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An Eighth-Order Kernel-Based Numerical Method for Linear Sixth-Order Boundary Value Problems

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Abstract. This study focuses on developing a numerical algorithm with eighth-order convergence for sixth-order boundary value problems (BVPs). The method begins by constructing basis functions that inherently satisfy the boundary conditions, achieved through the use of reproducing kernel functions (RKFs). Based on these functions, a novel collocation method with eighth-order convergence is then proposed to address such problems. A series of numerical experiments are conducted, and comparisons with existing approaches demonstrate the superiority of the proposed technique.

AMS (MOS) Subject Classification Codes: 34B15; 65L10; 65L11
Key Words: Kernel functions; sixth order BVPs; numerical approaches

1. Introduction

Higher order boundary value problems (BVPs) have extensive applications in biology, physics and engineering. Due to their significant applications, there is a clear need for efficient numerical methods capable of solving these problems. Based on Quasi-Newton's method and the kernel technique, Xu et al. [18] proposed a new numerical approach for fourth order BVPs. Costabile and Napoli [3] introduced a general collocation technique for high even order differential equations. Noor and Mohyud-Din [13] proposed the variational iteration technique for higher order BVPs. Viswanadham and Ballem [17] introduced a cubic b-splines based Galerkin approach for fourth order BVPs. Using reproducing kernel theory, Geng [5] solved nonlinear fourth order BVPs.Khalid and Naeem [9] proposed Quartic B-spline approaches for fourth order BVPs. Iqbal et al.[7] introduced B-spline numerical algorithms for sixth order BVPs. Siddiqi and Akram[15] proposed septic spline methods for sixth order BVPs.

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Islam et.al [8] presented non-polynomial splines techniquesfor sixth order BVPs. Pervaiz et al. [14] presented non-polynomial spline tech-nique for sixth order BVPs. Noor and Din [12] proposed homotopy perturbation methods for sixth order BVPs. Khandelwal and Sultana [11] proposed a parametric septic splines technique for linear sixth-order BVPs. Sohaib et al. [16] introduced a Legendre wavelet collocation method for sixth-order BVPs. Khalid et al. [10] solved linear sixth order BVPs by using cubic B-spline. By combining iterative technique and finite dif-ference technique, Dang and Dang [4] developed an efficient technique for the solution of sixth-order nonlinear BVPs. Akram and Rehman [1] solved eighth order BVPs in reproducing kernel space. Geng et al. [6] proposed sixth order kernel functions approach for nonlinear fourth order BVPs. Amin [2] introduced Haar wavelet algorithms for sixth-order BVPs. Existing numerical methods can effectively solve higher-order boundary value problems; however, their limitation lies in the need to employ a large number of nodes to achieve high accuracy. This requirement leads to significantly increased computational costs. Our investigation focuses on the sixth-order BVPs given by:

$$\begin{aligned} &Lv(x) = g(x), \ \ 0 < x < 1, \\ &v(0) = \alpha_1, v(1) = \alpha_2, v'(0) = \beta_1, v'(1) = \beta_2, v''(0) = \gamma_1, v''(1) = \gamma_2, \end{aligned} \tag{1. 1}$$

where $Lv(x) = v^{(6)}(x) + a_5(x)v^{(5)}(x) + a_4(x)v^{(4)}(x) + a_3(x)v^{(3)}(x) + a_2(x)v''(x) + a_1(x)v'(x) + a_0 v(x)$, $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$ and $g(x) \in C[0,1]$.

Through the definition of an auxiliary function

$$y(x) = v(x) - \rho(x), \tag{1. 2}$$

where $\rho(x) = \mu_1 + \mu_2 x + \mu_3 x^2 + \mu_4 x^3 + \mu_5 x^4 + \mu_6 x^5$ and satisfies $\rho(0) = \alpha_1, \ \rho(1) = \alpha_2, \rho'(0) = \beta_1, \ \rho'(1) = \beta_2, \rho''(0) = \gamma_1, \ \rho''(1) = \gamma_2$.

