-\psi(\zeta)} + c_2 e^{-2\psi(\zeta)}, \qquad c_2 \neq 0. \tag{4. 20}$$

$$\begin{aligned} w^2 = {} & c_2^2 e^{-4\psi(\zeta)} + 2c_1c_2 e^{-3\psi(\zeta)} + \left(c_1^2 + 2c_0c_2\right) e^{-2\psi(\zeta)} \\ & + 2c_0c_1 e^{-\psi(\zeta)} + c_0^2. \end{aligned} \tag{4. 21}$$

$$\begin{aligned} w'' = {} & 6c_2 e^{-4\psi(\zeta)} + \left(10c_2\lambda + 2c_1\right) e^{-3\psi(\zeta)} \\ & + \left(3c_1\lambda + 8c_2\mu + 4c_2\lambda^2\right) e^{-2\psi(\zeta)} \\ & + \left(6c_2\lambda\mu + 2c_1\mu + c_1\lambda^2\right) e^{-\psi(\zeta)} \\ & + c_1\lambda\mu + 2c_2\mu^2. \end{aligned} \tag{4. 22}$$

By plugging Eq. ( 4. 21 ) and Eq. ( 4. 22 ) into Eq. ( 4. 19 ) and collecting the coefficients of like powers of $e^{-\psi(\zeta)}$, we get

$$\begin{aligned} e^0 : \quad & 2c_0 - 2c_0c + c_0^2 + 2c_1\lambda\mu c + 4c_2\mu^2 c = 0, \\ e^1 : \quad & 2c_1 - 2c_1c + 2c_0c_1 + 12c_2\lambda\mu c + 4c_1\mu c + 2c_1\lambda^2 c = 0, \\ e^2 : \quad & 2c_2 - 2c_2c + 2c_0c_2 + c_1^2 + 16c_2\mu c + 8c_2\lambda^2 c + 6c_1\lambda c = 0, \\ e^3 : \quad & 2c_1c_2 + 4c_1c + 20c_2\lambda c = 0, \\ e^4 : \quad & 12c_2c + c_2^2 = 0. \end{aligned}$$

The nonlinear system of algebraic equations that has been solved for the required parameter by using MATLAB

$$c_0 = \frac{-12\mu}{4\mu - \lambda^2 + 1}, \quad c_1 = \frac{-12\lambda}{4\mu - \lambda^2 + 1}, \quad c_2 = \frac{-12}{4\mu - \lambda^2 + 1}, \quad c = \frac{1}{4\mu - \lambda^2 + 1}. \tag{4. 23}$$

By inserting Eq. ( 4. 23 ) into Eq. ( 4. 20 ) it yields:

$$w(\zeta) = \frac{-12\mu}{4\mu - \lambda^2 + 1} - \frac{12\lambda}{4\mu - \lambda^2 + 1} e^{-\psi(\zeta)} - \frac{12}{4\mu - \lambda^2 + 1} e^{-2\psi(\zeta)}. \tag{4. 24}$$

where,

$$\zeta = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{1}{4\mu - \lambda^2 + 1} \frac{t^\beta}{\Gamma(1+\beta)}.$$

By inserting the solutions of Eq. ( 3. 8 ) into Eq. ( 4. 24 ), we obtain different types of soliton solutions of Eq. ( 1. 1 ).
Case 1. For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$

$$w_1(\zeta) = \frac{12}{\lambda^2 - 4\mu - 1} \left[ \mu - \frac{2\mu\lambda}{\sqrt{\lambda^2 - 4\mu} \tanh\left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} (\zeta + k) \right) + \lambda} + \frac{4\mu^2}{\left( \sqrt{\lambda^2 - 4\mu} \tanh\left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} (\zeta + k) \right) + \lambda \right)^2} \right]. \tag{4. 25}$$

and

$$w_2(\zeta) = \frac{12}{\lambda^2 - 4\mu - 1} \left[ \mu - \frac{2\mu\lambda}{\sqrt{\lambda^2 - 4\mu} \coth\left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} (\zeta + k) \right) + \lambda} + \frac{4\mu^2}{\left( \sqrt{\lambda^2 - 4\mu} \tanh\left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} (\zeta + k) \right) + \lambda \right)^2} \right]. \tag{4. 26}$$

