

**IoT-Driven Entropy-SWARA-MARCOS Decision Analytics Integrating Linear
Diophantine Fuzzy BCC Ideals**

Received
23 April, 2026

Revised
31 July, 2026

Accepted
21 August, 2026

Published Online
01 October, 2026

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Abstract. This research introduces a novel fuzzy model that is capable to endure incredulity when streamlining waste management in smart cities in a way that is capable of improving environmental protection, cost-efficiency and sustainability. Industry 5.0 has enhanced the pace of technological development and improved computational intelligence to allow more efficient decision-making in the case of uncertainty by tackling problems in industrial and engineering fields. Traditional models do not necessarily deal with the imprecision and vagueness of the real-world complex situations. Linear Diophantine fuzzy (LiDF) control parameters generalized the fuzzy set theory to BCC-algebras and uncertainty in smart waste management (SWM) and recycling. It is suggested a strong multi-criteria decision-making (MCDM) model that incorporates the use of entropy regarding the objective weight of alternatives, SWARA regarding the subjective weight of alternatives based on the input of experts, and MARCOS regarding the ranking of alternatives by compromise. The Entropy-SWARA-MARCOS Model, which is driven by the IoT, is validated through a sensitivity analysis to examine how varying weights of criteria impact the final rankings, and through a comparative analysis to evaluate the proposed method to existing MCDM techniques. The suggested model presents a versatile decision-making process which is more reliable and feasible, offering policymakers and urban planners a valuable instrument for applying intelligent, resilient and forward-looking solutions to complex and uncertain smart city waste management and recycling situations.

AMS (MOS) Subject Classification Codes: 90B50, 03B52; 03E72

Key Words: Internet of Things; Entropy-SWARA-MARCOS; Linear Diophantine Fuzzy Set; Smart City Waste Management; Recycling; Decision Analytics.

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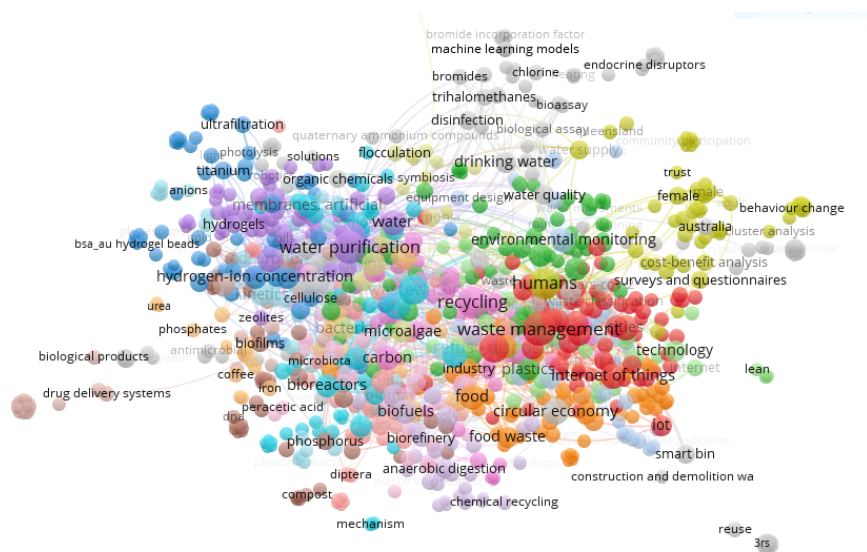


FIGURE 1. Visualizing, clustering and co-occurrence of SWM in other fields

1. INTRODUCTION

The fast growth of urban populations and industrial processes has increased the complexity and urgency of efficient decision support system, particularly in the context of smart cities that aim to achieve the interdependent goals of sustainability, circularity, digital transformation integrating computational intelligence and fuzzy machine learning [3, 4, 7]. The increasing quantity, types, and toxic nature of waste (such as municipal solid waste, e-waste and industrial byproducts) pose serious environmental and human health hazards, circular economy of medical waste challenges for urban planners and policymakers [38]. The traditional approaches to waste management that can be described as the use of a static linear model and manual process are becoming inadequate in addressing the dynamic needs of modern waste ecosystems [25, 26]. The legacy systems have typically lacked the ability to monitor in real-time, make adaptive plans, and make decisions based on data, which are essential towards achieving sustainability goals set by international systems such as the United Nations Sustainable Development Goals (SDGs) and the European Green Deal [16]. Considering such constraints, the implementation of innovative digital solutions has turned out to be a key solution to the present-day waste management systems. Blockchain technology can be used to ensure the secure, transparent, and tamper-free tracking of waste and its origin to the disposal, increasing accountability and reducing illegal dumping or data editing. AI and, in particular, predictive and self-learning algorithms can improve the process of sorting, route optimization, and resource recovery in a significantly faster and more accurate manner [51]. Figure 1 demonstrates visualization, clustering and co-occurrence of SWM in various fields. The Internet of Things (IoT) allows to gather data in real-time using smart bins, sensors, and edge devices, facilitating predictive maintenance, dynamic

scheduling, and monitoring of emissions. The metaverse and immersive technologies offer new platforms to engage with citizens and educate the population. Such technologies allow users to communicate with virtual models of waste systems, participate in gamified sustainability challenges, and gain a better insight into how their consumption behaviors impact the environment. Such digital innovations are changing the plastic waste governance as it enables decentralized, transparent, intelligent, and participatory systems [46]. Although such technologies have a significant potential, their usage and evaluation require the creation of extensive decision-making frameworks that can coordinate a multitude of conflicting criteria, address uncertainty, and develop policies with informed decision-making [15]. This growing complexity necessitates the creation of a new type of MCDM models which are able to successfully combine the technical performance of innovative solutions with the subjective judgments of domain experts when faced with circumstances of imprecision and ambiguity.

The decision to select the most appropriate technological solution among a number of intelligent waste management solutions is a multi-dimensional problem of decision making [27]. The reasons the alternatives vary include technological maturity, cost structure, environmental impact, social acceptance, data security, regulatory compliance and scaling. Therefore, multiple and often incompatible evaluation criteria have to be evaluated at the same time. This is further complicated by the inherent uncertainty and subjectivity of expert judgments especially when the assessments involve use of vague language words or where experts are hesitant or incomplete in their information. The traditional decision-making models, particularly the models that are founded on the explicit logic or the models that are founded on simple weighting systems, are not sufficient to deal with the complexity and uncertainty. The above-mentioned models, therefore, can result in rankings that are either unstable or overly sensitive to starting assumptions, which limits their usefulness in non-experimental and dynamic real-world situations and multiple-attribute decision-making (MADM).

In this research, a hybrid decision support framework has been suggested that can be used to overcome current limitations. It combines three known techniques: the entropy technique used to determine objective weights on the basis of variability of data, the SWARA technique used to systematically obtain the subjective preferences of experts, and the MARCOS technique [50] used to rank alternatives using a compromise-driven process that evaluates the utility of each alternative relative to ideal and anti-ideal solutions. The combination of these approaches enables them to consider the full evaluation process not only based on quantitative indicators of the performance but also on the qualitative expertise. Proposed framework uses LiDFSs to make the model more robust to deal with imprecision, vagueness, and expert system for decision analytics.

1.1. A Brief Review of Fuzzy Modeling. The study of algebraic structures related to logical processes was the focus of Komori's earlier algebraic frameworks, which were improved upon by BCC-algebras [33]. Later, W. A. Dudek redefined BCC-algebras by introducing a dual version of Komori's original definition, highlighting the properties of proper BCC-algebras, and offering a more generalized structure [21]. In [24], Hao examined certain associated features and presented the idea of ideals in a BCC algebra. The

components of fuzzy group theory were developed by Rosenfeld in [47]. The combined concepts of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, first presented by Pu and Liu [39], were used by Bhakat and Das [11] to introduce a new type of fuzzy subgroup, the $(\in, \in \vee q)$ -fuzzy subgroup. Later on, Bhakat [12] proposed the idea of $(\in \vee q)$ -level subsets. According to a natural equivalence on fuzzy subsets, Murali [37] suggested the definition of a fuzzy point belonging to a fuzzy subset. It was discovered that the $(\in, \in \vee q)$ -fuzzy subgroup is the most feasible extension of Rosenfeld's fuzzy subgroup. In 1999, Bhakat [13] introduced the notions of $(\in, \in \vee q)$ -level subsets, $(\in, \in \vee q)$ -fuzzy normal, quasi-normal, and maximal subgroups. According to Jun's definition in [29], (α, β) -fuzzy ideals in BCK/BCI-algebras. Zulfikar presented the idea of (α, β) -fuzzy positive implicative ideals in BCK-algebras in [55]. In [30], Jun defined $(\in, \in \vee q)$ -fuzzy subalgebras in BCK/BCI-algebras. The $(\in, \in \vee q)$ -fuzzy ideal in BCI-algebra was discussed in [53]. In [55] Zulfikar introduced the concept of sub-implicative (α, β) -fuzzy ideals in BCH-algebras. The seminal article of Zadeh on fuzzy sets (FS) [52] has completely revolutionized the decision-making processes under ambiguity. The mathematical system that he developed has a way of assimilating and conveying the bewildering information, hence it is a valuable instrument in handling the complexity of the decision-making processes. The simplicity framework by Zadeh provides a comprehensive and comprehensible way of defining and managing imprecision. This is a demonstration that the element of uncertainty is always common in the decision-making process. A further development of this framework is the proposal by Atanassov [8] of further development of intuitionistic fuzzy sets (IFS). Membership and non-membership characteristics can be combined by means of these sets. This invention makes fuzzy sets more adaptable, and thus, they can deal with the inherent complexity of decision-making scenarios. Riaz and Hashmi [42] suggested LiDFSs in 2019 to address these limitations. They also developed a more adaptable structure of managing uncertainty with the help of standard parameters. LiDFS is a new approach to real-life scenarios, optimizing computational efficiency and modeling uncertainty analysis in making the reasonable decision. The application of LiDFS has received much interest in recent decision-making research due to its ability to deal with a large amount of uncertainty and inaccurate information. They introduced new LiDFS-based aggregation operations to MCDM situations. Their approach was strong in integrating disorienting information on various criteria, giving a practical approach to assess complex decision-making issues. This study contributes to the field through the development of organized operators and the empirical test of their operation. Hayat et al. [43] introduced the idea of intuitionistic fuzzy credibility implicative ideals with application to supply chain efficiency evaluation. Surya et al. [44] introduced new extension of entropy using q-rung linear Diophantine fuzzy hypersoft sets for wastewater treatment technology. Gul and Aydogdu [45] proposed new extensions of distance and entropy measures for LDFS information. Muhiuddin et al. [36] defined LiDF ideals in BCK-algebras. Ayub et al. [10] introduced fuzzy relations and rough set approximations to the LiDFS theoretical framework to deal with the multi-criteria group decision-making case. Their method enhanced the accuracy of the decision outcomes, as they effectively overcame ambiguity in expert opinions, which is not well considered in traditional fuzzy models. The hybrid model came in handy in situations where there were conflicting data and situations that required group assessments. Aydogdu et al. [9] applied the LiDF-ARAS method to select data storage systems in the LiDFS

environment and proposed new measures of entropy and similarity in the article. Their method showed the significance of estimating uncertainty with advanced information measures to allow more accurate comparisons of options. LiDFS-ARAS is a good illustration of the versatility of these fuzzy models in practical contexts, especially in the technological sector.

The entropy technique has also found wide application in recent MCDM research to objectively estimate the weight of criteria in terms of data variability. The hybrid decision-making framework that Cui et al. [18] suggested to evaluate the investment opportunity in petroleum basins combines the analytic hierarchy process (AHP) with the entropy-weighted fuzzy TOPSIS. This model optimizes the objectivity of the investment decision-making in uncertain conditions by combining both the geological and economic aspects. Zhou et al. [54] created an intuitionistic fuzzy-based approach to data fabric solution selection, combining entropy, ELECTRE and AHP. Their approach was useful in multi-criteria assessments in dynamic digital infrastructures. In a bid to ensure the use of maximum energy in smart manufacturing systems, Anjum et al. [6] optimized the use of entropy by combining it with the CRADIS ranking algorithm in an intuitionistic fuzzy framework. The research focused on operational efficiency and sustainability by making use of sensitivity-based validation. Roszkowska et al. [48] studied how various normalization techniques impact the entropy-based weights of the Hellwig method and showed that the normalization decision can strongly affect the ultimate rankings, especially in the sustainability context. Atenidegbe and Mogaji [14] investigated groundwater uncertainty and risk factors using fuzzy TOPSIS-entropy. Kalbar [32] proposed technology assessment for wastewater management using MADM framework. Such studies emphasize the growing application of entropy-enhanced hybrid MCDM techniques in addressing complex decision-making challenges related to fuzzy optimization, infrastructure development, and sustainability.

To determine the subjective weights of criteria based on expert assessment, Kersulienė et al. [31] first presented the SWARA approach. Decision-makers find SWARA to be a more effective option than the Analytic Hierarchy Process (AHP) since it requires fewer computations and does not rely on pairwise comparisons. In order to solve challenging real-world issues, SWARA has been extensively used and combined with various decision-making methodologies over time. To identify and assess obstacles in humanitarian supply chain management, for instance, Agarwal et al. [2] used a hybrid approach that combined SWARA and WASPAS. The CoCoSo approach, SWARA, and Interval Rough Numbers (IRN) were all used by Bouraima et al. [17] to create an integrated framework for assessing different railway systems from the standpoint of sustainable development. To evaluate suppliers for healthcare testing facilities, Debnath et al. [19] used a combined SWARA-WASPAS model, where SWARA was used to establish the weights of sustainability-related criteria. Sarac et al. [49] demonstrate the adaptability of SWARA by employing the SWARA-WASPAS technique to select the optimal vegan cake sample based on physicochemical and sensory characteristics. Deveci et al. [20] employed a Fermatean fuzzy SWARA technique to evaluate the hazards affecting sustainable mining operations. These examples demonstrate the applicability of the SWARA method to the different industries and show that it can be effective in both qualitative and quantitative decision-making settings.

A recent multi-criteria decision-making technique is the MARCOS method, which creates a compromise rating through comparison of the alternatives with an ideal solution and a

suboptimal solution. It has been able to be used successfully in different fields because of its flexibility. Jayakumar et al. [28] demonstrated the effectiveness of the MARCOS method in dealing with uncertainty and group-based decision-making through its application combined with LiDFSs to prioritise risks posed by the IoT. In another study, Abdulla et al. [1] used machine learning and MARCOS to assess and select suppliers, emphasizing the flexibility of the strategy in the context of data-driven environments to demonstrate the use of the fuzzy modeling in the selection of optimal alternative. Liu [35] used a framework of interval MARCOS with mixed objective weighting to choose renewable desalination technologies under the conditions of ambiguous data. The contributions highlight the ability of the MARCOS method to handle a range of criteria and imprecise information in a wide range of decision-making situations. All these studies point towards the growing trend of the incorporation of different weighting and ranking methods into fuzzy and intuitionistic fuzzy systems to enhance realism, transparency, and soundness in decision support systems. This paper fills an important research gap in relating these methods to more sophisticated fuzzy algebraic models, namely Diophantine fuzzy sets, through the introduction of $(\bar{\alpha}, \bar{\beta})$ -LiDF Ideals into a hybrid Entropy-SWARA-MARCOS model.

1.2. Research Gap. Although the study of the fuzzy set theory and its extensions have been growing, the integration of LiDF sets into the context of BCC-algebras has been largely under-considered, particularly as to ideal structures based on generalized membership relations like $(\bar{\alpha}, \bar{\beta})$. Most of the existing literature focuses on classical fuzzy concepts or intuitionistic fuzzy frameworks and does not consider the combination of LiDF modeling as it is needed to deal with algebraic uncertainty in numerical precision. Moreover, many of the MCDM techniques have been applied to complex decision-making problems, but lack an in-depth structure that incorporates both objective and subjective weighting schemes combined with a complex algebraically-based fuzzy basis. In addition, real-world applications of hybrid MCDM models rarely explore the potential of fuzzy algebraic frameworks in improving the management of uncertainties, especially in technology-driven aspects like smart waste management. The gaps identified provide a rationale for the importance of a unified framework that would serve to deepen the theoretical insight into fuzzy BCC-algebras as well as demonstrate their applicability in real-life intelligent decision-making systems.

The proposed Entropy-SWARA-MARCOS hybrid fuzzy model has a set of research gaps that involve methodological, theoretical, practical, and data-driven gaps as follows.

(1) **Inadequate handling of multi-source uncertainty in Industry 5.0**

The existing SWM models do not consider multi-source uncertainty, imprecision, and vagueness in dynamic smart city environments, and overlook the human-centric and intelligent manner of integrating Industry 5.0. There are only a limited number of works that generalize fuzzy modeling to LiDF BCC ideal to provide the algebraic structure and uncertainty in decision-making.

(2) **Neglect of IoT-driven real-time data visualization in decision analysis**

LiDF sets effectively model complexity of uncertainty, but there is no research on the application of LiDFs in the implementation of smart city waste management systems using IoT. The incorporation of IoT data streams in fuzzy-MCDM models of adaptive and data-driven SWM decision-making is immature.

- (3) **Insufficient integration of objective and subjective weights for ranking**
There is no single framework that integrates Entropy (objective), SWARA (subjective) and MARCOS (ranking) within the LiDF environment of SWM. The methods are applied separately or in part, without a hybrid fuzzy-based approach, in previous studies.
- (4) **Limited empirical validation with real-world data**
Hybrid fuzzy-MCDM models are not validated on real IoT or sensor-based urban waste data-sets in many researches.
- (5) **Inadequate comparative analysis with conventional MCDM approaches**
There is a lack of studies that compare the suggested models to existing MCDM methods to confirm their strength and stability when faced with uncertainty. The sensitivity analysis is also usually absent, which diminishes the credibility in the practical application.
- (6) **Weak link between human-centric computational intelligence**
The current models do not have a strong relationship between intelligent SWM optimization and sustainability and resource efficiency in line with UN SDGs. The industry 5.0 emphasis on human-machine cooperation is widely neglected, with the existing methods focusing on either theory without practice or practice without theory- practice - hybrid LiDF-MCDM models can be applied to fill this gap.

1.3. Research Motivation and Objectives. The inspiration behind the research is the increasing need across the globe to have sustainable waste management practices that are in line with the environmental protection, resource efficiency, and development of smart cities. Conventional waste management systems are not very effective in responding to the growing complexity of waste production, integration of technology and making decisions in the face of uncertainty. Due to the fast-evolving development of intelligent technologies and data-driven systems, the intelligent, efficient, and flexible decision-support models that can process imprecise and subjective expert information are highly sought after. To address this issue, the research aims at developing a new fuzzy hybrid MCDM model on LiDF sets and BCC-algebraic frameworks. This method does not only improve the accuracy and versatility of expert judgments, but also plays a theoretical role by further expanding fuzzy algebra by introducing LiDF BCC ideals. Ultimately, the research aims to provide a scientifically robust and practically applicable tool for evaluating and selecting innovative waste management and recycling technologies that support sustainable development objectives. This study aims to develop a comprehensive hybrid decision support framework for assessing sustainable waste management and recycling in uncertain conditions. The study specifically aims to gain the following goals.

- (1) To examine and evaluate a variety of innovative SWM and recycling solutions aimed at promoting sustainable development. The integration of linear Diophantine fuzzy algebraic structure into the decision model enhances the management of expert hesitation and linguistic uncertainty during the evaluation process.
- (2) Develop a comprehensive multi-criteria evaluation framework incorporating environmental, economic, operational, cost-efficiency, and smart performance indicators.

- (3) Construct a novel fuzzy hybrid MCDM model integrating the entropy method for objective weighting, the SWARA method for subjective expert-based weighting, and the MARCOS method for ranking alternatives.
- (4) Computational intelligence strengthens Industry 5.0 by utilizing control parameters of LiDF sets and LiDF algebraic structures, which efficiently capture and represent uncertainty, hesitation, and vagueness.
- (5) Extend fuzzy algebra theoretically by introducing and analyzing the concept of LiDF BCC ideals within the framework of BCC-algebras.
- (6) The validation of the IoT-driven Entropy–SWARA–MARCOS model is conducted through a sensitivity analysis to examine the impact of varying criteria weights on the final ranking results. The robustness of proposed framework is based on the following key components.
 - Entropy — for determining objective criteria weights based on data dispersion.
 - SWARA — for deriving subjective weights through expert judgment.
 - MARCOS — for achieving robust, compromise-based ranking of alternatives.
 - IoT Integration — enabling real-time monitoring, scheduling, collection, and analysis of waste-related data to enhance SWM efficiency and recycling.
- (7) Assess the applicability and effectiveness of the proposed model through a case-based evaluation of alternative waste management technologies using expert judgments. Guidance for policymakers, urban planners, and system designers to facilitate informed, transparent, and intelligent decision-making in uncertain, multi-dimensional contexts.

This paper is organized into different sections. Section 2 presents the fundamental concepts of fuzzy sets and systems, offering the essential background knowledge necessary for this study. In Section 3, Linear Diophantine fuzzy points in LiDFS are discussed, including $(\bar{\alpha}, \bar{\beta})$ -LiDF ideals and $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDF ideals. In Section 4, $(\bar{\alpha}, \bar{\beta})$ -Linear Diophantine Fuzzy BCC ideals are proposed, and $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDF BCC ideals are examined. In Section 5, the Entropy–SWARA–MARCOS model is developed for MCDM applications using LiDFSs. Section 6 demonstrates the applicability of the proposed framework through a real-world case study, accompanied by sensitivity and comparative analyses, and discusses the model's limitations. Finally, Section 7 concludes the paper by summarizing the key findings and outlining directions for future research.

2. PRELIMINARIES

This section includes some basic definitions that are essential to our next research. This research work explores the algebraic structures of BCC-algebras [33, 21, 22], utilizing their axioms to generalize fuzzy algebras further and establish strong theoretical foundations and practical applications for real-world problems.

Definition 1. BCC-Algebra [33]

Let $\mathcal{C}$ be a set. A BCC-algebra is an algebra $(\mathcal{C}, *, 0)$ of type $(2, 0)$ if the following criteria are true for all $\eta, \zeta, \rho \in \mathcal{C}$.

- (BCC-1) $((\eta * \zeta) * (\rho * \zeta)) * (\eta * \rho) = 0$
 (BCC-2) $\eta * \eta = 0$

(BCC-3) $0 * \eta = 0$

(BCC-4) $\eta * 0 = \eta$

(BCC-5) $\eta * \zeta = 0$ and $\zeta * \eta = 0 \implies \eta = \zeta$

Definition 2. [21] If $(\mathcal{C}, *, 0)$ is a BCC-algebra, then $(\mathcal{C}, \leq)$ is a partially ordered set as we may define a binary relation $\leq$ on $\mathcal{C}$ by $\eta \leq \zeta$ if and only if $\eta * \zeta = 0$.

Proposition 2.1. [22] In BCC-algebra $\mathcal{C}$, the following statements are true.

- (i) $(\eta * \zeta) * \eta = 0$
- (ii) $\eta \leq \zeta \implies \eta * \rho \leq \zeta * \rho$
- (iii) $\eta \leq \zeta \implies \rho * \eta \leq \rho * \zeta$
- (iv) $(\eta * \zeta) * (\rho * \zeta) \leq \eta * \rho$

Riaz and Hashmi [42] suggested the novel concept of Linear Diophantine Fuzzy Set (LiDFS) integrating control parameters or reference parameters.

Definition 3. [42] A LiDFS in the universe Θ is defined as

$$\Psi_d = \left\{ (L, (A^{\gamma_{\Psi}}(L), B^{\gamma_{\Psi}}(L)), (C^{\gamma_{\Psi}}(L), D^{\gamma_{\Psi}}(L))) \mid L \in \Theta \right\}, \quad (2.1)$$

where $A^{\gamma_{\Psi}}(L)$ and $B^{\gamma_{\Psi}}(L)$ denote the MD and NMD, respectively, and $C^{\gamma_{\Psi}}(L)$ and $D^{\gamma_{\Psi}}(L)$ represent the control parameters, subject to the constraints

$$0 \leq C^{\gamma_{\Psi}}(L) + D^{\gamma_{\Psi}}(L) \leq 1, \quad \text{and} \quad 0 \leq A^{\gamma_{\Psi}}(L)C^{\gamma_{\Psi}}(L) + B^{\gamma_{\Psi}}(L)D^{\gamma_{\Psi}}(L) \leq 1, \forall L \in \Theta.$$

The hesitancy component can be assessed as follows

$$\Pi_d(L) = 1 - (A^{\gamma_{\Psi}}(L)C^{\gamma_{\Psi}}(L) + B^{\gamma_{\Psi}}(L)D^{\gamma_{\Psi}}(L)).$$

Muhiuddin et al. [36] presented the idea of LiDF ideals in BCK-algebras.