Then the solution of the above equation (1.1) can be easily reduced to the solution of following BVPs

$$\begin{cases} Ly(x) = f(x), \ \ 0 < x < 1, \\ y(0) = y(1) = 0, y'(0) = y'(1) = 0, y''(0) = y''(1) = 0. \end{cases} \tag{1. 3}$$

In the work, based on the RKF in RKHS $W^5[0,1]$, a novel numerical technique will be proposed for the above linear sixth order BVPs. It has higher convergence order and its convergence order can reach to $O(h^8)$.

2. NUMERICAL APPROACH

As the foundation of our numerical approach, the related theory on RKHS shall be introduced.

Let E be a set. The RKF of a Hilbert function space H over E, denoted by $K : E \times E \to \mathbb{R}$ satisfies

$$\begin{aligned} &(1)\ K(\cdot,\ s) \in H, for\ all\ s \in E, \\ &(2)\ r(s) = (r(\cdot),\ K(\cdot,\ s)), for\ all\ s \in E\ and\ all\ r \in H. \end{aligned}$$

We define the space $W^5[0,1]$ as the collection of functions $y(x) \in C^4[0,1]$ whose fifth derivative $y^{(5)}$ is in $L^2[0,1]$, whose fourth derivative is absolutely continuous, and which satisfy the homogeneous boundary conditions $y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0$.

Its inner products is given by

$$(y_1(x), y_2(x)))_5 = \sum_{i=0}^{2} y_1^{(i)}(0)y_2^{(i)}(0) + \sum_{i=0}^{1} y_1^{(i)}(1)y_2^{(i)}(1) + \int_0^1 y_1^{(5)}y_2^{(5)}dx. \quad (2.\ 1)$$

Theorem 2.1. *$W^5[0,1]$ forms a RKHS. Its corresponding RKF admits the following expression:*

$$K(t,s) = \begin{cases} Q_1(t,s), s \leq t \\ Q_1(s,t), s > t, \end{cases}$$

where $Q_1(t,s) = [s(s^8(3t^4-4t^3+1) - 9s^7(t-1)^2t(2t+1) + 36s^6(t-1)^2t^2 + 3s^3t(t^8 - 6t^7 + 12t^6 - 42t^4 + 635100t^3 - 907225t^2 + 30240t + 241920) - s^2t(4t^8 - 27t^7 + 72t^6 - 84t^5 + 2721675t^3 - 3991720t^2 + 181440t + 1088640) + 90720s(t-1)^2t^2 + 362880(t-1)^2t(2t+1))]/362880.$

For each fixed $s \in [0,1]$, the RKF $K(x,s)$ belongs to $C^8[0,1]$. Furthermore, it is a piecewise polynomial of degree nine, i.e., a spline of order nine.

Through the use of the kernel-based basis and the careful selection of collocation points, we construct a novel high-precision collocation scheme. Consider a partition $T_N : 0 = x_1 < x_2 < \ldots < x_N = 1$ of the interval $[0,1]$, $\eta_k = (x_k, x_{k+1})$, $h_k = x_{k+1} - x_k$ with $h = \max_k h_k$. Let $P_m(x)$ denote the vector space of polynomials of degree at most m.

The basis functions are then constructed in terms of the RKF $K(x,s)$ and its derivatives with respect to s as follows:

$$\psi_k(x) = \begin{cases} K(x, x_{k+1}), & 1 \leq k \leq N-2, \\ \frac{\partial K(x,s)}{\partial s} \mid_{s=x_{k-N+3}}, & N-1 \leq k \leq 2N-4, \\ \frac{\partial^2 K(x,s)}{\partial s^2} \mid_{s=x_{k-2N+5}}, & 2N-3 \leq k \leq 3N-6, \\ \frac{\partial^3 K(x,s)}{\partial s^3} \mid_{s=x_{k-3N+7}}, & 3N-5 \leq k \leq 4N-8, \\ x^3(1-x)^3, & k = 4N-7, \\ x^4(1-x)^3, & k = 4N-6, \\ x^5(1-x)^3, & k = 4N-5, \\ x^6(1-x)^3, & k = 4N-4. \end{cases}$$

We define the finite-dimensional space S_N as the span of the basis functions:

$$\{\psi_j(x), j = 1, 2, \cdots, 4N-4\}.$$

This space coincides with the set of piecewise polynomials of degree nine that are globally five times continuously differentiable.

