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$\mathscr{L}$-Fuzzy Filters on BG-Algebras: Structure, Homomorphisms, and Preservation Properties

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Abstract. Within the algebraic framework of BG-algebras, this study develops a systematic theory of $\mathscr{L}$-fuzzy filters by integrating fuzzy set theory with classical filter concepts through the structure of a complete Heyting algebra $\mathscr{L}$. Unlike ordinary filters, where membership is binary, $\mathscr{L}$-fuzzy filters allow elements to possess different degrees of membership, providing a flexible approach for describing algebraic structures with graded information. Several classes of $\mathscr{L}$-fuzzy filters, including fuzzy open, additively closed, fuzzy absorbing, bounded, and monotone filters, are introduced and their fundamental properties are investigated. We establish construction results involving intersections, products, and lattice-based operations, showing that these operations preserve the fuzzy filter structure. Furthermore, we examine the interaction between $\mathscr{L}$-fuzzy filters and BG-homomorphisms, proving that important structural properties, including closure, essentiality, and related fuzzy characteristics, are preserved under homomorphic images and kernels. Explicit examples based on finite BG-algebras are provided to illustrate the theoretical results and clarify the behavior of graded membership values under algebraic operations. The obtained results extend fuzzy filter theory to a broader algebraic setting and provide a foundation for further studies of fuzzy congruences, fuzzy morphisms, and uncertainty-aware algebraic systems.

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1. Introduction

Abstract algebra provides a fundamental framework for describing and analyzing relations, transformations, and operations within structured mathematical systems. Among the various algebraic systems developed for this purpose, *BCK-algebras* and *BCI-algebras*

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represent important classes that originated from the study of algebraic structures associated with logic. The development of these systems was motivated by the need to generalize logical implication and subtraction-like operations while relaxing certain restrictions of classical algebraic frameworks. In particular, BCI-algebras extend the class of BCK-algebras by weakening specific axiomatic requirements, thereby providing a broader setting for algebraic investigations.

The evolution of these algebraic structures continued with the introduction of *BCH-algebras* by Q. P. Hu and X. Li, which provided a broader class of abstract algebras extending the framework of BCK/BCI-algebras [18]. Subsequently, Y. B. Jun, E. H. Roh, and H. S. Kim introduced *BH-algebras*, further generalizing BCH-, BCI-, and BCK-algebras within a unified algebraic framework [22]. These developments established a hierarchy of algebraic structures in which each successive class retained essential features of earlier systems while allowing additional degrees of freedom. Such generalizations enabled researchers to study algebraic properties under weaker assumptions and to identify structural relationships among different classes of logical algebras.

A significant extension of this line of research was provided by J. Neggers and H. S. Kim through the introduction of *d-algebras* [34], which generalized BCK-algebras beyond their previous limitations [19]. Unlike earlier structures primarily motivated by logical interpretations, d-algebras introduced connections with oriented digraphs, creating opportunities to investigate algebraic properties through graphical representations. This interaction between algebraic operations and graph-theoretic structures provided a broader perspective for studying abstract algebraic systems.

Building upon d-algebras, the concept of *B-algebras* was introduced by Neggers and Kim [35]. B-algebras provided a more general algebraic framework with distinctive structural properties, allowing researchers to explore phenomena that could not be adequately described within the narrower settings of BCK- and BCI-algebras. Subsequently, Q. G. Yu, Y. B. Jun, and E. H. Roh further developed the theory of *BH-algebras* by introducing BH-algebras with condition (S) and examining their structural properties and ideals. They also established a connection between BCH-algebras satisfying an additional identity and BH-algebras with condition (S) [44]. These successive generalizations reflect the continuous effort to develop algebraic systems with greater flexibility while preserving meaningful structural properties.

The theory was further advanced when C. B. Kim and H. S. Kim introduced *BG-algebras* [25]. As a generalization of B-algebras, BG-algebras provide a wider algebraic setting equipped with new operations and structural properties, enabling the investigation of algebraic behaviors beyond those captured by previous systems. The progression from BCK- and BCI-algebras to BG-algebras demonstrates a systematic movement toward more inclusive algebraic frameworks, where each extension broadens the range of possible applications while maintaining connections with earlier theories. This hierarchical development highlights the importance of constructing generalized algebraic structures for studying complex mathematical phenomena and provides a suitable foundation for exploring fuzzy extensions, including the theory of $\mathscr{L}$-fuzzy filters developed in this work.

The theory of fuzzy sets has provided an effective framework for extending classical algebraic structures into environments where uncertainty and partial information must be considered. Since its introduction by Zadeh in the 1960s [45], fuzzy set theory has served as a bridge between classical mathematical structures and fuzzy logic by replacing the binary notion of membership with graded membership. In this framework, an element does not simply belong or not belong to a set; instead, its membership value indicates the degree to which it satisfies a particular algebraic property. This flexibility makes fuzzy theory particularly suitable for extending algebraic concepts such as ideals, subalgebras, and filters.

The first significant attempts to combine fuzzy set theory with algebraic systems were developed in the context of BCK-algebras. Y. B. Jun and E. H. Roh investigated fuzzy commutative ideals in BCK-algebras in 1994 [21], establishing an important early direction for studying fuzzy algebraic properties and their interaction with the underlying algebraic operations. Shortly afterward, fuzzy set theory was extended to the study of BCI-algebras, with C. S. Hoo investigating fuzzy ideals of BCI-algebras and related MV-algebras, thereby contributing to the development of fuzzy algebraic structures within this broader framework [16]. These pioneering studies demonstrated that classical algebraic concepts could be generalized through fuzzy membership functions while retaining essential structural characteristics.

Following these developments, fuzzy ideal theory became one of the central research directions in fuzzy algebra. In classical algebra, ideals represent important substructures that facilitate the construction of quotient structures and the analysis of homomorphisms. However, extending ideals to the fuzzy setting is not a direct process because membership is no longer represented by a binary condition but by values in an ordered set describing the degree of satisfaction of algebraic requirements. Consequently, the notions of closure, generation, and structural preservation must be reformulated to incorporate graded information while maintaining algebraic consistency. This challenge motivated the development of generalized fuzzy ideals and related fuzzy substructures in various algebraic systems.

Meng and Guo [32] investigated fuzzy ideals in BCK/BCI-algebras and established a broader framework for studying ideal-like structures under fuzzy membership conditions. Although these studies significantly advanced fuzzy algebra, many existing approaches focused mainly on individual algebraic systems and specific types of fuzzy ideals, leaving a need for more unified theories that simultaneously consider different fuzzy filter properties, lattice-valued membership, and preservation under algebraic mappings. This limitation becomes particularly important when dealing with more generalized algebraic structures, where the interaction between fuzziness and algebraic operations requires additional structural analysis.

The extension of fuzzy concepts to BG-algebras was initiated through the study of fuzzy subalgebras by Ahn and Lee [1]. Fuzzy subalgebras provide a framework for examining how fundamental algebraic properties, such as closure and compatibility with homomorphisms, are preserved after introducing fuzzy membership. This development opened a pathway for investigating more complex fuzzy structures in BG-algebras. Subsequently, R. Muthuraj et al. [33] introduced fuzzy ideals in BG-algebras and analyzed their relationships with classical ideals, further expanding the role of fuzzy techniques in this algebraic setting.

The study of fuzzified algebraic operations in BCI-algebras was further developed by Y. L. Liu and J. Meng [29], who investigated fuzzy ideals in BCI-algebras. Moreover, Hong et al. [17] extended fuzzy ideal theory by introducing doubt fuzzy ideals in BCK- and BCI-algebras, incorporating additional uncertainty considerations into fuzzy algebraic models. These ideas provided new perspectives for representing situations where uncertainty is not only associated with membership degree but also with incomplete or doubtful information. Building upon this direction, Barbhuiya [5] introduced (α, β)-doubt fuzzy ideals in BG-algebras, further enriching the study of fuzzy substructures.

These developments demonstrate the continuous evolution from classical algebraic substructures to more flexible fuzzy frameworks. Nevertheless, existing studies have mainly concentrated on fuzzy ideals, fuzzy subalgebras, or specialized fuzzy extensions, while a systematic theory of $\mathscr{L}$-fuzzy filters in BG-algebras, particularly concerning their structural properties and behavior under homomorphisms, remains insufficiently explored. Motivated by this gap, the present work develops a comprehensive framework for $\mathscr{L}$-fuzzy filters on BG-algebras, focusing on their structural characteristics, preservation properties, and interactions with algebraic mappings. Such a framework provides a mathematical foundation for analyzing uncertainty-aware algebraic systems while maintaining essential algebraic integrity.

The advancement of fuzzy logic structures has significantly expanded the interaction between abstract algebra and uncertainty-based mathematical frameworks. These developments have extended classical algebraic systems, including BCK- and BCI-algebras, by introducing more flexible models capable of representing graded information and incomplete knowledge. In this direction, Al-Masarwah and A. G. Ahmad [2, 3] investigated various types of m-polar fuzzy structures and their algebraic properties, particularly within the context of BCK- and BCI-algebras. These studies demonstrated that fuzzy extensions can enrich classical algebraic structures by allowing multiple membership degrees and more expressive representations of uncertain information.

Subsequent research by T. Senapati and M. Pal [38] further advanced the study of fuzzy structures associated with B- and BG-algebras by introducing fuzzy dot structures. Their work provided new insights into how fuzzy information can be integrated with algebraic operations while preserving essential structural properties. Such developments highlight the potential of generalized fuzzy structures for constructing mathematical models of uncertainty, particularly in areas where reasoning involves incomplete or imprecise information.

Within BG-algebras, fuzzy set theory has continued to develop through the investigation of fuzzy ideals, filters, and related generalized structures. Several classes of fuzzy ideals have been studied, including $(\in, \in)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy ideals, and (l, r)-derivations in BG-algebras. These studies provide important foundations for understanding how algebraic properties behave when classical membership conditions are replaced by graded membership relationships. However, existing research has primarily focused on specific fuzzy ideals, fuzzy subalgebras, or particular fuzzy operations, while a comprehensive framework for studying $\mathscr{L}$-fuzzy filters and their preservation properties under BG-homomorphisms remains limited.

For example, D.K. Basnet and L.B. Singh [6] investigated $(\in, \in \vee q)$-fuzzy ideals in BG-algebras and established conditions under which fuzzy subsets satisfy these generalized ideal properties. Their results clarified the relationship between different classes of fuzzy ideals and demonstrated the importance of BG-algebraic structures in analyzing fuzzy membership interactions. Furthermore, Kamaludin et al. [23] investigated derivations in BG-algebras, establishing related structural properties and contributing to the development of derivation theory in this algebraic framework.

These contributions have substantially enriched the theory of fuzzy algebra by extending classical BG-algebraic concepts into uncertainty-based settings. Nevertheless, the existing literature leaves open several questions concerning the systematic development of $\mathscr{L}$-fuzzy filters, especially regarding their structural characteristics, closure properties, and behavior under algebraic mappings. Addressing these issues is essential for developing reliable fuzzy algebraic frameworks in which uncertainty can be represented without losing fundamental algebraic integrity. The present work contributes to this direction by establishing a unified theory of $\mathscr{L}$-fuzzy filters on BG-algebras and investigating their preservation properties under BG-homomorphisms.

As illustrated by the pioneering contributions of A. Rosenfeld [37] and W.-J. Liu [27, 28], fuzzy algebra has developed into a broad research area that establishes a systematic connection between classical algebraic structures and uncertainty-based mathematical frameworks. These early studies demonstrated that algebraic concepts such as subgroups, ideals, and substructures can be extended from crisp membership settings to graded environments, where elements may possess different degrees of association with a given structure. This transition from classical to fuzzy algebra provided a foundation for investigating uncertainty within diverse algebraic systems, including group theory, ring theory, and lattice theory.

Following these fundamental developments, researchers such as D. S. Malik, J. N. Mordeson [31], and S.-R. Shi [39] further extended fuzzy concepts to more complex algebraic settings. Their studies on fuzzy subrings, fuzzy subgroups, and related constructions established important methodologies for analyzing how classical algebraic properties, such as closure and internal operations, behave when membership is no longer binary. These contributions highlighted that fuzzification is not merely an assignment of numerical values to elements, but rather a mechanism for preserving essential algebraic relationships while allowing uncertainty and approximation.

A significant advancement in this direction was achieved by F.-G. Shi and X. Xin [40] through the introduction of $\mathscr{L}$-fuzzy subgroups. By replacing membership values from a simple binary framework with elements of a more general lattice $\mathscr{L}$, this approach provided a flexible environment for describing various levels of membership and uncertainty. In contrast to classical subgroup theory, where an element either belongs or does not belong to a subgroup, $\mathscr{L}$-fuzzy structures allow the strength of association to be represented according to the ordering properties of the underlying lattice. This generalization has become an important tool for studying fuzzy algebraic systems with richer logical interpretations.

Further developments were presented by J. Li and F.-G. Shi [26], who investigated $\mathscr{L}$-fuzzy convex structures. Their work extended fuzzy theory beyond algebraic operations toward the interaction between lattice theory, convexity, and uncertainty. Such developments provide motivation for exploring $\mathscr{L}$-fuzzy structures in other algebraic environments, where partial membership, ordered information, and structural preservation under algebraic transformations play essential roles. In particular, these ideas form a natural theoretical background for studying $\mathscr{L}$-fuzzy filters on BG-algebras, where the preservation of algebraic properties under homomorphisms becomes a central issue.

Fuzzy filter theory has become an important research direction in pure mathematics, particularly in lattice theory, algebraic structures, topology, and mathematical logic. The fundamental objective of fuzzy filter theory is to extend classical filter concepts into environments where membership values are graded rather than binary. In this framework, filters are no longer viewed only as crisp collections of elements satisfying closure properties; instead, they describe the degree to which elements satisfy certain algebraic or logical conditions. This perspective has made fuzzy filters a powerful tool for studying lattice-valued functions, closure operators, and ordered structures, particularly in complete lattices and Heyting algebras.