Definition 4. A LiDFS $\Psi_d = ((A^{\gamma_{\Psi}}, B^{\gamma_{\Psi}}), (C^{\gamma_{\Psi}}, D^{\gamma_{\Psi}}))$ of $\mathcal{C}$ is called LiDF ideal (LiDFI) if for all $\eta, \xi \in \mathcal{C}$, the following conditions hold

- (L1) $A^{\gamma_{\Psi}}(0) \geq A^{\gamma_{\Psi}}(\eta)$, $B^{\gamma_{\Psi}}(0) \leq B^{\gamma_{\Psi}}(\eta)$ and $C^{\gamma_{\Psi}}(0) \geq C^{\gamma_{\Psi}}(\eta)$, $D^{\gamma_{\Psi}}(0) \leq D^{\gamma_{\Psi}}(\eta)$
- (L2) $A^{\gamma_{\Psi}}(\eta) \geq A^{\gamma_{\Psi}}(\eta * \zeta) \wedge A^{\gamma_{\Psi}}(\zeta)$ and $B^{\gamma_{\Psi}}(\eta) \leq B^{\gamma_{\Psi}}(\eta * \zeta) \vee B^{\gamma_{\Psi}}(\zeta)$
- (L3) $C^{\gamma_{\Psi}}(\eta) \geq C^{\gamma_{\Psi}}(\eta * \zeta) \wedge C^{\gamma_{\Psi}}(\zeta)$ and $D^{\gamma_{\Psi}}(\eta) \leq D^{\gamma_{\Psi}}(\eta * \zeta) \vee D^{\gamma_{\Psi}}(\zeta)$.

Definition 5. [42] The LiDFS with the following expression

$$\Psi_d = \{(\sqsupset, (1, 0), (1, 0)) \mid L \in L\} \quad (2.2)$$

is called absolute LiDFS, and

$$\Psi_d = \{(\sqsupset, (0, 1), (0, 1)) \mid L \in L\} \quad (2.3)$$

is called Null LiDFS.

Definition 6. [42] Let $\Psi_d = ((A^{\gamma_{\Psi}}, B^{\gamma_{\Psi}}), (C^{\gamma_{\Psi}}, D^{\gamma_{\Psi}}))$ represent a Linear Diophantine Fuzzy Number (LiDFN). A scoring function (SF) can be defined through the mapping $P : \text{LiDFN}(Z) \rightarrow [-1, 1]$, which is given by:

$$S_D = \frac{1}{2} [(A^{\gamma_{\Psi}} - B^{\gamma_{\Psi}}) + (C^{\gamma_{\Psi}} - D^{\gamma_{\Psi}})] \quad (2.4)$$

where $\text{LiDFN}(Z)$ is an assembling of LiDFNs on Z .

Definition 7. [42] The mapping $\psi : \text{LiDFN}(Z) \rightarrow [0, 1]$ can be used to define the accuracy function (AF), which is given as

$$\psi_D = \frac{1}{2} \left[\left(\frac{A^{\gamma\Psi} + B^{\gamma\Psi}}{2} \right) + (C^{\gamma\Psi} + D^{\gamma\Psi}) \right] \quad (2.5)$$

Definition 8. [42] Let $\mathcal{N}_i = ((A^{\gamma_i}, B^{\gamma_i}), (C^{\gamma_i}, D^{\gamma_i}))$ be a set of LiDFN, and let $\Xi = (\Xi_1, \Xi_2, \dots, \Xi_n)$ be a weight vector with $i = 1, 2, \dots, n$, where each $\Xi_i \in [0, 1]$ and $\sum_{i=1}^n \Xi_i = 1$. The linear Diophantine fuzzy weighted average (LiDFWA) and linear Diophantine fuzzy weighted geometric (LiDFWG) operators are defined as follows:

$$\begin{aligned} \text{LiDFWA}(\mathfrak{D}_1, \mathfrak{D}_2, \dots, \mathfrak{D}_n) &= \bigoplus_{i=1}^n \Xi_i \mathfrak{D}_i \\ &= \left(\left(1 - \prod_{i=1}^n (1 - A^{\gamma_i})^{\Xi_i}, \prod_{i=1}^n B^{\gamma_i \Xi_i} \right), \left(1 - \prod_{i=1}^n (1 - C^{\gamma_i})^{\Xi_i}, \prod_{i=1}^n D^{\gamma_i \Xi_i} \right) \right) \\ \text{LiDFWG}(\mathfrak{D}_1, \mathfrak{D}_2, \dots, \mathfrak{D}_n) &= \bigotimes_{i=1}^n \mathfrak{D}_i^{\Xi_i} \\ &= \left(\left(\prod_{i=1}^n A^{\gamma_i \Xi_i}, 1 - \prod_{i=1}^n (1 - B^{\gamma_i})^{\Xi_i} \right), \left(\prod_{i=1}^n C^{\gamma_i \Xi_i}, 1 - \prod_{i=1}^n (1 - D^{\gamma_i})^{\Xi_i} \right) \right) \end{aligned}$$

3. LINEAR DIOPHANTINE FUZZY POINTS IN LiDFS

Definition 9. Let $\Psi_d = ((A^{\gamma\Psi}, B^{\gamma\Psi}), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ be an LiDFS. For any constants $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$ such that $0 \leq \Phi\partial + \lambda\vartheta \leq 1$ and $0 \leq \partial + \vartheta \leq 1$, define the $((\Phi, \partial), (\lambda, \vartheta))$ -cut of Ψ_d as follows:

$$\Psi_d^{(\Phi, \partial; \lambda, \vartheta)} = \{\eta \in \mathcal{C} : A^{\gamma\Psi}(\eta) \geq \Phi, B^{\gamma\Psi}(\eta) \leq \partial, C^{\gamma\Psi}(\eta) \geq \lambda, D^{\gamma\Psi}(\eta) \leq \vartheta\} \quad (3.6)$$

This is also known as linear Diophantine level subset (LDLS).

Definition 10. A LiDFS $\Psi_d = ((A^{\gamma\Psi}, B^{\gamma\Psi}), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ of the form

$$\eta_{(\Phi, \partial, \lambda, \vartheta)}(\zeta) = \begin{cases} ((\Phi, \partial), (\lambda, \vartheta)) & \text{if } \zeta = \eta, \\ ((0, 1), (0, 1)) & \text{if } \zeta \neq \eta \end{cases}$$

is said to be an LiDF point with support η and value $(\Phi, \partial, \lambda, \vartheta)$ and denoted by $\eta_{(\Phi, \partial, \lambda, \vartheta)} \Psi_d$. For an LiDF point $\eta_{(\Phi, \partial, \lambda, \vartheta)}$ and LiDF set Ψ_d in a $\mathcal{C}$, Consider the symbol $\eta_{(\Phi, \partial, \lambda, \vartheta)} \in \Psi_d$ (resp. $\eta_{(\Phi, \partial, \lambda, \vartheta)} q \Psi_d$) called linear Diophantinally belong to (resp. quasi-coincide).

If

$$A^{\gamma\Psi}(\eta) \geq \Phi, B^{\gamma\Psi}(\eta) \leq \partial, C^{\gamma\Psi}(\eta) \geq \lambda, D^{\gamma\Psi}(\eta) \leq \vartheta \quad (\text{resp. } A^{\gamma\Psi}(\eta) + \Phi > 1, B^{\gamma\Psi}(\eta) + \partial < 1, C^{\gamma\Psi}(\eta) + \lambda > 1, D^{\gamma\Psi}(\eta) + \vartheta < 1).$$

By the symbol $\eta_{(\Phi, \partial, \lambda, \vartheta)} q \Psi_d$ we mean that $A^{\gamma\Psi}(\eta) + \Phi + k > 1, B^{\gamma\Psi}(\eta) + \partial + k < 1, C^{\gamma\Psi}(\eta) + \lambda + k > 1, D^{\gamma\Psi}(\eta) + \vartheta + k < 1$ where $k \in (0, 1)$. We used the symbol $\eta_{\Phi} \in A^{\gamma\Psi}$ implies $A^{\gamma\Psi}(\eta) \geq \Phi$, $\eta_{\partial} \in B^{\gamma\Psi}$ implies $B^{\gamma\Psi}(\eta) \leq \partial$ and $\eta_{\lambda} \in C^{\gamma\Psi}$ implies $C^{\gamma\Psi}(\eta) \geq \lambda$, $\eta_{\vartheta} \in D^{\gamma\Psi}$ implies $D^{\gamma\Psi}(\eta) \leq \vartheta$, in the whole paper. To see that: $\eta_{(\Phi, \partial, \lambda, \vartheta)} \in \vee q \Psi_d$ we mean $\eta_{(\Phi, \partial, \lambda, \vartheta)} \in \Psi_d$ or $\eta_{(\Phi, \partial, \lambda, \vartheta)} q \Psi_d$.

Theorem 1. The LiDFS Ψ_d of BCC algebra $\mathcal{C}$ is an LiDF BCC ideal of $\mathcal{C}$ if and only if the set $\Psi_d^{(\Phi, \partial; \lambda, \vartheta)} \neq \emptyset$ for all $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$, is either empty or an BCC ideal of $\mathcal{C}$.

Proof: Straightforward.

3.1. $(\bar{\alpha}, \bar{\beta})$ -Linear Diophantine Fuzzy ideals. In this section, we propose and discuss $(\bar{\alpha}, \bar{\beta})$ -LiDF ideals,

$$\text{where } \bar{\alpha}, \bar{\beta} \in \{\bar{\epsilon}, \bar{q}_k, \bar{\epsilon} \vee \bar{q}_k, \bar{\epsilon} \wedge \bar{q}_k\}, \quad \bar{\alpha} \neq \bar{\epsilon} \wedge \bar{q}_k$$

Definition 11. A LiDF set $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ of a BCC-algebra $\mathcal{C}$ is called a $(\bar{\alpha}, \bar{\beta})$ LiDF ideal, where $\bar{\alpha} \neq \bar{\epsilon} \wedge \bar{q}_k$, if it satisfies the following conditions: for all $\Phi, \Phi_1, \Phi_2, \lambda, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial, \partial_1, \partial_2, \vartheta, \vartheta_1, \vartheta_2 \in [0, 1)$ and $\eta, \zeta \in \mathcal{C}$.

$$(LD1) \quad 0_\Phi \bar{\alpha} A^\gamma_\Psi \Rightarrow \eta_\Phi \bar{\beta} A^\gamma_\Psi, 0_\partial [\bar{\alpha}] B^\gamma_\Psi \Rightarrow \eta_\partial [\bar{\beta}] B^\gamma_\Psi, 0_\lambda \bar{\alpha} C^\gamma_\Psi \Rightarrow \eta_\lambda \bar{\beta} C^\gamma_\Psi, 0_\vartheta [\bar{\alpha}] D^\gamma_\Psi \Rightarrow \eta_\vartheta [\bar{\beta}] D^\gamma_\Psi$$

$$(LD2) \quad \eta_{\Phi_1 \wedge \Phi_2} \bar{\alpha} A^\gamma_\Psi \implies (\eta * \zeta)_{\Phi_1} \bar{\beta} A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\beta} A^\gamma_\Psi \text{ and } \eta_{\partial_1 \vee \partial_2} [\bar{\alpha}] B^\gamma_\Psi \implies (\eta * \zeta)_{\partial_1} [\bar{\beta}] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\beta}] B^\gamma_\Psi$$

$$(LD3) \quad \eta_{\lambda_1 \wedge \lambda_2} \bar{\alpha} C^\gamma_\Psi \implies (\eta * \zeta)_{\lambda_1} \bar{\beta} C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\beta} C^\gamma_\Psi \text{ and } \eta_{\vartheta_1 \vee \vartheta_2} [\bar{\alpha}] D^\gamma_\Psi \implies (\eta * \zeta)_{\vartheta_1} [\bar{\beta}] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\beta}] D^\gamma_\Psi$$

Theorem 2. The LiDFS $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ of a BCC-algebra $\mathcal{C}$ is a LiDFI of $\mathcal{C}$ if and only if Ψ_d is an $(\bar{\epsilon}, \bar{\epsilon})$ -LiDFI of $\mathcal{C}$.

Proof. Let $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ be LiDFI of a BCC-algebra $\mathcal{C}$. Suppose $0_\Phi \bar{\epsilon} A^\gamma_\Psi, 0_\partial [\bar{\epsilon}] B^\gamma_\Psi$ and $0_\lambda \bar{\epsilon} C^\gamma_\Psi, 0_\vartheta [\bar{\epsilon}] D^\gamma_\Psi$ for $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$. Then $A^\gamma_\Psi(0) < \Phi, B^\gamma_\Psi(0) > \partial$ and $C^\gamma_\Psi(0) < \lambda, D^\gamma_\Psi(0) > \vartheta$.

Since Ψ_d is an LiDFI, So we have

$$A^\gamma_\Psi(0) \geq A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) \leq B^\gamma_\Psi(\eta) \text{ and } C^\gamma_\Psi(0) \geq C^\gamma_\Psi(\eta), D^\gamma_\Psi(0) \leq D^\gamma_\Psi(\eta),$$

Thus

$$A^\gamma_\Psi(\eta) < \Phi, B^\gamma_\Psi(\eta) > \partial \text{ and } C^\gamma_\Psi(\eta) < \lambda, D^\gamma_\Psi(\eta) > \vartheta$$

which implies

$$\eta_\Phi \bar{\epsilon} A^\gamma_\Psi, \eta_\partial [\bar{\epsilon}] B^\gamma_\Psi \text{ and } \eta_\lambda \bar{\epsilon} C^\gamma_\Psi, \eta_\vartheta [\bar{\epsilon}] D^\gamma_\Psi$$

Suppose $\eta \in \mathcal{C}$ and $\Phi_1, \Phi_2, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial_1, \partial_2, \vartheta_1, \vartheta_2 \in [0, 1)$ such that

$$\eta_{\Phi_1 \wedge \Phi_2} \bar{\epsilon} A^\gamma_\Psi, \eta_{\partial_1 \vee \partial_2} [\bar{\epsilon}] B^\gamma_\Psi \text{ and } \eta_{\lambda_1 \wedge \lambda_2} \bar{\epsilon} C^\gamma_\Psi, \eta_{\vartheta_1 \vee \vartheta_2} [\bar{\epsilon}] D^\gamma_\Psi$$

Then

$$A^\gamma_\Psi(\eta) < \Phi_1 \wedge \Phi_2, B^\gamma_\Psi(\eta) > \partial_1 \vee \partial_2 \text{ and } C^\gamma_\Psi(\eta) < \lambda_1 \wedge \lambda_2, D^\gamma_\Psi(\eta) > \vartheta_1 \vee \vartheta_2$$

So by given condition

$$\Phi_1 \wedge \Phi_2 > A^\gamma_\Psi(\eta) \geq A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta),$$

$$\partial_1 \vee \partial_2 < B^\gamma_\Psi(\eta) \leq B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta),$$

$$\lambda_1 \wedge \lambda_2 > C^\gamma_\Psi(\eta) \geq C^\gamma_\Psi(\eta * \zeta) \wedge C^\gamma_\Psi(\zeta),$$

$$\vartheta_1 \vee \vartheta_2 < D^\gamma_\Psi(\eta) \leq D^\gamma_\Psi(\eta * \zeta) \vee D^\gamma_\Psi(\zeta).$$

This implies that

$$A^\gamma_\Psi(\eta * \zeta) \leq \Phi_1, A^\gamma_\Psi(\zeta) \leq \Phi_2 \text{ and } B^\gamma_\Psi(\eta * \zeta) \geq \partial_1, B^\gamma_\Psi(\zeta) \geq \partial_2$$

$$C^\gamma_\Psi(\eta * \zeta) \leq \lambda_1, C^\gamma_\Psi(\zeta) \leq \lambda_2 \text{ and } D^\gamma_\Psi(\eta * \zeta) \geq \vartheta_1, D^\gamma_\Psi(\zeta) \geq \vartheta_2$$

that is,

$$(\eta * \zeta)_{\Phi_1} \bar{\epsilon} A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\epsilon} A^\gamma_\Psi \text{ and } (\eta * \zeta)_{\partial_1} [\bar{\epsilon}] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\epsilon}] B^\gamma_\Psi$$

$$(\eta * \zeta)_{\lambda_1} \bar{\epsilon} C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\epsilon} C^\gamma_\Psi \text{ and } (\eta * \zeta)_{\vartheta_1} [\bar{\epsilon}] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\epsilon}] D^\gamma_\Psi$$

This shows that Ψ_d is an $(\overline{\in}, \overline{\in})$ -LiDFI of $\mathcal{C}$.

Conversely, Suppose that Ψ_d is an $(\overline{\in}, \overline{\in})$ -LiDFI of $\mathcal{C}$. We have to show that Ψ_d is a linear Diophantine fuzzy ideal (LiDFI) of $\mathcal{C}$. Then there exists $\eta \in \mathcal{C}$ such that

$$A^\gamma_\Psi(0) < A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) > B^\gamma_\Psi(\eta) \text{ and } C^{\gamma\Psi}(0) < C^{\gamma\Psi}(\eta), D^{\gamma\Psi}(0) > D^{\gamma\Psi}(\eta)$$

Choose $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$ such that $A^\gamma_\Psi(0) < \Phi \leq A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) > \partial \geq B^\gamma_\Psi(\eta)$ and $C^{\gamma\Psi}(0) < \lambda \leq C^{\gamma\Psi}(\eta), D^{\gamma\Psi}(0) > \vartheta \geq D^{\gamma\Psi}(\eta)$ Then

$$0_\Phi \overline{\in} A^\gamma_\Psi, 0_\partial \overline{\in} B^\gamma_\Psi$$

$$\text{and } 0_\lambda \overline{\in} C^{\gamma\Psi}, 0_\vartheta \overline{\in} D^{\gamma\Psi} \text{ but } \eta_\Phi \in A^\gamma_\Psi$$

$$, \eta_\partial \in B^\gamma_\Psi$$

$$\text{and } \eta_\lambda \in C^{\gamma\Psi}$$

$$, \eta_\vartheta \in D^{\gamma\Psi} \text{ a contradiction. Hence}$$

$$A^\gamma_\Psi(0) \geq A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) \leq B^\gamma_\Psi(\eta) \text{ and } C^{\gamma\Psi}(0) \geq C^{\gamma\Psi}(\eta), D^{\gamma\Psi}(0) \leq D^{\gamma\Psi}(\eta)$$

for all $\eta \in \mathcal{C}$. Now there exist $\eta, \zeta \in \mathcal{C}$ such that $A^\gamma_\Psi(\eta) < A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta)$, $B^\gamma_\Psi(\eta) > B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta)$ and $C^{\gamma\Psi}(\eta) < C^{\gamma\Psi}(\eta * \zeta) \wedge C^{\gamma\Psi}(\zeta)$, $D^{\gamma\Psi}(\eta) > D^{\gamma\Psi}(\eta * \zeta) \vee D^{\gamma\Psi}(\zeta)$ Select $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$ such that

$$A^\gamma_\Psi(\eta) < \Phi \leq A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta)$$

$$B^\gamma_\Psi(\eta) > \partial \geq B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta)$$

$$C^{\gamma\Psi}(\eta) < \lambda \leq C^{\gamma\Psi}(\eta * \zeta) \wedge C^{\gamma\Psi}(\zeta)$$

$$D^{\gamma\Psi}(\eta) > \vartheta \geq D^{\gamma\Psi}(\eta * \zeta) \vee D^{\gamma\Psi}(\zeta)$$

Then

$$\eta_\Phi \overline{\in} A^\gamma_\Psi \text{ but } (\eta * \zeta)_\Phi \in A^\gamma_\Psi \text{ and } \zeta_\Phi \in A^\gamma_\Psi$$

$$\eta_\partial \overline{\in} B^\gamma_\Psi \text{ but } (\eta * \zeta)_\partial \in B^\gamma_\Psi \text{ and } \zeta_\partial \in B^\gamma_\Psi$$

$$\eta_\lambda \overline{\in} C^{\gamma\Psi} \text{ but } (\eta * \zeta)_\lambda \in C^{\gamma\Psi} \text{ and } \zeta_\lambda \in C^{\gamma\Psi}$$

$$\eta_\vartheta \overline{\in} D^{\gamma\Psi} \text{ but } (\eta * \zeta)_\vartheta \in D^{\gamma\Psi} \text{ and } \zeta_\vartheta \in D^{\gamma\Psi}$$

which is a contradiction. Therefore,

$$A^\gamma_\Psi(\eta) \geq A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta),$$

$$B^\gamma_\Psi(\eta) \leq B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta),$$

$$C^{\gamma\Psi}(\eta) \geq C^{\gamma\Psi}(\eta * \zeta) \wedge C^{\gamma\Psi}(\zeta),$$

$$D^{\gamma\Psi}(\eta) \leq D^{\gamma\Psi}(\eta * \zeta) \vee D^{\gamma\Psi}(\zeta). \text{ Therefore, } \Psi_d \text{ is a LiDFI of } \mathcal{C}.$$

3.2. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ -LiDF Ideals. In this section, we define $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ -LiDF ideals of $\mathcal{C}$ as a special case of $(\overline{\alpha}, \overline{\beta})$ -LiDF ideals, and discuss several results.

Definition 12. A LiDF set $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ of a BCC-algebra $\mathcal{C}$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ -LiDF ideal, if it satisfies the following conditions: for all

$\Phi, \Phi_1, \Phi_2, \lambda, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial, \partial_1, \partial_2, \vartheta, \vartheta_1, \vartheta_2 \in [0, 1)$ and $\eta, \zeta \in \mathcal{C}$.

$$(EL1) \quad 0_\Phi \overline{\in} A^\gamma_\Psi \Rightarrow \eta_\Phi \overline{\in} \vee \overline{q_k} A^\gamma_\Psi, 0_\partial \overline{\in} B^\gamma_\Psi \Rightarrow \eta_\partial \overline{\in} \vee \overline{q_k} B^\gamma_\Psi, 0_\lambda \overline{\in} C^{\gamma\Psi} \Rightarrow \eta_\lambda \overline{\in} \vee \overline{q_k} C^{\gamma\Psi}, 0_\vartheta \overline{\in} D^{\gamma\Psi} \Rightarrow \eta_\vartheta \overline{\in} \vee \overline{q_k} D^{\gamma\Psi}$$

$$\begin{aligned}
(\text{EL2}) \quad & \eta_{\Phi_1 \wedge \Phi_2} \bar{\in} A^\gamma_\Psi \implies (\eta * \zeta)_{\Phi_1} \bar{\in} \vee \overline{q_k} A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\in} \vee \overline{q_k} A^\gamma_\Psi \text{ and } \eta_{\partial_1 \vee \partial_2} [\bar{\in}] B^\gamma_\Psi \implies \\
& (\eta * \zeta)_{\partial_1} [\bar{\in}] \vee [\overline{q_k}] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\in}] \vee [\overline{q_k}] B^\gamma_\Psi \\
(\text{EL3}) \quad & \eta_{\lambda_1 \wedge \lambda_2} \bar{\in} C^\gamma_\Psi \implies (\eta * \zeta)_{\lambda_1} \bar{\in} \vee \overline{q_k} C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\in} \vee \overline{q_k} C^\gamma_\Psi \text{ and } \eta_{\vartheta_1 \vee \vartheta_2} [\bar{\in}] D^\gamma_\Psi \implies \\
& (\eta * \zeta)_{\vartheta_1} [\bar{\in}] \vee [\overline{q_k}] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\in}] \vee [\overline{q_k}] D^\gamma_\Psi
\end{aligned}$$

Lemma 3.3. For a LiDFS $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ the Conditions (EL1-EL3) are equivalent to following conditions for all $\eta, \zeta \in \mathcal{C}$.

$$\begin{aligned}
(\text{ED1}) \quad & \max \left\{ A^\gamma_\Psi(0), \frac{1-k}{2} \right\} \geq A^\gamma_\Psi(\eta), (\min \left\{ B^\gamma_\Psi(0), \frac{1-k}{2} \right\} \leq B^\gamma_\Psi(\eta) \text{ and } \\
& \max \left\{ C^\gamma_\Psi(0), \frac{1-k}{2} \right\} \geq C^\gamma_\Psi(\eta), \min \left\{ D^\gamma_\Psi(0), \frac{1-k}{2} \right\} \leq D^\gamma_\Psi(\eta) \\
(\text{ED2}) \quad & \max \left\{ A^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} \geq \min \left\{ A^\gamma_\Psi(\eta * \zeta), A^\gamma_\Psi(\zeta) \right\} \text{ and } \\
& \min \left\{ B^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} \leq \max \left\{ B^\gamma_\Psi(\eta * \zeta), B^\gamma_\Psi(\zeta) \right\} \\
(\text{ED3}) \quad & \max \left\{ C^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} \geq \min \left\{ C^\gamma_\Psi(\eta * \zeta), C^\gamma_\Psi(\zeta) \right\} \text{ and } \\
& \min \left\{ D^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} \leq \max \left\{ D^\gamma_\Psi(\eta * \zeta), D^\gamma_\Psi(\zeta) \right\}
\end{aligned}$$

Proof. (EL1) $\iff$ (ED1)

The proof is obvious.