Given that S_N has a dimension of $4N-4$, a total of $4N-4$ collocation nodes are required. These nodes are specifically defined as:

$$Z_{4N-4} = \{z_{ki} = x_k + \mu_i h_k \mid i = 1, 2, , 3, 4, 1 \leq k \leq N-1\},$$

where the parameters μ_i correspond to the Gauss-Legendre nodes on the interval $[0,1]$: $\mu_1 = \frac{1}{2}\left(1 - \sqrt{\frac{1}{35}\left(15 + 2\sqrt{30}\right)}\right)$, $\mu_2 = \frac{1}{2}\left(1 - \sqrt{\frac{1}{35}\left(15 - 2\sqrt{30}\right)}\right)$,
$\mu_3 = \frac{1}{2}\left(1 + \sqrt{\frac{1}{35}\left(15 - 2\sqrt{30}\right)}\right)$ and $\mu_4 = \frac{1}{2}\left(1 + \sqrt{\frac{1}{35}\left(15 + 2\sqrt{30}\right)}\right)$.

The approximate solution to (1.2) is sought within the space S_N and expressed in the form:

$$y_N(x) = \sum_{k=1}^{4N-4} c_k \psi_k(x). \tag{2. 2}$$

We enforce the governing equation at each node of the set Z_{4N-4} for the approximate solution $y_N(x)$, which leads to the following collocation system:

$$Ly_N(s) = \sum_{k=1}^{4N-4} c_k L\psi_k(s) = f(s),\ s \in Z_{4N-4}. \tag{2. 3}$$

The solution of this linear system yields the coefficients $\{c_k\}$, which completely determines the approximate solution $y_N(x)$.

3. Analysis of convergence order

This section is devoted to the analysis of the convergence order for our numerical scheme.

Theorem 3.1. *Assume that* $\{a_j(x)\}_{j=1}^{5}$ *and* $f(x) \in C^8[0,1]$. *We have*

$$\| y_N - y \|_\infty \leq c\, h^8,$$

where $\| y \|_\infty = \max\limits_{x\in[0,1]} |y(x)|$.

Proof. (1.2) can be rewrited as

$$y(x) = \int_0^1 G(x,t) f(t) dt,$$

where $G(x,s)$ represents the Green's function. Observe that

$$\begin{aligned} y_N(x) - y(x) \quad &= \textstyle\int_0^1 G(x,s)[Ly_N(s) - Ly(s)]ds, \\ &= \textstyle\int_0^1 G(x,s)[Ly_N(s) - f(s)]ds. \end{aligned} \tag{3. 1}$$

Denoting $Ly_N(s)$ by $f_N(s)$, (3.4) is simplified to

$$y_N(x) - y(x) = \int_0^1 G(x,s)[f_N(s) - f(s)]ds.$$

Given that four collocation points $z_{k1},\ z_{k2},\ z_{k3}$ and z_{k4} located on $[x_k, x_{k+1}]$, we have $Ly_N(z_{ki}) = f(z_{ki}),\ i = 1,2,3,4$, that is, $f_N(z_{ki}) = f(z_{ki}),\ i = 1,2,3,4$. Then

$$\begin{aligned} f_N(s) - f(s) \quad &= \tfrac{f_N''''(\tau) - f''''(\tau)}{4!} \prod_{i=1}^{4}(s - z_{ki}), \\ &\triangleq F_l''''(\tau) \prod_{i=1}^{4}(s - z_{ki}), \end{aligned} \tag{3. 2}$$

where $\tau \in [x_k, x_{k+1}]$. Set $D_k(x,s) = G(x,s)F_k''''(\tau)$. Thus,

$$\int_{x_k}^{x_{k+1}} G(x,s)[f_N(s) - f(s)]ds = \int_{x_k}^{x_{k+1}} D_l(x,s) \prod_{i=1}^{4}(s - z_{ki})ds.$$