Case 2. For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$

$$w_3(\zeta) = \frac{12}{\lambda^2 - 4\mu - 1} \left[ \mu + \frac{2\mu\lambda}{\sqrt{4\mu - \lambda^2} \tan\left( \frac{\sqrt{4\mu - \lambda^2}}{2} (\zeta + k) \right) - \lambda} + \frac{4\mu^2}{\left( \sqrt{4\mu - \lambda^2} \tan\left( \frac{\sqrt{4\mu - \lambda^2}}{2} (\zeta + k) \right) - \lambda \right)^2} \right]. \tag{4. 27}$$

$$w_4(\zeta) = \frac{12}{\lambda^2 - 4\mu - 1}\left[\mu + \frac{2\mu\lambda}{\sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\zeta+k)\right)-\lambda} + \frac{4\mu^2}{\left(\sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\zeta+k)\right)-\lambda\right)^2}\right]. \tag{4. 28}$$

Case 3. For $\lambda^2 - 4\mu > 0$, $\mu = 0$ and $\lambda \neq 0$

$$w_5(\zeta) = \frac{12\lambda^2 e^{\lambda(\zeta+k)}}{(\lambda^2-1)\left(e^{\lambda(\zeta+k)}-1\right)^2}, \tag{4. 29}$$

where

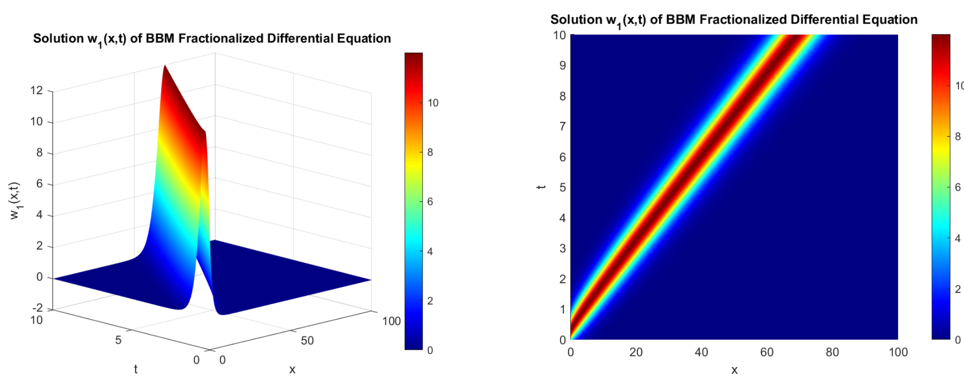
$$\zeta = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{1}{\lambda^2-1}\frac{t^\beta}{\Gamma(1+\beta)}.$$

Case 4. For $\lambda^2 - 4\mu = 0$, $\mu \neq 0$ and $\lambda \neq 0$

$$w_6(\zeta) = \frac{12\lambda^2}{(\lambda^2-4\mu-1)\left(\lambda(\zeta+k)+2\right)^2}. \tag{4. 30}$$

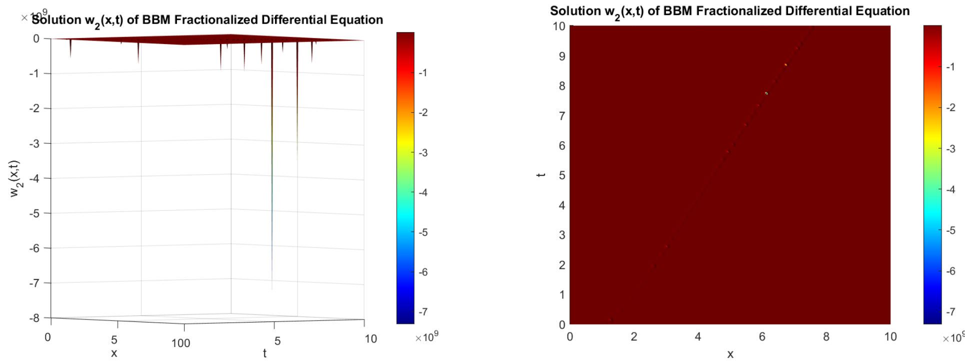
Case 5. For $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda = 0$

$$w_7(\zeta) = \frac{-12}{\zeta+k}. \tag{4. 31}$$



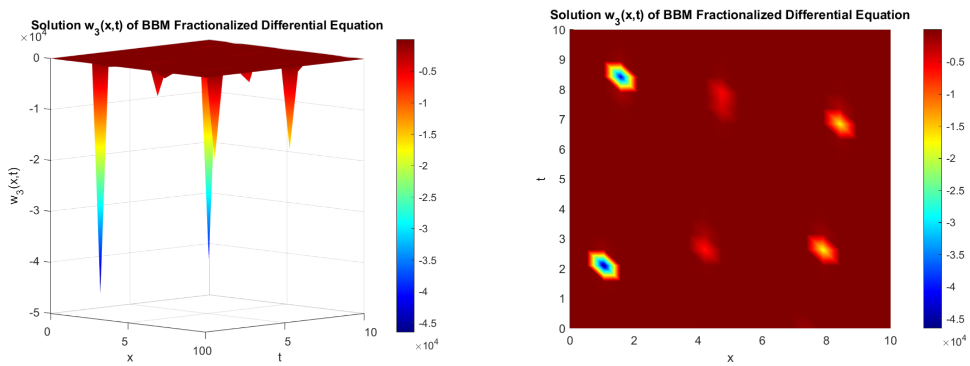

(A) (B)