(EL2-EL3) $\Rightarrow$ (ED2-ED3).

Let $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ be an LiDFS of $\mathcal{C}$. On the Contrary, assume that there exist $\eta, \zeta \in \mathcal{C}$ and choose $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$ such that $\max \left\{ A^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} < \Phi = \min \left\{ A^\gamma_\Psi(\eta * \zeta), A^\gamma_\Psi(\zeta) \right\}$, $\min \left\{ B^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} > \partial = \max \left\{ B^\gamma_\Psi(\eta * \zeta), B^\gamma_\Psi(\zeta) \right\}$, $\max \left\{ C^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} < \lambda = \min \left\{ C^\gamma_\Psi(\eta * \zeta), C^\gamma_\Psi(\zeta) \right\}$, $\min \left\{ D^\gamma_\Psi(\eta), \frac{1-k}{2} \right\} > \vartheta = \max \left\{ D^\gamma_\Psi(\eta * \zeta), D^\gamma_\Psi(\zeta) \right\}$

Then $\Phi, \lambda \in (\frac{1-k}{2}, 1]$ and $\partial, \vartheta \in [\frac{1-k}{2}, 1)$ such that

$$\begin{aligned}
& \eta_\Phi \bar{\in} A^\gamma_\Psi \text{ but } (\eta * \zeta)_\Phi \in A^\gamma_\Psi, \zeta_\Phi \in A^\gamma_\Psi \text{ and } \eta_\partial [\bar{\in}] B^\gamma_\Psi \text{ but } (\eta * \zeta)_\partial [\bar{\in}] B^\gamma_\Psi, \zeta_\partial [\bar{\in}] B^\gamma_\Psi \\
& \eta_\lambda \bar{\in} C^\gamma_\Psi \text{ but } (\eta * \zeta)_\lambda \in C^\gamma_\Psi, \zeta_\lambda \in C^\gamma_\Psi \text{ and } \eta_\vartheta [\bar{\in}] D^\gamma_\Psi \text{ but } (\eta * \zeta)_\vartheta [\bar{\in}] D^\gamma_\Psi, \zeta_\vartheta [\bar{\in}] D^\gamma_\Psi
\end{aligned}$$

It follows that

$$\begin{aligned}
& (\eta * \zeta)_\Phi \overline{q_k} A^\gamma_\Psi, \zeta_\Phi \overline{q_k} A^\gamma_\Psi \text{ and } (\eta * \zeta)_\partial [\overline{q_k}] B^\gamma_\Psi, \zeta_\partial [\overline{q_k}] B^\gamma_\Psi \\
& (\eta * \zeta)_\lambda \overline{q_k} C^\gamma_\Psi, \zeta_\lambda \overline{q_k} C^\gamma_\Psi \text{ and } (\eta * \zeta)_\vartheta [\overline{q_k}] D^\gamma_\Psi, \zeta_\vartheta [\overline{q_k}] D^\gamma_\Psi
\end{aligned}$$

Then

$$\begin{aligned}
& A^\gamma_\Psi(\eta * \zeta) + \Phi + k \leq 1 \text{ or } A^\gamma_\Psi(\zeta) + \Phi + k \leq 1 \text{ and } B^\gamma_\Psi(\eta * \zeta) + \partial + k \geq 1 \\
& \text{or } B^\gamma_\Psi(\zeta) + \partial + k \geq 1 \\
& C^\gamma_\Psi(\eta * \zeta) + \lambda + k \leq 1 \text{ or } C^\gamma_\Psi(\zeta) + \lambda + k \leq 1 \text{ and } D^\gamma_\Psi(\eta * \zeta) + \vartheta + k \geq 1 \\
& \text{or } D^\gamma_\Psi(\zeta) + \vartheta + k \geq 1
\end{aligned}$$

As

$$\begin{aligned}
& \Phi \leq A^\gamma_\Psi(\eta * \zeta), \Phi \leq A^\gamma_\Psi(\zeta) \text{ and } \partial \geq B^\gamma_\Psi(\eta * \zeta), \partial \geq B^\gamma_\Psi(\zeta) \\
& \lambda \leq C^\gamma_\Psi(\eta * \zeta), \lambda \leq C^\gamma_\Psi(\zeta) \text{ and } \vartheta \geq D^\gamma_\Psi(\eta * \zeta), \vartheta \geq D^\gamma_\Psi(\zeta)
\end{aligned}$$

It follows that

$$\Phi \leq \frac{1-k}{2}, \partial \geq \frac{1-k}{2} \text{ and } \lambda \leq \frac{1-k}{2}, \vartheta \geq \frac{1-k}{2}$$

a contradiction. So we have

$$\begin{aligned} \max \{A^\gamma_\Psi(\eta), \frac{1-k}{2}\} &\geq \min \{A^\gamma_\Psi(\eta * \zeta), A^\gamma_\Psi(\zeta)\}, \\ \min \{B^\gamma_\Psi(\eta), \frac{1-k}{2}\} &\leq \max \{B^\gamma_\Psi(\eta * \zeta), \zeta^\partial_\gamma(\zeta)\}, \\ \max \{C^\gamma_\Psi(\eta), \frac{1-k}{2}\} &\geq \min \{C^\gamma_\Psi(\eta * \zeta), C^\gamma_\Psi(\zeta)\}, \\ \min \{D^\gamma_\Psi(\eta), \frac{1-k}{2}\} &\leq \max \{D^\gamma_\Psi(\eta * \zeta), D^\gamma_\Psi(\zeta)\}. \end{aligned}$$

(ED2-ED3) $\Rightarrow$ (EL2-EL3)

Suppose $\eta \in \mathcal{C}$ and $\Phi_1, \Phi_2, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial_1, \partial_2, \vartheta_1, \vartheta_2 \in [0, 1]$ such that

$$\eta_{\Phi_1 \wedge \Phi_2} \bar{\in} A^\gamma_\Psi, \eta_{\partial_1 \vee \partial_2} [\bar{\in}] B^\gamma_\Psi \text{ and } \eta_{\lambda_1 \wedge \lambda_2} \bar{\in} C^\gamma_\Psi, \eta_{\vartheta_1 \vee \vartheta_2} [\bar{\in}] D^\gamma_\Psi$$

Then

$$A^\gamma_\Psi(\eta) < \Phi_1 \wedge \Phi_2, B^\gamma_\Psi(\eta) > \partial_1 \vee \partial_2 \text{ and } C^\gamma_\Psi(\eta) < \lambda_1 \wedge \lambda_2, D^\gamma_\Psi(\eta) > \vartheta_1 \vee \vartheta_2$$

Case (I)

$$A^\gamma_\Psi(\eta) \geq \frac{1-k}{2}, B^\gamma_\Psi(\eta) \leq \frac{1-k}{2} \text{ and } C^\gamma_\Psi(\eta) \geq \frac{1-k}{2}, D^\gamma_\Psi(\eta) \leq \frac{1-k}{2}$$

then by (ED2-ED3), we have

$$\begin{aligned} A^\gamma_\Psi(\eta) &\geq A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta) \text{ and } B^\gamma_\Psi(\eta) \leq B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta) \\ C^\gamma_\Psi(\eta) &\geq C^\gamma_\Psi(\eta * \zeta) \wedge C^\gamma_\Psi(\zeta) \text{ and } D^\gamma_\Psi(\eta) \leq D^\gamma_\Psi(\eta * \zeta) \vee D^\gamma_\Psi(\zeta) \end{aligned}$$

Thus

$$\begin{aligned} A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta) &< \Phi_1 \wedge \Phi_2 \text{ and } B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta) > \partial_1 \vee \partial_2 \\ C^\gamma_\Psi(\eta * \zeta) \wedge C^\gamma_\Psi(\zeta) &< \lambda_1 \wedge \lambda_2 \text{ and } D^\gamma_\Psi(\eta * \zeta) \vee D^\gamma_\Psi(\zeta) > \vartheta_1 \vee \vartheta_2 \end{aligned}$$

Consequently

$$\begin{aligned} A^\gamma_\Psi(\eta * \zeta) &< \Phi_1 \text{ or } A^\gamma_\Psi(\zeta) < \Phi_2 \text{ and } B^\gamma_\Psi(\eta * \zeta) > \partial_1 \text{ or } B^\gamma_\Psi(\zeta) > \partial_2 \\ C^\gamma_\Psi(\eta * \zeta) &< \lambda_1 \text{ or } C^\gamma_\Psi(\zeta) < \lambda_2 \text{ and } D^\gamma_\Psi(\eta * \zeta) > \vartheta_1 \text{ or } D^\gamma_\Psi(\zeta) > \vartheta_2 \end{aligned}$$

It follows that

$$\begin{aligned} (\eta * \zeta)_{\Phi_1} \bar{\in} A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\in} A^\gamma_\Psi \text{ and } (\eta * \zeta)_{\partial_1} [\bar{\in}] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\in}] B^\gamma_\Psi \\ (\eta * \zeta)_{\lambda_1} \bar{\in} C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\in} C^\gamma_\Psi \text{ and } (\eta * \zeta)_{\vartheta_1} [\bar{\in}] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\in}] D^\gamma_\Psi \end{aligned}$$

Case (II)

$$A^\gamma_\Psi(\eta) < \frac{1-k}{2}, B^\gamma_\Psi(\eta) > \frac{1-k}{2} \text{ and } C^\gamma_\Psi(\eta) < \frac{1-k}{2}, D^\gamma_\Psi(\eta) > \frac{1-k}{2}$$

then by (ED2-ED3), we have

$$\begin{aligned} \frac{1-k}{2} &\geq A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta) \text{ and } \frac{1-k}{2} \leq B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta) \\ \frac{1-k}{2} &\geq C^\gamma_\Psi(\eta * \zeta) \wedge C^\gamma_\Psi(\zeta) \text{ and } \frac{1-k}{2} \leq D^\gamma_\Psi(\eta * \zeta) \vee D^\gamma_\Psi(\zeta) \end{aligned}$$

Suppose

$$\begin{aligned} (\eta * \zeta)_{\Phi_1} \in A^\gamma_\Psi, \zeta_{\Phi_2} \in A^\gamma_\Psi \text{ and } (\eta * \zeta)_{\partial_1} [\in] B^\gamma_\Psi, \zeta_{\partial_2} [\in] B^\gamma_\Psi \\ (\eta * \zeta)_{\lambda_1} \in C^\gamma_\Psi, \zeta_{\lambda_2} \in C^\gamma_\Psi \text{ and } (\eta * \zeta)_{\vartheta_1} [\in] D^\gamma_\Psi, \zeta_{\vartheta_2} [\in] D^\gamma_\Psi \end{aligned}$$

Then

$$A^\gamma_\Psi(\eta * \zeta) \geq \Phi_1 \text{ or } A^\gamma_\Psi(\zeta) \geq \Phi_2 \text{ and } B^\gamma_\Psi(\eta * \zeta) \leq \partial_1 \text{ or } B^\gamma_\Psi(\zeta) \leq \partial_2$$

$$C^{\gamma\Psi}(\eta * \zeta) \geq \lambda_1 \text{ or } C^{\gamma\Psi}(\zeta) \geq \lambda_2 \text{ and } D^{\gamma\Psi}(\eta * \zeta) \leq \vartheta_1 \text{ or } D^{\gamma\Psi}(\zeta) \leq \vartheta_2$$

Thus

$$\Phi_1 \wedge \Phi_2 \leq \frac{1-k}{2}, \partial_1 \vee \partial_2 \geq \frac{1-k}{2} \text{ and } \lambda_1 \wedge \lambda_2 \leq \frac{1-k}{2}, \vartheta_1 \wedge \vartheta_2 \geq \frac{1-k}{2}$$

Hence

$$A^\gamma_\Psi(\eta * \zeta) \wedge A^\gamma_\Psi(\zeta) + (\Phi_1 \wedge \Phi_2) + k \leq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

$$B^\gamma_\Psi(\eta * \zeta) \vee B^\gamma_\Psi(\zeta) + (\partial_1 \vee \partial_2) + k \geq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

$$C^{\gamma\Psi}(\eta * \zeta) \wedge C^{\gamma\Psi}(\zeta) + (\lambda_1 \wedge \lambda_2) + k \leq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

$$D^{\gamma\Psi}(\eta * \zeta) \vee D^{\gamma\Psi}(\zeta) + (\vartheta_1 \vee \vartheta_2) + k \geq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

So

$$(\eta * \zeta)_{\Phi_1} \overline{q_k} A^\gamma_\Psi \text{ or } \zeta_{\Phi_2} \overline{q_k} A^\gamma_\Psi \text{ and } (\eta * \zeta)_{\partial_1} [\overline{q_k}] B^\gamma_\Psi \text{ or } \zeta_{\partial_2} [\overline{q_k}] B^\gamma_\Psi$$

$$(\eta * \zeta)_{\lambda_1} \overline{q_k} C^{\gamma\Psi} \text{ or } \zeta_{\lambda_2} \overline{q_k} C^{\gamma\Psi} \text{ and } (\eta * \zeta)_{\vartheta_1} [\overline{q_k}] D^{\gamma\Psi} \text{ or } \zeta_{\vartheta_2} [\overline{q_k}] D^{\gamma\Psi}$$

Hence

$$(\eta * \zeta)_{\Phi_1} \overline{\in} \vee \overline{q_k} A^\gamma_\Psi \text{ or } \zeta_{\Phi_2} \overline{\in} \vee \overline{q_k} A^\gamma_\Psi \text{ and } (\eta * \zeta)_{\partial_1} [\overline{\in}] \vee [\overline{q_k}] B^\gamma_\Psi \text{ or } \zeta_{\partial_2} [\overline{\in}] \vee [\overline{q_k}] B^\gamma_\Psi$$

$$(\eta * \zeta)_{\lambda_1} \overline{\in} \vee \overline{q_k} C^{\gamma\Psi} \text{ or } \zeta_{\lambda_2} \overline{\in} \vee \overline{q_k} C^{\gamma\Psi} \text{ and } (\eta * \zeta)_{\vartheta_1} [\overline{\in}] \vee [\overline{q_k}] D^{\gamma\Psi} \text{ or } \zeta_{\vartheta_2} [\overline{\in}] \vee [\overline{q_k}] D^{\gamma\Psi}$$

Therefore, the conditions (EL2-EL3) are satisfied.

Theorem 3. A LiDF set $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ of a BCC-algebra $\mathcal{C}$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ -LiDF ideal, iff it satisfies the following conditions:

- (ED1) $\max \{A^\gamma_\Psi(0), \frac{1-k}{2}\} \geq A^\gamma_\Psi(\eta), (\min \{B^\gamma_\Psi(0), \frac{1-k}{2}\} \leq B^\gamma_\Psi(\eta)$
and $\max \{C^{\gamma\Psi}(0), \frac{1-k}{2}\} \geq C^{\gamma\Psi}(\eta), \min \{D^{\gamma\Psi}(0), \frac{1-k}{2}\} \leq D^{\gamma\Psi}(\eta)$
- (ED2) $\max \{A^\gamma_\Psi(\eta), \frac{1-k}{2}\} \geq \min \{A^\gamma_\Psi(\eta * \zeta), A^\gamma_\Psi(\zeta)\}$ and
 $\min \{B^\gamma_\Psi(\eta), \frac{1-k}{2}\} \leq \max \{B^\gamma_\Psi(\eta * \zeta), \varsigma^\partial_\gamma(\zeta)\}$
- (ED3) $\max \{C^{\gamma\Psi}(\eta), \frac{1-k}{2}\} \geq \min \{C^{\gamma\Psi}(\eta * \zeta), C^{\gamma\Psi}(\zeta)\}$ and
 $\min \{D^{\gamma\Psi}(\eta), \frac{1-k}{2}\} \leq \max \{D^{\gamma\Psi}(\eta * \zeta), D^{\gamma\Psi}(\zeta)\}$

Example 1. Consider the BCC-algebra $\mathfrak{B} = \{0, a, b, c, d\}$ as defined in Table 1, and let $k = 0.2$. Define a LiDFS $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ of $\mathcal{C}$ as follows:

$$\Psi_d(\eta) = \begin{cases} ((0.7, 0.2), (0.8, 0.1)) & \text{if } \eta = 0, \\ ((0.6, 0.3), (0.7, 0.2)) & \text{if } \eta = a, \\ ((0.5, 0.5), (0.5, 0.4)) & \text{if } \eta = b, \\ ((0.5, 0.6), (0.5, 0.5)) & \text{if } \eta = c, \\ ((0.5, 0.7), (0.5, 0.4)) & \text{if } \eta = d. \end{cases}$$

Then Ψ_d is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDFI of $\mathcal{C}$, as it satisfies all conditions of Theorem 3.

TABLE 1. BCC-Algebra

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	d	0

4. $(\bar{\alpha}, \bar{\beta})$ -LINEAR DIOPHANTINE FUZZY BCC IDEALS

In this section, we propose and discuss $(\bar{\alpha}, \bar{\beta})$ -LiDF ideals,

where $\bar{\alpha}, \bar{\beta} \in \{\bar{\epsilon}, \bar{q}_k, \bar{\epsilon} \vee \bar{q}_k, \bar{\epsilon} \wedge \bar{q}_k\}$, $\bar{\alpha} \neq \bar{\epsilon} \wedge \bar{q}_k$

Definition 13. A LiDF set $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ of a BCC-algebra $\mathcal{C}$ is called a $(\bar{\alpha}, \bar{\beta})$ -LiDF BCC ideal, where $\bar{\alpha} \neq \bar{\epsilon} \wedge \bar{q}_k$, if it satisfies condition LD1 and the following conditions: for all $\Phi, \Phi_1, \Phi_2, \lambda, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial, \partial_1, \partial_2, \vartheta, \vartheta_1, \vartheta_2 \in [0, 1]$ and $\eta, \zeta, \rho \in \mathcal{C}$.

- (LD4) $(\eta * \rho)_{\Phi_1 \wedge \Phi_2} \bar{\alpha} A^\gamma_\Psi \implies ((\eta * \zeta) * \rho)_{\Phi_1} \bar{\beta} A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\beta} A^\gamma_\Psi$ and
 $(\eta * \rho)_{\partial_1 \vee \partial_2} [\bar{\alpha}] B^\gamma_\Psi \implies ((\eta * \zeta) * \rho)_{\partial_1} [\bar{\beta}] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\beta}] B^\gamma_\Psi$
 (LD5) $(\eta * \rho)_{\lambda_1 \wedge \lambda_2} \bar{\alpha} C^\gamma_\Psi \implies ((\eta * \zeta) * \rho)_{\lambda_1} \bar{\beta} C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\beta} C^\gamma_\Psi$ and
 $(\eta * \rho)_{\vartheta_1 \vee \vartheta_2} [\bar{\alpha}] D^\gamma_\Psi \implies ((\eta * \zeta) * \rho)_{\vartheta_1} [\bar{\beta}] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\beta}] D^\gamma_\Psi$

Theorem 4. The LiDFS $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ of a BCC-algebra $\mathcal{C}$ is a LiDF BBC ideal (LiDFBI) of $\mathcal{C}$ if and only if Ψ_d is an $(\bar{\epsilon}, \bar{\epsilon})$ -LiDFBI of $\mathcal{C}$.

Proof. Let $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^\gamma_\Psi, D^\gamma_\Psi))$ be LiDFBI of a BCC-algebra $\mathcal{C}$. Suppose $0_\Phi \bar{\epsilon} A^\gamma_\Psi, 0_\partial [\bar{\epsilon}] B^\gamma_\Psi$ and $0_\lambda \bar{\epsilon} C^\gamma_\Psi, 0_\vartheta [\bar{\epsilon}] D^\gamma_\Psi$ for $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1]$. Then $A^\gamma_\Psi(0) < \Phi, B^\gamma_\Psi(0) > \partial$ and $C^\gamma_\Psi(0) < \lambda, D^\gamma_\Psi(\eta) > \vartheta$.

Since Ψ_d is an LiDFBI, So we have

$$A^\gamma_\Psi(0) \geq A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) \leq B^\gamma_\Psi(\eta) \text{ and } C^\gamma_\Psi(0) \geq C^\gamma_\Psi(\eta), D^\gamma_\Psi(0) \leq D^\gamma_\Psi(\eta),$$

Thus

$$A^\gamma_\Psi(\eta) < \Phi, B^\gamma_\Psi(\eta) > \partial \text{ and } C^\gamma_\Psi(\eta) < \lambda, D^\gamma_\Psi(\eta) > \vartheta$$

which implies

$$\eta_\Phi \bar{\epsilon} A^\gamma_\Psi, \eta_\partial [\bar{\epsilon}] B^\gamma_\Psi \text{ and } \eta_\lambda \bar{\epsilon} C^\gamma_\Psi, \eta_\vartheta [\bar{\epsilon}] D^\gamma_\Psi$$

Suppose $\eta, \zeta, \rho \in \mathcal{C}$ and $\Phi_1, \Phi_2, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial_1, \partial_2, \vartheta_1, \vartheta_2 \in [0, 1]$ such that

$$(\eta * \rho)_{\Phi_1 \wedge \Phi_2} \bar{\epsilon} A^\gamma_\Psi, (\eta * \rho)_{\partial_1 \vee \partial_2} [\bar{\epsilon}] B^\gamma_\Psi \text{ and } (\eta * \rho)_{\lambda_1 \wedge \lambda_2} \bar{\epsilon} C^\gamma_\Psi, (\eta * \rho)_{\vartheta_1 \vee \vartheta_2} [\bar{\epsilon}] D^\gamma_\Psi$$

Then

$$A^\gamma_\Psi(\eta * \rho) < \Phi_1 \wedge \Phi_2, B^\gamma_\Psi(\eta * \rho) > \partial_1 \vee \partial_2 \text{ and } C^\gamma_\Psi(\eta * \rho) < \lambda_1 \wedge \lambda_2, D^\gamma_\Psi(\eta * \rho) > \vartheta_1 \vee \vartheta_2$$

So by given condition

$$\Phi_1 \wedge \Phi_2 > A^\gamma_\Psi(\eta * \rho) \geq A^\gamma_\Psi((\eta * \zeta) * \rho) \wedge A^\gamma_\Psi(\zeta) \text{ and } \partial_1 \vee \partial_2 < B^\gamma_\Psi(\eta * \rho) \leq B^\gamma_\Psi((\eta * \zeta) * \rho) \vee B^\gamma_\Psi(\zeta)$$

$$\lambda_1 \wedge \lambda_2 > C^{\gamma\Psi}(\eta * \rho) \geq C^{\gamma\Psi}((\eta * \zeta) * \rho) \wedge C^{\gamma\Psi}(\zeta) \text{ and } \vartheta_1 \vee \vartheta_2 < D^{\gamma\Psi}(\eta * \rho) \leq D^{\gamma\Psi}((\eta * \zeta) * \rho) \vee D^{\gamma\Psi}(\zeta)$$

This implies that

$$A^\gamma_\Psi((\eta * \zeta) * \rho) \leq \Phi_1, A^\gamma_\Psi(\zeta) \leq \Phi_2 \text{ and } B^\gamma_\Psi((\eta * \zeta) * \rho) \geq \partial_1, B^\gamma_\Psi(\zeta) \geq \partial_2$$

$$C^{\gamma\Psi}((\eta * \zeta) * \rho) \leq \lambda_1, C^{\gamma\Psi}(\zeta) \leq \lambda_2 \text{ and } D^{\gamma\Psi}((\eta * \zeta) * \rho) \geq \vartheta_1, D^{\gamma\Psi}(\zeta) \geq \vartheta_2$$

that is,

$$((\eta * \zeta) * \rho)_{\Phi_1} \bar{\in} A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\in} A^\gamma_\Psi \text{ and } ((\eta * \zeta) * \rho)_{\partial_1} [\bar{\in}] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\in}] B^\gamma_\Psi$$

$$((\eta * \zeta) * \rho)_{\lambda_1} \bar{\in} C^{\gamma\Psi}, \zeta_{\lambda_2} \bar{\in} C^{\gamma\Psi} \text{ and } ((\eta * \zeta) * \rho)_{\vartheta_1} [\bar{\in}] D^{\gamma\Psi}, \zeta_{\vartheta_2} [\bar{\in}] D^{\gamma\Psi}$$

This shows that Ψ_d is an $(\bar{\in}, \bar{\in})$ -LiDFBI of $\mathcal{C}$.