Expanding $D_k(x,s)$ at z_{k1} using Taylor's formula gives

$$\begin{aligned} D_k(x,s) = \quad & D_k(x,z_{k1}) + \tfrac{\partial D_k}{\partial s}(x,z_{k1})(s-z_{k1}) + \tfrac{1}{2}\tfrac{\partial^2 D_k}{\partial s^2}(x,z_{k1})(s-z_{k1})^2 + \\ & \tfrac{1}{3!}\tfrac{\partial^3 D_k}{\partial s^3}(x,z_{k1})(s-z_{k1})^3 + \tfrac{1}{4!}\tfrac{\partial^4 D_k}{\partial s^4}(x,\tau)(s-z_{k1})^4. \end{aligned}$$

Considering a set of four Gauss points z_{k1}, z_{k2}, z_{k3} and z_{k4} on $[x_k, x_{k+1}]$, for all polynomials $q(s)$ whose degree does not exceed seven, the Gaussian quadrature formula $\sum\limits_{i=1}^{4}\omega_i g(z_{ki})$ satisfies

$$\int_{x_k}^{x_{k+1}} q(s)ds = \sum_{i=1}^{4}\omega_i q(z_{ki}).$$

Furthermore, we have

$$\begin{aligned} \int_{x_k}^{x_{k+1}} D_k(x,s)\prod_{i=1}^{4}(s-z_{ki})\,ds = \frac{1}{4!}\int_{x_k}^{x_{k+1}} & \frac{\partial^4 D_k}{\partial s^4}(x,\tau) \\ & \times (s-z_{k1})^5(s-z_{k2}) \\ & \times (s-z_{k3})(s-z_{k4})\,ds. \end{aligned} \tag{3. 3}$$

Putting $s = th_k + x_k$, it follows that

$$\begin{aligned} & \textstyle\int_{x_k}^{x_{k+1}} \frac{\partial^4 D_k}{\partial s^4}(x,\tau)(s-z_{k1})^5(s-z_{k2})(s-z_{k3})(s-z_{k4})ds \\ & = h_k^9 \textstyle\int_0^1 \frac{\partial^4 D_k}{\partial s^4}((x,\overline{\tau}))(t-\mu_1)^5(t-\mu_2)(t-\mu_3)(t-\mu_4)dt. \end{aligned} \tag{3. 4}$$

It follows from $\{a_j(x)\}_{j=1}^5$ and $f \in C^8[0,1]$ that $\int_0^1 [\frac{\partial^4 D_k}{\partial s^4}(x,\overline{\tau})]^2 ds$ is bounded, that is

$$\int_0^1 [\frac{\partial^4 D_k}{\partial s^4}(x,\overline{\tau})]^2 ds \le \beta.$$

Through the application of Schwarz's inequality, we have

$$\begin{aligned} & |\textstyle\int_0^1 \frac{\partial^4 D_k}{\partial s^4}((x,\overline{\tau}))(t-\mu_1)^5(t-\mu_2)(t-\mu_3)(t-\mu_4)dt|^2 \\ & \le \textstyle\int_0^1 [\frac{\partial^4 D_k}{\partial s^4}(x,\overline{\tau})]^2 dt \int_0^1 [(t-\mu_1)^5(t-\mu_2)(t-\mu_3)(t-\mu_4)]^2 dt \\ & \le \tfrac{1}{44100}\beta. \end{aligned} \tag{3. 5}$$

Then

$$|\int_{x_k}^{x_{k+1}} D_k(x,\tau)\prod_{i=1}^{4}(s-z_{ki})ds| \le \frac{1}{4!}\sqrt{\frac{1}{44100}\beta}\,h^9. \tag{3. 6}$$