FIGURE 1. Wave Solution of BBM with $\mu = 0.8$, $\lambda = 2$, $k = 1$, $\alpha = 0.8$, $\beta = 0.8$



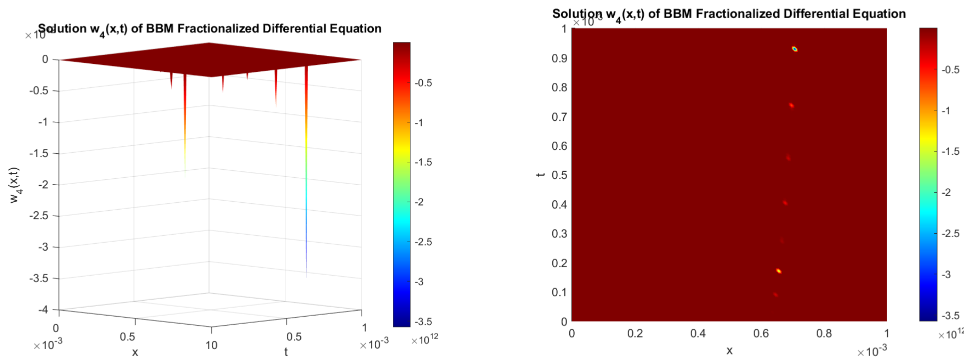

(A) (B)

FIGURE 2. Wave Solution of BBM with $\mu = 2.9, \lambda = 2.5, k = 1, \alpha = 0.2, \beta = 0.5$



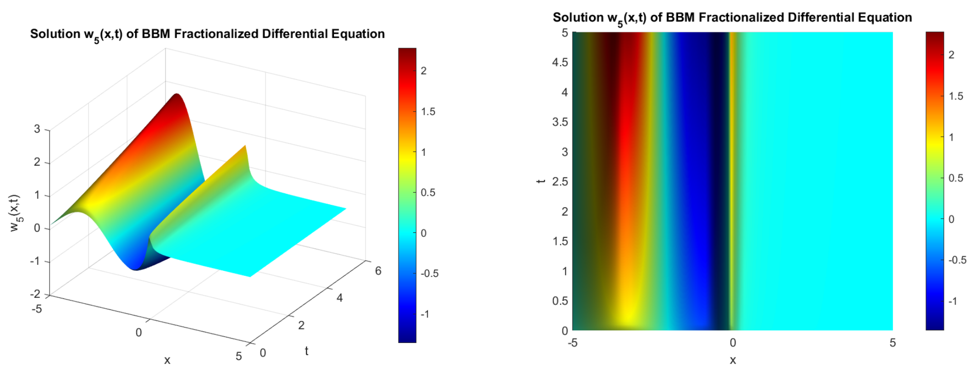

(A) (B)

FIGURE 3. Wave Solution of BBM with $\mu = 3$, $\lambda = 2$, $k = 2$, $\alpha = 0.8$, $\beta = 0.8$



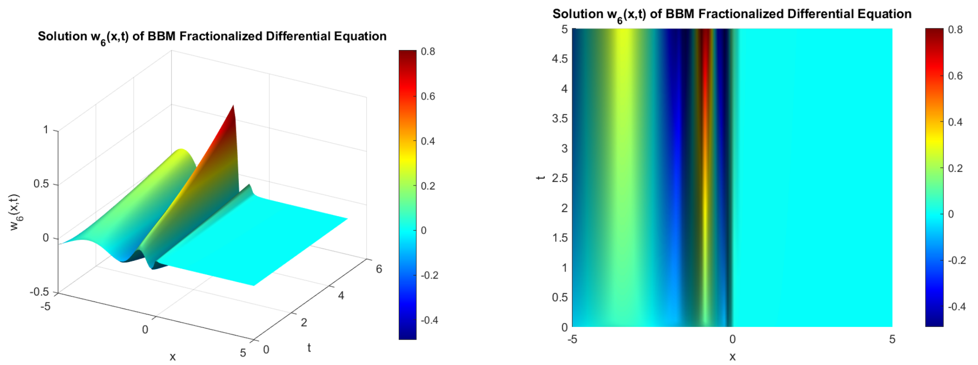

(A) (B)