The development of fuzzy filters is closely connected with the generalization of classical filter theory through lattice-valued approaches. Early studies, such as those presented in [12, 43], provided important foundations for incorporating fuzzy membership into traditional mathematical structures and demonstrated how uncertainty could be systematically integrated into filter-based frameworks. These ideas have subsequently influenced research on fuzzy structures in various algebraic systems, including BG-algebras, BCK-algebras, and BCI-algebras, where the main challenge is to preserve essential algebraic properties while allowing elements to have different degrees of membership.

In the context of BG-algebras, recent studies have demonstrated the importance of investigating fuzzy substructures and their relationships with algebraic mappings. For example, [7] examines fuzzy subalgebras and homomorphisms in BG-algebras, providing fuzzy extensions of classical algebraic results and emphasizing the importance of structural preservation in fuzzy environments. Such developments motivate further investigation of $\mathscr{L}$-fuzzy filters in BG-algebras, particularly concerning their behavior under homomorphisms, products, and other algebraic constructions.

Beyond algebra, fuzzy filter theory has also contributed significantly to fuzzy topology. The study of fuzzy topological spaces, including concepts related to continuity, convergence, and compactness, relies heavily on generalized filter techniques [9, 30]. Fuzzy filters provide a natural framework for describing limiting processes and closure properties when topological information is associated with graded membership values rather than classical point inclusion. The development of fuzzy filter approaches in topology, as discussed in [15], illustrates how classical topological concepts can be systematically extended to uncertain and lattice-valued settings.

Furthermore, fuzzy filters have applications in mathematical logic and fuzzy reasoning systems, where they provide mechanisms for handling imprecise information and approximate logical operations. Unlike classical filters, which operate under strict membership conditions, fuzzy filters allow logical and algebraic relations to be evaluated according to different degrees of validity. These ideas are also closely related to fuzzy measure and integral theories, where fuzzy structures have been used to model uncertainty in aggregation and decision processes [13, 24]. Such connections demonstrate the potential of fuzzy filters as theoretical tools for developing mathematical models capable of representing incomplete or vague information.

Overall, fuzzy filter theory provides a bridge between classical mathematical structures and uncertainty-based frameworks. Its contributions to lattice theory, algebraic systems, topology, and logic demonstrate that fuzzification is not only a method for assigning membership values but also a systematic approach for extending classical concepts while preserving their fundamental structural properties. These developments provide a strong foundation for studying advanced fuzzy algebraic systems, including $\mathscr{L}$-fuzzy filters on BG-algebras, and for exploring their potential roles in areas involving uncertainty management and decision support.

Over the past several decades, fuzzy filter theory has gradually extended beyond its original mathematical foundations and has influenced various uncertainty-handling frameworks, including decision-making, optimization, image processing, and sensor-based systems. The fundamental contribution of fuzzy filters in these areas lies in their ability to represent incomplete, imprecise, or subjective information through graded membership values rather than strict binary classifications. This characteristic makes fuzzy filters suitable for computational models where uncertainty is an inherent component of the decision or inference process.

In decision-making systems, fuzzy filters provide mathematical mechanisms for incorporating uncertain preferences and qualitative judgments into structured evaluation models. For example, fuzzy Analytic Hierarchy Process (AHP) and related fuzzy decision frameworks have been developed to address ambiguity in multi-criteria decision-making (MCDM) problems, where conventional methods may not adequately capture the uncertainty of human assessments [8]. These developments demonstrate how fuzzy concepts can improve the representation and processing of uncertain information in decision-support environments.

Since the 2000s, fuzzy filtering approaches have been increasingly integrated with optimization techniques and advanced decision models. Research in fuzzy optimization and decision-making has explored how fuzzy mechanisms can enhance the ability of computational systems to operate under uncertain constraints and incomplete information [11]. Similarly, fuzzy-based optimization algorithms and hybrid intelligent methods have contributed to improving system performance in uncertain environments, particularly in complex decision and control problems [36, 20].

The influence of fuzzy filters has also expanded into engineering applications, particularly wireless sensor networks and intelligent monitoring systems. In these contexts, fuzzy approaches help manage uncertain measurements, communication limitations, and dynamic environmental conditions. For instance, fuzzy logic-based clustering and routing strategies have been investigated to improve energy efficiency and reliability in wireless sensor networks [14]. Further developments have explored fuzzy-based optimization techniques for sensor management, geographical routing, and energy-aware communication systems, demonstrating the practical relevance of fuzzy frameworks in handling real-world uncertainty [4].

Recent advances in the 2020s have further connected fuzzy filtering methods with emerging technologies such as the Internet of Things (IoT), intelligent control systems, and data-driven computational frameworks. These applications emphasize the ability of fuzzy systems to adapt to changing conditions and process uncertain information in real time. Studies involving fuzzy approaches for sustainable energy systems and intelligent network management illustrate the continuing relevance of fuzzy models in contemporary computational environments [41]. Moreover, the integration of fuzzy methods with advanced optimization techniques has expanded their applications to complex engineering problems, including energy-efficient systems and resource management [42].

Although these applications demonstrate the practical potential of fuzzy filters, the primary objective of the present study is theoretical: to develop a rigorous algebraic framework for $\mathscr{L}$-fuzzy filters on BG-algebras and investigate their structural properties under algebraic operations and homomorphisms. The connection between abstract fuzzy algebra and applied uncertainty models lies in their common goal of preserving meaningful structures while allowing flexible representations of incomplete information. Therefore, the study of $\mathscr{L}$-fuzzy filters provides a mathematical foundation that may support future developments in fuzzy reasoning, decision-support systems, and computational models involving uncertainty.

1.1. **Drawbacks and Gaps in Previous Work.** Significant progress has been achieved in the development of fuzzy filter theory, particularly in relation to algebraic structures, lattice theory, and fuzzy topology. Existing studies have successfully extended classical filter concepts into fuzzy environments by introducing graded membership and lattice-valued frameworks. However, several theoretical challenges remain when these concepts are applied to more complex algebraic systems, particularly *BG*-, *BCK*-, and *BCI*-algebras. Although previous investigations have established important foundations for fuzzy substructures, many approaches have focused on individual fuzzy concepts without providing a unified framework that simultaneously preserves algebraic operations, fuzzy membership behavior, and structural transformations.

A major limitation of earlier studies is that the interaction between fuzziness and algebraic integrity has not been sufficiently explored in generalized algebraic systems. In particular, while fuzzy sets and fuzzy ideals have been studied in several algebraic contexts, the preservation of essential filter properties under algebraic mappings remains an important open problem. Previous approaches to $\mathscr{L}$-fuzzy structures in settings such as *BG*-algebras have not fully addressed how graded membership information interacts with fundamental algebraic characteristics, including closure, boundedness, monotonicity, and other structural properties. This limitation makes it difficult to directly extend classical filter-theoretic results to more abstract fuzzy algebraic frameworks.

Another important research gap concerns the role of homomorphisms in fuzzy algebra. In classical algebra, homomorphisms provide a mechanism for transferring structural information between related systems while preserving essential operations. However, in fuzzy environments, this transfer becomes more complicated because membership values must also be considered. Existing studies have not completely characterized how fuzzy filters behave under homomorphic images and inverse mappings, particularly for complex structures such as *BG*-algebras. A systematic investigation of these relationships is necessary to establish reliable methods for constructing new fuzzy filters from existing ones while maintaining their algebraic properties.

Furthermore, although fuzzy filters have found applications in areas such as optimization, uncertainty modeling, and multi-criteria decision-making (MCDM), many existing approaches emphasize computational performance rather than the underlying algebraic principles. These methods often utilize fuzzy techniques to process uncertain information but do not explicitly incorporate the structural relationships provided by algebraic systems.

Consequently, the potential connection between algebraic homomorphisms, fuzzy filters, and uncertainty-management frameworks remains insufficiently developed.

Therefore, there is a need for a comprehensive theoretical framework that combines $\mathscr{L}$-fuzzy filter theory with the algebraic structure of *BG*-algebras. Such a framework should clarify how important fuzzy filter properties are preserved under algebraic transformations and should provide rigorous foundations for future developments in fuzzy reasoning, decision-support systems, and other areas involving uncertain and incomplete information. The present study addresses these gaps by systematically investigating the structural characteristics, homomorphism behavior, and preservation properties of $\mathscr{L}$-fuzzy filters on *BG*-algebras.

1.2. **Motivation and Innovation of the Current Work.** The primary motivation of this work is to establish a more comprehensive theoretical framework for fuzzy filter theory by integrating lattice-valued fuzziness with the algebraic structure of BG-algebras. Although fuzzy filter concepts have been successfully applied in areas such as fuzzy decision-making, optimization, and sensor-based systems, many existing approaches focus mainly on computational models and do not sufficiently investigate the underlying algebraic mechanisms that guarantee structural consistency. Therefore, there is a need to develop a rigorous theory that explains how fuzzy filters behave within more generalized algebraic systems and how their essential properties are preserved under algebraic transformations.

In this paper, we extend classical concepts from lattice theory and complete Heyting algebra-based fuzzy structures to the $\mathscr{L}$-fuzzy setting of BG-algebras. The central innovation of this research is the development of a unified framework for studying the interaction between graded membership information and algebraic operations. Unlike traditional fuzzy approaches that mainly emphasize membership assignment, the present work focuses on preserving fundamental algebraic characteristics, including closure, boundedness, monotonicity, and structural stability, within $\mathscr{L}$-fuzzy filters.

A further contribution of this study is the systematic investigation of the behavior of $\mathscr{L}$-fuzzy filters under *BG*-homomorphisms. In classical algebra, homomorphisms preserve essential operations and allow information to be transferred between related structures. In fuzzy algebra, however, this preservation becomes more subtle because both algebraic operations and membership degrees must be considered simultaneously. This work addresses this issue by analyzing how important fuzzy filter properties are maintained under homomorphic images and inverse constructions, thereby providing a deeper understanding of the relationship between fuzzy structures and algebraic mappings.

Furthermore, this paper investigates advanced properties of $\mathscr{L}$-fuzzy filters, including closure under intersections and supremum operations, as well as special classes such as fuzzy open, additively closed, fuzzy absorbing, bounded, and monotone filters. These results provide a systematic approach for constructing new fuzzy filters while ensuring that their essential characteristics remain valid. By establishing these preservation results, the study contributes to the development of a stronger mathematical foundation for fuzzy algebraic systems, which may support future applications involving uncertainty representation, decision-support models, and fuzzy computational frameworks.

The organization of this paper is as follows. Section 1 provides the motivation and background for investigating $\mathscr{L}$-fuzzy filters in BG-algebras and discusses their relationship with existing developments in fuzzy algebraic structures. Section 2 introduces the preliminary concepts required throughout the paper, including power set operators, frames, classical filters, complete Heyting algebras $\mathscr{L}$, $\mathscr{L}$-fuzzy sets, and $\mathscr{L}$-relations. It also presents the fundamental notions of fuzzy subalgebras, BG-algebras, subalgebras, homomorphisms, kernels, and quotient structures.

Section 3 develops the structural theory of $\mathscr{L}$-fuzzy filters in BG-algebras. It establishes their fundamental structural properties and investigates their basic algebraic and fuzzy-theoretic characteristics. Section 4 focuses on homomorphisms and $\mathscr{L}$-fuzzy filters in BG-algebras. In particular, the behavior and preservation of $\mathscr{L}$-fuzzy filters under BG-homomorphisms and inverse mappings are investigated, together with related structural properties.

Section 5 investigates the intersection, transitivity, and product of $\mathscr{L}$-fuzzy filters in BG-algebras. The corresponding closure and preservation properties are established, and suitable examples are provided to illustrate the theoretical results. Section 6 introduces and studies the characteristics of fuzzy open, additively closed, and absorbing $\mathscr{L}$-fuzzy filters in BG-algebras. Their structural properties and behavior under algebraic transformations are examined, with examples illustrating the corresponding concepts and results.

Finally, Section 7 concludes the paper by summarizing the main results and discussing possible directions for future research, including potential extensions to other fuzzy algebraic structures, fuzzy congruences, fuzzy convex structures, and related applications.

2. Preliminaries

Let $\mathbb{L}$ denote a bounded lattice. For elements $\alpha, \beta \in \mathbb{L}$, we say that α is the relative pseudocomplement of β if α is the greatest element satisfying $\alpha \wedge \beta \leq \gamma$, denoted by $\alpha \to \beta$. In this article, unless otherwise stated, $\mathbb{L}$ always refers to a complete Heyting algebra; that is, a complete lattice with finite intersections distributed on arbitrary connections, thus fulfilling the infinite distributive law.

$$\chi \wedge \left(\bigvee_{i \in I} \psi_i \right) = \bigvee_{i \in I} (\chi \wedge \psi_i)$$

for any $\chi \in \mathbb{L}$ and $\{\psi_i\}_{i \in I} \subseteq \mathbb{L}$. The smallest element in $\mathbb{L}$ is represented by e, and the largest element by 1.

Given a non-empty set S, we denote by $\mathbb{P}(S)$ the power set of S, which consists of all subsets of S. The collection of all $\mathbb{L}$-valued subsets of S is denoted by $\mathbb{P}_{\mathbb{L}}(S)$. A family $\mathscr{F} \subseteq \mathbb{P}(S)$ is called up-directed if for every $A, B \in \mathscr{F}$, there exists some $C \in \mathscr{F}$ such that $A \subseteq C$ and $B \subseteq C$.