Conversely, Suppose that Ψ_d is an $(\bar{\in}, \bar{\in})$ -LiDFBI of $\mathcal{C}$. We have to show that Ψ_d is a Linear Diophantine fuzzy BCC ideal (LiDFBI) of $\mathcal{C}$. Then there exists $\eta \in \mathcal{C}$ such that

$$A^\gamma_\Psi(0) < A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) > B^\gamma_\Psi(\eta) \text{ and } C^{\gamma\Psi}(0) < C^{\gamma\Psi}(\eta), D^{\gamma\Psi}(0) > D^{\gamma\Psi}(\eta)$$

Choose $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$ such that $A^\gamma_\Psi(0) < \Phi \leq A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) > \partial \geq B^\gamma_\Psi(\eta)$ and $C^{\gamma\Psi}(0) < \lambda \leq C^{\gamma\Psi}(\eta), D^{\gamma\Psi}(0) > \vartheta \geq D^{\gamma\Psi}(\eta)$.

Then $0_\Phi \bar{\in} A^\gamma_\Psi, 0_\partial [\bar{\in}] B^\gamma_\Psi$ and $0_\lambda \bar{\in} C^{\gamma\Psi}, 0_\vartheta [\bar{\in}] D^{\gamma\Psi}$ but $\eta_\Phi \in A^\gamma_\Psi, \eta_\partial [\in] B^\gamma_\Psi$

and $\eta_\lambda \in C^{\gamma\Psi}, \eta_\vartheta [\in] D^{\gamma\Psi}$ a contradiction. Hence

$$A^\gamma_\Psi(0) \geq A^\gamma_\Psi(\eta), B^\gamma_\Psi(0) \leq B^\gamma_\Psi(\eta) \text{ and } C^{\gamma\Psi}(0) \geq C^{\gamma\Psi}(\eta), D^{\gamma\Psi}(0) \leq D^{\gamma\Psi}(\eta)$$

for all $\eta \in \mathcal{C}$. Now there exist $\eta, \zeta, \rho \in \mathcal{C}$ such that $A^\gamma_\Psi(\eta * \rho) < A^\gamma_\Psi((\eta * \zeta) * \rho) \wedge A^\gamma_\Psi(\zeta)$, $B^\gamma_\Psi(\eta * \rho) > B^\gamma_\Psi((\eta * \zeta) * \rho) \vee B^\gamma_\Psi(\zeta)$ and

$$C^{\gamma\Psi}(\eta * \rho) < C^{\gamma\Psi}((\eta * \zeta) * \rho) \wedge C^{\gamma\Psi}(\zeta), D^{\gamma\Psi}(\eta * \rho) > D^{\gamma\Psi}((\eta * \zeta) * \rho) \vee D^{\gamma\Psi}(\zeta)$$

Select $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1)$ such that

$$A^\gamma_\Psi(\eta * \rho) < \Phi \leq A^\gamma_\Psi((\eta * \zeta) * \rho) \wedge A^\gamma_\Psi(\zeta)$$

$$B^\gamma_\Psi(\eta * \rho) > \partial \geq B^\gamma_\Psi((\eta * \zeta) * \rho) \vee B^\gamma_\Psi(\zeta)$$

$$C^{\gamma\Psi}(\eta * \rho) < \lambda \leq C^{\gamma\Psi}((\eta * \zeta) * \rho) \wedge C^{\gamma\Psi}(\zeta)$$

$$D^{\gamma\Psi}(\eta * \rho) > \vartheta \geq D^{\gamma\Psi}((\eta * \zeta) * \rho) \vee D^{\gamma\Psi}(\zeta)$$

Then

$$(\eta * \rho)_\Phi \bar{\in} A^\gamma_\Psi \text{ but } ((\eta * \zeta) * \rho)_\Phi \in A^\gamma_\Psi \text{ and } \zeta_\Phi \in A^\gamma_\Psi$$

$$(\eta * \rho)_\partial [\bar{\in}] B^\gamma_\Psi \text{ but } ((\eta * \zeta) * \rho)_\partial [\in] B^\gamma_\Psi \text{ and } \zeta_\partial [\in] B^\gamma_\Psi$$

$$(\eta * \rho)_\lambda \bar{\in} C^{\gamma\Psi} \text{ but } ((\eta * \zeta) * \rho)_\lambda \in C^{\gamma\Psi} \text{ and } \zeta_\lambda \in C^{\gamma\Psi}$$

$$(\eta * \rho)_\vartheta [\bar{\in}] D^{\gamma\Psi} \text{ but } ((\eta * \zeta) * \rho)_\vartheta [\in] D^{\gamma\Psi} \text{ and } \zeta_\vartheta [\in] D^{\gamma\Psi}$$

which is a contradiction. Therefore

$$A^\gamma_\Psi(\eta * \rho) \geq A^\gamma_\Psi((\eta * \zeta) * \rho) \wedge A^\gamma_\Psi(\zeta) \text{ and } B^\gamma_\Psi(\eta * \rho) \leq B^\gamma_\Psi((\eta * \zeta) * \rho) \vee B^\gamma_\Psi(\zeta)$$

$$C^{\gamma\Psi}(\eta * \rho) \geq C^{\gamma\Psi}((\eta * \zeta) * \rho) \wedge C^{\gamma\Psi}(\zeta) \text{ and } D^{\gamma\Psi}(\eta * \rho) \leq D^{\gamma\Psi}((\eta * \zeta) * \rho) \vee D^{\gamma\Psi}(\zeta)$$

Therefore, Ψ_d is a LiDF BCC ideal (LiDFBI) of $\mathcal{C}$.

4.1. $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q_k})$ -LiDF BCC Ideals. In this section, we define $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q_k})$ -LiDF BCC ideals of $\mathcal{C}$ as a special case of $(\overline{\alpha}, \overline{\beta})$ -LiDF BCC ideals, and discuss several results.

Definition 14. A LiDF set $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ of a BCC-algebra $\mathcal{C}$ is called an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q_k})$ -LiDF BCC ideal, if it satisfies condition (EL1) and following conditions: for all $\Phi, \Phi_1, \Phi_2, \lambda, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial, \partial_1, \partial_2, \vartheta, \vartheta_1, \vartheta_2 \in [0, 1]$ and $\eta, \zeta, \rho \in \mathcal{C}$.

$$(EL4) \quad (\eta * \rho)_{\Phi_1 \wedge \Phi_2} \overline{\epsilon} A^\gamma_\Psi \implies ((\eta * \zeta) * \rho)_{\Phi_1} \overline{\epsilon} \vee \overline{q_k} A^\gamma_\Psi, \quad \zeta_{\Phi_2} \overline{\epsilon} \vee \overline{q_k} A^\gamma_\Psi \text{ and } (\eta * \rho)_{\partial_1 \vee \partial_2} [\overline{\epsilon}] B^\gamma_\Psi \implies ((\eta * \zeta) * \rho)_{\partial_1} [\overline{\epsilon}] \vee [\overline{q_k}] B^\gamma_\Psi, \quad \zeta_{\partial_2} [\overline{\epsilon} \vee \overline{q_k}] B^\gamma_\Psi$$

$$(EL5) \quad (\eta * \rho)_{\lambda_1 \wedge \lambda_2} \overline{\epsilon} C^{\gamma\Psi} \implies ((\eta * \zeta) * \rho)_{\lambda_1} \overline{\epsilon} \vee \overline{q_k} C^{\gamma\Psi}, \quad \zeta_{\lambda_2} \overline{\epsilon} \vee \overline{q_k} C^{\gamma\Psi} \text{ and } (\eta * \rho)_{\vartheta_1 \vee \vartheta_2} [\overline{\epsilon}] D^{\gamma\Psi} \implies ((\eta * \zeta) * \rho)_{\vartheta_1} [\overline{\epsilon}] \vee [\overline{q_k}] D^{\gamma\Psi}, \quad \zeta_{\vartheta_2} [\overline{\epsilon}] \vee [\overline{q_k}] D^{\gamma\Psi}$$

Lemma 4.2. For a LiDFS $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ the Conditions (EL4-EL5) are equivalent to following conditions for all $\eta, \zeta, \rho \in \mathcal{C}$.

$$(ED4) \quad \max \left\{ A^\gamma_\Psi(\eta * \rho), \frac{1-k}{2} \right\} \geq \min \left\{ A^\gamma_\Psi((\eta * \zeta) * \rho), A^\gamma_\Psi(\zeta) \right\} \text{ and } \min \left\{ B^\gamma_\Psi(\eta * \rho), \frac{1-k}{2} \right\} \leq \max \left\{ B^\gamma_\Psi((\eta * \zeta) * \rho), B^\gamma_\Psi(\zeta) \right\}$$

$$(ED5) \quad \max \left\{ C^{\gamma\Psi}(\eta * \rho), \frac{1-k}{2} \right\} \geq \min \left\{ C^{\gamma\Psi}((\eta * \zeta) * \rho), C^{\gamma\Psi}(\zeta) \right\} \text{ and } \min \left\{ D^{\gamma\Psi}(\eta * \rho), \frac{1-k}{2} \right\} \leq \max \left\{ D^{\gamma\Psi}((\eta * \zeta) * \rho), D^{\gamma\Psi}(\zeta) \right\}$$

Proof. (EL4-EL5) $\Rightarrow$ (ED4-ED5).

Let $\Psi_d = ((A^\gamma_\Psi, B^\gamma_\Psi), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ be an LiDFS of $\mathcal{C}$. On the Contrary, assume that there exist $\eta, \zeta, \rho \in \mathcal{C}$ and choose $\Phi, \lambda \in (0, 1]$ and $\partial, \vartheta \in [0, 1]$ such that

$$\max \left\{ A^\gamma_\Psi(\eta * \rho), \frac{1-k}{2} \right\} < \Phi = \min \left\{ A^\gamma_\Psi((\eta * \zeta) * \rho), A^\gamma_\Psi(\zeta) \right\},$$

$$\min \left\{ B^\gamma_\Psi(\eta * \rho), \frac{1-k}{2} \right\} > \partial = \max \left\{ B^\gamma_\Psi((\eta * \zeta) * \rho), B^\gamma_\Psi(\zeta) \right\}$$

$$\max \left\{ C^{\gamma\Psi}(\eta * \rho), \frac{1-k}{2} \right\} < \lambda = \min \left\{ C^{\gamma\Psi}((\eta * \zeta) * \rho), C^{\gamma\Psi}(\zeta) \right\},$$

$$\min \left\{ D^{\gamma\Psi}(\eta * \rho), \frac{1-k}{2} \right\} > \vartheta = \max \left\{ D^{\gamma\Psi}((\eta * \zeta) * \rho), D^{\gamma\Psi}(\zeta) \right\}$$

Then $\Phi, \lambda \in (\frac{1-k}{2}, 1]$ and $\partial, \vartheta \in [\frac{1-k}{2}, 1)$ such that

$$(\eta * \rho)_\Phi \overline{\epsilon} A^\gamma_\Psi, \quad ((\eta * \zeta) * \rho)_\Phi \overline{\epsilon} A^\gamma_\Psi, \quad \zeta_\Phi \overline{\epsilon} A^\gamma_\Psi, \quad (\eta * \rho)_{\partial} [\overline{\epsilon}] B^\gamma_\Psi, \quad ((\eta * \zeta) * \rho)_{\partial} [\overline{\epsilon}] B^\gamma_\Psi, \quad \zeta_{\partial} [\overline{\epsilon}] B^\gamma_\Psi$$

$$(\eta * \rho)_\lambda \overline{\epsilon} C^{\gamma\Psi}, \quad ((\eta * \zeta) * \rho)_\lambda \overline{\epsilon} C^{\gamma\Psi}, \quad \zeta_\lambda \overline{\epsilon} C^{\gamma\Psi} \text{ and } (\eta * \rho)_{\vartheta} [\overline{\epsilon}] D^{\gamma\Psi}, \quad ((\eta * \zeta) * \rho)_{\vartheta} [\overline{\epsilon}] D^{\gamma\Psi}, \quad \zeta_{\vartheta} [\overline{\epsilon}] D^{\gamma\Psi}$$

It follows that

$$((\eta * \zeta) * \rho)_\Phi \overline{q_k} A^\gamma_\Psi, \quad \zeta_\Phi \overline{q_k} A^\gamma_\Psi \text{ and } ((\eta * \zeta) * \rho)_{\partial} [\overline{q_k}] B^\gamma_\Psi, \quad \zeta_{\partial} [\overline{q_k}] B^\gamma_\Psi$$

$$((\eta * \zeta) * \rho)_\lambda \overline{q_k} C^{\gamma\Psi}, \quad \zeta_\lambda \overline{q_k} C^{\gamma\Psi} \text{ and } ((\eta * \zeta) * \rho)_{\vartheta} [\overline{q_k}] D^{\gamma\Psi}, \quad \zeta_{\vartheta} [\overline{q_k}] D^{\gamma\Psi}$$

Then

$$A^\gamma_\Psi((\eta * \zeta) * \rho) + \Phi + k \leq 1 \text{ or } A^\gamma_\Psi(\zeta) + \Phi + k \leq 1 \text{ and } B^\gamma_\Psi((\eta * \zeta) * \rho) + \partial + k \geq 1 \text{ or } B^\gamma_\Psi(\zeta) + \partial + k \geq 1$$

$$C^{\gamma\Psi}((\eta * \zeta) * \rho) + \lambda + k \leq 1 \text{ or } C^{\gamma\Psi}(\zeta) + \lambda + k \leq 1 \text{ and } D^{\gamma\Psi}((\eta * \zeta) * \rho) + \vartheta + k \geq 1 \text{ or } D^{\gamma\Psi}(\zeta) + \vartheta + k \geq 1$$

As

$$\Phi \leq A^\gamma_\Psi((\eta * \zeta) * \rho), \quad \Phi \leq A^\gamma_\Psi(\zeta) \text{ and } \partial \geq B^\gamma_\Psi((\eta * \zeta) * \rho), \quad \partial \geq B^\gamma_\Psi(\zeta)$$

$$\lambda \leq C^{\gamma\Psi}((\eta * \zeta) * \rho), \quad \lambda \leq C^{\gamma\Psi}(\zeta) \text{ and } \vartheta \geq D^{\gamma\Psi}((\eta * \zeta) * \rho), \quad \vartheta \geq D^{\gamma\Psi}(\zeta)$$

It follows that

$$\Phi \leq \frac{1-k}{2}, \partial \geq \frac{1-k}{2} \quad \text{and} \quad \lambda \leq \frac{1-k}{2}, \vartheta \geq \frac{1-k}{2}$$

a contradiction. So $\max \{A^\gamma_\Psi(\eta * \rho), \frac{1-k}{2}\} \geq \min \{A^\gamma_\Psi((\eta * \zeta) * \rho), A^\gamma_\Psi(\zeta)\}$
 $\min \{B^\gamma_\Psi(\eta * \rho), \frac{1-k}{2}\} \leq \max \{B^\gamma_\Psi((\eta * \zeta) * \rho), \zeta^\partial_\gamma(\zeta)\}$
 $\max \{C^\gamma_\Psi(\eta * \rho), \frac{1-k}{2}\} \geq \min \{C^\gamma_\Psi((\eta * \zeta) * \rho), C^\gamma_\Psi(\zeta)\}$
 $\min \{D^\gamma_\Psi(\eta * \rho), \frac{1-k}{2}\} \leq \max \{D^\gamma_\Psi((\eta * \zeta) * \rho), D^\gamma_\Psi(\zeta)\}$

(ED4-ED5) $\Rightarrow$ (EL4-EL5)

Suppose $\eta, \zeta, \rho \in \mathcal{C}$ and $\Phi_1, \Phi_2, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial_1, \partial_2, \vartheta_1, \vartheta_2 \in [0, 1]$ such that

$$(\eta * \rho)_{\Phi_1 \wedge \Phi_2} \bar{\in} A^\gamma_\Psi, (\eta * \rho)_{\partial_1 \vee \partial_2} [\bar{\in}] B^\gamma_\Psi \quad \text{and} \quad (\eta * \rho)_{\lambda_1 \wedge \lambda_2} \bar{\in} C^\gamma_\Psi, (\eta * \rho)_{\vartheta_1 \vee \vartheta_2} [\bar{\in}] D^\gamma_\Psi$$

Then

$$A^\gamma_\Psi(\eta * \rho) < \Phi_1 \wedge \Phi_2, B^\gamma_\Psi(\eta * \rho) > \partial_1 \vee \partial_2 \quad \text{and} \quad C^\gamma_\Psi(\eta * \rho) < \lambda_1 \wedge \lambda_2, D^\gamma_\Psi(\eta * \rho) > \vartheta_1 \vee \vartheta_2$$

Case (I)

$$A^\gamma_\Psi(\eta * \rho) \geq \frac{1-k}{2}, B^\gamma_\Psi(\eta * \rho) \leq \frac{1-k}{2} \quad \text{and} \quad C^\gamma_\Psi(\eta * \rho) \geq \frac{1-k}{2}, D^\gamma_\Psi(\eta * \rho) \leq \frac{1-k}{2}$$

then by (ED4-ED5), we have

$$A^\gamma_\Psi(\eta * \rho) \geq A^\gamma_\Psi((\eta * \zeta) * \rho) \wedge A^\gamma_\Psi(\zeta) \quad \text{and} \quad B^\gamma_\Psi(\eta * \rho) \leq B^\gamma_\Psi((\eta * \zeta) * \rho) \vee B^\gamma_\Psi(\zeta)$$

$$C^\gamma_\Psi(\eta * \rho) \geq C^\gamma_\Psi((\eta * \zeta) * \rho) \wedge C^\gamma_\Psi(\zeta) \quad \text{and} \quad D^\gamma_\Psi(\eta * \rho) \leq D^\gamma_\Psi((\eta * \zeta) * \rho) \vee D^\gamma_\Psi(\zeta)$$

Thus

$$A^\gamma_\Psi((\eta * \zeta) * \rho) \wedge A^\gamma_\Psi(\zeta) < \Phi_1 \wedge \Phi_2 \quad \text{and} \quad B^\gamma_\Psi((\eta * \zeta) * \rho) \vee B^\gamma_\Psi(\zeta) > \partial_1 \vee \partial_2$$

$$C^\gamma_\Psi((\eta * \zeta) * \rho) \wedge C^\gamma_\Psi(\zeta) < \lambda_1 \wedge \lambda_2 \quad \text{and} \quad D^\gamma_\Psi((\eta * \zeta) * \rho) \vee D^\gamma_\Psi(\zeta) > \vartheta_1 \vee \vartheta_2$$

Consequently

$$A^\gamma_\Psi((\eta * \zeta) * \rho) < \Phi_1 \quad \text{or} \quad A^\gamma_\Psi(\zeta) < \Phi_2 \quad \text{and} \quad B^\gamma_\Psi((\eta * \zeta) * \rho) > \partial_1 \quad \text{or} \quad B^\gamma_\Psi(\zeta) > \partial_2$$

$$C^\gamma_\Psi((\eta * \zeta) * \rho) < \lambda_1 \quad \text{or} \quad C^\gamma_\Psi(\zeta) < \lambda_2 \quad \text{and} \quad D^\gamma_\Psi((\eta * \zeta) * \rho) > \vartheta_1 \quad \text{or} \quad D^\gamma_\Psi(\zeta) > \vartheta_2$$

It follows that

$$((\eta * \zeta) * \rho)_{\Phi_1} \bar{\in} A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\in} A^\gamma_\Psi \quad \text{and} \quad ((\eta * \zeta) * \rho)_{\partial_1} [\bar{\in}] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\in}] B^\gamma_\Psi$$

$$((\eta * \zeta) * \rho)_{\lambda_1} \bar{\in} C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\in} C^\gamma_\Psi \quad \text{and} \quad ((\eta * \zeta) * \rho)_{\vartheta_1} [\bar{\in}] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\in}] D^\gamma_\Psi$$

Case (II)

$$A^\gamma_\Psi(\eta * \rho) < \frac{1-k}{2}, B^\gamma_\Psi(\eta * \rho) > \frac{1-k}{2} \quad \text{and}$$

$$C^\gamma_\Psi(\eta * \rho) < \frac{1-k}{2}, D^\gamma_\Psi(\eta * \rho) > \frac{1-k}{2}$$

then by (ED4-ED5), we have

$$\frac{1-k}{2} \geq A^\gamma_\Psi((\eta * \zeta) * \rho) \wedge A^\gamma_\Psi(\zeta) \quad \text{and} \quad \frac{1-k}{2} \leq B^\gamma_\Psi((\eta * \zeta) * \rho) \vee B^\gamma_\Psi(\zeta)$$

$$\frac{1-k}{2} \geq C^\gamma_\Psi((\eta * \zeta) * \rho) \wedge C^\gamma_\Psi(\zeta) \quad \text{and} \quad \frac{1-k}{2} \leq D^\gamma_\Psi((\eta * \zeta) * \rho) \vee D^\gamma_\Psi(\zeta)$$

Suppose

$$((\eta * \zeta) * \rho)_{\Phi_1} \in A^\gamma_\Psi, \zeta_{\Phi_2} \in A^\gamma_\Psi \quad \text{and} \quad ((\eta * \zeta) * \rho)_{\partial_1} [\in] B^\gamma_\Psi, \zeta_{\partial_2} [\in] B^\gamma_\Psi$$

$$((\eta * \zeta) * \rho)_{\lambda_1} \in C^\gamma_\Psi, \zeta_{\lambda_2} \in C^\gamma_\Psi \quad \text{and} \quad ((\eta * \zeta) * \rho)_{\vartheta_1} [\in] D^\gamma_\Psi, \zeta_{\vartheta_2} [\in] D^\gamma_\Psi$$

Then

$$A^\gamma_\Psi((\eta * \zeta) * \rho) \geq \Phi_1 \quad \text{or} \quad A^\gamma_\Psi(\zeta) \geq \Phi_2 \quad \text{and} \quad B^\gamma_\Psi((\eta * \zeta) * \rho) \leq \partial_1 \quad \text{or} \quad B^\gamma_\Psi(\zeta) \leq \partial_2$$

$$C^{\gamma\Psi}((\eta * \zeta) * \rho) \geq \lambda_1 \text{ or } C^{\gamma\Psi}(\zeta) \geq \lambda_2 \text{ and } D^{\gamma\Psi}((\eta * \zeta) * \rho) \leq \vartheta_1 \text{ or } D^{\gamma\Psi}(\zeta) \leq \vartheta_2$$

Thus

$$\Phi_1 \wedge \Phi_2 \leq \frac{1-k}{2}, \partial_1 \vee \partial_2 \geq \frac{1-k}{2} \text{ and } \lambda_1 \wedge \lambda_2 \leq \frac{1-k}{2}, \vartheta_1 \wedge \vartheta_2 \geq \frac{1-k}{2}$$

Hence

$$A^{\gamma\Psi}((\eta * \zeta) * \rho) \wedge A^{\gamma\Psi}(\zeta) + (\Phi_1 \wedge \Phi_2) + k \leq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

$$B^{\gamma\Psi}((\eta * \zeta) * \rho) \vee B^{\gamma\Psi}(\zeta) + (\partial_1 \vee \partial_2) + k \geq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

$$C^{\gamma\Psi}((\eta * \zeta) * \rho) \wedge C^{\gamma\Psi}(\zeta) + (\lambda_1 \wedge \lambda_2) + k \leq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

$$D^{\gamma\Psi}((\eta * \zeta) * \rho) \vee D^{\gamma\Psi}(\zeta) + (\vartheta_1 \vee \vartheta_2) + k \geq \left(\frac{1-k}{2}\right) + \left(\frac{1-k}{2}\right) + k = 1$$

So

$$((\eta * \zeta) * \rho)_{\Phi_1} \overline{q_k} A^{\gamma\Psi} \text{ or } \zeta_{\Phi_2} \overline{q_k} A^{\gamma\Psi} \text{ and } ((\eta * \zeta) * \rho)_{\partial_1} [\overline{q_k}] A^{\gamma\Psi} \text{ or } \zeta_{\partial_2} [\overline{q_k}] A^{\gamma\Psi}$$

$$((\eta * \zeta) * \rho)_{\lambda_1} \overline{q_k} C^{\gamma\Psi} \text{ or } \zeta_{\lambda_2} \overline{q_k} C^{\gamma\Psi} \text{ and } ((\eta * \zeta) * \rho)_{\vartheta_1} [\overline{q_k}] D^{\gamma\Psi} \text{ or } \zeta_{\vartheta_2} [\overline{q_k}] D^{\gamma\Psi}$$

Hence

$$\begin{aligned} & ((\eta * \zeta) * \rho)_{\Phi_1} \overline{\in} \vee \overline{q_k} A^{\gamma\Psi} \text{ or } \zeta_{\Phi_2} \overline{\in} \vee \overline{q_k} A^{\gamma\Psi} \text{ and} \\ & ((\eta * \zeta) * \rho)_{\partial_1} [\overline{\in}] \vee [\overline{q_k}] B^{\gamma\Psi} \text{ or } \zeta_{\partial_2} [\overline{\in}] \vee [\overline{q_k}] B^{\gamma\Psi} \\ & ((\eta * \zeta) * \rho)_{\lambda_1} \overline{\in} \vee \overline{q_k} C^{\gamma\Psi} \text{ or } \zeta_{\lambda_2} \overline{\in} \vee \overline{q_k} C^{\gamma\Psi} \text{ and} \\ & ((\eta * \zeta) * \rho)_{\vartheta_1} [\overline{\in}] \vee [\overline{q_k}] D^{\gamma\Psi} \text{ or } \zeta_{\vartheta_2} [\overline{\in}] \vee [\overline{q_k}] D^{\gamma\Psi} \end{aligned}$$

Therefore, the conditions (EL4-EL5) are satisfied.