FIGURE 4. Wave Solution of BBM with $\mu = 2.5$, $\lambda = 2.4$, $k = 1$, $\alpha = 0.2$, $\beta = 0.5$



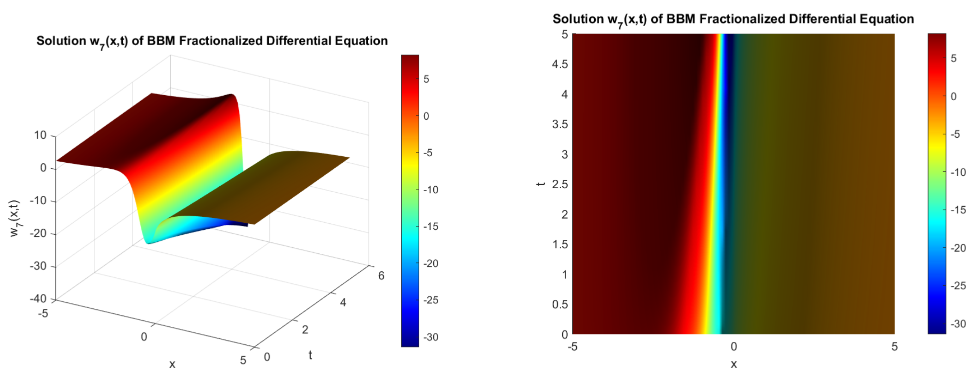

(A) (B)

FIGURE 5. Wave Solution of BBM with $\mu = 0$, $\lambda = 3$, $k = 1$, $\alpha = 0.5$, $\beta = 0.5$




(A) (B)

FIGURE 6. Wave Solution of BBM with $\mu = 1$, $\lambda = 3$, $k = 1$, $\alpha = 0.5$, $\beta = 0.5$




(A) (B)

FIGURE 7. Wave Solution of BBM with $\mu = 0$, $\lambda = 0$, $k = 1$, $\alpha = 0.8$, $\beta = 0.8$

4.2. **Space-Time Fractional**$(2{+}1)$ **Dimensional Extended Kadomtsev Petviashvili Equation (STFEKPE).** The extended KP equation is the generalization of KP equation. It is provided as follows and characterizes the development of dispersive, weakly nonlinear waves in a two-dimensional medium.

$$D_x^{\alpha}\left(D_t^{\gamma}w+6wD_x^{\alpha}w+D_x^{3\alpha}w\right)-D_y^{2\beta}w+\lambda D_t^{2\gamma}w+\mu D_t^{\gamma}(D_y^{\beta}w)=0,\quad 0<\alpha,\beta,\gamma\leq 1,\ t>0. \tag{4. 32}$$

The following nonlinear ODE is obtained by entering the traveling wave transformation provided in Eq.( 3. 5 ) and its necessary derivatives using the chain rule into Eq. ( 4. 32 ).

$$\left(-\vartheta w'+6ww'+w'''\right)'-w''+\lambda\vartheta^2w''-\mu\vartheta w''=0.$$

By integrating the above equation w.r.t $\zeta$ and by considering the constant of integration to be zero, it yields:

$$w''+(\lambda\vartheta^2-\mu\vartheta-\vartheta-1)w+3w^2=0. \tag{4. 33}$$

Let us assume the solution of Eq.( 4. 33 ) as follows:

$$w(\zeta)=\sum_{i=0}^{m}c_ie^{-i\psi(\zeta)},\qquad c_m\neq 0. \tag{4. 34}$$

Furthermore, the value of a non-negative integer $m$ can be found by using homogeneous balance as follows:

$$2+m=2m\quad\Rightarrow\quad m=2. \tag{4. 35}$$

By expanding Eq.( 4. 34 ) for $m = 2$ we get the following results Thus, from Eq.( 4. 35 ):

$$w(\zeta)=c_0+c_1e^{-\psi(\zeta)}+c_2e^{-2\psi(\zeta)},\quad c_2\neq 0. \tag{4. 36}$$

$$\begin{aligned} w^2=&\,c_2^2e^{-4\psi(\zeta)}+2c_1c_2e^{-3\psi(\zeta)}+\left(c_1^2+2c_0c_2\right)e^{-2\psi(\zeta)}\\ &+2c_0c_1e^{-\psi(\zeta)}+c_0^2. \end{aligned} \tag{4. 37}$$

$$\begin{aligned} w''=&\,6c_2e^{-4\psi(\zeta)}+\left(10c_2\lambda+2c_1\right)e^{-3\psi(\zeta)}\\ &+\left(3c_1\lambda+8c_2\mu+4c_2\lambda^2\right)e^{-2\psi(\zeta)}\\ &+\left(6c_2\mu\lambda+2c_1\mu+c_1\lambda^2\right)e^{-\psi(\zeta)}\\ &+c_1\mu\lambda+2c_2\mu^2. \end{aligned} \tag{4. 38}$$