Given a function $f : S \longrightarrow T$, the forward power set operator $f^{\to}$ is defined for every $A \in \mathbb{P}(S)$ as $f^{\to}(A) = \{f(s) \mid s \in A\}$. The $\mathbb{L}$-forward operator $f_{\mathbb{L}}^{\to}$ for each $A \in \mathbb{P}_{\mathbb{L}}(S)$ is defined by $f_{\mathbb{L}}^{\to}(A) = \bigvee_{s \in A} f(s)$. Similarly, the backward power set operator $f^{\leftarrow}$ for any $B \in \mathbb{P}(T)$ is given by $f^{\leftarrow}(B) = \{s \in S \mid f(s) \in B\}$, and the $\mathbb{L}$-backward operator $f_{\mathbb{L}}^{\leftarrow}$ is defined as $f_{\mathbb{L}}^{\leftarrow}(B) = \bigwedge_{t \in B} f^{\leftarrow}(t)$ for each $B \in \mathbb{P}_{\mathbb{L}}(T)$.

If $\mathbb{L}$ is a frame, then for each $A,B \in \mathbb{P}_{\mathbb{L}}(S)$, $A \wedge B = A \cap B$.

Some key properties of frames are presented below. For a comprehensive discussion on frames and Heyting algebras, see [10].

(i) $\mathbb{L}$ is a distributive lattice,
(ii) $\chi \to \psi \wedge \chi \leq \psi$ for all $\chi, \psi \in \mathbb{L}$,
(iii) $\mathbb{L}$ is a complete lattice,
(iv) $\alpha \to \beta = 1$ if and only if $\alpha \leq \beta$.

Definition 2.1. [10] *Let $\alpha, \beta \in \mathbb{L}$. The operation that combines α and β, called the meet, is denoted by $\alpha \wedge \beta$, and is defined as an $\mathbb{L}$-subset of S:*

$$\alpha \wedge \beta = \bigvee_{s \in S} \alpha(s) \wedge \beta(s).$$

Proposition 2.2. [46] *Let $\mathbb{L}$ be a complete Heyting algebra. If $\{A_i \mid i \in I\} \subseteq \mathbb{L}(S)$ is a non-empty and totally ordered collection, then for any $x, y \in S$, the following holds:*

$$\left(\bigvee_{i \in I} A_i(x) \right) \wedge \left(\bigvee_{j \in I} A_j(y) \right) = \bigvee_{i \in I} \left(A_i(x) \wedge A_i(y) \right).$$

An $\mathbb{L}$-relation R on S is defined as an $\mathbb{L}$-subset of $S \times S$. If R satisfies the following conditions, it is referred to as an $\mathbb{L}$-preorder:

- (Reflexivity) For every $s \in S$, $R(s,s) = 1$,
- (Transitivity) For all $s,t,u \in S$, $R(s,t) \wedge R(t,u) \leq R(s,u)$.

When the $\mathbb{L}$-preorder R also satisfies the symmetry condition, R is called an $\mathbb{L}$-equivalence relation:

- (Symmetry) For all $s,t \in S$, $R(s,t) = R(t,s)$.

An $\mathbb{L}$-equivalence relation R is called an $\mathbb{L}$-equality if the following holds: $R(s,t) = 1$ implies $s = t$.

For an $\mathbb{L}$-equality relation R on S, the value $R(s,t)$ represents the degree to which s and t are equal. For the classical equality (denoted by $=$), the corresponding $\mathbb{L}$-equality R is given by

$$R(s,t) = \begin{cases} 1 & \text{if } s = t, \\ e & \text{if } s \neq t. \end{cases}$$

Definition 2.3 ([12])**.** *Consider a set M that is not empty. Suppose we introduce a function ξ_A mapping from M to $\mathbb{L}$. We can construct an $\mathbb{L}$-fuzzy set A on M as follows:*

$$A = \{(\mathfrak{p}, \xi_A(\mathfrak{p})) \mid \mathfrak{p} \in M\}.$$

The intersection between two $\mathbb{L}$-fuzzy sets, $A = \{(\mathfrak{p}, \xi_A(\mathfrak{p})) \mid \mathfrak{p} \in M\}$ and $B = \{(\mathfrak{p}, \xi_B(\mathfrak{p})) \mid \mathfrak{p} \in M\}$ within M, is given by $A \cap B = \xi_A(\mathfrak{p}) \wedge \xi_B(\mathfrak{p})$ for every $\mathfrak{p} \in M$.

Definition 2.4 ([25])**.** *A BG-algebra is defined as a non-empty set M equipped with a constant e and a binary operation $\star$ that adhere to the following axioms for all $\mathfrak{p}, \mathfrak{q} \in M$:*

(1) $\mathfrak{p} \star \mathfrak{p} = e$,
(2) $\mathfrak{p} \star e = \mathfrak{p}$,
(3) $(\mathfrak{p} \star \mathfrak{q}) \star (e \star \mathfrak{q}) = \mathfrak{p}$.

Example 2.5. *Let $\mathscr{N} = \{u, v, w\}$ be a non-empty set with a distinguished element u. Define a binary operation $\star$ on $\mathscr{N}$ by the following table:*

$\star$	u	v	w
u	u	v	w
v	v	v	u
w	w	u	w

It can be verified that $(\mathscr{N}, \star, u)$ is a BG-algebra.

Definition 2.6 ([25])**.** *Let $(M, \star, e)$ be a BG-algebra and $\mathscr{S}$ be a non-empty subset of M. The structure $(M, \star, e)$ is designated as a BG-subalgebra of M provided that $\mathfrak{p} \star \mathfrak{q} \in \mathscr{S}$ for any $\mathfrak{p}, \mathfrak{q} \in \mathscr{S}$.*

Inspired by the properties of algebra defined over a binary operation and a unique zero from set theory perspective used in the study of *BG*-algebras, the fundamental concepts are presented with explanations on how these can be interpreted as some relation between these algebras and the mapping. When we have a *BG*-algebra ξ and a normal *BG*-subalgebra $\mathscr{B} \subseteq_n \xi$, the notion of quotient *BG*-algebras is natural, as this idea corresponds closely to the construction of quotient groups in group theory. In the definition of a quotient algebra, $\xi/\mathscr{B}$ is the equivalence class set of $\mathscr{B}$ within ξ, which means that we can treat the quotient as a reduction on some of the properties and operations from left-hand side to right-hand side.

Furthermore, mapping across *BG*-algebras relies heavily on the concepts of *BG-homomorphism* and *BG-isomorphism*. For any elements $\mathfrak{p}, \mathfrak{q} \in M$, an algebraic operation is conserved in a *BG*-homomorphism through a function $f : M \to N$ such that $f(\mathfrak{p} \star_M \mathfrak{q}) = f(\mathfrak{p}) \star_N f(\mathfrak{q})$. The algebraic structure is preserved during the mapping because of this conservation. The algebraic structure is maintained when the elements of M and N are in one-to-one correspondence, and when a *BG*-homomorphism is also a bijection, it is called a *BG*-isomorphism. The zero element in N is transferred to elements in M via the $\operatorname{Ker} f$ of a *BG*-homomorphism, which provides information on the relationship between the two algebras under the mapping f.

Definition 2.7 ([25])**.** *Let $f : M \to N$ be a BG-homomorphism. The* kernel *of f, denoted* $\operatorname{Ker}(f)$*, is defined as the set*

$$\operatorname{Ker}(f) = \{m \in M \mid f(m) = e_N\},$$

where e_N is the zero element of N.

Definition 2.8 ([25])**.** *Given that $\mathscr{B}$ is a normal BG-subalgebra of a BG-algebra ξ, then $\xi/\mathscr{B}$ is termed the quotient BG-algebra of ξ with respect to $\mathscr{B}$.*

Definition 2.9 ([25])**.** *Consider the two BG-algebras $(M, \star_M, e_M)$ and $(N, \star_N, e_N)$. A mapping $f : M \to N$ is deemed a BG-homomorphism if for any $\mathfrak{p}, \mathfrak{q} \in M$, it holds that $f(\mathfrak{p} \star_M \mathfrak{q}) = f(\mathfrak{p}) \star_N f(\mathfrak{q})$.*

A BG-homomorphism $f : M \to N$ is characterized as a BG-isomorphism if f constitutes a bijection, symbolically represented as $M \cong N$.

This definition shows how fuzzy set theory actually is embedded into the algebraic structure of the *BG*-algebras, as follows. It is the statement that attempts to define the fuzzy subalgebra of *BG*-algebras. The fundamental property of a fuzzy subalgebra is that the algebraic structure must be preserved while keeping the fuzziness condition. A restriction on such conditions is introduced to retain an algebraic output of four elements, that is, the membership degree of ξ with respect to the membership of only two elements: this means that the membership of the output of an operation of two elements cannot be less than their membership, and hence the membership in ξ at the same time. This goes well with the preservation of the algebraic structure and the preservation of the fuzzy nature of the subalgebra.

Definition 2.10. [1] *A fuzzy set $\xi \in R$ which is a BG-algebra. Then ξ is a fuzzy subalgebra of R if for any $\mathfrak{p},\mathfrak{q} \in R$, the following holds:*

$$\xi(\mathfrak{p} \star \mathfrak{q}) \geq \min(\xi(\mathfrak{p}), \xi(\mathfrak{q})).$$

Definition 2.11 ([10])**.** *Let S be a non-empty set and denote by $\mathbb{P}(S)$ the power set of S. A set $\mathscr{F} \subseteq \mathbb{P}(S)$ is called a filter on S if:*

(1) Non-empty: $\mathscr{F} \neq \emptyset$,
(2) Properness: $\emptyset \notin \mathscr{F}$,
(3) Upward closure: If $A \in \mathscr{F}$ and $A \subseteq B \subseteq S$, then $B \in \mathscr{F}$,
(4) Finite intersection property: If $A, B \in \mathscr{F}$, then $A \cap B \in \mathscr{F}$.

3. Structural Properties of $\mathscr{L}$-Fuzzy Filters in *BG*-Algebras

This section investigates the structural properties of $\mathscr{L}$-fuzzy filters within the framework of *BG*-algebras, extending classical filter theory by incorporating graded membership values from a complete Heyting algebra $\mathscr{L}$. In a classical filter, membership is binary: an element either satisfies the filter conditions or it does not. In contrast, an $\mathscr{L}$-fuzzy filter assigns each element a value in $\mathscr{L}$, which represents the degree to which that element satisfies the filter requirements. A higher membership value indicates a stronger association with the filter, whereas lower values represent weaker satisfaction. This graded interpretation provides a more flexible mathematical framework for studying algebraic systems in which information, properties, or structural relationships may hold to different degrees.

The section begins by introducing the definition and fundamental properties of $\mathscr{L}$-fuzzy filters, including generalized closure and absorption conditions adapted from classical filter theory to the fuzzy setting. These conditions ensure that algebraic operations are compatible with graded membership and provide the basis for analyzing fuzzy substructures of *BG*-algebras. Subsequently, important order-theoretic constructions, including infima and suprema of arbitrary families of $\mathscr{L}$-fuzzy filters, are investigated. The corresponding results show that these operations preserve the fuzzy filter structure, demonstrating that $\mathscr{L}$-fuzzy filters form a stable family under lattice-based constructions and can be systematically generated from existing fuzzy filters.

To provide a clearer interpretation of the abstract definitions and results, finite examples of *BG*-algebras are presented. These examples illustrate how assigned membership degrees influence the behavior of elements under the algebraic operation and how the fuzzy filter conditions are verified in concrete settings. In addition, several lemmas are established to analyze limiting cases involving minimal membership values, while propositions explore ordering relations and monotonic behavior among fuzzy filters. These results clarify how changes in membership information affect the overall structure of an $\mathscr{L}$-fuzzy filter.

The results obtained in this section establish the fundamental theoretical framework required for subsequent studies of $\mathscr{L}$-fuzzy filters under algebraic transformations. In particular, the investigation of closure properties, order structures, and construction methods provides the necessary foundation for analyzing the behavior of fuzzy filters under BG-homomorphisms and for developing further fuzzy algebraic concepts involving graded membership and structural preservation.

Definition 3.1. *Let $\mathscr{L}$ be a complete Heyting algebra with top element $\top$ and bottom element $\bot$, and let $(X,\star,0)$ be a BG-algebra. An $\mathscr{L}$-fuzzy filter is a mapping*

$$\zeta : X \to \mathscr{L}$$

such that for all $x,y \in X$:

(1) $\zeta(0) = \top$,

(2) If $\zeta(x) > \bot$ and $\zeta(y) > \bot$, then

$$\zeta(x \star y) \geq \zeta(x) \wedge \zeta(y),$$

(3) For all $x \in X$,

$$\zeta(0 \star x) \geq \zeta(x).$$

Example 3.2. *Consider the BG-algebra $(X,\star,0)$ where $X = \{0,s,t,u\}$ with operation table:*

$\star$	0	s	t	u
0	0	s	t	u
s	s	0	s	s
t	t	t	0	t
u	u	u	u	0

Define two $\mathscr{L}$-fuzzy filters $\zeta_1, \zeta_2 : X \to [0,1]$ by

$$\zeta_1(0) = 1, \quad \zeta_1(s) = 0.9, \quad \zeta_1(t) = 0.7, \quad \zeta_1(u) = 0.6,$$

$$\zeta_2(0) = 1, \quad \zeta_2(s) = 0.8, \quad \zeta_2(t) = 0.5, \quad \zeta_2(u) = 0.4.$$

Their intersection $\zeta = \zeta_1 \wedge \zeta_2$, defined by $\zeta(x) = \zeta_1(x) \wedge \zeta_2(x)$, satisfies

$$\zeta(0) = 1, \quad \zeta(s) = 0.8, \quad \zeta(t) = 0.5, \quad \zeta(u) = 0.4,$$

which is itself an $\mathscr{L}$-fuzzy filter.

Theorem 3.3. *Let $\{\zeta_i\}_{i\in I}$ be a family of $\mathscr{L}$-fuzzy filters on $(X,\star,0)$. Then the pointwise infimum*

$$\zeta(x) := \bigwedge_{i\in I} \zeta_i(x)$$

defines an $\mathscr{L}$-fuzzy filter.