Therefore,

$$\begin{aligned} \| y_N - y \|_\infty \quad & \le \sum_{k=1}^{N-1} |\textstyle\int_{x_k}^{x_{k+1}} D_k(x,\tau)\prod\limits_{i=1}^{4}(s-z_{ki})ds|, \\ & \le \tfrac{N-1}{4!}\sqrt{\tfrac{1}{44100}\beta}\,h^9, \\ & \le c\,h^8, \end{aligned}$$

with a positive constant c. □

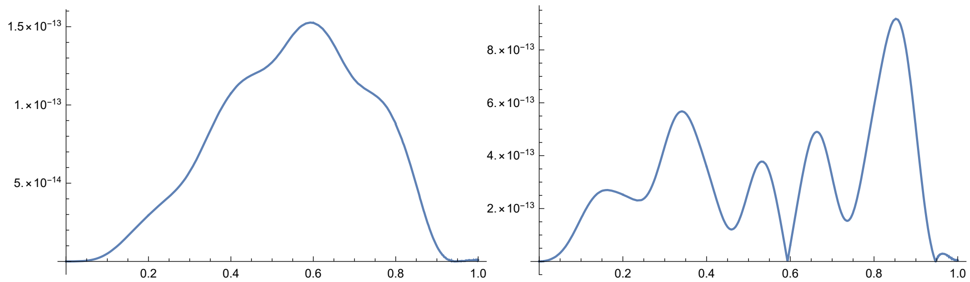


FIGURE 1. Absolute errors $|v_6(x) - v(x)|$ (left) and $|v_6'(x) - v'(x)|$ (right) for Test 4.1.

4. NUMERICAL TESTS

To validate the accuracy and convergence order of the proposed method, four numerical test cases are presented. In the following experiments, We select $x_l = \frac{l-1}{N-1}$ for $l = 1, 2, \ldots, N$.

Test 4.1

Consider the sixth order BVP used in [15, 8, 11, 4]

$$\begin{cases} v^{(6)}(x) + xv(x) = f(x), \\ v(0) = 0,\ v(1) = 0,\ v'(0) = -1,\ v'(1) = 2\sin 1,\ v''(0) = 0,\ v''(1) = 2\sin 1 + 4\cos 1, \end{cases}$$

where $f(x) = 30\sin x + 12x\cos x$. Its exact solution is $v(x) = (x^2 - 1)\sin x$. Table 1 provides the numerical results from present method and those from [15, 8, 11, 4]. The convergence order is also listed in Table 2. Figures 1 and 2 depict the absolute errors for the approximate solution and its corresponding derivatives when $N = 6$.

TABLE 1. Results of maximum absolute errors for Test 4.1

h	Method in [15]	Method in [8]	Method in [11]	Method in [4]	Present method
$\frac{1}{8}$	8.25×10^{-6}	8.45×10^{-11}	3.14×10^{-11}	2.27×10^{-10}	3.77×10^{-15}
$\frac{1}{16}$	1.13×10^{-6}	2.75×10^{-11}	5.76×10^{-12}	1.91×10^{-11}	1.11×10^{-16}

TABLE 2. Results of numerical solution and its derivative for Test 4.1

N	E_N	Convergence order
3	2.55×10^{-10}	–
5	8.65×10^{-13}	8.20
9	3.77×10^{-15}	7.84
17	1.11×10^{-16}	5.08

Test 4.2

Consider the sixth order BVP used in [11]

$$\begin{cases} v^{(6)}(x) + v(x) = f(x), \\ v(0) = 0,\ v(1) = 0,\ v'(0) = -1,\ v'(1) = 2\sin 1,\ v''(0) = 0,\ v''(1) = 2\sin 1 + 4\cos 1, \end{cases}$$