By plugging Eq.( 4. 36 ), Eq.( 4. 37 ) and Eq.( 4. 38 ) into Eq.( 4. 33 ) and by collecting the coefficients of like powers of it, $e^{-\psi(\zeta)}$ it yields

$$\begin{aligned} e^0:&\quad c_1\mu\lambda+2c_2\mu^2+c_0\lambda\vartheta^2-c_0\mu\vartheta-c_0\vartheta-c_0+3c_0^2=0,\\ e^1:&\quad 2c_1\mu+c_1\lambda^2+2c_2\mu\lambda+4c_2\mu\lambda+c_1\lambda\vartheta^2-c_1\mu\vartheta-c_1\vartheta-c_1+6c_0c_1=0,\\ e^2:&\quad 3c_1\lambda+8c_2\mu+4c_2\lambda^2+c_2\lambda\vartheta^2-c_2\mu\vartheta-c_2\vartheta-c_2+3c_1^2+6c_0c_2=0,\\ e^3:&\quad 2c_1+10c_2\lambda+6c_1c_2=0,\\ e^4:&\quad 6c_2+3c_2^2=0. \end{aligned}$$

The above nonlinear system can be solved for required parameters $c_0, c_1, c_2$, and $\vartheta$ it yields

$$\begin{aligned} c_0 &= -2\mu, \\ c_1 &= -2\lambda, \\ c_2 &= -2, \\ \vartheta &= \mu + \frac{\sqrt{4\lambda + 16\lambda\mu - 4\lambda^3 + \mu^2 + 2\mu + 1}\ + 1}{2\lambda}. \end{aligned} \tag{4. 39}$$

By plugging Eq.( 4. 39 ) into Eq.( 4. 36 ) we obtained.

$$w(\zeta) = -2\mu - 2\lambda e^{-\psi(\zeta)} - 2e^{-2\psi(\zeta)}. \tag{4. 40}$$

Where

$$\zeta = \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{y^{\beta}}{\Gamma(1+\beta)} - \frac{\mu + \frac{\sqrt{4\lambda+16\lambda\mu-4\lambda^3+\mu^2+2\mu+1}+1}{2\lambda} t^{\gamma}}{\Gamma(1+\gamma)}.$$

plugging the generalized solutions of Eq.( 3. 8 ) into Eq.( 4. 40 ), we obtained different types of soliton solutions of Eq.( 4. 32 ) as under:
Case 1. For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$:

$$\begin{aligned} w_1(\zeta) = -2\mu &+ \frac{4\mu\lambda}{\sqrt{\lambda^2 - 4\mu}\ \tanh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}(\zeta + k)\Big) + \lambda} \\ &- \frac{8\mu^2}{\Big(\sqrt{\lambda^2 - 4\mu}\ \tanh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}(\zeta + k)\Big) + \lambda\Big)^2}. \end{aligned} \tag{4. 41}$$

$$\begin{aligned} w_2(\zeta) = -2\mu &+ \frac{4\mu\lambda}{\sqrt{\lambda^2 - 4\mu}\ \coth\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}(\zeta + k)\Big) + \lambda} \\ &- \frac{8\mu^2}{\Big(\sqrt{\lambda^2 - 4\mu}\ \coth\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}(\zeta + k)\Big) + \lambda\Big)^2}. \end{aligned} \tag{4. 42}$$

Case 2. For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$:

$$\begin{aligned} w_3(\zeta) = -2\mu &- \frac{4\mu\lambda}{\sqrt{4\mu - \lambda^2}\ \tan\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}(\zeta + k)\Big) - \lambda} \\ &- \frac{8\mu^2}{\Big(\sqrt{4\mu - \lambda^2}\ \tan\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}(\zeta + k)\Big) - \lambda\Big)^2}. \end{aligned} \tag{4. 43}$$

$$w_4(\zeta) = -2\mu - \frac{4\mu\lambda}{\sqrt{4\mu-\lambda^2}\,\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\zeta+k)\right)-\lambda} - \frac{8\mu^2}{\left(\sqrt{4\mu-\lambda^2}\,\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\zeta+k)\right)-\lambda\right)^2}. \tag{4. 44}$$