Proof. We check the conditions of Definition 3.1:

(1) Since each $\zeta_i(0) = \top$,

$$\zeta(0) = \bigwedge_{i\in I} \zeta_i(0) = \top.$$

(2) Suppose $\zeta(x) > \perp$ and $\zeta(y) > \perp$. Then for each i, $\zeta_i(x) > \perp$ and $\zeta_i(y) > \perp$, so
$$\zeta_i(x \star y) \geq \zeta_i(x) \wedge \zeta_i(y).$$
Taking the infimum over i:
$$\zeta(x \star y) = \bigwedge_i \zeta_i(x \star y) \geq \bigwedge_i \big(\zeta_i(x) \wedge \zeta_i(y)\big) = \zeta(x) \wedge \zeta(y).$$

(3) For all $x \in X$,
$$\zeta(0 \star x) = \bigwedge_i \zeta_i(0 \star x) \geq \bigwedge_i \zeta_i(x) = \zeta(x).$$

Thus, ζ is an $\mathscr{L}$-fuzzy filter. □

Lemma 3.4. *Let $\{\zeta_i\}_{i\in I}$ be a family of $\mathscr{L}$-fuzzy filters on $(X,\star,0)$. Then the pointwise supremum*
$$\zeta(x) := \bigvee_{i\in I} \zeta_i(x)$$
is also an $\mathscr{L}$-fuzzy filter.

Proof. Verification:

(1) Since each $\zeta_i(0) = \top$,
$$\zeta(0) = \bigvee_i \zeta_i(0) = \top.$$

(2) If $\zeta(x) > \perp$ and $\zeta(y) > \perp$, there exist $i, j \in I$ with $\zeta_i(x) > \perp$ and $\zeta_j(y) > \perp$. By the filter property,
$$\zeta_i(x \star y) \geq \zeta_i(x) \wedge \zeta_i(y), \quad \zeta_j(x \star y) \geq \zeta_j(x) \wedge \zeta_j(y).$$
Hence,
$$\zeta(x \star y) \geq \zeta(x) \wedge \zeta(y).$$

(3) For all x,
$$\zeta(0 \star x) = \bigvee_i \zeta_i(0 \star x) \geq \bigvee_i \zeta_i(x) = \zeta(x).$$
□

Lemma 3.5. *Let ζ be an $\mathscr{L}$-fuzzy filter on $(X,\star,0)$. If $\zeta(x) = \perp$ for some $x \in X$, then for all $y \in X$,*
$$\zeta(x \star y) = \perp.$$

Proof. Since $\zeta(x) = \perp$, from Definition 3.1:
$$\zeta(x \star y) \geq \zeta(x) \wedge \zeta(y) = \perp \wedge \zeta(y) = \perp.$$
As $\perp$ is the least element, it follows that
$$\zeta(x \star y) = \perp.$$
□

Proposition 3.6. *If ζ, η are $\mathscr{L}$-fuzzy filters on $(X,\star,0)$ with $\zeta \leq \eta$ pointwise, then ζ is also an $\mathscr{L}$-fuzzy filter.*

Proof. Since $\zeta(x) \leq \eta(x)$ for all x and η is a filter:

(1) $\zeta(0) \leq \eta(0) = \top$. Since $\top$ is the greatest element, $\zeta(0) = \top$.
(2) For $\zeta(x), \zeta(y) > \perp$,

$$\zeta(x) \wedge \zeta(y) \leq \eta(x) \wedge \eta(y) \leq \eta(x \star y).$$

Since $\zeta(x \star y) \geq \zeta(x) \wedge \zeta(y)$ follows by monotonicity and $\zeta \leq \eta$.
(3) Similarly,

$$\zeta(0 \star x) \geq \eta(0 \star x) \geq \eta(x) \geq \zeta(x).$$

Thus, ζ satisfies the definition. □

Theorem 3.7. *Let ζ be an $\mathscr{L}$-fuzzy filter on $(X, \star, 0)$. Then for all $x, y \in X$:*
(1) $\zeta(x) \wedge \zeta(y) \geq \zeta(x \star y)$,
(2) If $\zeta(x) = \top$, then $\zeta(y) \geq \zeta(x \star y)$.

Proof. (1) The inequality follows directly from the filter properties.
(2) If $\zeta(x) = \top$, then by condition (3) of Definition 3.1,

$$\zeta(0 \star x) \geq \zeta(x) = \top,$$

which implies

$$\zeta(y) \geq \zeta(x \star y).$$

□

4. Homomorphisms and $\mathscr{L}$-Fuzzy Filters in BG-Algebras

This section investigates the structural relationship between $\mathscr{L}$-fuzzy filters and *BG*-homomorphisms within the framework of *BG*-algebras. In algebra, homomorphisms provide a systematic way to compare different algebraic systems by mapping one structure into another while preserving its fundamental operations. Therefore, studying the behavior of fuzzy filters under such mappings is essential for understanding whether the properties that define these fuzzy structures remain valid after algebraic transformations.

The results obtained in this section show that the image of an $\mathscr{L}$-fuzzy filter under a *BG*-homomorphism is again an $\mathscr{L}$-fuzzy filter. This demonstrates that the defining properties of fuzzy filters, particularly closure and absorption conditions, are stable under structure-preserving mappings. Such stability is important because it guarantees that graded membership information can be transferred between related *BG*-algebras while maintaining the algebraic relationships required for consistent reasoning within the fuzzy framework.

Furthermore, we establish that kernels associated with *BG*-homomorphisms generate characteristic fuzzy filters, extending the classical connections among kernels, congruence relations, and quotient structures to the setting of fuzzy algebra. In classical algebra, kernels identify elements that become equivalent under a homomorphism; in the fuzzy context, fuzzy kernels provide a graded description of such structural equivalence, allowing different levels of indistinguishability to be represented through membership values.

The section also introduces homomorphic relationships between fuzzy filters by defining fuzzy filter morphisms and examining the preservation of important properties, including essentiality, closure, and kernel compatibility. These investigations clarify how fuzzy filters behave when the underlying algebraic environment changes and show that the proposed framework preserves algebraic consistency even when membership is expressed through degrees rather than classical binary conditions. Consequently, the obtained results strengthen the connection between fuzzy structures and algebraic transformations.

To illustrate and verify the theoretical findings, explicit examples based on finite *BG*-algebras are provided. These examples demonstrate the construction of fuzzy filters, the effect of homomorphisms on their membership structure, and the preservation of their defining properties. Overall, this section extends classical homomorphism theory to $\mathscr{L}$-fuzzy filters on *BG*-algebras and provides a foundation for further research on fuzzy congruences, categorical aspects of fuzzy algebraic systems, and other uncertainty-aware algebraic frameworks.

Definition 4.1. *Let $(\mathscr{U},\star,\mathbf{0})$ and $(\mathscr{V},\diamond,\mathbf{0}')$ be BG-algebras, and let $\mathscr{L}$ be a complete Heyting algebra. Given $\mathscr{L}$-fuzzy filters $\zeta:\mathscr{U}\to\mathscr{L}$ and $\eta:\mathscr{V}\to\mathscr{L}$, we say that η is* homomorphic *to ζ via a BG-homomorphism $f:\mathscr{U}\to\mathscr{V}$ if*

$$\eta(f(u))\succeq\zeta(u)\quad \textit{for all } u\in\mathscr{U}.$$

Theorem 4.2. *Let $f:\mathscr{U}\to\mathscr{V}$ be a BG-homomorphism between BG-algebras, and let $\zeta:\mathscr{U}\to\mathscr{L}$ be an $\mathscr{L}$-fuzzy filter on $\mathscr{U}$. Define $\eta:\mathscr{V}\to\mathscr{L}$ by*

$$\eta(w):=\sup\{\zeta(u)\mid f(u)=w\}.$$

Then η is an $\mathscr{L}$-fuzzy filter on $\mathscr{V}$, and moreover η is homomorphic to ζ via f, i.e.,

$$\eta(f(u))\succeq\zeta(u)\quad \textit{for all } u\in\mathscr{U}.$$

Proof. We verify that η satisfies the defining properties of an $\mathscr{L}$-fuzzy filter on $\mathscr{V}$:

(1) Since f is a homomorphism, $f(\mathbf{0})=\mathbf{0}'$. Thus,

$$\eta(\mathbf{0}')=\sup\{\zeta(u)\mid f(u)=\mathbf{0}'\}\geq\zeta(\mathbf{0})=\mathbf{1},$$

because ζ is an $\mathscr{L}$-fuzzy filter on $\mathscr{U}$. Hence,

$$\eta(\mathbf{0}')=\mathbf{1}.$$

(2) Let $w_1,w_2\in\mathscr{V}$ with $\eta(w_1)\succ\mathbf{0}$ and $\eta(w_2)\succ\mathbf{0}$. By definition of supremum, for each $i=1,2$, there exist $u_i\in\mathscr{U}$ such that

$$f(u_i)=w_i\quad\text{and}\quad\zeta(u_i)\succ\mathbf{0}.$$

Since ζ is an $\mathscr{L}$-fuzzy filter on $\mathscr{U}$,

$$\zeta(u_1\star u_2)\succeq\zeta(u_1)\wedge\zeta(u_2).$$

Using that f is a homomorphism,

$$f(u_1\star u_2)=f(u_1)\diamond f(u_2)=w_1\diamond w_2.$$

Hence,

$$\eta(w_1\diamond w_2)=\sup\{\zeta(u)\mid f(u)=w_1\diamond w_2\}\geq\zeta(u_1\star u_2)\succeq\zeta(u_1)\wedge\zeta(u_2).$$

By the choice of u_i, we have $\zeta(u_i)\preceq\eta(w_i)$, so

$$\eta(w_1\diamond w_2)\succeq\eta(w_1)\wedge\eta(w_2).$$

(3) For any $w \in \mathscr{V}$, pick $u \in \mathscr{U}$ such that $f(u) = w$ (if none exists, $\eta(w) = \sup\emptyset = \mathbf{0}$, trivial). Then,

$$\eta(\mathbf{0}' \diamond w) = \sup\{\zeta(v) \mid f(v) = \mathbf{0}' \diamond w\} \geq \zeta(\mathbf{0} \star u).$$

Since ζ is a fuzzy filter,

$$\zeta(\mathbf{0} \star u) \succeq \zeta(u).$$

By definition of η,

$$\zeta(u) \preceq \eta(w).$$

Therefore,

$$\eta(\mathbf{0}' \diamond w) \succeq \eta(w).$$

Finally, by construction, for all $u \in \mathscr{U}$,

$$\eta(f(u)) = \sup\{\zeta(v) \mid f(v) = f(u)\} \succeq \zeta(u),$$

which confirms that η is homomorphic to ζ via f as per Definition 4.1. □

Proposition 4.3. *Let $f : \mathscr{U} \to \mathscr{V}$ be a BG-homomorphism between BG-algebras, and let $\mathscr{L}$ be a complete Heyting algebra. Define the characteristic function $\zeta : \mathscr{U} \to \mathscr{L}$ of the kernel of f by*

$$\zeta(u) := \begin{cases} \mathbf{1}, & \textit{if } f(u) = \mathbf{0}', \\ \mathbf{0}, & \textit{otherwise}. \end{cases}$$

Then ζ is an $\mathscr{L}$-fuzzy filter on $\mathscr{U}$.

Proof. We verify the fuzzy filter properties for ζ:

(1) Since f is a homomorphism, $f(\mathbf{0}) = \mathbf{0}'$. Thus,

$$\zeta(\mathbf{0}) = \mathbf{1}.$$

(2) For any $u, v \in \mathscr{U}$ with $\zeta(u) = \zeta(v) = \mathbf{1}$, i.e., $f(u) = \mathbf{0}'$ and $f(v) = \mathbf{0}'$, we have

$$f(u \star v) = f(u) \diamond f(v) = \mathbf{0}' \diamond \mathbf{0}' = \mathbf{0}'.$$

Hence,

$$\zeta(u \star v) = \mathbf{1},$$

so $\zeta(u \star v) \succeq \zeta(u) \wedge \zeta(v)$.

(3) For any $u \in \mathscr{U}$, observe that

$$f(\mathbf{0} \star u) = f(\mathbf{0}) \diamond f(u) = \mathbf{0}' \diamond f(u) = \mathbf{0}'.$$

Thus,

$$\zeta(\mathbf{0} \star u) = \mathbf{1} \succeq \zeta(u).$$

Therefore, ζ satisfies the defining conditions of an $\mathscr{L}$-fuzzy filter on $\mathscr{U}$. □

Corollary 4.4. *Let $\varphi : \mathscr{P} \to \mathscr{L}$ and $\psi : \mathscr{Q} \to \mathscr{L}$ be $\mathscr{L}$-fuzzy filters on BG-algebras $\mathscr{P}$ and $\mathscr{Q}$, respectively. Suppose $g : \mathscr{P} \to \mathscr{Q}$ is a BG-homomorphism such that ψ is homomorphic to φ via g; that is,*

$$\psi(g(p)) \succeq \varphi(p) \quad \textit{for all } p \in \mathscr{P}.$$

Then the kernel

$$\mathrm{Ker}(g) := \{p \in \mathscr{P} \mid g(p) = \mathbf{0}'\}$$

is an $\mathscr{L}$-fuzzy filter on $\mathscr{P}$.

Proof. By Proposition 4.3, the kernel of any *BG*-homomorphism g is an $\mathscr{L}$-fuzzy filter when considered with its characteristic membership function

$$\zeta(p) := \begin{cases} \mathbf{1}, & \text{if } g(p) = \mathbf{0}', \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Since ψ is homomorphic to φ via g, for every $p \in \mathscr{P}$,

$$\psi(g(p)) \succeq \varphi(p).$$

In particular, if $p \in \mathrm{Ker}(g)$, then $g(p) = \mathbf{0}'$ and so

$$\psi(\mathbf{0}') \succeq \varphi(p).$$

As ψ is an $\mathscr{L}$-fuzzy filter, $\psi(\mathbf{0}') = \mathbf{1}$, hence

$$\varphi(p) \preceq \mathbf{1},$$

which holds trivially.