Theorem 5. A LiDF set $\Psi_d = ((A^{\gamma\Psi}, B^{\gamma\Psi}), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ of a BCC-algebra $\mathcal{C}$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ - LiDF BCC ideal, iff it satisfies the condition (ED1) and the conditions (ED4-ED5).

Example 2. Consider a BCC-algebra $\mathcal{C} = \{0, a, b, c\}$ defined by Table 2 and let $k = 0.1$. Now define a LiDFS $\Psi_d = ((A^{\gamma\Psi}, B^{\gamma\Psi}), (C^{\gamma\Psi}, D^{\gamma\Psi}))$ as:

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	a	0

TABLE 2. BCC-algebra $\mathcal{C}$

$$\Psi_d(\eta) = \begin{cases} ((0.6, 0.3), (0.5, 0.1)) & \text{if } \eta = 0, \\ ((0.5, 0.3), (0.4, 0.2)) & \text{if } \eta = a, \\ ((0.4, 0.3), (0.3, 0.3)) & \text{if } \eta = b, \\ ((0.4, 0.6), (0.3, 0.4)) & \text{if } \eta = c. \end{cases}$$

Then Ψ_d is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDF BCC Ideal (LiDFBI) of $\mathcal{C}$, as it satisfies all conditions of Theorem 5.

Theorem 6. Every $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDF BCC Ideal of BCC algebra $\mathcal{C}$ is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDF Ideal of $\mathcal{C}$.

Proof: Let Ψ_d is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDF BCC Ideal of BCC-algebra $\mathcal{C}$. Let $\eta, \zeta, \rho \in \mathcal{C}$ and $\Phi_1, \Phi_2, \lambda_1, \lambda_2 \in (0, 1]$ and $\partial_1, \partial_2, \vartheta_1, \vartheta_2 \in [0, 1]$. Then, by taking $\rho = 0$ in (EL4-EL5) of Definition 14, we have

$$\begin{aligned} (\eta * 0)_{\Phi_1 \wedge \Phi_2} \bar{\epsilon} A^\gamma_\Psi &\implies ((\eta * \zeta) * 0)_{\Phi_1} \bar{\epsilon} \vee \bar{q}_k A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\epsilon} \vee \bar{q}_k A^\gamma_\Psi \\ (\eta * 0)_{\partial_1 \vee \partial_2} [\bar{\epsilon}] B^\gamma_\Psi &\implies ((\eta * \zeta) * 0)_{\partial_1} [\bar{\epsilon}] \vee [\bar{q}_k] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\epsilon}] \vee [\bar{q}_k] B^\gamma_\Psi \\ (\eta * 0)_{\lambda_1 \wedge \lambda_2} \bar{\epsilon} C^\gamma_\Psi &\implies ((\eta * \zeta) * 0)_{\lambda_1} \bar{\epsilon} \vee \bar{q}_k C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\epsilon} \vee \bar{q}_k C^\gamma_\Psi \\ (\eta * 0)_{\vartheta_1 \vee \vartheta_2} [\bar{\epsilon}] D^\gamma_\Psi &\implies ((\eta * \zeta) * 0)_{\vartheta_1} [\bar{\epsilon}] \vee [\bar{q}_k] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\epsilon}] \vee [\bar{q}_k] D^\gamma_\Psi \vee \vartheta_2 \end{aligned}$$

In a BCC-algebra, $0 * \eta = 0$ and $\eta * 0 = \eta$. Then we obtain.

$$\begin{aligned} \eta_{\Phi_1 \wedge \Phi_2} \bar{\epsilon} A^\gamma_\Psi &\implies (\eta * \zeta)_{\Phi_1} \bar{\epsilon} \vee \bar{q}_k A^\gamma_\Psi, \zeta_{\Phi_2} \bar{\epsilon} \vee \bar{q}_k A^\gamma_\Psi \\ \eta_{\partial_1 \vee \partial_2} [\bar{\epsilon}] B^\gamma_\Psi &\implies (\eta * \zeta)_{\partial_1} [\bar{\epsilon}] \vee [\bar{q}_k] B^\gamma_\Psi, \zeta_{\partial_2} [\bar{\epsilon}] \vee [\bar{q}_k] B^\gamma_\Psi \\ \eta_{\lambda_1 \wedge \lambda_2} \bar{\epsilon} C^\gamma_\Psi &\implies (\eta * \zeta)_{\lambda_1} \bar{\epsilon} \vee \bar{q}_k C^\gamma_\Psi, \zeta_{\lambda_2} \bar{\epsilon} \vee \bar{q}_k C^\gamma_\Psi \\ \eta_{\vartheta_1 \vee \vartheta_2} [\bar{\epsilon}] D^\gamma_\Psi &\implies (\eta * \zeta)_{\vartheta_1} [\bar{\epsilon}] \vee [\bar{q}_k] D^\gamma_\Psi, \zeta_{\vartheta_2} [\bar{\epsilon}] \vee [\bar{q}_k] D^\gamma_\Psi \vee \vartheta_2 \end{aligned}$$

Hence, Ψ_d satisfies (EL2-EL3) of the definition 12. Combining with (EL1) of definition 12 implies that Ψ_d is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -LiDFI of $\mathcal{C}$.

Note that Example 2 clearly demonstrates that the converse of Theorem 6 does not hold in general.

5. IOT-DRIVEN ENTROPY-SWARA-MARCOS MODEL

This study presents a comprehensive MCDM framework for the evaluation of r advanced waste management alternatives, denoted as $R^\Theta = \{R^\Theta_1, R^\Theta_2, \dots, R^\Theta_r\}$, with respect to a set of s sustainability-oriented decision criteria $P^\Theta = \{P^\Theta_1, P^\Theta_2, \dots, P^\Theta_s\}$. The proposed framework is designed to support decision-makers, urban planners, and sustainability analysts in identifying the most suitable waste management strategies under uncertainty and vague expert knowledge. First, the Entropy method is employed to derive objective weights for each criterion by analyzing the dispersion of values across alternatives, thereby identifying influential criteria based on inherent variability. The SWARA technique facilitates the capture of subjective preferences and expert priorities, enabling flexible adjustments of criterion importance via an iterative pairwise comparison process. The objective and subjective weights are aggregated to create a balanced weighting scheme that incorporates both data-driven insights and expert intuition. The MARCOS method is ultimately employed to rank the alternatives. A diverse panel of k domain experts engages in all stages of the decision-making process, facilitating a balanced, comprehensive, and expert-informed evaluation. Expert judgments are first articulated using linguistic terms and subsequently transformed into LiDF values, as demonstrated in Table 4. The qualifications and expertise of the decision-makers are presented in Table 3.

TABLE 3. Decision makers involved in the evaluation of smart waste management alternatives

Profession	Role and Responsibility	Experience and Qualification
Urban Sustainability Expert	Evaluates environmental regeneration, carbon footprint reduction, and alignment with smart city sustainability goals.	10+ years in urban planning and climate policy, Ph.D. in Environmental Science.
AI Data Analytics Specialist	Assesses the implementation potential of Auto-AI in predictive waste analytics and intelligent system adaptability.	8 years in AI-driven infrastructure projects, M.Sc. in Artificial Intelligence.
Blockchain Technologist	Reviews the transparency, traceability, and data integrity of blockchain-enabled waste management solutions.	6 years in blockchain integration and DLT security, M.Sc. in Computer Science.
Circular Economy Consultant	Advises on operational scalability, resource efficiency, and circularity of waste-to-resource solutions.	12+ years in sustainable development, MBA in Circular Economy Management.

Algorithm: Entropy-SWARA-MARCOS Model

Step 1: During the initial phase of the decision-making process, a panel of experts assesses each alternative plastic recycling technology based on established criteria. The performance of each alternative is systematically assessed, and evaluations are documented in an organised manner. This assessment is structured as a matrix, with rows indicating the alternatives and columns representing the criteria.

Step 2: Compute the $\varrho_{i,j}^C$ matrix along with the associated score values according to

$$\begin{matrix}
 & \mathfrak{R}_1^C & \mathfrak{R}_2^C & \dots & \mathfrak{R}_s^C \\
 \begin{matrix} R_1^\Theta \\ R_2^\Theta \\ \vdots \\ R_r^\Theta \end{matrix} & \begin{bmatrix} \langle A^\gamma_{\Psi_{11}}, B^\gamma_{\Psi_{11}} \rangle, \langle C^\gamma_{\Psi_{11}}, D^\gamma_{\Psi_{11}} \rangle & \langle A^\gamma_{\Psi_{12}}, B^\gamma_{\Psi_{12}} \rangle, \langle C^\gamma_{\Psi_{12}}, D^\gamma_{\Psi_{12}} \rangle & \dots & \langle A^\gamma_{\Psi_{1s}}, B^\gamma_{\Psi_{1s}} \rangle, \langle C^\gamma_{\Psi_{1s}}, D^\gamma_{\Psi_{1s}} \rangle \\ \langle A^\gamma_{\Psi_{21}}, B^\gamma_{\Psi_{21}} \rangle, \langle C^\gamma_{\Psi_{21}}, D^\gamma_{\Psi_{21}} \rangle & \langle A^\gamma_{\Psi_{22}}, B^\gamma_{\Psi_{22}} \rangle, \langle C^\gamma_{\Psi_{22}}, D^\gamma_{\Psi_{22}} \rangle & \dots & \langle A^\gamma_{\Psi_{2s}}, B^\gamma_{\Psi_{2s}} \rangle, \langle C^\gamma_{\Psi_{2s}}, D^\gamma_{\Psi_{2s}} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle A^\gamma_{\Psi_{r1}}, B^\gamma_{\Psi_{r1}} \rangle, \langle C^\gamma_{\Psi_{r1}}, D^\gamma_{\Psi_{r1}} \rangle & \langle A^\gamma_{\Psi_{r2}}, B^\gamma_{\Psi_{r2}} \rangle, \langle C^\gamma_{\Psi_{r2}}, D^\gamma_{\Psi_{r2}} \rangle & \dots & \langle A^\gamma_{\Psi_{rs}}, B^\gamma_{\Psi_{rs}} \rangle, \langle C^\gamma_{\Psi_{rs}}, D^\gamma_{\Psi_{rs}} \rangle \end{bmatrix}
 \end{matrix}$$

Equation (5.7).

$$\varrho_{i,j}^C = \frac{[(A^\gamma_{\xi_{i,j}} - B^\gamma_{\xi_{i,j}}) + (C^\gamma_{\xi_{i,j}} - D^\gamma_{\xi_{i,j}})]}{2}, \tilde{S} \in [0, 1] \quad (5.7)$$

Step 3: Equation (5.8) serves to normalise the softmax function $o_{ij}^{\tau,\ell}$. The function may be normalized according to Equation (5.9).

TABLE 4. Linguistic terms for evaluating advanced smart waste management alternatives

Linguistic Term	Abbreviation	Description	LiDFNs
Highly Impactful	HI	Signifies a major transformation in waste efficiency and smart integration	$(\langle 0.91, 0.06 \rangle, \langle 0.89, 0.07 \rangle)$
Strongly Green	SG	Contributes significantly to sustainability and emission reduction targets	$(\langle 0.83, 0.14 \rangle, \langle 0.81, 0.12 \rangle)$
Operationally Optimized	OO	Enhances energy savings and process effectiveness	$(\langle 0.76, 0.19 \rangle, \langle 0.73, 0.16 \rangle)$
Technologically Suitable	TS	Aligns well with existing digital and IoT infrastructures	$(\langle 0.71, 0.24 \rangle, \langle 0.66, 0.21 \rangle)$
Moderate Compatibility	MC	Offers partial integration with current processes and policies	$(\langle 0.66, 0.29 \rangle, \langle 0.61, 0.26 \rangle)$
Policy Aligned	PA	Conforms to regulatory and compliance requirements	$(\langle 0.61, 0.34 \rangle, \langle 0.56, 0.31 \rangle)$
Emerging Concept	EC	Innovative but not fully deployed or tested at scale	$(\langle 0.56, 0.39 \rangle, \langle 0.51, 0.36 \rangle)$
Niche Applicable	NA	Suitable for specific or limited use cases only	$(\langle 0.46, 0.49 \rangle, \langle 0.41, 0.44 \rangle)$
Not Recommended	NR	Fails to meet sustainability or technological viability standards	$(\langle 0.31, 0.59 \rangle, \langle 0.26, 0.59 \rangle)$

$$\beta_{ij}^{\alpha, \ell} = \frac{\exp\left(\frac{\theta_{i,j}^c}{\tau}\right)}{\sum_{j=1}^v \exp\left(\frac{\theta_{i,j}^c}{\tau}\right)} \quad i = 1, 2, \dots, r; \quad j = 1, 2, \dots, s; \quad (5.8)$$

$$\Xi_{ij}^T = \frac{\beta_{ij}^{\alpha, \ell}}{\sum_{j=1}^v \beta_{ij}^{\alpha, \ell}} \quad i = 1, 2, \dots, u; \quad j = 1, 2, \dots, v; \quad (5.9)$$

Step 4: The linear Diophantine fuzzy softmax Hamacher weighted average (LiDFSHWA) operator is Utilized to process the evaluation data from each decision maker, resulting in aggregated values. The procedure is executed utilizing Equation (5.10).

$$LiDFSHWA(\chi^{\wp}_1, \chi^{\wp}_2, \dots, \chi^{\wp}_n) = \bigoplus_{j=1}^n \wp_{ij} \chi^{\wp}_j$$

$$= \left(\frac{\prod_{j=1}^n (1 + (\lambda^{\wp} - 1)A^{\gamma}_{\wp_j})^{\wp_{ij}} - \prod_{j=1}^n (1 - A^{\gamma}_{\wp_j})^{\wp_{ij}}}{\prod_{j=1}^n (1 + (\lambda^{\wp} - 1)A^{\gamma}_{\wp_j})^{\wp_{ij}} + (\lambda^{\wp} - 1) \prod_{j=1}^n (1 - A^{\gamma}_{\wp_j})^{\wp_{ij}}}, \frac{\prod_{j=1}^n B^{\gamma}_{\wp_j} \wp_{ij}}{\prod_{j=1}^n (1 + (\lambda^{\wp} - 1)(1 - B^{\gamma}_{\wp_j}))^{\wp_{ij}} + (\lambda^{\wp} - 1) \prod_{j=1}^n B^{\gamma}_{\wp_j} \wp_{ij}}, \frac{\lambda^{\wp} \prod_{j=1}^n C^{\gamma}_{\wp_j} \wp_{ij}}{\prod_{j=1}^n (1 + (\lambda^{\wp} - 1)(1 - C^{\gamma}_{\wp_j}))^{\wp_{ij}} + (\lambda^{\wp} - 1) \prod_{j=1}^n C^{\gamma}_{\wp_j} \wp_{ij}}, \frac{\lambda^{\wp} \prod_{j=1}^n D^{\gamma}_{\wp_j} \wp_{ij}}{\prod_{j=1}^n (1 + (\lambda^{\wp} - 1)(1 - D^{\gamma}_{\wp_j}))^{\wp_{ij}} + (\lambda^{\wp} - 1) \prod_{j=1}^n D^{\gamma}_{\wp_j} \wp_{ij}} \right) \quad (5.10)$$

Step 5

For every LiDFN $\mathfrak{I}_{i,j}$, define the scoring matrix $\phi^v_{r \times s}$ as follows:

$$\phi^v_{i,j} = \frac{[(A^{\gamma}_{\xi_{i,j}} - B^{\gamma}_{\xi_{i,j}}) + (C^{\gamma}_{\xi_{i,j}} - D^{\gamma}_{\xi_{i,j}})]}{2}. \quad (5.11)$$

Entropy Method: A mechanism Utilized in MCDM is the entropy approach, which determines the weights of criteria based on their entropy values. It determines each criterion's level of uncertainty or unpredictability and assigns weights appropriately.

Step 6: Transforming the input data into a range of 0 to 1 for a decision-making matrix is known as data normalisation. Normalisation is a crucial simplification approach when needs call for a variety of numerical values. The entropy approach was normalized for the benefit type criterion using Equation 5.12 and for the cost type criterion using Equation 5.13.

$$T_{ij} = \frac{\phi^v_{ij} - \min(\phi^v_{ij})}{\max(\phi^v_{ij}) - \min(\phi^v_{ij})} \quad (5.12)$$

where $i = 1, 2, \dots, q$; $j = 1, 2, \dots, p$

$$T_{ij} = \frac{\max_i(\phi^v_{ij}) - \phi^v_{ij}}{\max_i(\phi^v_{ij}) - \min(\phi^v_{ij})} \quad i = 1, 2, \dots, q; \quad j = 1, 2, \dots, p \quad (5.13)$$

Step 7: In this phase, the standardized value is computed using Equation 5.14.

$$\rho_{ij} = \frac{T_{ij}}{\sum_{i=1}^m (T_{ij})} \quad i = 1, 2, \dots, q; \quad j = 1, 2, \dots, p \quad (5.14)$$

Step 8: Equation 5.15 describes how to determine the entropy value using the standardised value.

$$\Psi_j = -h \sum_{i=1}^m (\rho_{ij} \ln(\rho_{ij})) \quad j = 1, 2, \dots, p; \quad (5.15)$$

Where $h = \frac{1}{\ln(m)}$.

Step 9: Prior to calculating the weights of the j th criteria, the degree of divergence ϖ_j for each criterion is calculated using equation 5.16.

$$\varpi_j = 1 - \Psi_j \quad j = 1, 2, \dots, p; \quad (5.16)$$

Step 10: Weights are calculated by using Equation 5.17.

$$V_j = \frac{\varpi_j}{\sum_{j=1}^n (\varpi_j)} \quad j = 1, 2, \dots, p; \quad (5.17)$$

such that, $V_j \in [0, 1]$ and $\sum_{i=1}^n V_j = 1$. There are several zero entries in the measured data collection ρ_{ij} . Therefore, in circumstances where $\rho_{ij} = 0$, the restriction $\rho_{ij} \ln(\rho_{ij}) = 0$ is imposed while computing Ψ_j . For such cases, this modification raises the associated weight by decreasing the entropy value. To prevent zero values in the normalized data set, the updated standardisation process adapted from Equation 5.14 is shown in Equation 5.18.

$$\rho_{ij} = \frac{T_{ij} + C}{\sum_{i=1}^m (T_{ij} + C)} \quad i = 1, 2, \dots, q; \quad j = 1, 2, \dots, p \quad (5.18)$$

where C is a constant satisfying the condition,

$$T_{ij} + C > 0$$

SWARA Method: Step 11: Determine the weights derived from Equation 5.17's entropy.

Step 12: DMs begin by prioritising factors according to their perceived relative significance. The criteria coefficient (Φ^τ) for each decision-maker is calculated using Equation 5.19.

$$\Phi^\tau_j = \begin{cases} 1 & \text{if } j = 1 \\ V_j + 1 & \text{if } j > 1 \end{cases} \quad j = 1, \dots, p \quad (5.19)$$

Step 13: At this stage, the starting weight of the criteria for each alternative is determined using equation 5.20.

$$\xi^\nu_j = \begin{cases} 1 & \text{if } j = 1 \\ \frac{\xi^\nu_j}{\Phi^\tau_j} & \text{if } j > 1 \end{cases} \quad j = 1, \dots, p \quad (5.20)$$

Step 14 The relative weight of the criterion for each choice is calculated using equation 5.21.

$$F^W_j = \frac{\xi^\nu_j}{\sum_{j=1}^n \xi^\nu_j} \quad j = 1, \dots, p \quad (5.21)$$

MARCOS Method: Step 15: Considering the kind of criterion, determine the best and worst alternatives for each criterion. Equation (5.22) is used to identify the row of best options, denoted by D . Equation (5.23) defines the row of worst alternatives, which is denoted by Φ .

$$D_j = \begin{cases} H(\mathfrak{P})_{ij}^+ & \text{if } \mathfrak{U}_j \text{ is benefit type} \\ H(\mathfrak{P})_{ij}^- & \text{if } \mathfrak{U}_j \text{ is cost type} \\ i = 1, 2, \dots, u; \quad j = 1, 2, \dots, v \end{cases} \quad (5.22)$$

$$\Pi_j = \begin{cases} H(\mathfrak{P})_{ij}^- & \text{if } \mathfrak{U}_j \text{ is benefit type} \\ H(\mathfrak{P})_{ij}^+ & \text{if } \mathfrak{U}_j \text{ is cost type} \\ i = 1, 2, \dots, u; \quad j = 1, 2, \dots, v \end{cases} \quad (5.23)$$

Step 16: Introduce two additional rows, D and Π , to create an expanded aggregated score matrix.

Step 17: Apply normalisation to the extended matrix to derive the new normalized matrix according to Equation (5.24).

$$\vartheta^S_{ij} = \begin{cases} \frac{H(\P)_{ij}}{H(\P)_{bj}} & \text{if } \mathfrak{U}_j \text{ is benefit type} \\ \frac{H(\P)_{bj}}{H(\P)_{ij}} & \text{if } \mathfrak{U}_j \text{ is cost type} \\ \iota = 1, 2, \dots, u; j = 1, 2, \dots, v \end{cases} \quad (5.24)$$

Step 18: To derive the weighted normalized matrix, multiply the aggregated weights from Step 14 by the corresponding entries of the normalized matrix.

$$\sigma_{ij} = \vartheta^S_{ij} * F^W_j \quad \iota = 1, 2, \dots, u; j = 1, 2, \dots, v \quad (5.25)$$

Step 19: Calculate the utility degrees for each alternative represented by $K^{\gamma+}$ and $K^{\gamma-}$ as specified in Equation (5.26) and Equation (5.27).

$$K^{\gamma+}_i = \frac{\Re_i}{\rho} \quad \iota = 1, 2, \dots, u; \quad (5.26)$$

$$K^{\gamma-}_i = \frac{\Re_i}{\chi} \quad \iota = 1, 2, \dots, u; \quad (5.27)$$

Where $\Re_i = \sum_{j=1}^t \sigma_{ij}$, $\rho = \sum_{j=1}^t H(\P)_{bj}$ and $\chi = \sum_{j=1}^t H(\P)_{wj}$

Step 20: Determining the utility function of alternatives, denoted as $F^\lambda(A)$, as outlined in Equation (5.28). In this context, $F^\lambda(K^{\gamma-}_i)$ denotes the utility function associated with Π , while $F^\lambda(K^{\gamma+}_i)$ represents the utility function for D .

$$AS_i = (K^{\gamma+}_i + K^{\gamma-}_i) \frac{(1 + F^\lambda(K^{\gamma+}_i))^T (1 + F^\lambda(K^{\gamma-}_i))^{1-T} - (1 - F^\lambda(K^{\gamma+}_i))^T (1 - F^\lambda(K^{\gamma-}_i))^{1-T}}{(1 + F^\lambda(K^{\gamma+}_i))^T (1 + F^\lambda(K^{\gamma-}_i))^{1-T} + (1 - F^\lambda(K^{\gamma+}_i))^T (1 - F^\lambda(K^{\gamma-}_i))^{1-T}} \quad (5.28)$$

where,

$$F^\lambda(K^{\gamma+}_i) = \frac{K^{\gamma-}_i}{K^{\gamma+}_i + K^{\gamma-}_i} \quad \iota = 1, 2, \dots, u;$$

and,

$$F^\lambda(K^{\gamma-}_i) = \frac{K^{\gamma+}_i}{K^{\gamma+}_i + K^{\gamma-}_i} \quad \iota = 1, 2, \dots, u;$$

The Entropy-SWARA-MARCOS framework is expressed in Figure 2.