Case 3. For $\lambda^2 - 4\mu > 0$, $\mu = 0$ and $\lambda \neq 0$:

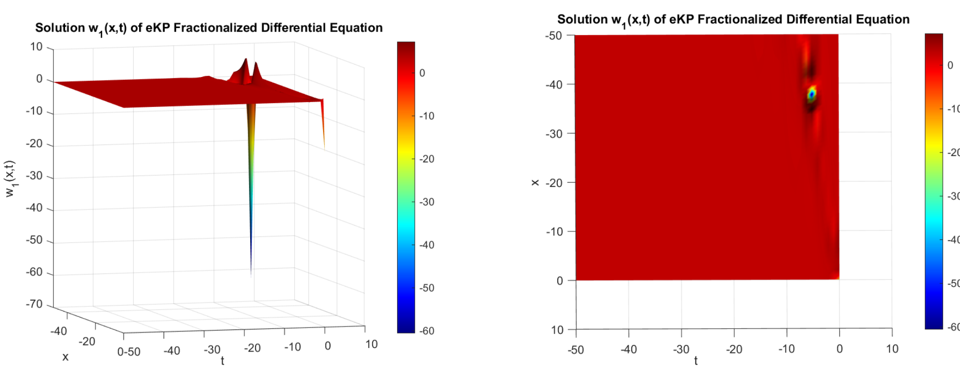
$$w_5(\zeta) = -\frac{2\lambda^2 e^{\lambda(\zeta+k)}}{\left(e^{\lambda(\zeta+k)}-1\right)^2}. \tag{4. 45}$$

Case 4. For $\lambda^2 - 4\mu = 0$, $\mu \neq 0$ and $\lambda \neq 0$:

$$w_6(\zeta) = -2\mu + \frac{\lambda^3(\zeta+k)}{\lambda(\zeta+k)+2} - \frac{\lambda^4(\zeta+k)^2}{2(\lambda(\zeta+k)+2)^2}. \tag{4. 46}$$

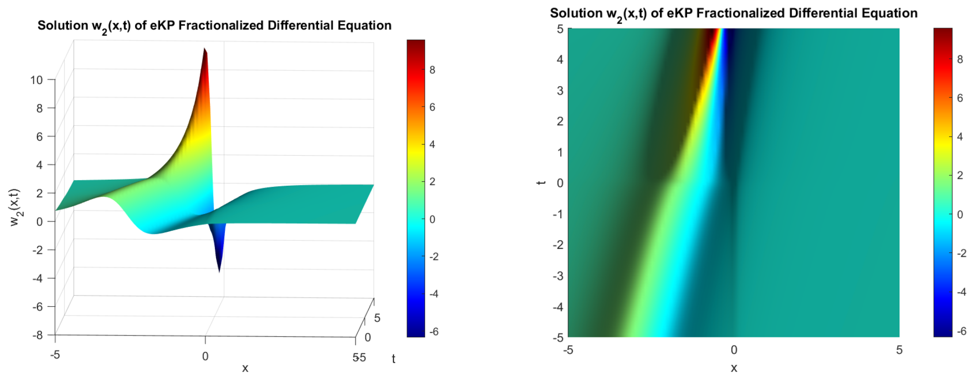
Case 5. For $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda = 0$:

$$w_7(\zeta) = -\frac{2}{(\zeta+k)^2}. \tag{4. 47}$$



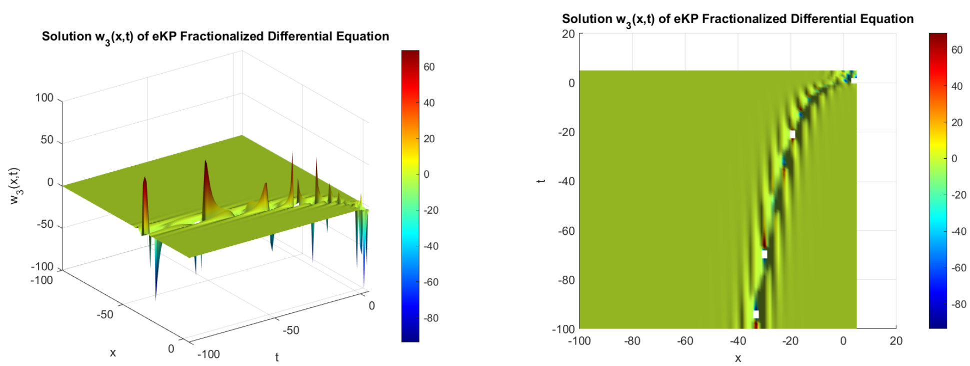

(A) (B)