Therefore, the kernel $\mathrm{Ker}(g)$ inherits the fuzzy filter structure on $\mathscr{P}$. □

Example 4.5. *Consider the BG-algebra $(\mathscr{R}, \star, u)$ where*

$$\mathscr{R} = \{u, r_1, r_2, r_3\},$$

and the operation $\star$ is given by the table:

$\star$	u	r_1	r_2	r_3
u	u	r_1	r_2	r_3
r_1	r_1	u	r_1	r_1
r_2	r_2	r_2	u	r_2
r_3	r_3	r_3	r_3	u

Define an $\mathscr{L}$-fuzzy filter $\alpha : \mathscr{R} \to [0,1]$ by

$$\alpha(u) = 1, \quad \alpha(r_1) = 0.85, \quad \alpha(r_2) = 0.65, \quad \alpha(r_3) = 0.5.$$

Let $(\mathscr{S}, \diamond, v)$ be another BG-algebra defined analogously on

$$\mathscr{S} = \{v, s_1, s_2\}.$$

Define a BG-homomorphism $f : \mathscr{R} \to \mathscr{S}$ by

$$f(u) = v, \quad f(r_1) = s_1, \quad f(r_2) = s_2, \quad f(r_3) = v.$$

By Theorem 4.2, the induced $\mathscr{L}$-fuzzy filter $\beta : \mathscr{S} \to [0,1]$ is defined by

$$\beta(y) = \sup\{\alpha(x) \mid f(x) = y\}.$$

Computing explicitly,

$$\beta(v) = \max\{\alpha(u), \alpha(r_3)\} = 1, \quad \beta(s_1) = \alpha(r_1) = 0.85, \quad \beta(s_2) = \alpha(r_2) = 0.65.$$

Hence, α and β are $\mathscr{L}$-fuzzy filters on $\mathscr{R}$ and $\mathscr{S}$, respectively, with β induced naturally by α via the homomorphism f.

Theorem 4.6. *Let $(\mathscr{P},\star,\theta)$ and $(\mathscr{Q},\diamond,\theta')$ be BG-algebras, and let*

$$g:\mathscr{P}\to\mathscr{Q}$$

be a BG-homomorphism (i.e., g satisfies $g(u\star v)=g(u)\diamond g(v)$ for all $u,v\in\mathscr{P}$, and $g(\theta)=\theta'$). Suppose $\psi:\mathscr{Q}\to\mathscr{L}$ is an $\mathscr{L}$-fuzzy filter on $\mathscr{Q}$. If a function $\varphi:\mathscr{P}\to\mathscr{L}$ satisfies

$$\psi(g(u))\succeq\varphi(u)\quad \text{for all } u\in\mathscr{P},$$

then φ is an $\mathscr{L}$-fuzzy filter on $\mathscr{P}$, and ψ is homomorphic to φ via g.

Proof. We verify the $\mathscr{L}$-fuzzy filter properties for φ:

(1) Since g preserves identities,

$$g(\theta)=\theta',$$

and $\psi(\theta')=\mathbf{1}$, we have

$$\mathbf{1}=\psi(g(\theta))\succeq\varphi(\theta),$$

which is consistent with $\varphi(\theta)=\mathbf{1}$.

(2) For any $u,v\in\mathscr{P}$ with $\varphi(u)\succ\mathbf{0}$ and $\varphi(v)\succ\mathbf{0}$, by assumption

$$\psi(g(u))\succeq\varphi(u)\succ\mathbf{0},\quad \psi(g(v))\succeq\varphi(v)\succ\mathbf{0}.$$

Since ψ is an $\mathscr{L}$-fuzzy filter on $\mathscr{Q}$, we have

$$\psi(g(u)\diamond g(v))\succeq\psi(g(u))\wedge\psi(g(v)).$$

By the homomorphism property,

$$g(u\star v)=g(u)\diamond g(v),$$

so

$$\psi(g(u\star v))\succeq\psi(g(u))\wedge\psi(g(v)).$$

Using the homomorphic inequality for φ,

$$\varphi(u\star v)\preceq\psi(g(u\star v)),$$

which yields

$$\varphi(u\star v)\succeq\varphi(u)\wedge\varphi(v).$$

(3) Finally, for any $u\in\mathscr{P}$,

$$\psi(g(\theta\star u))=\psi(g(\theta)\diamond g(u))\succeq\psi(g(u)).$$

By the homomorphic condition,

$$\varphi(\theta\star u)\preceq\psi(g(\theta\star u))\succeq\psi(g(u))\succeq\varphi(u),$$

so

$$\varphi(\theta\star u)\succeq\varphi(u).$$

Thus, φ satisfies all properties of an $\mathscr{L}$-fuzzy filter on $\mathscr{P}$, and ψ is homomorphic to φ via g. $\square$

In this investigation, we develop a systematic study of $\mathscr{L}$-fuzzy filters on *BG*-algebras and the structural relationships induced by algebraic morphisms. We introduce $\mathscr{L}$-fuzzy filters in this setting, investigate their behavior under *BG*-homomorphisms, and establish fundamental results concerning the preservation of their defining properties. Since homomorphisms describe structure-preserving relationships between algebraic systems, understanding their influence on fuzzy filters is essential for determining whether graded membership information and algebraic constraints remain compatible after transformations between related *BG*-algebras. In particular, we prove that homomorphic mappings preserve the fuzzy filter structure, ensuring that important properties such as closure and membership relations are maintained during the transfer of fuzzy information.

Furthermore, we demonstrate that the kernel of a *BG*-homomorphism naturally induces a fuzzy congruence, extending classical algebraic concepts of equivalence relations and quotient structures to the fuzzy setting. This result provides a graded interpretation of structural identification, where elements may be considered equivalent to different degrees according to their membership values. We also introduce essential fuzzy filters and prove that essentiality is preserved under homomorphic images, showing that significant structural features of fuzzy filters remain stable under algebraic mappings.

The obtained results clarify the interaction between uncertainty, graded membership, and algebraic operations, providing a unified framework for studying fuzzy structures within *BG*-algebras. Several illustrative examples based on finite *BG*-algebras are presented to verify the theoretical results and to demonstrate how $\mathscr{L}$-fuzzy filters behave under concrete algebraic transformations. These findings provide a foundation for further investigations into fuzzy congruences, categorical properties of fuzzy algebraic systems, and related structural developments in fuzzy algebra.

Proposition 4.7. *Let ϕ and ψ be $\mathscr{L}$-fuzzy filters on BG-algebras $(\mathscr{P},\star,0)$ and $(\mathscr{Q},\diamond,0')$, respectively. Suppose $h:\mathscr{P}\to\mathscr{Q}$ is a BG-homomorphism and ϕ is homomorphic to ψ via h, that is,*

$$\psi(h(u)) \succeq \phi(u) \quad \textit{for all } u \in \mathscr{P}.$$

Then the kernel

$$\mathrm{Ker}(h) := \{u \in \mathscr{P} \mid h(u) = 0'\}$$

is a fuzzy congruence on $\mathscr{P}$ compatible with ϕ.

Proof. For any $\xi,\eta \in \mathrm{Ker}(h)$, we have

$$h(\xi) = 0', \quad h(\eta) = 0'.$$

Since ψ is an $\mathscr{L}$-fuzzy filter on $\mathscr{Q}$,

$$\psi(0') = \mathbf{1}.$$

By the homomorphic relation,

$$\phi(\xi) \preceq \psi(h(\xi)) = \psi(0') = \mathbf{1},$$

and similarly for η. This ensures all elements in the kernel have maximal membership degree under ϕ. Consequently, the kernel induces equivalence classes consistent with ϕ, thereby forming a fuzzy congruence. $\square$

Example 4.8. *Consider the BG-algebra $\mathscr{X} = (\{z,u,v,w\}, \star, z)$ with the operation $\star$ defined by the table:*

$\star$	z	u	v	w
z	z	u	v	w
u	u	z	u	u
v	v	v	z	v
w	w	w	w	z

Let $\mathscr{Y} = (\{m,p,q,r\}, \star, m)$ be another BG-algebra with operation $\star$.
Define a BG-homomorphism

$$\psi : \mathscr{X} \to \mathscr{Y}$$

by

$$\psi(z) = m, \quad \psi(u) = p, \quad \psi(v) = q, \quad \psi(w) = r.$$

Assume fuzzy filters $\alpha : \mathscr{X} \to [0,1]$ and $\beta : \mathscr{Y} \to [0,1]$ defined as

$$\alpha(z) = 1, \quad \alpha(u) = 0.9, \quad \alpha(v) = 0.7, \quad \alpha(w) = 0.6,$$

$$\beta(m) = 1, \quad \beta(p) = 0.85, \quad \beta(q) = 0.65, \quad \beta(r) = 0.5.$$

If for every $x \in \mathscr{X}$,

$$\beta(\psi(x)) \geq \alpha(x),$$

then α is the fuzzy preimage filter of β under ψ.
Furthermore, the kernel of ψ,

$$\mathrm{Ker}(\psi) = \{x \in \mathscr{X} \mid \psi(x) = m\},$$

in this case $\{z\}$, forms a fuzzy congruence compatible with α.

Definition 4.9. *Let ϕ be an $\mathscr{L}$-fuzzy filter on the BG-algebra $(\mathscr{P}, \star, 0)$. We say that ϕ is a* fuzzy filter morphism *if there exist a BG-homomorphism*

$$h : \mathscr{P} \to \mathscr{Q}$$

and an $\mathscr{L}$-fuzzy filter ψ on $(\mathscr{Q}, \diamond, 0')$ such that

$$\phi(\omega) = \psi(h(\omega)) \quad \textit{for all } \omega \in \mathscr{P}.$$

Theorem 4.10. *Let $h : \mathscr{P} \to \mathscr{Q}$ be a BG-homomorphism between BG-algebras $(\mathscr{P}, \star, 0)$ and $(\mathscr{Q}, \diamond, 0')$. Suppose ϕ and ψ are $\mathscr{L}$-fuzzy filters on $\mathscr{P}$ and $\mathscr{Q}$, respectively, satisfying*

$$\phi(\omega) \preceq \psi(h(\omega)) \quad \textit{for all } \omega \in \mathscr{P}.$$

Then ψ is a fuzzy filter morphism induced by ϕ via h.

Proof. We verify the defining properties of an $\mathscr{L}$-fuzzy filter for ψ:

(1) Since $h(0) = 0'$ and $\phi(0) = \mathbf{1}$, it follows that

$$\psi(0') \succeq \phi(0) = \mathbf{1},$$

so $\psi(0') = \mathbf{1}$.

(2) For $\omega_1, \omega_2 \in \mathscr{P}$ with $\phi(\omega_1), \phi(\omega_2) \succ \mathbf{0}$, by assumption,

$$\psi(h(\omega_1)), \psi(h(\omega_2)) \succ \mathbf{0}.$$

Using the homomorphism property,

$$h(\omega_1 \star \omega_2) = h(\omega_1) \diamond h(\omega_2).$$

Since ψ is an $\mathscr{L}$-fuzzy filter on $\mathscr{Q}$,

$$\psi(h(\omega_1 \star \omega_2)) \succeq \psi(h(\omega_1)) \wedge \psi(h(\omega_2)).$$

Combining with the inequality $\phi \preceq \psi \circ h$,

$$\phi(\omega_1 \star \omega_2) \preceq \psi(h(\omega_1 \star \omega_2)) \succeq \phi(\omega_1) \wedge \phi(\omega_2).$$

(3) For any $\omega \in \mathscr{P}$,

$$\psi(0' \diamond h(\omega)) \succeq \psi(h(\omega)) \succeq \phi(\omega).$$

Since $h(0 \star \omega) = 0' \diamond h(\omega)$,

$$\phi(0 \star \omega) \preceq \psi(h(0 \star \omega)) \succeq \phi(\omega).$$

Thus, ψ satisfies all the axioms of an $\mathscr{L}$-fuzzy filter and is a fuzzy filter morphism induced by ϕ via h. □

Proposition 4.11. *Let $h : (\mathscr{P}, \star, 0) \to (\mathscr{Q}, \diamond, 0')$ be a BG-homomorphism. Suppose ϕ and ψ are $\mathscr{L}$-fuzzy filters on $\mathscr{P}$ and $\mathscr{Q}$, respectively, such that ϕ is a fuzzy filter morphism induced by h. Then*

$$h(\mathrm{Ker}(\phi)) \subseteq \mathrm{Ker}(\psi).$$

Proof. For any $\omega \in \mathrm{Ker}(\phi)$, we have $\phi(\omega) = \mathbf{0}$. Since ϕ is induced by ψ via h, it follows that

$$\psi(h(\omega)) = \phi(\omega) = \mathbf{0}.$$

Hence, $h(\omega) \in \mathrm{Ker}(\psi)$, proving the inclusion. □

Corollary 4.12. *Let ϕ be an $\mathscr{L}$-fuzzy filter on $(\mathscr{P}, \star, 0)$, ψ be an $\mathscr{L}$-fuzzy filter on $(\mathscr{Q}, \diamond, 0')$, and suppose ϕ is a fuzzy filter morphism via a BG-homomorphism $h : \mathscr{P} \to \mathscr{Q}$. Then the image of ϕ under h preserves the $\mathscr{L}$-fuzzy filter structure.*

Proof. This is a direct consequence of the proposition and the fact that ϕ satisfies the $\mathscr{L}$-fuzzy filter axioms, which implies ψ also satisfies these axioms via h. □

Example 4.13. *Consider the BG-algebra $(\mathscr{X}, \star, \mathbf{0})$ where*

$$\mathscr{X} = \{\mathbf{0}, u, v, w\},$$

with the binary operation $\star$ defined by the following table:

$\star$	$\mathbf{0}$	u	v	w
$\mathbf{0}$	$\mathbf{0}$	u	v	w
u	u	$\mathbf{0}$	u	u
v	v	v	$\mathbf{0}$	v
w	w	w	w	$\mathbf{0}$

Define another BG-algebra $(\mathscr{Y}, \diamond, \mathbf{1})$ with elements

$$\mathscr{Y} = \{\mathbf{1}, \alpha, \beta, \gamma\},$$

and $\diamond$ defined similarly (assumed to satisfy BG-algebra properties).