6. CASE STUDY: SMART CITY WASTE MANAGEMENT AND RECYCLING

Modern urban environments are experiencing a dramatic increase in the amount and complexity of municipal solid waste that is the result of accelerated industrialization, urbanization and population growth. New digital technologies, such as blockchain traceability, Auto-AI prediction models, metaverse simulations, and IoT-based monitoring, hold great potential in improving the efficiency, transparency, and sustainability of waste management systems. Nevertheless, decision-makers have significant difficulties in evaluating and selecting the most suitable technological interventions. The issues can be linked to the

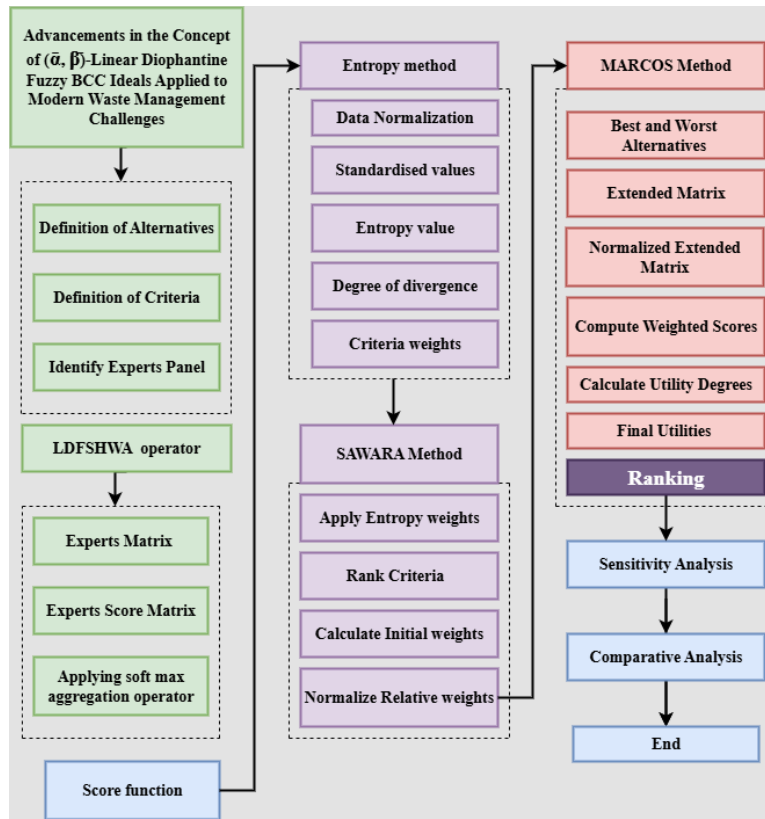


FIGURE 2. Framework of proposed methodology

multi-dimensional nature of the problem, which involves multiple and, in many cases, opposing criteria, such as environmental impact, technological feasibility, economic viability, public acceptance, energy efficiency, cybersecurity, and regulatory compliance. The subjectivity, vagueness, and ambiguity of expert judgments complicate the evaluation process, particularly the prediction of the performance of emerging or experimental technologies. The prevailing decision-making models often fail to suitably represent uncertainty and hesitation in a systematized and mathematically strict manner. Conventional paradigms tend to rely on either objective data or subjective views, with no harmonious approach to the two sides. There is a gap between algebraically enriched models, which can process vague linguistic information and remain logically consistent and computational, and such models can be found. A strong, flexible and open decision support system is necessary to deal with these problems. This system has to deal with complicated decision criteria, be able to combine objective and subjective appraisals, and be able to model uncertainty and imprecision in the expert judgments. This paper introduces a model which is a hybrid of a new hybrid MCDM method, that is, a combination of Entropy, SWARA and MARCOS methods. The aim is to enhance the realism, reliability, and interpretability of the selection

process regarding next-generation waste management alternatives of smart and sustainable urban settings.

6.1. SWM Alternatives. The eight alternatives presented are advanced, technology-driven solutions for contemporary waste management systems. Each alternative demonstrates advancements in automation, data science, digital infrastructure, or policy modeling, and is assessed based on sustainability and operational criteria.

(1) $R^{\ominus}_1$: **Smart waste-to-energy systems**

Processes of converting waste that cannot be recycled to usable energy by means of sophisticated thermal, biological, or chemical treatment. With the help of real time information of IoT and AI systems, optimizes combustion, anaerobic digestion or pyrolysis to increase yield and reduce emissions.

(2) $R^{\ominus}_2$: **Metaverse-based public awareness platforms**

The metaverse gamified environments will inform and involve citizens in sustainable waste management. Users are introduced to virtual city settings that replicate the recycling processes, energy use and environmental impacts and therefore increase awareness and behavioral change through incentives and feedback.

(3) $R^{\ominus}_3$: **Digital twin for urban waste simulation**

An electronic model of the real waste management system Used to analyze the scenario and predict performance. This model will model real-time urban waste flows, treatment capacity and policy interventions and enable planners to visualize and optimize the performance of a system before it is implemented in the real world.

(4) $R^{\ominus}_4$: **IoT-Enabled waste monitoring**

IoT sensors integrated into bins and facilities for the monitoring of waste levels, contamination, and environmental conditions. Devices gather data regarding fill levels, temperature, and petrol emissions, which is relayed to a central dashboard for predictive route planning, maintenance notifications, and emissions management.

(5) $R^{\ominus}_5$: **Auto-AI predictive sorting systems**

Self-improvement AI algorithms are employed to automate and optimise waste classification in real time. Employs computer vision and deep learning techniques to rapidly identify and categorise waste materials. The system persistently acquires knowledge from sensor data to enhance precision and efficiency autonomously.

(6) $R^{\ominus}_6$: **Blockchain-based waste traceability**

A decentralized digital ledger system that will track the lifecycle of waste materials. Blockchain can ensure the safe records of waste origin, type, transport, and processing through smart contracts to increase accountability and reduce illegal dumping or fraud in recycling systems.

(7) $R^{\ominus}_7$: **AI-driven circular economy analyzer**

Artificial intelligence solutions to evaluate garbage streams with the aim of reusing, recycling, and recovering materials. This work analyses lifecycle information and demand supply and proposes circular materials to enable industries to minimize landfills and improve resources efficiency.

(8) R^{Θ}_8 : **Cloud-based integrated decision platform**

A centralized software platform that would facilitate policy, logistic and stakeholder inputs in the waste management ecosystem. Gathers data on sensors, administration databases, and citizens to support making informed decisions, analyzing scenarios and real-time cooperation between municipalities and service providers.

6.2. SWM Evaluation criterion. The decision-making process considers ten key criteria that cover sustainability, technological, economic, and social aspects of smart waste management alternatives.

- P^{Θ}_1 : **Technological readiness level (TRL)**
Shows how mature the technology is, field-testing performance, and how deployable it is. Checks that the selected solutions are viable to be reliably implemented in real-world smart city situations.
- P^{Θ}_2 : **Environmental regeneration potential**
How much the alternative is able to reduce waste-related emissions, pollution, and destruction of ecosystems. Enables long-term ecological balance and helps to achieve climate action and biodiversity conservation goals.
- P^{Θ}_3 : **Energy efficiency**
Assesses the success of the solution in cutting down on the energy usage during the process of operation. Directly affects the sustainability of the environment and the costs of operation.
- P^{Θ}_4 : **Cybersecurity and data integrity**
Assesses data management security and reliability of digital systems, including blockchain and IoT technologies. Protect sensitive waste, logistics and user data against cyberattacks or tampering.
- P^{Θ}_5 : **Integration capability with smart infrastructure**
Talks about how the alternative can be integrated with intelligent systems including digital twins, IoT, and cloud-based systems. Improves system interoperability, facilitates real-time control and decision making based on data.
- P^{Θ}_6 : **Regulatory compliance and policy alignment**
Identifies the level of consistency that the solution has with the current waste management activities, data requirements and environmental policies. Non-compliance may result in legal, moral or administrative problems during the deployment process.
- P^{Θ}_7 : **Circularity and resource recovery rate**
Measures the ability of the system to increase reuse, recycling, and material recovery. Enables a circular economy and reduces use of virgin resources and landfills.
- P^{Θ}_8 : **Operational scalability**
The ability of the solution to expand or replicate to other urban areas, regions or networks. Critical to widespread application and long-lasting impact not just in pilot research.

- P^{Θ}_9 : **Economic viability**
Assesses lifecycle costs, return on investment, and cost-benefit trade-offs associated with the implementation of the alternative. Assesses financial sustainability and the acceptance of policy-makers within the context of budget constraints.
- P^{Θ}_{10} : **Social acceptance and engagement**
Assesses the likelihood of public and stakeholder adoption and support for the solution. Citizen participation and trust are critical for the effectiveness of behavior-driven technologies.

6.3. Decision Making Process and Discussion. Step 1: The decision makers employed the terminology outlined in Table 4 to express their ideas, as reflected in the responses recorded in Table 5.

Step 2: Equation (5.7) is used to calculate the score values and $\varrho_{i,j}^{\subset}$ matrices for each matrix that decision makers submit.

Step 3: Using Equation (5.8), the softmax function for every matrix supplied by DMs is calculated. Equation (5.9) is used to normalize the findings, using $\tau = 2$. $\Xi_{ij}^{\mu,1} =$

$$\Xi_{ij}^{\mu,1} = \begin{bmatrix} 0.2303 & 0.3465 & 0.3229 & 0.3003 & 0.2571 & 0.2395 & 0.2329 & 0.2314 & 0.2302 & 0.2303 \\ 0.2279 & 0.3132 & 0.2929 & 0.2446 & 0.2380 & 0.2298 & 0.2273 & 0.2281 & 0.2280 & 0.2280 \\ 0.2681 & 0.2332 & 0.2576 & 0.2670 & 0.2725 & 0.2674 & 0.2672 & 0.2679 & 0.2682 & 0.2681 \\ 0.2539 & 0.2440 & 0.2687 & 0.2580 & 0.2538 & 0.2526 & 0.2547 & 0.2544 & 0.2540 & 0.2539 \\ 0.2558 & 0.2469 & 0.2246 & 0.2546 & 0.2568 & 0.2557 & 0.2562 & 0.2560 & 0.2558 & 0.2558 \\ 0.2564 & 0.2474 & 0.2541 & 0.2580 & 0.2607 & 0.2581 & 0.2574 & 0.2567 & 0.2564 & 0.2564 \\ 0.2549 & 0.2459 & 0.2314 & 0.2528 & 0.2561 & 0.2555 & 0.2554 & 0.2549 & 0.2549 & 0.2549 \\ 0.2542 & 0.2542 & 0.2603 & 0.2542 & 0.2542 & 0.2542 & 0.2543 & 0.2543 & 0.2542 & 0.2542 \\ 0.2444 & 0.2990 & 0.3427 & 0.2682 & 0.2520 & 0.2437 & 0.2437 & 0.2456 & 0.2444 & 0.2444 \\ 0.2129 & 0.3599 & 0.3079 & 0.2821 & 0.2184 & 0.2153 & 0.2131 & 0.2138 & 0.2128 & 0.2128 \\ 0.2740 & 0.2382 & 0.2632 & 0.2127 & 0.2610 & 0.2719 & 0.2733 & 0.2738 & 0.2740 & 0.2740 \\ 0.2526 & 0.2427 & 0.2672 & 0.2566 & 0.2632 & 0.2604 & 0.2531 & 0.2529 & 0.2526 & 0.2526 \\ 0.2633 & 0.2542 & 0.1729 & 0.2387 & 0.2597 & 0.2624 & 0.2638 & 0.2635 & 0.2634 & 0.2633 \\ 0.2562 & 0.2472 & 0.2538 & 0.2690 & 0.2571 & 0.2550 & 0.2558 & 0.2557 & 0.2562 & 0.2562 \\ 0.2530 & 0.2441 & 0.2569 & 0.2496 & 0.2530 & 0.2528 & 0.2530 & 0.2529 & 0.2529 & 0.2530 \\ 0.2552 & 0.2552 & 0.2488 & 0.2552 & 0.2552 & 0.2552 & 0.2552 & 0.2552 & 0.2552 & 0.2552 \\ 0.2711 & 0.2794 & 0.2063 & 0.2560 & 0.2756 & 0.2760 & 0.2742 & 0.2699 & 0.2711 & 0.2711 \\ 0.2902 & 0.0963 & 0.1740 & 0.2329 & 0.2739 & 0.2837 & 0.2870 & 0.2906 & 0.2904 & 0.2903 \\ 0.2354 & 0.2990 & 0.3070 & 0.3276 & 0.2491 & 0.2399 & 0.2376 & 0.2360 & 0.2353 & 0.2354 \\ 0.2550 & 0.2451 & 0.2698 & 0.2591 & 0.2477 & 0.2499 & 0.2531 & 0.2539 & 0.2548 & 0.2550 \\ 0.2485 & 0.2400 & 0.2934 & 0.2594 & 0.2446 & 0.2460 & 0.2483 & 0.2483 & 0.2483 & 0.2485 \\ 0.2522 & 0.2434 & 0.3043 & 0.2523 & 0.2570 & 0.2538 & 0.2532 & 0.2525 & 0.2522 & 0.2522 \\ 0.2549 & 0.2460 & 0.2314 & 0.2529 & 0.2560 & 0.2554 & 0.2553 & 0.2550 & 0.2549 & 0.2549 \\ 0.2543 & 0.2543 & 0.2604 & 0.2543 & 0.2543 & 0.2543 & 0.2541 & 0.2542 & 0.2542 & 0.2543 \\ 0.2542 & 0.0751 & 0.1281 & 0.1754 & 0.2153 & 0.2408 & 0.2492 & 0.2531 & 0.2542 & 0.2542 \\ 0.2689 & 0.2306 & 0.2252 & 0.2404 & 0.2697 & 0.2711 & 0.2726 & 0.2675 & 0.2687 & 0.2689 \\ 0.2225 & 0.2296 & 0.1721 & 0.1927 & 0.2175 & 0.2209 & 0.2219 & 0.2223 & 0.2225 & 0.2225 \\ 0.2384 & 0.2681 & 0.1943 & 0.2263 & 0.2353 & 0.2371 & 0.2391 & 0.2388 & 0.2385 & 0.2384 \\ 0.2324 & 0.2589 & 0.3091 & 0.2473 & 0.2389 & 0.2360 & 0.2317 & 0.2322 & 0.2325 & 0.2324 \\ 0.2353 & 0.2619 & 0.1878 & 0.2207 & 0.2252 & 0.2330 & 0.2336 & 0.2351 & 0.2353 & 0.2353 \\ 0.2372 & 0.2641 & 0.2802 & 0.2447 & 0.2350 & 0.2362 & 0.2364 & 0.2372 & 0.2372 & 0.2372 \\ 0.2363 & 0.2363 & 0.2305 & 0.2363 & 0.2363 & 0.2363 & 0.2364 & 0.2364 & 0.2363 & 0.2363 \end{bmatrix}$$

TABLE 5. Evaluations of all four decision-makers

DM	P^Θ_1	P^Θ_2	P^Θ_3	P^Θ_4	P^Θ_5	P^Θ_6	P^Θ_7	P^Θ_8	P^Θ_9	P^Θ_{10}
Decision-maker 1										
R^Θ_1	HI	SG	OO	TS	MC	PA	EC	NA	NR	HI
R^Θ_2	SG	OO	TS	MC	PA	EC	NA	NR	HI	SG
R^Θ_3	OO	TS	MC	PA	EC	NA	NR	HI	SG	OO
R^Θ_4	TS	MC	PA	EC	NA	NR	HI	SG	OO	TS
R^Θ_5	MC	PA	EC	NA	NR	HI	SG	OO	TS	MC
R^Θ_6	PA	EC	NA	NR	HI	SG	OO	TS	MC	PA
R^Θ_7	EC	NA	NR	HI	SG	OO	TS	MC	PA	EC
R^Θ_8	NA	NR	HI	SG	OO	TS	MC	PA	EC	NA
Decision-maker 2										
R^Θ_1	SG	HI	MC	TS	PA	EC	OO	NA	NR	SG
R^Θ_2	HI	OO	SG	EC	MC	PA	TS	NR	HI	NA
R^Θ_3	OO	TS	NR	PA	NA	HI	MC	SG	OO	EC
R^Θ_4	TS	MC	PA	HI	SG	NA	EC	NR	TS	OO
R^Θ_5	MC	NR	TS	PA	EC	NA	HI	SG	OO	MC
R^Θ_6	PA	EC	SG	NR	HI	TS	OO	NA	MC	PA
R^Θ_7	EC	TS	NA	HI	SG	OO	PA	MC	NR	EC
R^Θ_8	NA	HI	NR	SG	OO	TS	MC	PA	EC	NA
Decision-maker 3										
R^Θ_1	OO	PA	SG	HI	TS	MC	NR	NA	EC	OO
R^Θ_2	NR	TS	EC	PA	HI	SG	NA	OO	MC	TS
R^Θ_3	HI	OO	SG	EC	MC	TS	PA	NR	NA	HI
R^Θ_4	TS	MC	PA	NR	HI	SG	OO	EC	NA	TS
R^Θ_5	MC	HI	EC	NR	TS	PA	SG	OO	NA	MC
R^Θ_6	PA	SG	NA	NR	EC	HI	MC	TS	OO	PA
R^Θ_7	EC	NA	NR	SG	TS	OO	HI	MC	PA	EC
R^Θ_8	NA	NR	HI	SG	OO	PA	EC	TS	MC	NA
Decision-maker 4										
R^Θ_1	NR	TS	HI	MC	PA	SG	OO	NA	EC	HI
R^Θ_2	MC	PA	OO	SG	TS	HI	NR	EC	NA	PA
R^Θ_3	SG	EC	NA	HI	NR	MC	TS	PA	OO	SG
R^Θ_4	OO	NA	PA	TS	MC	EC	SG	NR	HI	OO
R^Θ_5	TS	HI	EC	PA	SG	NR	MC	NA	OO	TS
R^Θ_6	MC	NR	TS	HI	PA	SG	EC	NA	OO	MC
R^Θ_7	PA	SG	EC	NR	TS	HI	NA	OO	MC	PA
R^Θ_8	NA	HI	NR	SG	OO	TS	MC	PA	EC	NA

Step 4: To get the aggregated values, apply the LiDFSHWA operator to each decision maker's assessment data. To do this, the Equation (5.10) is used. Table 7 presents the findings.

Step 5: Equation (5.11) was used to generate the score matrix $\phi^v_{i,j}$ as follows.

$$\varrho_{\iota,j}^1 = \begin{bmatrix} 0.8350 & 0.6900 & 0.5700 & 0.4600 & 0.3600 & 0.2600 & 0.1600 & -0.0300 & -0.3050 & 0.8350 \\ 0.6900 & 0.5700 & 0.4600 & 0.3600 & 0.2600 & 0.1600 & -0.0300 & -0.3050 & 0.8350 & 0.6900 \\ 0.5700 & 0.4600 & 0.3600 & 0.2600 & 0.1600 & -0.0300 & -0.3050 & 0.8350 & 0.6900 & 0.5700 \\ 0.4600 & 0.3600 & 0.2600 & 0.1600 & -0.0300 & -0.3050 & 0.8350 & 0.6900 & 0.5700 & 0.4600 \\ 0.3600 & 0.2600 & 0.1600 & -0.0300 & -0.3050 & 0.8350 & 0.6900 & 0.5700 & 0.4600 & 0.3600 \\ 0.2600 & 0.1600 & -0.0300 & -0.3050 & 0.8350 & 0.6900 & 0.5700 & 0.4600 & 0.3600 & 0.2600 \\ 0.1600 & -0.0300 & -0.3050 & 0.8350 & 0.6900 & 0.5700 & 0.4600 & 0.3600 & 0.2600 & 0.1600 \\ -0.0300 & -0.3050 & 0.8350 & 0.6900 & 0.5700 & 0.4600 & 0.3600 & 0.2600 & 0.1600 & -0.0300 \end{bmatrix}$$

$$\varrho_{\iota,j}^2 = \begin{bmatrix} 0.6900 & 0.8350 & 0.3600 & 0.4600 & 0.2600 & 0.1600 & 0.5700 & -0.0300 & -0.3050 & 0.6900 \\ 0.8350 & 0.5700 & 0.6900 & 0.1600 & 0.3600 & 0.2600 & 0.4600 & -0.3050 & 0.8350 & -0.0300 \\ 0.5700 & 0.4600 & -0.3050 & 0.2600 & -0.0300 & 0.8350 & 0.3600 & 0.6900 & 0.5700 & 0.1600 \\ 0.4600 & 0.3600 & 0.2600 & 0.8350 & 0.6900 & -0.0300 & 0.1600 & -0.3050 & 0.4600 & 0.5700 \\ 0.3600 & -0.3050 & 0.4600 & 0.2600 & 0.1600 & -0.0300 & 0.8350 & 0.6900 & 0.5700 & 0.3600 \\ 0.2600 & 0.1600 & 0.6900 & -0.3050 & 0.8350 & 0.4600 & 0.5700 & -0.0300 & 0.3600 & 0.2600 \\ 0.1600 & 0.4600 & -0.0300 & 0.8350 & 0.6900 & 0.5700 & 0.2600 & 0.3600 & -0.3050 & 0.1600 \\ -0.0300 & 0.8350 & -0.3050 & 0.6900 & 0.5700 & 0.4600 & 0.3600 & 0.2600 & 0.1600 & -0.0300 \end{bmatrix}$$

$$\varrho_{\iota,j}^3 = \begin{bmatrix} 0.5700 & 0.2600 & 0.6900 & 0.8350 & 0.4600 & 0.3600 & -0.3050 & -0.0300 & 0.1600 & 0.5700 \\ -0.3050 & 0.4600 & 0.1600 & 0.2600 & 0.8350 & 0.6900 & -0.0300 & 0.5700 & 0.3600 & 0.4600 \\ 0.8350 & 0.5700 & 0.6900 & 0.1600 & 0.3600 & 0.4600 & 0.2600 & -0.3050 & -0.0300 & 0.8350 \\ 0.4600 & 0.3600 & 0.2600 & -0.3050 & 0.8350 & 0.6900 & 0.5700 & 0.1600 & -0.0300 & 0.4600 \\ 0.3600 & 0.8350 & 0.1600 & -0.3050 & 0.4600 & 0.2600 & 0.6900 & 0.5700 & -0.0300 & 0.3600 \\ 0.2600 & 0.6900 & -0.0300 & -0.3050 & 0.1600 & 0.8350 & 0.3600 & 0.4600 & 0.5700 & 0.2600 \\ 0.1600 & -0.0300 & -0.3050 & 0.6900 & 0.4600 & 0.5700 & 0.8350 & 0.3600 & 0.2600 & 0.1600 \\ -0.0300 & -0.3050 & 0.8350 & 0.6900 & 0.5700 & 0.2600 & 0.1600 & 0.4600 & 0.3600 & -0.0300 \end{bmatrix}$$

$$\varrho_{\iota,j}^4 = \begin{bmatrix} -0.3050 & 0.4600 & 0.8350 & 0.3600 & 0.2600 & 0.6900 & 0.5700 & -0.0300 & 0.1600 & 0.8350 \\ 0.3600 & 0.2600 & 0.5700 & 0.6900 & 0.4600 & 0.8350 & -0.3050 & 0.1600 & -0.0300 & 0.2600 \\ 0.6900 & 0.1600 & -0.0300 & 0.8350 & -0.3050 & 0.3600 & 0.4600 & 0.2600 & 0.5700 & 0.6900 \\ 0.5700 & -0.0300 & 0.2600 & 0.4600 & 0.3600 & 0.1600 & 0.6900 & -0.3050 & 0.8350 & 0.5700 \\ 0.4600 & 0.8350 & 0.1600 & 0.2600 & 0.6900 & -0.3050 & 0.3600 & -0.0300 & 0.5700 & 0.4600 \\ 0.3600 & -0.3050 & 0.4600 & 0.8350 & 0.2600 & 0.6900 & 0.1600 & -0.0300 & 0.5700 & 0.3600 \\ 0.2600 & 0.6900 & 0.1600 & -0.3050 & 0.4600 & 0.8350 & -0.0300 & 0.5700 & 0.3600 & 0.2600 \\ -0.0300 & 0.8350 & -0.3050 & 0.6900 & 0.5700 & 0.4600 & 0.3600 & 0.2600 & 0.1600 & -0.0300 \end{bmatrix}$$