FIGURE 8. Wave Solution of EKP with $\mu = 1,\ \lambda = 1.5,\ k = 1,\ \alpha = 0.5,\ \beta = 0.9,\ \gamma = 0.5$



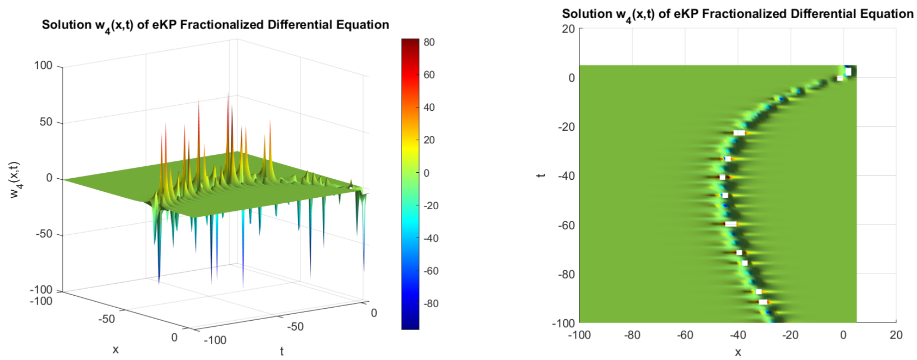

(A) (B)

FIGURE 9. Wave Solution of EKP with $\mu = 1.9,\ \lambda = 3,\ k = 1,\ \alpha = 0.8,\ \beta = 0.8,\ \gamma = 0.8$



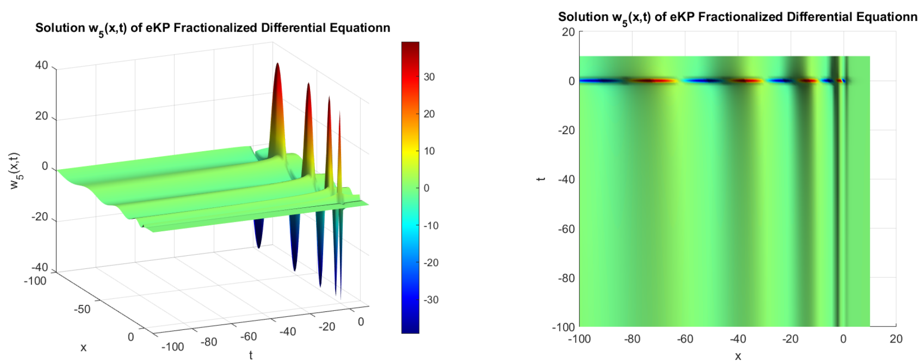

(A) (B)

FIGURE 10. Wave Solution of EKP with $\mu = 1.6,\ \lambda = 1,\ k = 1,\ \alpha = 0.9,\ \beta = 0.2,\ \gamma = 0.3$



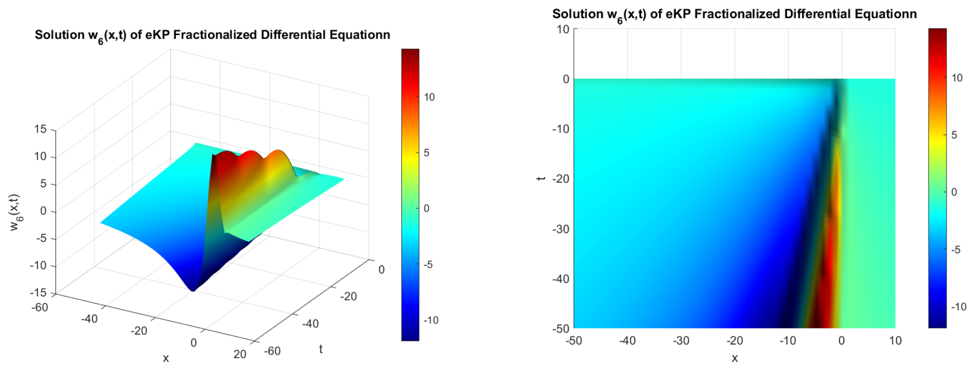

(A) (B)

FIGURE 11. Wave Solution of EKP with $\mu = 2.6,\ \lambda = 1.7,\ k = 1,\ \alpha = 0.5,\ \beta = 0.9,\ \gamma = 0.3$