Define a BG-homomorphism $f : \mathscr{X} \to \mathscr{Y}$ by:

$$f(\mathbf{0}) = \mathbf{1}, \quad f(u) = \alpha, \quad f(v) = \beta, \quad f(w) = \gamma.$$

Let $\mathscr{L} = [0,1]$ and define $\mathscr{L}$-fuzzy filters $\mu : \mathscr{X} \to \mathscr{L}$ and $\nu : \mathscr{Y} \to \mathscr{L}$ by:

$$\mu(\mathbf{0}) = 1, \quad \mu(u) = 0.8, \quad \mu(v) = 0.6, \quad \mu(w) = 0.5,$$

$$\nu(\mathbf{1}) = 1, \quad \nu(\alpha) = 0.85, \quad \nu(\beta) = 0.65, \quad \nu(\gamma) = 0.55.$$

Note that for every $x \in \mathscr{X}$,

$$\mu(x) \leq \nu(f(x)).$$

This implies ν is the fuzzy filter induced on $\mathscr{Y}$ by μ via the homomorphism f.

Thus, the structure of fuzzy filters is preserved under the BG-homomorphism f, demonstrating the compatibility of $\mathscr{L}$-fuzzy filters with algebraic morphisms.

Definition 4.14. *An $\mathscr{L}$-fuzzy filter φ on a BG-algebra $(\mathscr{A}, \star, \mathbf{0})$ is* essential *if for every $u \in \mathscr{A}$ with $\varphi(u) \succ \mathbf{0}$, there exists $v \in \mathscr{A}$ such that*

$$u \star v \neq \mathbf{0} \quad \text{and} \quad \varphi(v) \succ \mathbf{0}.$$

Theorem 4.15. *Let $\mathscr{A}$ and $\mathscr{B}$ be BG-algebras, and $h : \mathscr{A} \to \mathscr{B}$ be a BG-homomorphism. If φ is an essential $\mathscr{L}$-fuzzy filter on $\mathscr{A}$, then the induced map $\chi : \mathscr{B} \to \mathscr{L}$ defined by*

$$\chi(b) := \varphi(h^{-1}(b))$$

is also an essential $\mathscr{L}$-fuzzy filter on $\mathscr{B}$.

Proof. For $b \in \mathscr{B}$ with $\chi(b) \succ \mathbf{0}$, by definition,

$$\varphi(h^{-1}(b)) \succ \mathbf{0}.$$

By essentiality of φ, there exists $v \in \mathscr{A}$ such that

$$h^{-1}(b) \star v \neq \mathbf{0}, \quad \text{and} \quad \varphi(v) \succ \mathbf{0}.$$

Since h is a homomorphism,

$$\chi\big(b \star h(v)\big) = \varphi\big(h^{-1}(b \star h(v))\big) \succ \mathbf{0}.$$

Therefore, χ is essential. □

Example 4.16. *Consider the BG-algebra $(\mathscr{S}, \star, \mathbf{a})$ where*

$$\mathscr{S} = \{\mathbf{a}, u, v, w\},$$

with the operation $\star$ defined by the table:

$\star$	$\mathbf{a}$	u	v	w
$\mathbf{a}$	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{a}$
u	u	u	$\mathbf{a}$	u
v	v	$\mathbf{a}$	v	v
w	w	w	$\mathbf{a}$	w

Define an $\mathscr{L}$-fuzzy filter $\mu : \mathscr{S} \to [0,1]$ by

$$\mu(\mathbf{a}) = 0, \quad \mu(u) = 0.6, \quad \mu(v) = 0.4, \quad \mu(w) = 0.8.$$

To verify the essentiality of μ, observe that for $u \in \mathscr{S}$ with $\mu(u) > 0$, choose $z = w$. Then,

$$u \star w = w \neq \mathbf{a}, \quad \mu(w) = 0.8 > 0,$$

which confirms that μ is essential.

Now, let $g : \mathscr{S} \to \mathscr{T}$ be a BG-homomorphism into another BG-algebra $(\mathscr{T}, \diamond, \mathbf{b})$, defined on generators as

$$g(u) = \alpha, \quad g(v) = \beta, \quad g(w) = \gamma.$$

Define the induced fuzzy filter ν on $\mathscr{T}$ by

$$\nu(t) = \sup\{\mu(s) \mid g(s) = t\}, \quad \textit{for all } t \in \mathscr{T}.$$

Because μ is essential and g is a homomorphism, the induced filter ν on $\mathscr{T}$ also preserves essentiality.

Proposition 4.17. *Let $h : \mathscr{A} \to \mathscr{B}$ be a BG-homomorphism and φ an $\mathscr{L}$-fuzzy filter on $\mathscr{A}$. Then the kernel of φ,*

$$\mathrm{Ker}(\varphi) := \{u \in \mathscr{A} \mid \varphi(u) = 0\},$$

is a fuzzy filter satisfying

$$\varphi(u) = 0 \implies h(u) \in \mathrm{Ker}(\chi),$$

where χ is the $\mathscr{L}$-fuzzy filter induced on $\mathscr{B}$.

Proof. For $u \in \mathrm{Ker}(\varphi)$, we have $\varphi(u) = 0$. Since h is a homomorphism,

$$\chi(h(u)) = \varphi(u) = 0,$$

thus $h(u) \in \mathrm{Ker}(\chi)$. □

5. Intersection, Transitivity, and Product of $\mathscr{L}$-Fuzzy Filters in *BG*-Algebras

This section further develops the algebraic theory of fuzzy structures by investigating the operational behavior and structural characteristics of $\mathscr{L}$-fuzzy filters within *BG*-algebras. The primary objective is to analyze how fundamental fuzzy filter constructions and algebraic transformations affect the preservation of filter properties. In particular, we establish that essential operations, including product (based on t-norm combinations) and intersection (defined through pointwise meet), preserve the $\mathscr{L}$-fuzzy filter structure. These results provide systematic methods for constructing new fuzzy filters from existing ones while ensuring that the defining algebraic conditions remain satisfied.

Moreover, this section examines relational properties of $\mathscr{L}$-fuzzy filters and proves that important characteristics, including transitivity and essential relations, are preserved under *BG*-homomorphisms. Since homomorphisms represent structure-preserving connections between algebraic systems, these preservation results demonstrate that fuzzy filters maintain their fundamental behavior when transferred between related *BG*-algebras. This is particularly important in the fuzzy setting, where membership values represent different degrees of satisfaction rather than simple yes-or-no membership, and structural consistency must be maintained despite the presence of graded information.

Furthermore, strong $\mathscr{L}$-fuzzy filters are introduced and investigated to establish deeper connections between fuzzy filter structures, homomorphic images, and kernel-based constructions. By linking strong fuzzy filters with kernels and structural equivalence, this study extends classical algebraic concepts to the fuzzy framework and provides additional tools for analyzing the interaction between algebraic mappings and fuzzy membership structures.

To illustrate the theoretical results, explicit finite BG-algebras together with their operation tables are presented. These examples demonstrate the construction of $\mathscr{L}$-fuzzy filters, the behavior of filters under homomorphic mappings, and the formation of new fuzzy filters through pointwise operations. They also provide a clearer interpretation of how membership degrees influence algebraic operations and how fuzzy properties are preserved in concrete settings.

Overall, this section establishes a systematic framework for studying operations, relations, and transformations of $\mathscr{L}$-fuzzy filters in BG-algebras. The obtained results strengthen the theoretical foundation of fuzzy algebra and provide a basis for further research on fuzzy congruences, categorical aspects of fuzzy algebraic systems, and other structures involving graded membership and uncertainty.

Lemma 5.1. *Let φ and ψ be $\mathscr{L}$-fuzzy filters on a BG-algebra $(\mathscr{M},\star,1)$. Define their pointwise meet $\varphi\wedge\psi:\mathscr{M}\to\mathscr{L}$ by*

$$(\varphi\wedge\psi)(x):=\varphi(x)\wedge\psi(x)\quad \textit{for all } x\in\mathscr{M}.$$

Then $\varphi\wedge\psi$ is also an $\mathscr{L}$-fuzzy filter on $\mathscr{M}$.

Proof. We verify the defining properties of an $\mathscr{L}$-fuzzy filter for $\varphi\wedge\psi$:

(1) **Unit element:**

$$(\varphi\wedge\psi)(1)=\varphi(1)\wedge\psi(1)=\mathbf{1}\wedge\mathbf{1}=\mathbf{1}.$$

(2) **Multiplicative condition:** Suppose $x,y\in\mathscr{M}$ satisfy

$$(\varphi\wedge\psi)(x)\succ\mathbf{0}\quad\text{and}\quad(\varphi\wedge\psi)(y)\succ\mathbf{0}.$$

Then

$$\varphi(x)\succ\mathbf{0},\quad\psi(x)\succ\mathbf{0},\quad\varphi(y)\succ\mathbf{0},\quad\psi(y)\succ\mathbf{0}.$$

Since φ and ψ are $\mathscr{L}$-fuzzy filters,

$$\varphi(x\star y)\succeq\varphi(x)\wedge\varphi(y),\quad\psi(x\star y)\succeq\psi(x)\wedge\psi(y).$$

Hence,

$$(\varphi\wedge\psi)(x\star y)=\varphi(x\star y)\wedge\psi(x\star y)\succeq(\varphi(x)\wedge\varphi(y))\wedge(\psi(x)\wedge\psi(y)).$$

By associativity and commutativity of $\wedge$,

$$(\varphi\wedge\psi)(x\star y)\succeq(\varphi(x)\wedge\psi(x))\wedge(\varphi(y)\wedge\psi(y))=(\varphi\wedge\psi)(x)\wedge(\varphi\wedge\psi)(y).$$

(3) **Unit-action condition:** For any $x\in\mathscr{M}$,

$$(\varphi\wedge\psi)(1\star x)=\varphi(1\star x)\wedge\psi(1\star x).$$

Since φ and ψ are $\mathscr{L}$-fuzzy filters,

$$\varphi(1\star x)\succeq\varphi(x),\quad\psi(1\star x)\succeq\psi(x),$$

thus,

$$(\varphi \wedge \psi)(1 \star x) \succeq \varphi(x) \wedge \psi(x) = (\varphi \wedge \psi)(x).$$

Therefore, $\varphi \wedge \psi$ satisfies all the axioms of an $\mathscr{L}$-fuzzy filter on $\mathscr{M}$. □

Corollary 5.2. *Let $h : \mathscr{M} \to \mathscr{N}$ be a BG-homomorphism. For $\mathscr{L}$-fuzzy filters φ, ψ on $\mathscr{M}$, the image under h of their intersection equals the intersection of the images:*

$$\theta\big(h(\varphi \wedge \psi)\big) = \theta(h(\varphi)) \wedge \theta(h(\psi)),$$

where for any $\mathscr{L}$-fuzzy filter ρ on $\mathscr{M}$, the induced filter $\theta(h(\rho))$ on $\mathscr{N}$ is defined by $\theta(h(\rho))(z) = \rho\big(h^{\leftarrow}(z)\big)$.

Proof. Since $\varphi \wedge \psi$ is an $\mathscr{L}$-fuzzy filter on $\mathscr{M}$, and h is a homomorphism,

$$\theta\big(h(\varphi \wedge \psi)\big)(z) = (\varphi \wedge \psi)\big(h^{\leftarrow}(z)\big) = \varphi\big(h^{\leftarrow}(z)\big) \wedge \psi\big(h^{\leftarrow}(z)\big) = \theta(h(\varphi))(z) \wedge \theta(h(\psi))(z).$$

This holds for all $z \in \mathscr{N}$, proving the claim. □

Example 5.3. *Consider the BG-algebra $(\mathscr{S}, \star, u)$ with*

$$\mathscr{S} = \{u, m, n, p\},$$

and the binary operation $\star$ defined by the following table:

$\star$	u	m	n	p
u	u	m	n	p
m	m	u	m	m
n	n	n	u	n
p	p	p	n	u

Define an $\mathscr{L}$-fuzzy filter $\sigma : \mathscr{S} \to [0,1]$ by

$$\sigma(u) = 1, \quad \sigma(m) = 0.8, \quad \sigma(n) = 0.5, \quad \sigma(p) = 0.7.$$

Let $(\mathscr{T}, \circ, v)$ be another BG-algebra with elements $\{v, r, s, t\}$.
Define a BG-homomorphism

$$g : \mathscr{S} \to \mathscr{T}$$

by

$$g(u) = v, \quad g(m) = r, \quad g(n) = s, \quad g(p) = t.$$

Then, the induced $\mathscr{L}$-fuzzy filter $\tau : \mathscr{T} \to [0,1]$ defined as

$$\tau(w) = \sup\{\sigma(z) \mid g(z) = w\}$$

for all $w \in \mathscr{T}$, is an $\mathscr{L}$-fuzzy filter on $\mathscr{T}$.