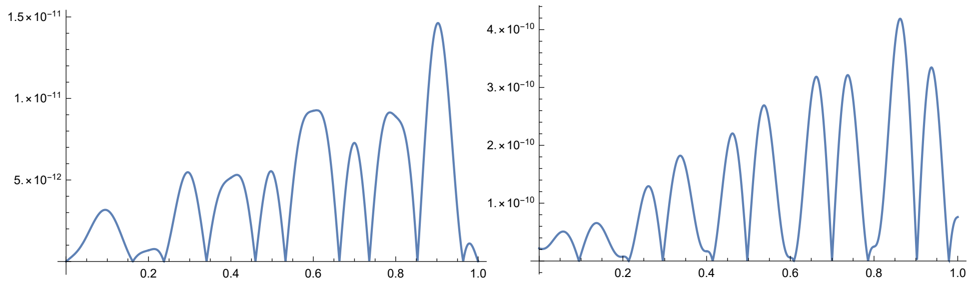


FIGURE 2. Absolute errors $|v_6''(x) - v''(x)|$ (left) and $|v_6'''(x) - v'''(x)|$ (right) for Test 4.1.

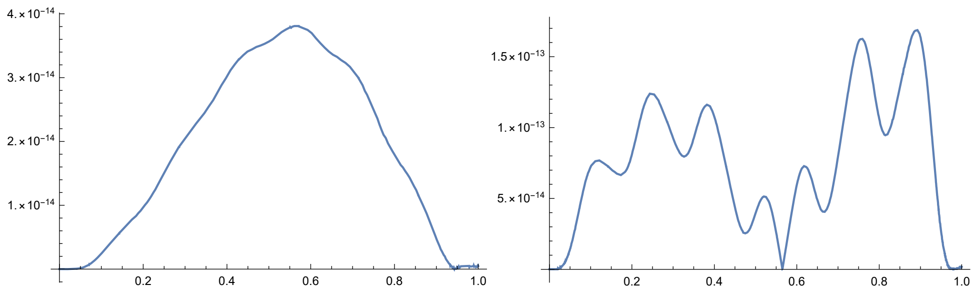


FIGURE 3. Absolute errors $|v_8(x) - v(x)|$ (left) and $|v_8'(x) - v'(x)|$ (right) for Test 4.2.

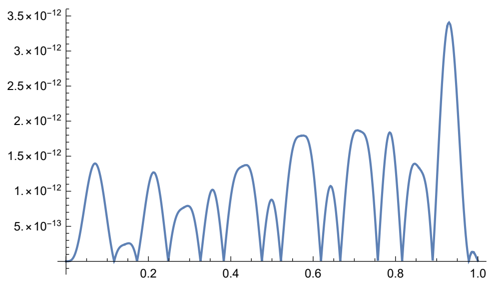
where $f(x) = -(x^3 + 11x + 24)e^x$. It has an exact solution of $v(x) = (x^2 - x)e^x$. Table 3 shows the numerical results from our method and those from [11]. The convergence order is also listed in Table 4. Taking $N = 8$, the absolute errors for the computed approximate solution and its corresponding derivatives are illustrated in Figures 3,4.

TABLE 3. Results of maximum absolute errors for Test 4.2

h	Method in [11]	Present method
$\frac{1}{8}$	9.39×10^{-11}	1.29×10^{-14}
$\frac{1}{16}$	7.52×10^{-12}	6.04×10^{-16}

TABLE 4. Results of numerical solution and its derivative for Test 4.2

N	E_N	Convergence order
3	7.09×10^{-10}	–
5	3.17×10^{-12}	7.80
9	1.29×10^{-14}	7.94
17	6.04×10^{-16}	4.41



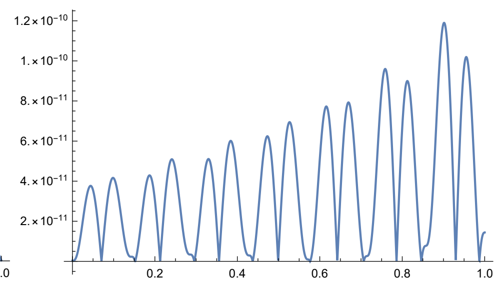


FIGURE 4. Absolute errors $|v_g''(x) - v''(x)|$ (left) and $|v_g'''(x) - v'''(x)|$ (right) for Test 4.2.