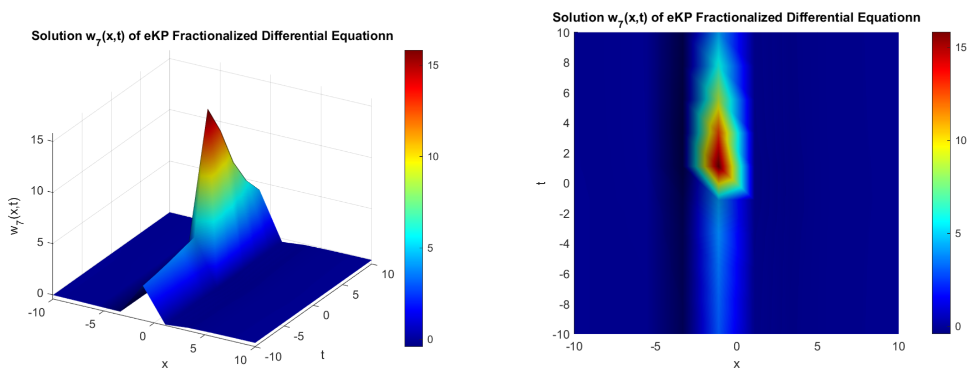

(A) (B)

FIGURE 12. Wave Solution of EKP with $\mu = 0,\ \lambda = 2.3,\ k = 1.3,\ \alpha = 0.5,\ \beta = 0.1,\ \gamma = 0.5$




(A) (B)

FIGURE 13. Wave Solution of EKP with $\mu = 0.9,\ \lambda = 1.3,\ k = 1,\ \alpha = 0.2,\ \beta = 0.1,\ \gamma = 0.1$




(A) (B)

FIGURE 14. Wave Solution of EKP with $\mu = 0,\ \lambda = 0,\ k = 1,\ \alpha = 0.9,\ \beta = 0.1,\ \gamma = 0.5$

## 5. NUMERICAL RESULTS AND DISCUSSION

This section's goal is to illustrate the many soliton solutions for the BBM and (2+1)-dimensional EKP problems graphically. MATLAB is used to solve for various parameters. From Figure 1, Figure 2 represents kink and antikink solitons, respectively. Figure 3 illustrates a bright soliton with a strong localized single peak that rapidly decays away from the center, while Figure 4 represents the dark soliton profile. Figure 5 shows periodic non-topological wave behavior, and Figure 6 demonstrates breather-type soliton. The interaction dynamics between topological modes are evident in Figure 7. Figure 8 represents a skyrmion-like localized wave with a smooth core and rapidly decaying tails, whereas Figure 9 shows a vortex-type pattern. Multi-peak bright soliton structures are shown in Figure 10, Figure 11 demonstrates a dispersive kink structure with oscillatory tails. A shock-like or steep-front solution is observed Figure 12, Figure 13 corresponds to a lump-type solution, while Figure 14 illustrates a hybrid periodic-localized structure, where localized peaks coexist with a periodic background. Notably, the sign of the wave's velocity dictates its direction: positive velocity corresponds to rightward movement, while negative velocity indicates leftward movement. Furthermore, specific parameters within the solutions directly influence the wave's amplitude and velocity. The graphical representations produced by our study effectively validate the accuracy and reliability of the $e^{-\psi(\zeta)}$expansion approach for generating these results.

## 6. CONCLUSIONS

In this article, we studied two different nonlinear STFPDEs that arise in wave phenomena named the fractional BBM and the fractional (2+1)-dimensional EKP equations. Both equations had not been studied before. In many branches of science and engineering, these equations are crucial. These equations are rewritten in terms of a well-known derivative operator named the Jumarie modified Riemann-Liouville derivative. The closed-form soliton solutions have been achieved in the form of transcendental functions by using the $e^{-\psi(\zeta)}$-expansion approach. The variety of exact soliton solutions obtained in this work are truly new with different parameters, groundbreaking, and will be very helpful to understand the physical behavior of these models and can be used in different fields. The importance of the $e^{-\psi(\zeta)}$expansion approach is that it is a short, effective, simple, and high-powered method for solving nonlinear equations of mathematical models. This method does not require any type of linearization, unnecessary assumptions, or perturbations that can alter the physical behavior of the problem and can be used for all types of problems directly.

## CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Sana Ullah Dehraj: Formulation, methodology, project ad-ministration, resources, su-pervision, interpretation, data analysis, review & editing.
Muhammad Hussain Baig and Sayam Munwer: Conceptualization, implementation, methodology, computation, comparison, validation, data analysis, original draft preparation, review & editing.
Deepak Umrao Sarwe: Proofreading, Language editing, and final manuscript preparation.

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Conflict of Interest: All authors confirmed that there is no conflict of interest regarding the publication of this research.
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