TABLE 6. Aggregated values by applying the LiDFSHWA

P^{ϱ_1}	P^{ϱ_2}	P^{ϱ_3}	P^{ϱ_4}	P^{ϱ_5}
(0.6930, 0.2566), (0.6636, 0.2454)	(0.6505, 0.2998), (0.6166, 0.2918)	(0.6035, 0.3499), (0.5712, 0.3186)	(0.6535, 0.2915), (0.6119, 0.2848)	(0.5070, 0.4449), (0.4618, 0.4143)
(0.5829, 0.3541), (0.5478, 0.3535)	(0.5826, 0.3636), (0.5458, 0.3287)	(0.5489, 0.4061), (0.5112, 0.3763)	(0.4594, 0.4955), (0.4166, 0.4672)	(0.6165, 0.3287), (0.5760, 0.3250)
(0.6807, 0.2633), (0.6488, 0.2534)	(0.5693, 0.3781), (0.5283, 0.3440)	(0.4959, 0.4499), (0.4598, 0.4263)	(0.4426, 0.5134), (0.4006, 0.4859)	(0.4241, 0.5318), (0.3821, 0.4994)
(0.5502, 0.3991), (0.5028, 0.3666)	(0.5028, 0.4498), (0.4581, 0.4199)	(0.4581, 0.4972), (0.4156, 0.4691)	(0.4966, 0.4361), (0.4554, 0.4399)	(0.6391, 0.3095), (0.6055, 0.3019)
(0.5028, 0.4498), (0.4581, 0.4199)	(0.5431, 0.3906), (0.5019, 0.3953)	(0.4596, 0.4945), (0.4161, 0.4655)	(0.3337, 0.6118), (0.2941, 0.5892)	(0.4090, 0.5339), (0.3655, 0.5113)
(0.4581, 0.4972), (0.4156, 0.4691)	(0.5236, 0.4338), (0.4887, 0.4054)	(0.4567, 0.5014), (0.4218, 0.4634)	(0.2225, 0.7032), (0.1863, 0.7032)	(0.6805, 0.2633), (0.6438, 0.2733)
(0.4156, 0.5419), (0.3747, 0.5153)	(0.4062, 0.5492), (0.3639, 0.5103)	(0.2580, 0.6785), (0.2209, 0.6654)	(0.7527, 0.1993), (0.7242, 0.2042)	(0.6354, 0.3221), (0.6036, 0.2929)
(0.3352, 0.6253), (0.2968, 0.5844)	(0.4398, 0.4810), (0.3987, 0.4931)	(0.6576, 0.2752), (0.6199, 0.2937)	(0.6799, 0.2823), (0.6562, 0.2553)	(0.6009, 0.3438), (0.5700, 0.3078)

TABLE 7. Aggregated values by applying the LiDFSHWA

P^{ϱ_6}	P^{ϱ_7}	P^{ϱ_8}	P^{ϱ_9}	$P^{\varrho_{10}}$
(0.4618, 0.4930), (0.4188, 0.4645)	(0.4101, 0.5290), (0.3725, 0.5056)	(0.3352, 0.6253), (0.2968, 0.5844)	(0.2948, 0.6423), (0.2562, 0.6318)	(0.6930, 0.2566), (0.6636, 0.2454)
(0.5358, 0.4212), (0.5006, 0.3925)	(0.4062, 0.5492), (0.3639, 0.5103)	(0.3720, 0.5570), (0.3355, 0.5389)	(0.7033, 0.2418), (0.6667, 0.2496)	(0.5344, 0.4203), (0.4965, 0.3860)
(0.5826, 0.3619), (0.5405, 0.3507)	(0.4007, 0.5437), (0.3582, 0.5223)	(0.5834, 0.3535), (0.5482, 0.3531)	(0.5402, 0.4130), (0.5078, 0.3766)	(0.6338, 0.3099), (0.5979, 0.3036)
(0.4442, 0.5029), (0.4098, 0.4762)	(0.6223, 0.3217), (0.5864, 0.3119)	(0.4534, 0.4934), (0.4177, 0.4713)	(0.4929, 0.4565), (0.4531, 0.4186)	(0.5665, 0.3812), (0.5245, 0.3474)
(0.5526, 0.3929), (0.5118, 0.3842)	(0.7179, 0.2381), (0.6914, 0.2287)	(0.6270, 0.3236), (0.5984, 0.2905)	(0.4927, 0.4567), (0.4528, 0.4188)	(0.5028, 0.4498), (0.4581, 0.4199)
(0.6905, 0.2596), (0.6567, 0.2529)	(0.5662, 0.3808), (0.5303, 0.3464)	(0.4867, 0.4645), (0.4410, 0.4288)	(0.5405, 0.4086), (0.5010, 0.3758)	(0.4581, 0.4972), (0.4156, 0.4691)
(0.6009, 0.3438), (0.5700, 0.3078)	(0.6297, 0.3152), (0.5887, 0.3103)	(0.5028, 0.4498), (0.4581, 0.4199)	(0.3865, 0.5590), (0.3450, 0.5386)	(0.4156, 0.5419), (0.3747, 0.5153)
(0.5172, 0.4338), (0.4714, 0.4026)	(0.4714, 0.4828), (0.4279, 0.4538)	(0.4931, 0.4595), (0.4485, 0.4295)	(0.4485, 0.5069), (0.4060, 0.4788)	(0.3352, 0.6253), (0.2968, 0.5844)

$$\phi_{\iota,j}^v = \begin{bmatrix} 0.4273 & 0.3378 & 0.2531 & 0.3446 & 0.0548 & -0.0384 & -0.1260 & -0.2888 & -0.3616 & 0.4273 \\ 0.2115 & 0.2181 & 0.1389 & -0.0433 & 0.2694 & 0.1113 & -0.1448 & -0.1941 & 0.4394 & 0.1123 \\ 0.4064 & 0.1877 & 0.0398 & -0.0781 & -0.1125 & 0.2053 & -0.1536 & 0.2125 & 0.1292 & 0.3091 \\ 0.1437 & 0.0457 & -0.0463 & 0.0380 & 0.3166 & -0.0625 & 0.2876 & -0.0468 & 0.0355 & 0.1812 \\ 0.0457 & 0.1295 & -0.0422 & -0.2866 & -0.1353 & 0.1437 & 0.4712 & 0.3056 & 0.0351 & 0.0457 \\ -0.0463 & 0.0865 & -0.0431 & -0.4988 & 0.3938 & 0.4173 & 0.1846 & 0.0171 & 0.1286 & -0.0463 \\ -0.1334 & -0.1448 & -0.4325 & 0.5367 & 0.3120 & 0.2596 & 0.2964 & 0.0457 & -0.1831 & -0.1334 \\ -0.2888 & -0.0678 & 0.3543 & 0.3992 & 0.2596 & 0.0761 & -0.0187 & 0.0263 & -0.0656 & -0.2888 \end{bmatrix}$$

Working Steps of Entropy Method: Step 6: Use Equations 5.12 and 5.13 to normalise the data.

$$\begin{bmatrix} 1.0000 & 1.0000 & 0.8714 & 0.8145 & 0.3593 & 0.9498 & 0.9558 & 0.0000 & 0.0000 & 1.0000 \\ 0.6987 & 0.7519 & 0.7262 & 0.4399 & 0.7648 & 0.6377 & 0.9859 & 0.1593 & 1.0000 & 0.5601 \\ 0.9709 & 0.6891 & 0.6003 & 0.4063 & 0.0432 & 0.4419 & 1.0000 & 0.8433 & 0.6127 & 0.8350 \\ 0.6040 & 0.3946 & 0.4909 & 0.5184 & 0.8540 & 1.0000 & 0.2940 & 0.4072 & 0.4957 & 0.6564 \\ 0.4671 & 0.5685 & 0.4961 & 0.2049 & 0.0000 & 0.5703 & 0.0000 & 1.0000 & 0.4952 & 0.4671 \\ 0.3387 & 0.4793 & 0.4949 & 0.0000 & 1.0000 & 0.0000 & 0.4588 & 0.5147 & 0.6119 & 0.3387 \\ 0.2170 & 0.0000 & 0.0000 & 1.0000 & 0.8454 & 0.3287 & 0.2798 & 0.5627 & 0.2229 & 0.2170 \\ 0.0000 & 0.1595 & 1.0000 & 0.8672 & 0.7463 & 0.7112 & 0.7842 & 0.5301 & 0.3695 & 0.0000 \end{bmatrix}$$

Step 7: Use Equation 5.18 to determine the standardised values ρ_{ij} for non-zero entries.

$$\begin{bmatrix} 0.0598 & 0.1865 & 0.1580 & 0.1593 & 0.0998 & 0.1678 & 0.1662 & 0.0624 & 0.0640 & 0.1858 \\ -0.1321 & 0.1557 & 0.1413 & 0.1139 & 0.1468 & 0.1317 & 0.1697 & 0.0822 & 0.1921 & 0.1313 \\ -1.0492 & 0.1478 & 0.1268 & 0.1098 & 0.0631 & 0.1090 & 0.1713 & 0.1676 & 0.1425 & 0.1653 \\ 1.3780 & 0.1112 & 0.1142 & 0.1234 & 0.1572 & 0.1736 & 0.0907 & 0.1131 & 0.1275 & 0.1432 \\ 1.4513 & 0.1328 & 0.1148 & 0.0854 & 0.0581 & 0.1239 & 0.0571 & 0.1871 & 0.1275 & 0.1198 \\ 1.4064 & 0.1218 & 0.1146 & 0.0606 & 0.1742 & 0.0579 & 0.1095 & 0.1266 & 0.1424 & 0.1039 \\ 0.2852 & 0.0622 & 0.0576 & 0.1818 & 0.1562 & 0.0959 & 0.0890 & 0.1325 & 0.0926 & 0.0888 \\ -0.1615 & 0.0820 & 0.1728 & 0.1657 & 0.1447 & 0.1402 & 0.1466 & 0.1285 & 0.1114 & 0.0619 \end{bmatrix}$$

Step 8: Calculate the entropy values. Equation Ψ_j using 5.15.

$$[0.9748 \quad 0.9772 \quad 0.9820 \quad 0.9756 \quad 0.9696 \quad 0.9791 \quad 0.9721 \quad 0.9767 \quad 0.9801 \quad 0.9771]$$

Step 9: Use Equation 5.16 to determine the degree of divergence ϖ_j for each criteria.

$$[0.0252 \quad 0.0228 \quad 0.0180 \quad 0.0244 \quad 0.0304 \quad 0.0209 \quad 0.0279 \quad 0.0233 \quad 0.0199 \quad 0.0229]$$

Step 10: Determine the weights V_j as per Equation 5.17.

$$[0.1071 \quad 0.0969 \quad 0.0762 \quad 0.1035 \quad 0.1292 \quad 0.0886 \quad 0.1182 \quad 0.0987 \quad 0.0844 \quad 0.0971]$$

Working Steps of SWARA Method: Step 11: Utilize weights derived from entropy in Step 10.

Step 12: Calculate the criteria coefficients (Φ^{τ}_j) as per Equation 5.19.

$$[1.0000 \quad 1.0969 \quad 1.0762 \quad 1.1035 \quad 1.1292 \quad 1.0886 \quad 1.1182 \quad 1.0987 \quad 1.0844 \quad 1.0971]$$

Step 13: Determine the initial weights (ξ^{ν}_j) by applying Equation 5.20.

$$[1.0000 \quad 0.9116 \quad 0.8471 \quad 0.7676 \quad 0.6798 \quad 0.6245 \quad 0.5585 \quad 0.5083 \quad 0.4687 \quad 0.4272]$$

Step 14: Calculate the relative weights (F^W_j) by applying Equation 5.21.

$$[0.1472 \quad 0.1342 \quad 0.1247 \quad 0.1130 \quad 0.1001 \quad 0.0919 \quad 0.0822 \quad 0.0748 \quad 0.0690 \quad 0.0629]$$

Working Steps of MARCOS: Step 15: The MARCOS approach for ranking alternatives identifies the optimal alternative, represented by the symbol D , which is determined using Equation (5.22). In a similar manner, the row indicating the least favourable alternative is represented by Π and is derived using Equation (5.23).

$$D_j = \begin{bmatrix} 0.4273 & 0.3378 & -0.4325 & 0.5367 & 0.3938 \\ 0.4173 & 0.4712 & 0.3056 & 0.4394 & 0.4273 \end{bmatrix}.$$

$$\Pi_j = \begin{bmatrix} -0.2888 & -0.1448 & 0.3543 & -0.4988 & -0.1353 \\ -0.0625 & -0.1536 & -0.2888 & -0.3616 & -0.2888 \end{bmatrix}.$$

Step 16: Thus, the two rows computed above augment the normalized matrix by incorporating additional rows. The extended normalized matrix is represented as ϑ^S_{ij} .

$$\eta^S_{ij} = \begin{bmatrix} 0.4273 & 0.3378 & 0.2531 & 0.3446 & 0.0548 & -0.0384 & -0.1260 & -0.2888 & -0.3616 & 0.4273 \\ 0.2115 & 0.2181 & 0.1389 & -0.0433 & 0.2694 & 0.1113 & -0.1448 & -0.1941 & 0.4394 & 0.1123 \\ 0.4064 & 0.1877 & 0.0398 & -0.0781 & -0.1125 & 0.2053 & -0.1536 & 0.2125 & 0.1292 & 0.3091 \\ 0.1437 & 0.0457 & -0.0463 & 0.0380 & 0.3166 & -0.0625 & 0.2876 & -0.0468 & 0.0355 & 0.1812 \\ 0.0457 & 0.1295 & -0.0422 & -0.2866 & -0.1353 & 0.1437 & 0.4712 & 0.3056 & 0.0351 & 0.0457 \\ -0.0463 & 0.0865 & -0.0431 & -0.4988 & 0.3938 & 0.4173 & 0.1846 & 0.0171 & 0.1286 & -0.0463 \\ -0.1334 & -0.1448 & -0.4325 & 0.5367 & 0.3120 & 0.2596 & 0.2964 & 0.0457 & -0.1831 & -0.1334 \\ -0.2888 & -0.0678 & 0.3543 & 0.3992 & 0.2596 & 0.0761 & -0.0187 & 0.0263 & -0.0656 & -0.2888 \\ 0.4273 & 0.3378 & -0.4325 & 0.5367 & 0.3938 & 0.4173 & 0.4712 & 0.3056 & 0.4394 & 0.4273 \\ -0.2888 & -0.1448 & 0.3543 & -0.4988 & -0.1353 & -0.0625 & -0.1536 & -0.2888 & -0.3616 & -0.2888 \end{bmatrix}$$

Step 17: The extended matrix is normalized according to Equation (5.24), resulting in the normalized matrix denoted as ϑ^S_{ij} .

$$\vartheta^{S(1)}_{ij} = \begin{bmatrix} 1.0000 & 1.0000 & -1.7088 & 0.6421 & 0.1391 & -0.0920 & -0.2674 & -0.9450 & -0.8230 & 1.0000 \\ 0.4950 & 0.6457 & -3.1137 & -0.0807 & 0.6840 & 0.2667 & -0.3073 & -0.6351 & 1.0001 & 0.2628 \\ 0.9511 & 0.5557 & -10.8667 & -0.1455 & -0.2856 & 0.4919 & -0.3259 & 0.6953 & 0.2941 & 0.7234 \\ 0.3363 & 0.1353 & 9.3412 & 0.0708 & 0.8039 & -0.1498 & 0.6103 & -0.1531 & 0.0808 & 0.4241 \\ 0.1070 & 0.3834 & 10.2487 & -0.5340 & -0.3435 & 0.3443 & 0.9999 & 0.9999 & 0.0799 & 0.1070 \end{bmatrix}$$

$$\vartheta^{S(2)}_{ij} = \begin{bmatrix} -0.1084 & 0.2561 & 10.0347 & -0.9294 & 0.9999 & 0.9999 & 0.3917 & 0.0560 & 0.2927 & -0.1084 \\ -0.3122 & -0.4287 & 1.0000 & 1.0000 & 0.7922 & 0.6220 & 0.6290 & 0.1495 & -0.4168 & -0.3122 \\ -0.6759 & -0.2007 & -1.2207 & 0.7438 & 0.6591 & 0.1823 & -0.0397 & 0.0861 & -0.1493 & -0.6759 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 0.9999 & 0.9999 & 0.9999 & 0.9999 & 1.0001 & 1.0000 \\ -0.6759 & -0.4287 & -1.2207 & -0.9294 & -0.3435 & -0.1498 & -0.3259 & -0.9450 & -0.8230 & -0.6759 \end{bmatrix}.$$

Step 18: The weighted normalized matrix, σ_{ij} , is derived by multiplying the final weights with the extended normalized matrix.

$$\sigma^{(1)}_{ij} = \begin{bmatrix} 0.1472 & 0.1342 & -0.2131 & 0.0726 & 0.0139 & -0.0085 & -0.0220 & -0.0707 & -0.0568 & 0.0629 \\ 0.0729 & 0.0866 & -0.3883 & -0.0091 & 0.0685 & 0.0245 & -0.0253 & -0.0475 & 0.0690 & 0.0165 \\ 0.1400 & 0.0746 & -1.3550 & -0.0164 & -0.0286 & 0.0452 & -0.0268 & 0.0520 & 0.0203 & 0.0455 \\ 0.0495 & 0.0182 & 1.1648 & 0.0080 & 0.0804 & -0.0138 & 0.0502 & -0.0115 & 0.0056 & 0.0267 \\ 0.0157 & 0.0514 & 1.2779 & -0.0603 & -0.0344 & 0.0317 & 0.0822 & 0.0748 & 0.0055 & 0.0067 \end{bmatrix}$$

$$\sigma^{(2)}_{ij} = \begin{bmatrix} -0.0160 & 0.0344 & 1.2512 & -0.1050 & 0.1001 & 0.0919 & 0.0322 & 0.0042 & 0.0202 & -0.0068 \\ -0.0460 & -0.0575 & 0.1247 & 0.1130 & 0.0793 & 0.0572 & 0.0517 & 0.0112 & -0.0288 & -0.0196 \\ -0.0995 & -0.0269 & -0.1522 & 0.0840 & 0.0660 & 0.0168 & -0.0033 & 0.0064 & -0.0103 & -0.0425 \\ 0.1472 & 0.1342 & 0.1247 & 0.1130 & 0.1001 & 0.0919 & 0.0822 & 0.0748 & 0.0690 & 0.0629 \\ -0.0995 & -0.0575 & -0.1522 & -0.1050 & -0.0344 & -0.0138 & -0.0268 & -0.0707 & -0.0568 & -0.0425 \end{bmatrix}.$$

Step 19: The utility value for each alternative is determined using Equations (5.26) and (5.27).

$$K^{\vee+}_i = [0.0412 \quad -0.0910 \quad -0.7229 \quad 0.9495 \quad 1.0000 \quad 0.9691 \quad 0.1965 \quad -0.1113]$$

$$K^{\vee-}_i = [0.0425 \quad -0.0940 \quad -0.7460 \quad 0.9799 \quad 1.0319 \quad 1.0000 \quad 0.2028 \quad -0.1148]$$

Step 20: Table 8 displays the usefulness values for the ideal, anti-ideal, and utility functions. The alternative appraisal scores are computed with $R^r = 0.5$. Thus the final ordering of alternatives is as given.

$$R^{\ominus}_5 \succ R^{\ominus}_6 \succ R^{\ominus}_4 \succ R^{\ominus}_7 \succ R^{\ominus}_1 \succ R^{\ominus}_2 \succ R^{\ominus}_8 \succ R^{\ominus}_3$$

Consequently, $R^{\ominus}_5$, which signifies intelligent road safety as an application of the Internet of Things, is identified as the most effective alternative.

6.4. Sensitivity Analysis. Table 9 provides a sensitivity analysis to assess the robustness of the proposed ranking results in relation to variations in the parameter T across different levels of λ^δ . This analysis examines the effects of variations in T (from 0.1 to 0.9) on the

TABLE 8. Final evaluation matrix

$K_i^{\gamma-}$	$K_i^{\gamma+}$	$F^{\lambda}(K^{\gamma+})$	$F^{\lambda}(K^{\gamma-})$	AS	Ranking
0.0425	0.0412	0.4921	0.5079	0.0418	5
-0.0940	-0.0910	0.4921	0.5079	-0.0925	6
-0.7460	-0.7229	0.4921	0.5079	-0.7345	8
0.9799	0.9495	0.4921	0.5079	0.9648	3
1.0319	1.0000	0.4921	0.5079	1.0161	1
1.0000	0.9691	0.4921	0.5079	0.9846	2
0.2028	0.1965	0.4921	0.5079	0.1997	4
-0.1148	-0.1113	0.4921	0.5079	-0.1131	7

calculated performance values ($R_i^{\ominus}$) and the resultant ranking order of alternatives. The results indicate a significant consistency in the rankings across all evaluated configurations. Despite the variations in both parameters T and λ^{δ} , the ranking sequence remains consistent: $R_5^{\ominus} \succ R_6^{\ominus} \succ R_4^{\ominus} \succ R_7^{\ominus} \succ R_1^{\ominus} \succ R_2^{\ominus} \succ R_8^{\ominus} \succ R_3^{\ominus}$. The consistent ranking order observed across all sensitivity scenarios demonstrates that the proposed decision-making method exhibits significant robustness and insensitivity to parameter variations. Modifications to the preference intensification parameter (λ^{δ}) or the threshold control parameter (T) do not alter the final ranking. The proximity of the computed performance values further indicates that small variations in inputs do not substantially influence the results. The observed tight clustering indicates that the method yields a stable evaluation, even in marginal preference scenarios. In practice, robustness is vital in a decision-making process because of the uncertainties in data and the tuning of the parameters, which can cause inconsistency. The results justify the dependability and uniformity of the suggested model in prioritizing alternatives under varying circumstances. The results of sensitivity analysis are presented in Figure 3 that represents line graphs that indicate how each alternative changes in performance with respect to parameter T at different fixed values of λ^{δ} . The alternatives have stable behaviors under different parameterization, and particular alternatives (e.g., $R_5^{\ominus}$ and $R_6^{\ominus}$) always perform better than others. To support this finding, the same data are presented as grouped bar plots in Figure 4: barplots. This form simplifies the task of comparing the extent of change of each alternative at each value of λ^{δ} as the temperature varies.

6.5. Comparative analysis. Comparative results have been offered in Table 10 indicating how various decision support systems can be effective and reliable in ranking alternatives. There is a common trend in the techniques used. Alternative $R_2^{\ominus}$ always ranks the highest based on the evaluation criteria, which underscores its superiority. The agreement between the different methods of MCDM, such as CRITIC-MARCOS, CRITIC-COPRAS, and CRITIC-MULTIMOORA, contributes to the conclusion that $R_2^{\ominus}$ is the most stable and robust one. The ranking of other alternatives does not show any significant changes, but $R_4^{\ominus}$ is always ranked second, which means that it is competitive. Increased variability is observed in the rankings of $R_3^{\ominus}$, $R_5^{\ominus}$ and $R_1^{\ominus}$ so that the preferences of the middle tier are sensitive to the choice of decision-making approaches. The identified differences highlight the importance of the method choice in determining the final results, particularly when it is necessary to distinguish between the options with the same level of performance.