Test 4.3

Consider the sixth order BVP used in [12, 16]

$$\begin{cases} v^{(6)}(x) - (1+\mu)v^{(4)}(x) + \mu v''(x) = \mu x, \\ v(0) = 1,\ v(1) = \sinh 1 + 7/6,\ v'(0) = 1, \\ v'(1) = \cosh 1 + 1,\ v''(0) = 0,\ v''(1) = \sinh 1 + 1, \end{cases}$$

where parameter μ is an arbitrary constant. It has an exact solution of $v(x) = \sinh x + \frac{x^3}{6} + 1$, which is independent of the parameter μ. Tables 5 and 6 illustrate the numerical outcomes of our method and the approach in [12, 16]. The absolute errors associated with the derived approximate solution and its corresponding derivatives are illustrated in Figures 5,6.

TABLE 5. A comparison of absolute errors across various methods for Test 4.3 with $\mu = 10$

x	HPM in [12]	Method in [16]	Present method(N=14)
0.1	1.20×10^{-6}	4.80×10^{-15}	6.66×10^{-16}
0.2	7.20×10^{-6}	4.20×10^{-15}	4.44×10^{-16}
0.3	1.70×10^{-5}	1.60×10^{-14}	2.22×10^{-16}
0.4	2.70×10^{-5}	3.80×10^{-14}	0
0.5	3.40×10^{-5}	4.30×10^{-14}	0
0.6	3.20×10^{-5}	4.00×10^{-14}	0
0.7	2.30×10^{-5}	2.10×10^{-14}	0
0.8	1.10×10^{-5}	8.70×10^{-15}	4.44×10^{-16}
0.9	2.20×10^{-6}	8.90×10^{-16}	4.44×10^{-16}

Test 4.4

Consider the follwoing sixth order BVP

$$\begin{cases} v^{(6)}(x) + e^x v'''(x) + \cos x v''(x) + 6v'(x) + xv(x) = f(x), \\ v(0) = 0,\ v(1) = e,\ v'(0) = 0,\ v'(1) = 4e,\ v''(0) = 0,\ v''(1) = 13e, \end{cases}$$

where $f(x)$ is selected such that the exact solution takes the form $v(x) = x^3e^x$. The absolute errors for the computed approximate solution and its corresponding derivatives are illustrated in Figure 7.

TABLE 6. A comparison of absolute errors across various methods for Test 4.3 with $\mu = 1000$

x	HPM in [12]	Method in [16]	Present method(N=14)
0.1	1.40×10^{-3}	7.80×10^{-15}	6.66×10^{-16}
0.2	1.00×10^{-2}	4.10×10^{-14}	0
0.3	3.20×10^{-2}	8.60×10^{-14}	4.44×10^{-16}
0.4	6.30×10^{-2}	1.00×10^{-13}	0
0.5	9.30×10^{-2}	9.60×10^{-14}	0
0.6	1.00×10^{-1}	6.20×10^{-14}	0
0.7	8.60×10^{-2}	4.00×10^{-14}	4.44×10^{-16}
0.8	4.70×10^{-2}	2.00×10^{-14}	2.22×10^{-16}
0.9	1.00×10^{-2}	4.40×10^{-15}	4.44×10^{-16}

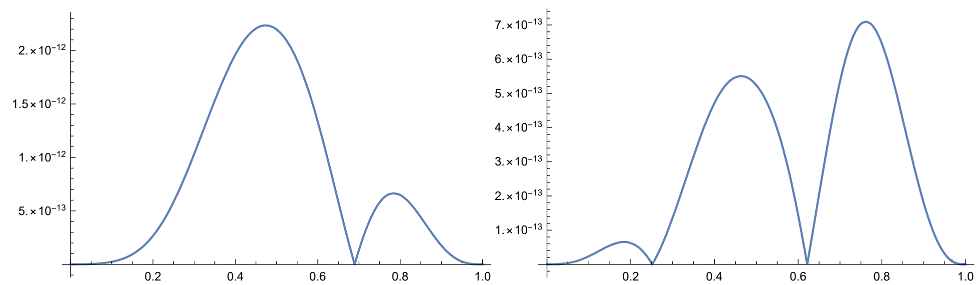


FIGURE 5. Absolute errors $|v_3(x) - v(x)|$ with $\mu = 10$(left) and $\mu = 1000$ (right) for Test 4.3.