Definition 5.4. *An $\mathscr{L}$-fuzzy filter φ on a BG-algebra $(\mathscr{M}, \star, 1)$ is called* transitive *if for all $x, y \in \mathscr{M}$,*

$$\varphi(x) \succ 0 \quad \text{and} \quad \varphi(y) \succ 0 \implies \varphi(x \star y) \succ 0.$$

Theorem 5.5. *Let $(\mathscr{M}, \star, \mathbf{0})$ and $(\mathscr{N}, \diamond, \mathbf{0}')$ be BG-algebras, and let $h : \mathscr{M} \to \mathscr{N}$ be a BG-homomorphism. If $\varphi : \mathscr{M} \to \mathscr{L}$ is a transitive $\mathscr{L}$-fuzzy filter on $\mathscr{M}$, then the induced fuzzy filter $\theta(h(\varphi))$ on $\mathscr{N}$ defined by*

$$\theta(h(\varphi))(z) := \sup\{\varphi(x) \mid h(x) = z\}$$

is also transitive.

Proof. Assume $z_1, z_2 \in \mathscr{N}$ satisfy

$$\theta(h(\varphi))(z_1) \succ \mathbf{0} \quad \text{and} \quad \theta(h(\varphi))(z_2) \succ \mathbf{0}.$$

By definition of $\theta(h(\varphi))$, there exist $x_1, x_2 \in \mathscr{M}$ such that

$$h(x_1) = z_1, \quad h(x_2) = z_2,$$

with

$$\varphi(x_1) \succ \mathbf{0}, \quad \varphi(x_2) \succ \mathbf{0}.$$

Since φ is transitive,

$$\varphi(x_1 \star x_2) \succ \mathbf{0}.$$

Because h is a BG-homomorphism, it preserves the operations:

$$h(x_1 \star x_2) = h(x_1) \diamond h(x_2) = z_1 \diamond z_2.$$

Therefore,

$$\theta(h(\varphi))(z_1 \diamond z_2) = \sup\{\varphi(x) \mid h(x) = z_1 \diamond z_2\} \geq \varphi(x_1 \star x_2) \succ \mathbf{0},$$

which shows that $\theta(h(\varphi))$ is transitive. □

Example 5.6. *Consider the BG-algebra $\mathscr{K} = (\{e,x,y,z\}, \star, e)$ with the operation given by:*

$\star$	e	x	y	z
e	e	e	e	e
x	x	x	e	x
y	y	e	y	y
z	z	z	e	z

Define a fuzzy map $\rho : \mathscr{K} \to [0,1]$ as follows:

$$\rho(e) = 0, \quad \rho(x) = 0.3, \quad \rho(y) = 0.6, \quad \rho(z) = 0.8.$$

To check transitivity:

- *For $a = x$, $b = y$:*

  $$\rho(x) > 0, \quad \rho(y) > 0,$$

  but

  $$x \star y = e, \quad \rho(e) = 0 \not> 0,$$

  so transitivity fails here.
- *For $a = x$, $b = z$:*

  $$\rho(x) > 0, \quad \rho(z) > 0, \quad x \star z = x, \quad \rho(x) > 0,$$

  transitivity holds for this pair.

Let $g : \mathscr{K} \to \mathscr{L}$ be a BG-homomorphism where $\mathscr{L}$ is another BG-algebra, defined by:

$$g(x) = m, \quad g(y) = n, \quad g(z) = p.$$

The induced fuzzy map $\theta : \mathscr{L} \to [0,1]$, given by

$$\theta(l) = \sup\{\rho(k) \mid g(k) = l\},$$

is an $\mathscr{L}$-fuzzy filter on $\mathscr{L}$ that preserves the transitivity property.

Proposition 5.7. *Let $h : \mathscr{M} \to \mathscr{N}$ be a homomorphism of BG-algebras. An $\mathscr{L}$-fuzzy filter μ on $\mathscr{M}$ is called* strong *if*

$$\forall x \in \mathscr{M}, \quad \mu(x) \leq \mu(e) \implies h(x) \in \mathrm{Ker}(\nu),$$

where ν is the image filter on $\mathscr{N}$. Then $\nu : \mathscr{N} \to \mathscr{L}$ is also a strong $\mathscr{L}$-fuzzy filter.

Proof. Assume $\mu(x) \leq \mu(e)$. By the kernel definition of ν:

$$h(x) \in \mathrm{Ker}(\nu) \implies \nu(h(x)) = 0.$$

Since h is a homomorphism and $\nu(w) = \mu\big(h^{\leftarrow}(w)\big)$, ν satisfies the strong filter condition

$$\nu(w) = 0 \implies w \in \mathrm{Ker}(\nu).$$

Hence, ν is strong. □

Lemma 5.8. *Let μ_1 and μ_2 be $\mathscr{L}$-fuzzy filters on a BG-algebra $(\mathscr{M}, \star, e)$. Define their* product filter *$\mu_1 \odot \mu_2 : \mathscr{M} \to \mathscr{L}$ by*

$$(\mu_1 \odot \mu_2)(x) := \mu_1(x) \cdot \mu_2(x),$$

where $\cdot$ is a commutative, associative, and monotone operation on $\mathscr{L}$ (e.g., a suitable t-norm).

Then $\mu_1 \odot \mu_2$ is also an $\mathscr{L}$-fuzzy filter on $\mathscr{M}$.

Proof. We verify the defining properties of an $\mathscr{L}$-fuzzy filter for $\mu_1 \odot \mu_2$:

(1) **Unit element:**

$$(\mu_1 \odot \mu_2)(e) = \mu_1(e) \cdot \mu_2(e) = 1 \cdot 1 = 1,$$

since μ_1 and μ_2 are filters.

(2) **Filter condition on product:**For any $x, y \in \mathscr{M}$ such that

$$(\mu_1 \odot \mu_2)(x) \succ \mathbf{0} \quad \text{and} \quad (\mu_1 \odot \mu_2)(y) \succ \mathbf{0},$$

we have

$$\mu_1(x) \succ \mathbf{0}, \quad \mu_2(x) \succ \mathbf{0}, \quad \mu_1(y) \succ \mathbf{0}, \quad \text{and} \quad \mu_2(y) \succ \mathbf{0}.$$

Since μ_1 and μ_2 are filters, ...

$$\mu_1(x \star y) \succeq \mu_1(x) \wedge \mu_1(y), \quad \mu_2(x \star y) \succeq \mu_2(x) \wedge \mu_2(y).$$

Using the monotonicity and distributive properties of $\cdot$,

$$(\mu_1 \odot \mu_2)(x \star y) = \mu_1(x \star y) \cdot \mu_2(x \star y) \succeq (\mu_1(x) \wedge \mu_1(y)) \cdot (\mu_2(x) \wedge \mu_2(y)).$$

Since $\cdot$ is monotone and commutes with $\wedge$ in a suitable way (e.g., if $\cdot$ is a t-norm), we get

$$(\mu_1 \odot \mu_2)(x \star y) \succeq (\mu_1(x) \cdot \mu_2(x)) \wedge (\mu_1(y) \cdot \mu_2(y)) = (\mu_1 \odot \mu_2)(x) \wedge (\mu_1 \odot \mu_2)(y).$$

(3) **Absorption property:** For any $x \in \mathscr{M}$,

$$(\mu_1 \odot \mu_2)(e \star x) = \mu_1(e \star x) \cdot \mu_2(e \star x) \succeq \mu_1(e) \cdot \mu_2(x) \wedge \mu_1(x) \cdot \mu_2(e).$$

Using $\mu_i(e) = 1$, this reduces to

$$(\mu_1 \odot \mu_2)(e \star x) \succeq \mu_2(x) \wedge \mu_1(x) \succeq (\mu_1 \odot \mu_2)(x).$$

Thus, $\mu_1 \odot \mu_2$ satisfies all axioms of an $\mathscr{L}$-fuzzy filter. □

Corollary 5.9. *Let $h : \mathscr{M} \to \mathscr{N}$ be a BG-homomorphism, and μ_1, μ_2 be $\mathscr{L}$-fuzzy filters on $\mathscr{M}$. Then their product filter satisfies*

$$\nu\big(h(\mu_1 \odot \mu_2)\big) = \nu\big(h(\mu_1)\big) \odot \nu\big(h(\mu_2)\big),$$

where $\nu(h(\mu_i))(z) := \mu_i\big(h^{\leftarrow}(z)\big)$.

Proof. Using Lemma 5.8 and the homomorphism property:

$$\nu\big(h(x)\big) = \mu_1\big(h^{\leftarrow}(h(x))\big) \cdot \mu_2\big(h^{\leftarrow}(h(x))\big) = (\mu_1 \odot \mu_2)(x).$$

□

Example 5.10. *Consider the BG-algebra $(\mathscr{S}, \star, e)$ where*

$$\mathscr{S} = \{e, a, b, c\},$$

and the operation $\star$ is defined by the following table:

$\star$	e	a	b	c
e	e	e	e	e
a	a	e	a	b
b	b	b	e	b
c	c	c	b	e

Define two $\mathscr{L}$-fuzzy filters $\mu, \nu : \mathscr{S} \to [0,1]$ by

$$\mu(e) = 1, \quad \mu(a) = 0.7, \quad \mu(b) = 0.5, \quad \mu(c) = 0.6,$$
$$\nu(e) = 1, \quad \nu(a) = 0.4, \quad \nu(b) = 0.8, \quad \nu(c) = 0.3.$$

Their product $\mu \odot \nu$ is computed pointwise by multiplication:

$$(\mu \odot \nu)(e) = 1 \times 1 = 1, \quad (\mu \odot \nu)(a) = 0.7 \times 0.4 = 0.28,$$
$$(\mu \odot \nu)(b) = 0.5 \times 0.8 = 0.40, \quad (\mu \odot \nu)(c) = 0.6 \times 0.3 = 0.18.$$

This example illustrates how combining two $\mathscr{L}$-fuzzy filters via the product operation yields another $\mathscr{L}$-fuzzy filter, preserving the algebraic structure within the BG-algebra.

6. Characteristics of Fuzzy Open, Additively Closed, and Absorbing $\mathscr{L}$-Fuzzy Filters in BG-Algebras

This section investigates the structural properties of three important classes of $\mathscr{L}$-fuzzy filters—fuzzy open, additively closed, and fuzzy absorbing filters—within the framework of BG-algebras. These filter classes extend classical notions of algebraic substructures by allowing membership to be represented through degrees in the lattice $\mathscr{L}$ rather than through a binary distinction between belonging and non-belonging. Therefore, they provide a more expressive framework for describing algebraic conditions that may hold with different levels of strength or certainty.

The concept of fuzzy open filters is introduced by analyzing the relationship between membership values and algebraic operations in BG-algebras. We prove that fuzzy openness is preserved under BG-homomorphisms, demonstrating that this structural property remains unchanged under mappings that preserve the algebraic operations. Such a result is important because it ensures that fuzzy filter characteristics are compatible with transformations between related algebraic systems and that the underlying graded membership information is preserved during these transformations.

Furthermore, we define additively closed $\mathscr{L}$-fuzzy filters and establish their invariance under homomorphic images. This property describes the behavior of filters in which elements having positive membership degrees produce algebraic combinations that continue to maintain positive membership. The result provides a clearer understanding of how membership conditions interact with the internal operation of a *BG*-algebra. We also introduce fuzzy absorbing filters, where the membership value of an element remains unchanged after combining it with elements of the algebra. This notion extends the classical absorption principle to the fuzzy setting and provides a framework for analyzing stability properties of fuzzy substructures.

In addition, the closure behavior of families of $\mathscr{L}$-fuzzy filters is studied through pointwise supremum operations. The obtained results show that collections of fuzzy filters can be combined to generate new fuzzy filters while preserving the fundamental axioms of the structure. Moreover, lemmas concerning elements with minimal membership value provide a detailed analysis of boundary cases and contribute to the completeness of the developed theory.

To illustrate the introduced concepts and verify the theoretical results, finite *BG*-algebras with explicit operation tables are presented. These examples demonstrate how different membership degrees interact with algebraic operations and how the proposed classes of fuzzy filters behave under structural mappings. They also provide an intuitive interpretation of the abstract definitions and preservation results.

Overall, this section extends the theory of $\mathscr{L}$-fuzzy filters on *BG*-algebras by introducing new filter categories and establishing their closure and preservation properties. The results strengthen the algebraic foundation of fuzzy filter theory and provide a basis for further studies involving fuzzy congruences, homomorphic structures, and other mathematical frameworks where graded membership and uncertainty are incorporated.

Definition 6.1. *Let $(\mathscr{M},\star,e)$ be a BG-algebra and $\mu:\mathscr{M}\to\mathscr{L}$ an $\mathscr{L}$-fuzzy filter on $\mathscr{M}$. The filter μ is said to be* fuzzy open *if for every $x,y\in\mathscr{M}$,*

$$\mu(x)\succ\mathbf{0}\implies\mu(x\star y)\succeq\mu(x).$$

Theorem 6.2. *Let $h:\mathscr{M}\to\mathscr{N}$ be a BG-homomorphism, and let μ be a fuzzy open $\mathscr{L}$-fuzzy filter on $\mathscr{M}$. Define the map $\nu:\mathscr{N}\to\mathscr{L}$ by*

$$\nu(z):=\mu\big(h^{\leftarrow}(z)\big).$$

Then ν is a fuzzy open $\mathscr{L}$-fuzzy filter on $\mathscr{N}$.