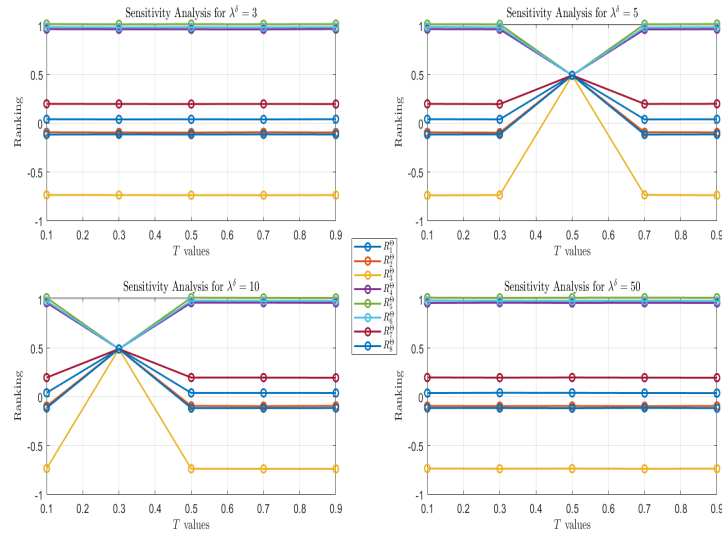
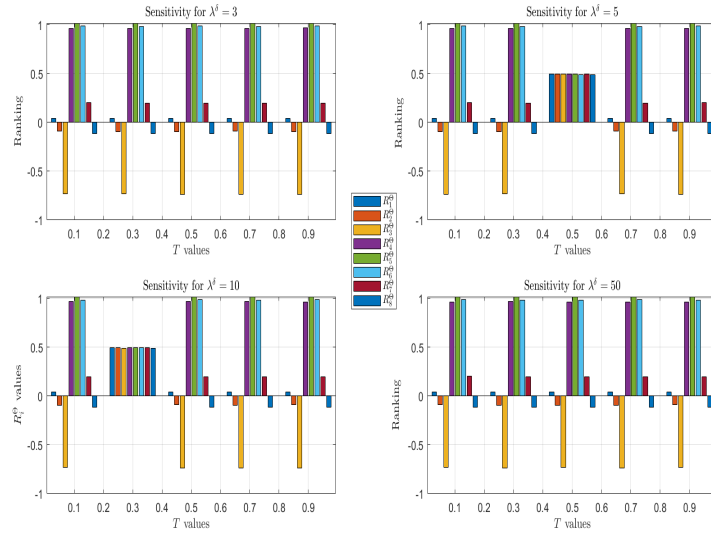


FIGURE 3. Line graphs illustrating the sensitivity of each alternative

FIGURE 4. Grouped bar plots showing the impact of T on the performance of alternatives

The RANCOM-ERUNS method based on the LOPCOW system is always reliable in giving

TABLE 9. The impact of the parameter T on the ranking

Techniques	T	(R^{Θ_1})	(R^{Θ_2})	(R^{Θ_3})	(R^{Θ_4})	(R^{Θ_5})	(R^{Θ_6})	(R^{Θ_7})	(R^{Θ_8})	Ranking
$\lambda^{\delta}=3$	$T=0.1$	0.0426	-0.0911	-0.7328	0.9653	1.0174	0.9860	0.2003	-0.1144	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.3$	0.0410	-0.0929	-0.7339	0.9640	1.0152	0.9831	0.1991	-0.1126	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.5$	0.0421	-0.0936	-0.7350	0.9636	1.0167	0.9852	0.1985	-0.1142	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.7$	0.0409	-0.0907	-0.7353	0.9631	1.0156	0.9839	0.1996	-0.1124	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.9$	0.0433	-0.0931	-0.7342	0.9665	1.0171	0.9854	0.1983	-0.1137	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
$\lambda^{\delta}=5$	$T=0.1$	0.0429	-0.0920	-0.7368	0.9652	1.0160	0.9867	0.2001	-0.1140	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.3$	0.0412	-0.0944	-0.7336	0.9638	1.0148	0.9845	0.1977	-0.1128	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.5$	0.4925	0.4922	0.4919	0.4923	0.4920	0.4916	0.4921	0.4918	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.7$	0.0405	-0.0909	-0.7323	0.9626	1.0153	0.9828	0.1993	-0.1145	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.9$	0.0424	-0.0917	-0.7355	0.9646	1.0172	0.9851	0.2005	-0.1130	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
$\lambda^{\delta}=10$	$T=0.1$	0.0431	-0.0928	-0.7330	0.9657	1.0164	0.9837	0.1989	-0.1122	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.3$	0.4920	0.4922	0.4916	0.4919	0.4923	0.4921	0.4924	0.4918	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.5$	0.0419	-0.0910	-0.7348	0.9658	1.0173	0.9849	0.1994	-0.1136	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.7$	0.0422	-0.0934	-0.7356	0.9662	1.0154	0.9836	0.1988	-0.1141	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.9$	0.0413	-0.0908	-0.7360	0.9641	1.0149	0.9853	0.1975	-0.1132	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
$\lambda^{\delta}=50$	$T=0.1$	0.0406	-0.0919	-0.7327	0.9628	1.0168	0.9863	0.2000	-0.1127	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.3$	0.0432	-0.0930	-0.7346	0.9655	1.0151	0.9842	0.1982	-0.1133	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.5$	0.0425	-0.0922	-0.7333	0.9633	1.0165	0.9834	0.1997	-0.1148	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.7$	0.0407	-0.0942	-0.7351	0.9647	1.0170	0.9847	0.1980	-0.1118	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$
	$T=0.9$	0.0404	-0.0913	-0.7338	0.9632	1.0163	0.9833	0.1976	-0.1143	$R^{\Theta_5} \succ R^{\Theta_6} \succ R^{\Theta_4} \succ R^{\Theta_7} \succ R^{\Theta_1} \succ R^{\Theta_2} \succ R^{\Theta_8} \succ R^{\Theta_3}$

a ranking system. The fact that the leading alternatives can be consistent across the methods implies that the suggested approach is robust and effective in offering reliable decision support. The fact that the final decisions in most cases are aligned means that the outcome of the decision does not greatly depend on the approach applied, which is a positive indicator of the stability and the strength of the suggested LOPCOW-based RANCOM-ERUNS method.

TABLE 10. Comparative evaluation of decision-making methods and their final rankings

Authors	Methods Applied	Rankings Achieved	Top Option
Haktanir and Kahraman [23]	CRITIC-REGIME	$R_5^A \succ R_6^A \succ R_4^A \succ R_2^A \succ R_1^A \succ R_7^A \succ R_8^A \succ R_3^A$	R_5^A
Razzaq et al. [41]	CRITIC-CoCoSo	$R_4^A \succ R_2^A \succ R_8^A \succ R_7^A \succ R_1^A \succ R_6^A \succ R_5^A \succ R_3^A$	R_4^A
Rani et al. [40]	CRITIC-MULTIMOORA	$R_5^A \succ R_6^A \succ R_4^A \succ R_7^A \succ R_1^A \succ R_2^A \succ R_8^A \succ R_3^A$	R_5^A
Krishankumar et al. [34]	CRITIC-MARCOS	$R_4^A \succ R_2^A \succ R_8^A \succ R_7^A \succ R_1^A \succ R_6^A \succ R_5^A \succ R_3^A$	R_4^A
Alkan and Kahraman [5]	TOPSIS	$R_5^A \succ R_6^A \succ R_4^A \succ R_7^A \succ R_1^A \succ R_2^A \succ R_8^A \succ R_3^A$	R_5^A
Proposed Framework	Entropy-SWARA-MARCOS	$R_5^A \succ R_6^A \succ R_4^A \succ R_7^A \succ R_1^A \succ R_2^A \succ R_8^A \succ R_3^A$	R_5^A

6.6. Discussion. The outcomes of the evaluation showed that there were substantial variations in the weights of decision criteria measured using the integrated Entropy-SWARA weighting method. Of the ten criteria, technological readiness level (P^{Θ_1}) achieved the largest weighted mean, which implies it had a major influence on determining the practicability of the smart waste management technologies implementation. Environmental regeneration potential (P^{Θ_2}) and energy efficiency (P^{Θ_3}) were closely the same and thus the emphasis on environmental sustainability and resource optimization in decision making. Resilience to cybersecurity and data integrity (P^{Θ_4}) and compatibility with smart infrastructure (P^{Θ_5}) were given a significant weight, particularly considering the nature of the assessed options as digital. On the other hand, the weights of social acceptance (P^{Θ_6}) were lower, which means that, as much as public engagement is necessary, technical and performance-based aspects take precedence in strategic planning. The MARCOS approach produced a powerful and clear ranking of the eight developed waste management

options. Auto-AI predictive sorting systems ($\mathcal{R}_5^A$) ranked highest because of the superior technological maturity, energy efficiency, and scalability. Waste traceability ($\mathcal{R}_6^A$) with blockchain earned second place, which is recognized to have a major impact on transparency, data integrity, and environmental compliance. Waste monitoring with IoT ($\mathcal{R}_4^A$) was ranked in the third place, showing great integration opportunities and business viability. The AI-driven circular economy analyzer ($\mathcal{R}_7^A$) and smart waste-to-energy systems ($\mathcal{R}_1^A$) were ranked fourth and fifth, respectively, demonstrating strong performance in circularity and regeneration potential. Although with an innovative potential, the Metaverse-based public awareness platforms ($\mathcal{R}_2^A$) were ranked sixth due to rather low scores on the level of technology readiness (TRL) and affordability. The seventh-ranked cloud-based integrated decision platform ($\mathcal{R}_8^A$) implies that it does not have the lowest performance in most aspects, yet it was not very energy-efficient. Lastly, the digital twin simulation ($\mathcal{R}_3^A$) ranked lowest in the ranking, primarily due to its less favorable environmental footprint and perceived complexity of integration, but it does not fail to meet the policy objectives. The final alternative ranking is: $\mathcal{R}_5^A \succ \mathcal{R}_6^A \succ \mathcal{R}_4^A \succ \mathcal{R}_7^A \succ \mathcal{R}_1^A \succ \mathcal{R}_2^A \succ \mathcal{R}_8^A \succ \mathcal{R}_3^A$.

7. CONCLUSION

The study presents a holistic decision support model in the evaluation of sophisticated waste management solutions as part of a sustainable city development model. Combining the objective Entropy approach with the subjective SWARA approach, and the use of the MARCOS method to achieve final ranking, creates a non-biased and transparent set-up of prioritizing solutions that fulfill multiple criteria that are often conflicting. The addition of $(\in, \in \vee q_k)$ -Linear Diophantine fuzzy sets will greatly enhance the model by reflecting accurately the uncertainty, imprecision and hesitation in expert judgments. The outcomes show that technologies are more effective in terms of technological, environmental, and operational. The ability of the frame work to incorporate both qualitative and quantitative inputs and its flexibility in addressing high- uncertainty contexts make it a useful instrument to policy-makers, urban planners and sustainability managers. The model exhibits various weaknesses including reliance on consistency of the expert input, the assumption of linear trade-offs in MARCOS approach as well as computational requirements which are high due to the inclusion of fuzzy logic. The suggested framework provides a strong foundation on future studies on smart infrastructure assessment that can help to create dynamic and real-time decision models that may guide the adoption of sustainable technology use in complex urban systems.

7.1. Limitations of the Proposed Model. The suggested decision-making framework suggests a composite method to assess smart waste management options, but there are a few limitations to consider. Subjective (SWARA) and objective (entropy) weighting techniques may introduce some inconsistencies when there is bias or lack of agreement in the expert judgments, especially in ambiguous language situations. Secondly, although $(\in, \in \vee q_k)$ -Linear Diophantine fuzzy sets enhance the capacity of the model to address uncertainty and hesitation, they also increase the level of computational complexity and could require the decision-makers to possess specialized mathematical expertise. The MARCOS method makes sure that the utility behaviors are linear and criteria are fully compensable when this may not be the case in the actual world, where criteria are interrelated, e.g. cost and energy

efficiency. The model is based on the quality and availability of expert evaluations, and this can be a scalability constraint where access to experts is scarce or when other domains of expertise are competing. Lastly, although the model turns out to be generalizable within a variety of smart waste management contexts, it lacks validation in the sense of actual implementation or dynamic feedback that can contribute to its adaptability and realism within highly dynamic urban environments.

7.2. Future direction.

- Apply both the supervised and non-supervised machine learning algorithms to predict the performance results of smart waste technologies, as well as to automatize expert decision making in the weight estimation and alternative ranking.
- Develop adaptive multi-criteria decision-making models to dynamically change weights and priorities in real-time, using streaming information of Internet of Things devices and cloud platforms as part of waste infrastructure.
- Use techniques such as fuzzy cognitive maps, or feature interaction analysis using machine learning to determine relationships and cause-effect relationships among decision criteria.
- Expand and explore the $(\in, \in \vee q_k)$ -LiDF set model by trying to integrate it with other soft computing models, including genetic algorithms and neural networks.
- Generalize the framework to multi-domain decision settings, such as energy, water, and transportation, in which smart waste management can be aligned to other sustainability goals.
- Test the practical effects and scalability of the model using a longitudinal study or pilot programs in smart cities.
- Combine MCDM results with interpretable machine learning, including SHAP and LIME, to provide insights that are understandable to stakeholders and increase trust in automated decision-making.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Fatima Rasool: Draft Original Writing, Data Curation, Methodology, Software. Asim Naseem: Review, and Editing, Conceptualization, Investigation. Muhammad Riaz: Formal Analysis, Review and Editing, Data Curation, Supervision, The final version of the manuscript has been read and approved by all authors.

DECLARATIONS

Conflict of interest: The the authors have no conflicts of interest.

Funding: This research received no funding.

Data Availability: No data is used in this paper.

Acknowledgment: We are thankful to the editor and reviewers for their valuable comments and suggestions to improve the quality of this manuscript.

REFERENCES

- [1] Abdulla, A., Baryannis, G., & Badi, I. (2023). An integrated machine learning and MARCOS method for supplier evaluation and selection. *Decision Analytics Journal*, 9, 100342.

- [2] Agarwal, S., Kant, R., & Shankar, R. (2020). Evaluating solutions to overcome humanitarian supply chain management barriers: A hybrid fuzzy SWARA—Fuzzy WASPAS approach. *Int. J. Disaster Risk Reduct.*, 51, 101838.
- [3] Akram, M., & Zhan, J. (2006). On sensible fuzzy ideals of BCK-algebras with respect to a t-conorm. *International Journal of Mathematics and Mathematical Sciences*, 2006, (1), 1-12.
- [4] Akram, M., Zahid, K., & Deveci, M. (2023). Multi-criteria group decision-making for optimal management of water supply with fuzzy ELECTRE-based outranking method. *Applied Soft Computing*, 143, 110403.
- [5] Alkan, N., & Kahraman, C. (2022). Circular intuitionistic fuzzy TOPSIS method: Pandemic hospital location selection. *Journal of Intelligent & Fuzzy Systems*, 42(1), 295-316.
- [6] Anjum, M., Kraiem, N., Min, H., Daradkeh, Y. I., Dutta, A. K., & Shahab, S. (2025). Integrating intuitionistic fuzzy and MCDM methods for sustainable energy management in smart factories. *PLOS One*, 20(1), e0315251.
- [7] Antiroiko, A. V. (2023). Smart circular cities: Governing the relationality, spatiality, and digitality in the promotion of circular economy in an urban region. *Sustainability*, 15(17), 12680.
- [8] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 20(1), 87-96.
- [9] Aydogdu, A., Gul, S., & Alniak, T. (2024). New information measures for linear Diophantine fuzzy sets and their applications with LiDF-ARAS on data storage system selection problem. *Expert Systems with Applications*, 252, 124135.
- [10] Ayub, S., Gul, R., Shabir, M., Hummdi, A. Y., Aljaedi, A., & Bassfar, Z. (2024). A novel approach to roughness of linear diophantine fuzzy sets by fuzzy relations and its application toward multi-criteria group decision-making. *IEEE Access*, 12, 53549–53564.
- [11] Bhakat, S. K. and P. Das. (1996) $(\in, \in \vee q)$ -fuzzy subgroups. *Fuzzy Sets and Systems*, 80, (1996), 359–368.
- [12] Bhakat, S. K. (1999). $(\in \vee q)$ -level subsets. *Fuzzy Sets and Systems*, 103, 529–533.
- [13] Bhakat, S.K.(2000) $(\in, \in \vee q)$ -fuzzy normal, quasi-normal and maximal subgroups. *Fuzzy Sets and Systems*, 112, (2000), 299–312.
- [14] Atenidegebe, O. F., & Mogaji, K. A. (2023). Modeling assessment of groundwater vulnerability to contamination risk in a typical basement terrain using TOPSIS-entropy developed vulnerability data mining technique. *Heliyon*, 9 (7), 1-25.
- [15] Biswas, T. R., Hossain, M. Z., & Comite, U. (2024). Role of Management Information Systems in Enhancing Decision-Making in Large-Scale Organizations. *Pacific Journal of Business Innovation and Strategy*, 1(1), 5–18.
- [16] Bonfante, A., Basile, A., & Bouma, J. (2020). Targeting the soil quality and soil health concepts when aiming for the United Nations Sustainable Development Goals and the EU Green Deal. *Soil*, 6(2), 453–466.
- [17] Bouraima, M. B., Qiu, Y., Stevic, Z., Simic, V. (2023). Assessment of alternative railway systems for sustainable transportation using an integrated IRN SWARA and IRN CoCoSo model. *Socio-Econ. Plan. Sci.* 86, 101475.
- [18] Cui, Z., Taiwo, O. L., & Aaron, P. M. (2024). An application of AHP and fuzzy entropy-TOPSIS methods to optimize upstream petroleum investment in representative African basins. *Scientific Reports*, 14(1), 6956.
- [19] Debnath, B., Bari, A.B.M.M., Haq, M. M., de Jesus Pacheco, D.A., Khan, M.A. (2023). An integrated stepwise weight assessment ratio analysis and weighted aggregated sum product assessment framework for sustainable supplier selection in the healthcare supply chains. *Supply Chain Anal.* , 1, 100001.
- [20] Deveci, M., Varouchakis, E.A., Brito-Parada, P.R., Mishra, A.R., Rani, P., Bolgkranou, M., Galtakis.(2023). M. Evaluation of risks impeding sustainable mining using Fermatean fuzzy score function based SWARA method. *Appl. Soft Comput.* 139, 110220.
- [21] Dudek, W. A. (1992). On proper BCC-algebras, *Bull. Inst. Math. Academia Sinica*, 20 , 137–150.
- [22] Dudek, W. A., & Zhang, X. H. (1998). On ideals and congruences in BCC-algebras, *Czechoslovak Mathematical Journal*, 48(123), 21–29.
- [23] Haktanir, E., & Kahraman, C. (2022). A novel picture fuzzy CRITIC & REGIME methodology: Wearable health technology application. *Engineering Applications of Artificial Intelligence*, 113, 104942.
- [24] Hao, J. (1998). Ideal Theory of BCC-algebras, *Sci. Math. Japon.* 3, 373–381.
- [25] Hashim, M., Tabassum, B., Khan, T., Khan, M., Hasan, A., & Khan, N. (2025). *Managing Hazardous Industrial Waste: Effective Treatment and Disposal Approaches*. In Management of Waste to Control Environmental Pollutions: Sustainability and Economic Feasibility (pp. 69–86). Cham: Springer Nature Switzerland.

- [26] Iratni, A., & Chang, N. B. (2019). Advances in control technologies for wastewater treatment processes: status, challenges, and perspectives. *IEEE/CAA Journal of Automatica Sinica*, 6(2), 337–363.
- [27] Ismail, Y., Adji, R., Santoso, A. D., Wahyono, S., Kholig, M. A., Handayani, T., ... & Tilottama, R. D. (2025). Multi-dimensional perspectives on sustainability in domestic waste management. *Global Journal of Environmental Science and Management*, 11(2), 747–770.
- [28] Jayakumar, V., Kannan, J., Kausar, N., Deveci, M., & Wen, X. (2024). Multicriteria group decision making for prioritizing IoT risk factors with linear diophantine fuzzy sets and MARCOS method. *Granular Computing*, 9(3), 56.
- [29] Jun, Y. B. (2004). On (α, β) -fuzzy ideals of BCK/BCI-algebras. *Sci. Math. Japon.*, 60, 613–617.
- [30] Jun, Y. B. (2009). Generalizations of $(\in, \in \vee q)$ -fuzzy subalgebras in BCK/BCI-algebras. *Comput. Math. Appl.*, 58, 1383–1390.
- [31] Kersulienė, V., Zavadskas, E. K., & Turskis, Z. (2010). Selection of rational dispute resolution method by applying new step-wise weight assessment ratio analysis (SWARA). *Journal of Business Economics and Management*, 11, (2), 243–258.
- [32] Kalbar, P. P., Karmakar, S., & Asolekar, S. R. (2012). Technology assessment for wastewater treatment using multiple-attribute decision-making. *Technology in Society*, 34,(4), 295–302.
- [33] Komori, Y. (1984). The class of BCC-algebras is not a variety, *Math. Japon.*, 29, 391–394.
- [34] Krishankumar, R., Mishra, A. R., Rani, P., Ecer, F., & Ravichandran, K. S. (2023). Assessment of zero-carbon measures for sustainable transportation in smart cities: a CRITIC-MARCOS framework based on Q-rung fuzzy preferences. *IEEE internet of things journal*, 10, (21), 18651–18662.
- [35] Liu, Z. (2023). Selecting renewable desalination using uncertain data: an MCDM framework combining mixed objective weighting and interval MARCOS. *Water Supply*, 23(4), 1571–1586.
- [36] Muhiuddin, G., Al-Tahan, M., Mahboob, A., Hoskova-Mayerova, S. and Al-Kaseasbeh, S.(2022). Linear Diophantine Fuzzy Set Theory Applied to BCK/BCI-Algebras. *Mathematics*, 10(12),2138, 2022.
- [37] Murali, V. (2004). Fuzzy points of equivalent fuzzy subsets, *Inform. Sci.*, 158 (2004), 277–288.
- [38] Chew, X., Khaw, K. W., Alnoor, A., Ferasso, M., Al Halbusi, H., & Muhsen, Y. R. (2023). Circular economy of medical waste: novel intelligent medical waste management framework based on extension linear Diophantine fuzzy FDOSM and neural network approach. *Environmental Science and Pollution Research*, 30(21), 60473–60499.
- [39] Pu, P. M. and Y. M. Liu. (1980). Fuzzy topology I: Neighbourhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.*, 76 (1980), 571–599.
- [40] Rani, P., Mishra, A. R., Krishankumar, R., Ravichandran, K. S., & Kar, S. (2021). Multi-criteria food waste treatment method selection using single-valued neutrosophic-CRITIC-MULTIMOORA framework. *Applied Soft Computing*, 111, 107657.
- [41] Razzaq, A., Riaz, M., & Aslam, M. (2024). Efficient picture fuzzy soft CRITIC-CoCoSo framework for supplier selection under uncertainties in Industry 4.0. *AIMS Mathematics*, 9(1), 665–701.
- [42] Riaz, M. and Hashmi, M. R. (2019). Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems. *Journal of Intelligent & Fuzzy Systems*, 37(4), 5417–5439.
- [43] Hayat, L., Naseem, A., & Riaz, M. (2025). Intuitionistic fuzzy credibility implicative ideals in BCK-algebra for efficiency evaluation, supply chain and risk analysis. *Punjab Univ. J. Math.*, 57, (10), 1066–1103.
- [44] Surya, A. N., Vimala, J., Kausar, N., Stevic, Z., & Shah, M. A. (2024). Entropy for q-rung linear Diophantine fuzzy hypersoft set with its application in MADM. *Scientific reports*, 14(1), 1–15.
- [45] S. Gul and A. Aydogdu, (2023). Novel distance and entropy definitions for linear Diophantine fuzzy sets and an extension of TOPSIS (LDF-TOPSIS). *Expert Systems*, 40(1), 1–22.
- [46] Akhtar, R. S. W., & Malik, M. A. (2026). Integrated Entropy-SWARA-ERUNS Algorithm for Optimization and Recycling of Plastic Waste Material using Bipolar Fuzzy Z-Numbers. *Punjab Univ. J. Math.*, 58(03), 259–295.
- [47] Rosenfeld, A. (1971). Fuzzy groups, *J. Math. Anal. Appl.*, 35, 512–517.
- [48] Roszkowska, E., & Wachowicz, T. (2024). Impact of normalization on entropy-based weights in Hellwig's method: a case study on evaluating sustainable development in the education area. *Entropy*, 26(5), 365.
- [49] Sarac, M. G., Dedebaş, T., Hastaoglu, E., Arslan, E.(2023). Influence of using scarlet runner bean flour on the production and physicochemical, textural, and sensorial properties of vegan cakes: WASPAS-SWARA techniques. *Int. J. Gastron. Food Sci.*, 27, 100489.

- [50] Stevic, Z., Pamucar, D., Puska, A., & Chatterjee, P. (2020). Sustainable supplier selection in healthcare industries using a new MCDM method: Measurement of alternatives and ranking according to COMpromise solution (MARCOS). *Computers & industrial engineering*, 140, 106231.
- [51] Wodecki, A. (2020). *Artificial Intelligence in Management: Self-learning and Autonomous Systems as Key Drivers of Value Creation*. Edward Elgar Publishing.
- [52] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.
- [53] Zhan, J., Jun, Y. B., & Davvaz, B. (2009). On $(\in, \in \vee q)$ -fuzzy ideals of BCI-algebras, *Iranian J. Fuzzy Syst.*, 6(1), 81–94.
- [54] Zhou, F., & Chen, T. Y. (2025). A synergetic intuitionistic fuzzy model combining AHP, entropy, and ELECTRE for data fabric solution selection. *Artificial Intelligence Review*, 58(5), 137.
- [55] Zulfqar, M. (2014). On sub-implicative (α, β) -fuzzy ideals of BCH-algebras, *Mathematical Reports*, 1, 141–161.