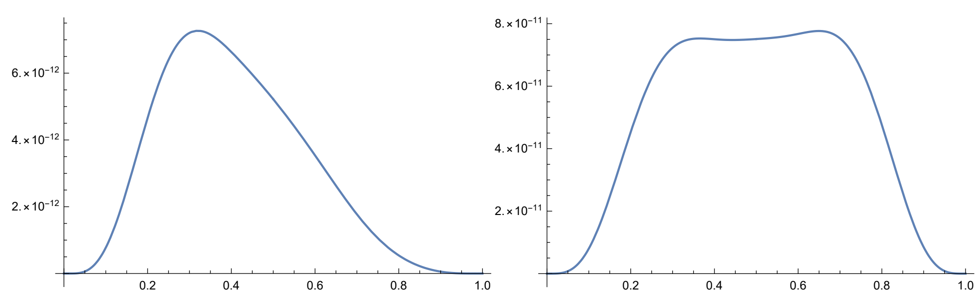
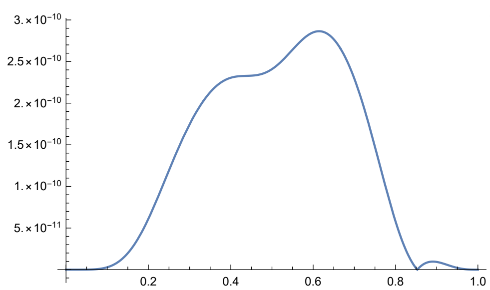


FIGURE 6. Absolute errors $|v_3(x) - v(x)|$ with $\mu = 10^4$(left) and $\mu = 10^6$ (right) for Test 4.3.

Remark: The results presented in Tables 1, 3, 5, and 6 demonstrate that the proposed method achieves higher accuracy compared to existing numerical approaches. The results in Tables 2 and 4 indicate that the proposed method exhibits eighth-order convergence. Figures 1-7 show that the proposed method achieves high accuracy with fewer nodes and also provides highly accurate derivatives of the approximate solution.



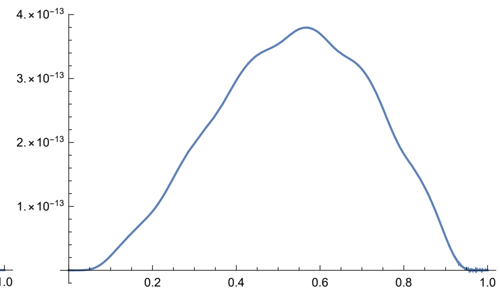


FIGURE 7. Absolute errors $|v_4(x)-v(x)|$ (left) and $|v_8(x)-v(x)|$ (right) for Test 4.4.

5. CONCLUSION

In this work, we have developed a novel numerical algorithm for solving sixth-order boundary value problems, achieving eighth-order convergence. Extensive numerical experiments demonstrate that the proposed technique outperforms existing methods in terms of accuracy and convergence order, confirming its effectiveness for linear sixth-order BVPs. The primary advantage of the proposed method lies in its high-order convergence, which allows for precise solutions with relatively coarse discretizations. Additionally, the use of reproducing kernels provides a systematic way to incorporate boundary conditions directly into the basis, enhancing the method's applicability to a wide range of linear problems. However, the approach also has some limitations. The performance of the method is sensitive to the choice of the reproducing kernel, and an inappropriate kernel may lead to reduced accuracy or stability issues. Future research directions include extending the method to nonlinear sixth-order boundary value problems.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

X.L.Li: Writing original draft.X.Y. Li: Conceptualization. F.Z. Geng: Methodology, Supervision. C.N. Li: Writing-review editing. All authors have read and agreed to the published version of the manuscript.

DECLARATIONS

Conflict of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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