Proof. Assume $\nu(z)\succ\mathbf{0}$ for some $z\in\mathscr{N}$. By definition,

$$\mu\big(h^{\leftarrow}(z)\big)\succ\mathbf{0}.$$

Since μ is fuzzy open, for any $y\in\mathscr{M}$,

$$\mu\big(h^{\leftarrow}(z)\star y\big)\succeq\mu\big(h^{\leftarrow}(z)\big).$$

In particular, taking $y=h^{\leftarrow}(z)$, we get

$$\mu\big(h^{\leftarrow}(z)\star h^{\leftarrow}(z)\big)\succeq\mu\big(h^{\leftarrow}(z)\big).$$

Because h is a homomorphism,

$$h^{\leftarrow}(z \diamond z) = h^{\leftarrow}(z) \star h^{\leftarrow}(z).$$

Therefore,

$$\nu(z \diamond z) = \mu\big(h^{\leftarrow}(z \diamond z)\big) = \mu\left(h^{\leftarrow}(z) \star h^{\leftarrow}(z)\right) \succeq \mu\big(h^{\leftarrow}(z)\big) = \nu(z).$$

Hence, ν is fuzzy open. □

Example 6.3. *Consider the BG-algebra $\mathscr{G} = (\{u, v, w, x\}, \star, u)$ with the operation table:*

$\star$	u	v	w	x
u	u	u	u	u
v	v	v	u	v
w	w	u	w	w
x	x	x	u	x

Define an $\mathscr{L}$-fuzzy filter $\alpha : \mathscr{G} \to [0,1]$ by

$$\alpha(u) = 1, \quad \alpha(v) = 0.6, \quad \alpha(w) = 0, \quad \alpha(x) = 0.8.$$

Verify fuzzy openness:

- *For $g = v$, $h = w$,*

$$\alpha(v \star w) = \alpha(u) = 1 \geq \alpha(v) = 0.6.$$

- *For $g = x$,*

$$\alpha(x \star x) = \alpha(x) = 0.8 \geq \alpha(x).$$

Let $f : \mathscr{G} \to \mathscr{H}$ be a BG-homomorphism defined by

$$f(u) = e', \quad f(v) = a, \quad f(w) = b, \quad f(x) = c,$$

where $(\mathscr{H}, \diamond, e')$ is another BG-algebra.

Then the induced map $\beta : \mathscr{H} \to [0,1]$ defined as

$$\beta(z) := \alpha\big(f^{\leftarrow}(z)\big)$$

is a fuzzy open $\mathscr{L}$-fuzzy filter on $\mathscr{H}$.

Proposition 6.4. *Let μ, ν be $\mathscr{L}$-fuzzy filters on $\mathscr{M}$, and $h : \mathscr{M} \to \mathscr{N}$ a BG-homomorphism. If*

$$\mu(h^{\leftarrow}(z)) \geq \nu(h^{\leftarrow}(z)), \quad \forall z \in \mathscr{N},$$

then

$$\mu(h^{\leftarrow}(z)) \geq \nu(z), \quad \forall z \in \mathscr{N}.$$

Proof. This follows directly from the ordering in $\mathscr{L}$ and the assumptions. □

Definition 6.5. *An $\mathscr{L}$-fuzzy filter μ on a BG-algebra $(\mathscr{M}, \circ, e)$ is said to be* additively closed *if for all $x, y \in \mathscr{M}$,*

$$\mu(x) \succ \mathbf{0} \quad \textit{and} \quad \mu(y) \succ \mathbf{0} \implies \mu(x \circ y) \succ \mathbf{0}.$$

Theorem 6.6. *Let μ be an $\mathscr{L}$-fuzzy filter on a BG-algebra $(\mathscr{M},\star,e)$, and let $h:\mathscr{M}\to\mathscr{N}$ be a BG-homomorphism. Define $\nu:\mathscr{N}\to\mathscr{L}$ by*

$$\nu(z) := \mu\big(h^{\leftarrow}(z)\big) \quad \textit{for all } z\in\mathscr{N}.$$

If μ is additively closed, then ν is also additively closed.

Proof. Assume $z_1, z_2 \in \mathscr{N}$ satisfy

$$\nu(z_1) \succ \mathbf{0} \quad \text{and} \quad \nu(z_2) \succ \mathbf{0}.$$

By definition of ν, this means

$$\mu\big(h^{\leftarrow}(z_1)\big) \succ \mathbf{0} \quad \text{and} \quad \mu\big(h^{\leftarrow}(z_2)\big) \succ \mathbf{0}.$$

Since μ is additively closed, we have

$$\mu\big(h^{\leftarrow}(z_1)\star h^{\leftarrow}(z_2)\big) \succ \mathbf{0}.$$

Using the homomorphism property of h,

$$h\big(h^{\leftarrow}(z_1)\star h^{\leftarrow}(z_2)\big) = h\big(h^{\leftarrow}(z_1)\big)\diamond h\big(h^{\leftarrow}(z_2)\big),$$

which implies

$$z_1\diamond z_2 = h\big(h^{\leftarrow}(z_1)\big)\diamond h\big(h^{\leftarrow}(z_2)\big).$$

Hence,

$$h^{\leftarrow}(z_1\diamond z_2) = h^{\leftarrow}(z_1)\star h^{\leftarrow}(z_2).$$

Therefore,

$$\nu(z_1\diamond z_2) = \mu\big(h^{\leftarrow}(z_1\diamond z_2)\big) = \mu\big(h^{\leftarrow}(z_1)\star h^{\leftarrow}(z_2)\big) \succ \mathbf{0}.$$

This shows that ν is additively closed. □

Example 6.7. *Consider the BG-algebra $\mathscr{S} = (\{u,v,w,x\},\diamond,u)$ where the operation $\diamond$ is given by the table:*

$\diamond$	u	v	w	x
u	u	u	u	u
v	v	v	u	v
w	w	u	w	w
x	x	x	u	x

Define an $\mathscr{L}$-fuzzy filter $\alpha:\mathscr{S}\to[0,1]$ by

$$\alpha(u)=0,\quad \alpha(v)=0.4,\quad \alpha(w)=0.3,\quad \alpha(x)=0.7.$$

Check additive closedness by taking $s=v$ and $t=x$:

$$\alpha(s)>0,\quad \alpha(t)>0,\quad s\diamond t = v\diamond x = v,$$

and

$$\alpha(s\diamond t)=\alpha(v)=0.4>0.$$

Hence, α satisfies the additive closedness condition.

Now, let $f:\mathscr{S}\to\mathscr{T}$ be a BG-homomorphism defined by

$$f(u)=e',\quad f(v)=a,\quad f(w)=b,\quad f(x)=c,$$

where $(\mathscr{T},\star,e')$ is another BG-algebra.

The induced fuzzy filter $\beta : \mathscr{T} \to [0,1]$ given by

$$\beta(z) := \alpha\big(f^{\leftarrow}(z)\big)$$

is also additively closed.

Definition 6.8. *Let μ be an $\mathscr{L}$-fuzzy filter on the BG-algebra $(\mathscr{M}, \star, e)$. Then μ is called* fuzzy absorbing *if for every $x, y \in \mathscr{M}$, whenever $\mu(x) \succ \mathbf{0}$, it holds that*

$$\mu(x \star y) = \mu(y).$$

Lemma 6.9. *Let $\mathscr{F} = \{\varphi_\lambda\}_{\lambda \in \Lambda}$ be a family of $\mathscr{L}$-fuzzy filters on a BG-algebra $(\mathscr{X}, \star, e)$. Then their pointwise supremum*

$$\varphi(x) := \bigvee_{\lambda \in \Lambda} \varphi_\lambda(x)$$

for all $x \in \mathscr{X}$, is also an $\mathscr{L}$-fuzzy filter on $\mathscr{X}$.

Proof. We verify the three defining properties of an $\mathscr{L}$-fuzzy filter for φ:

(1) **Unit condition:**

$$\varphi(e) = \bigvee_{\lambda \in \Lambda} \varphi_\lambda(e) = \bigvee_{\lambda \in \Lambda} 1 = 1.$$

(2) **Closure under $\star$:** For any $u, v \in \mathscr{X}$ such that $\varphi(u) \succ \mathbf{0}$ and $\varphi(v) \succ \mathbf{0}$, there exist indices $\lambda_1, \lambda_2 \in \Lambda$ with

$$\varphi_{\lambda_1}(u) \succ \mathbf{0}, \quad \varphi_{\lambda_2}(v) \succ \mathbf{0}.$$

Since each φ_λ is an $\mathscr{L}$-fuzzy filter, it follows that

$$\varphi_{\lambda_1}(u \star v) \succeq \varphi_{\lambda_1}(u) \wedge \varphi_{\lambda_1}(v), \quad \varphi_{\lambda_2}(u \star v) \succeq \varphi_{\lambda_2}(u) \wedge \varphi_{\lambda_2}(v).$$

Therefore,

$$\varphi(u \star v) = \bigvee_{\lambda \in \Lambda} \varphi_\lambda(u \star v) \succeq \varphi_{\lambda_1}(u \star v) \wedge \varphi_{\lambda_2}(u \star v) \succeq \varphi(u) \wedge \varphi(v).$$

(3) **Absorption condition:**

$$\varphi(e \star u) = \bigvee_{\lambda \in \Lambda} \varphi_\lambda(e \star u) \succeq \bigvee_{\lambda \in \Lambda} \varphi_\lambda(u) = \varphi(u).$$

Hence, φ satisfies all the axioms of an $\mathscr{L}$-fuzzy filter. □

Lemma 6.10. *Let φ be an $\mathscr{L}$-fuzzy filter on a BG-algebra $(\mathscr{X}, \star, e)$. If there exists an element $u \in \mathscr{X}$ such that $\varphi(u) = \mathbf{0}$, then for every $v \in \mathscr{X}$,*

$$\varphi(u \star v) = \mathbf{0}.$$

Proof. By the defining property of an $\mathscr{L}$-fuzzy filter (see Definition 3.1), for all $v \in \mathscr{X}$ we have

$$\varphi(u \star v) \succeq \varphi(u) \wedge \varphi(v).$$

Since $\varphi(u) = \mathbf{0}$, it follows that

$$\varphi(u \star v) \succeq \mathbf{0} \wedge \varphi(v) = \mathbf{0}.$$

Because $\mathbf{0}$ is the minimal element in $\mathscr{L}$, this implies

$$\varphi(u \star v) = \mathbf{0}$$

for every $v \in \mathscr{X}$. □

7. Conclusion and Future Work

This paper presents a systematic investigation of the structural properties and algebraic behavior of $\mathscr{L}$-fuzzy filters on *BG*-algebras, with particular emphasis on their interaction with *BG*-homomorphisms. By extending classical filter concepts to the framework of complete Heyting algebra-valued fuzzy sets, we establish a unified approach for describing algebraic substructures in which membership is represented by graded values rather than binary conditions. The developed framework provides a bridge between fuzzy set theory and the intrinsic algebraic operations of *BG*-algebras.

Several important classes of $\mathscr{L}$-fuzzy filters, including strong, open, additively closed, bounded, monotone, and fuzzy absorbing filters, have been introduced and analyzed. The results demonstrate that these fuzzy filters preserve essential structural properties under fundamental operations such as intersections, products, and lattice-based constructions. These findings extend classical filter theory by showing that graded membership information can coexist with algebraic closure requirements while maintaining consistency within the underlying *BG*-algebraic framework.

A major contribution of this work is the detailed study of the behavior of $\mathscr{L}$-fuzzy filters under *BG*-homomorphisms. We prove that important characteristics, including closure, boundedness, monotonicity, essentiality, and other filter-related properties, are preserved through algebraic mappings. These preservation results are significant because they ensure that fuzzy structural information can be transferred between related *BG*-algebras without losing the algebraic relationships encoded by the filters. Consequently, the proposed theory provides a reliable foundation for studying transformations, equivalences, and structural relationships among fuzzy algebraic systems.

Although the present study is primarily theoretical, the developed framework has potential relevance for uncertainty-aware mathematical models, where algebraic operations and graded information need to be considered simultaneously. In particular, the preservation results may provide useful mathematical tools for future investigations of fuzzy reasoning mechanisms, decision-support models, and computational systems whose structures can be represented through algebraic frameworks.

Several directions naturally arise from this study.

- **$\mathscr{L}$-Fuzzy Congruences and Quotient Structures:** A natural continuation of this work is the development of a systematic theory of $\mathscr{L}$-fuzzy congruences on *BG*-algebras. Such a theory would investigate how fuzzy filters induce equivalence relations, how quotient *BG*-algebras can be constructed in the presence of graded membership, and how homomorphic relationships between fuzzy quotient structures can be characterized. This direction may provide a categorical description of fuzzy algebraic systems and clarify the role of morphisms in preserving fuzzy equivalence.
- **Fuzzy Convex Structures on *BG*-Algebras:** The study of fuzzy convexity represents a natural extension because convex structures provide additional tools for describing combinations and interactions among elements with different membership degrees. Developing fuzzy convex structures compatible with *BG*-algebra

operations could establish new connections between fuzzy algebra, lattice theory, and generalized convex analysis.

- **Categorical Study of Fuzzy Morphisms:** Further research may focus on constructing appropriate categories whose objects are $\mathscr{L}$-fuzzy filters or fuzzy BG-algebras and whose morphisms preserve specific fuzzy structural properties. Such an approach would help classify different types of fuzzy homomorphisms and provide a clearer understanding of how algebraic information is transformed between fuzzy systems.
- **Applications to Uncertainty Modeling and Decision Systems:** Future studies may explore how the algebraic properties established here can be incorporated into specific uncertainty-management frameworks, such as fuzzy decision models with algebraically constrained information, optimization problems involving fuzzy structural conditions, or knowledge representation systems based on graded algebraic relations.
- **Extension to Other Algebraic Structures:** The proposed methods may also be investigated in other important algebraic settings, including residuated lattices, BL-algebras, and MV-algebras. Such extensions could reveal whether the preservation principles obtained for BG-algebras remain valid in broader classes of fuzzy algebraic systems.

Through these research directions, the theory of $\mathscr{L}$-fuzzy filters on BG-algebras can be further developed into a broader framework for studying fuzzy algebraic structures, their transformations, and their relationships with other mathematical theories involving uncertainty and graded information.

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Faisal Mehmood: Conceptualization, Methodology, Formal Analysis, Investigation, Writing–Original Draft, Writing–Review and Editing, Project Administration and Funding Acquisition. Heng Liu: Conceptualization, Formal Analysis, Investigation, Validation and Supervision.

DECLARATIONS

Conflict of Interest: The authors declare that they have no conflict of interest.

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