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Generalization of Special Functions and Explicit Form of Fractional Derivative of Rational Functions

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Abstract. The goal of this paper is to extend the classical fractional derivatives. For this purpose, it is introduced the new extended modified Bessel function and also given an important relation between this new function $I_\upsilon(q;x)$ and the confluent hypergeometric function ${}_1F_1(\alpha,\beta,x)$. Besides, it is used to generalize the hypergeometric, the confluent hypergeometric and the extended beta functions by using the new extended modified Bessel function. Also, the asymptotic formulae and the generating function of the extended modified Bessel function are obtained. The extensions of classical fractional derivatives are defined via extended modified Bessel function and, first time the fractional derivative of rational functions is explicitly given via complex partial fraction decomposition.

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1. Introduction

In the last two decades, several generalizations of the well-known special function have been introduced by different authors. In 1997, Chaudhry [6] have introduced the extension of Euler's beta function as

$$B_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt, \tag{1.1}$$
$$(Re(p)>0, Re(x)>0, Re(y)>0).$$

It is clear that the special case $p=0$ gives the Euler's beta function $B_0(x,y)=B(x,y)$.

Then, the authors in ([17]) extended beta functions and hypergeometric functions as

$$B_p^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\alpha;\beta;\frac{-p}{t(1-t)})dt, \tag{1.2}$$
$$(Re(p)>0, Re(x)>0, Re(y)>0, Re(\alpha)>0, Re(\beta)>0).$$

Lee et al. in ([13]) introduced the more generalized Beta type function as follows:

$$B_p^{(\alpha,\beta;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\alpha;\beta;\frac{-p}{t^m(1-t)^m})dt, \tag{1.3}$$
$$(Re(p)>0, Re(x)>0, Re(y)>0, Re(\alpha)>0, Re(\beta)>0).$$

Consequently, Luo et. al. in ([14]) generalized extended beta function (1.3) (as well as (1.1) and (1.2)) by introducing

$$B_{b;\rho;\lambda}^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\alpha;\beta;\frac{-b}{t^\rho\,(1-t)^\lambda})dt, \tag{1.4}$$
$$(\rho\geq 0, \lambda\geq 0, \min\{Re(\alpha), Re(\beta)\}>0, Re(x)>-Re(\rho\alpha), Re(y)>-Re(\lambda\alpha)).$$

Moreover, Parmar in ([19]) introduced very interesting special function consisting Bessel

function of second kind as

$$B_v(x,y;p)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}K_{v+\frac{1}{2}}(\frac{p}{t\,(1-t)})dt, \tag{1.5}$$
$$(Re(p)>0).$$

To improve the readability of the paper, representations of some notations should be men-

tioned. $J_v(x)$ is the Bessel function of the first kind; Γ is Eulerian Gamma function; $B(x,y)$ is Euler's beta function; ${}_1F_1(b;c;z)$ is confluent hypergeometric function; $D_z^\mu f(z)$ is Riemann-Liouville fractional derivative. The notation $I_v(q;x)$ will newly be used for representing new extended of modified Bessel functions. Besides, the notations $B_{v,q}^{(\mu,\sigma)}(x,y;p)$ and $F_{v,q;p}^{(\mu,\sigma)}(a,b;c;z)$ will be used for representing new extended beta-hypergeometric function and new extended confluent hypergeometric function respectively.

Finally, we refer the papers ([15]) for more properties of extended Gauss hypergeometric and extended confluent hypergeometric functions. We also suggest the papers [7]) and ([8]) for some important extensions of special functions.

In this paper, we introduce extended special functions as generalizations of modified Bessel Functions, Beta functions, hypergeometric functions and confluent hypergeometric functions. We would like to mention an interesting remark from Qadir [21] that explains the importance of generalization of the special functions as "Notice that the generalization of the other special functions has proved even more useful than the separate special functions themselves”. We refer the paper [21] for more details about generalization of the special functions. Based on Qadir’s important explanation, the papers [5], [20] and [2] are suggested as examples of some important applications of special functions.

2. EXTENSION OF SPECIAL FUNCTIONS

In this section, we introduce special functions which will be generalization of the functions (1.1)-(1.5).

2.1. **Extended Modified Bessel Function.** We here introduce new extended of modified Bessel functions as follows.

Definition 2.1. *The function*

$$I_v(q;x) = \frac{\left(\frac{x}{2}\right)^v}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)} \int_{-1}^{1} (1-t^2)^{v-\frac{1}{2}} (1-t)^{q-\frac{1}{2}} \exp\left(-x\left(t-1\right)\right) dt, \tag{2.6}$$

$$(Re(v+q) > 0, Re(v) > \frac{-1}{2}).$$

is called extended modified Bessel function whenever integral exists.

The generalized special function (2.6) introduced in definition 1 provides the most general form of many important special functions in the literature. As a result, it will be shown that the function (2.6) can be applied more easily and effectively in the applications. Consequently, the generalized special function (2.6) will be used in the next subsection to define another generalized special function for extending Hypergeometric and Beta functions.

It is clear that the function (2.6) reduces to Bessel function when $q = \frac{1}{2}$. Explicitly, $I_v(\frac{1}{2};x) = \exp(x) I_v(x)$.

Corollary 2.2. *We have the following integral representation for $I_v(q;x)$:*

$$I_v(q;x) = \frac{\left(\frac{x}{2}\right)^v 2^{2v+q-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)} \int_{0}^{1} t^{v+q-1} \left(1-t\right)^{v-\frac{1}{2}} \exp(2xt) \mathrm{d}t, \tag{2.7}$$

$$(Re(v+q) > 0, Re(v) > \frac{-1}{2}).$$

Proof. By using the transformation $t \to 1-2t$, the statement can be obtain. □

Theorem 2.3. *The extended modified Bessel function $I_v(q;x)$ has power series representation as follows:*

$$I_v(q;x) = \frac{\left(\frac{x}{2}\right)^v}{2^{q+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(2v+2q+2n)}{\Gamma(v+q+n+\frac{1}{2})\,\Gamma(2v+q+n+\frac{1}{2})\,n!} \left(\frac{x}{2}\right)^n. \tag{2.8}$$

Proof. From the representation (2.7), we can write the following relation consisting of the power series of the function $\exp(2xt)$

$$\begin{aligned}\int_0^1 t^{\upsilon+q-1}\,(1-t)^{\upsilon-\frac{1}{2}}\exp(2xt)\,dt &= \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}\int_0^1 t^{\upsilon+q+n-1}\,(1-t)^{\upsilon-\frac{1}{2}}\ dt\\ &= \sum_{n=0}^{\infty}B\left(\upsilon+q+n,\upsilon+\frac{1}{2}\right)\frac{(2x)^n}{n!} \qquad (2.9)\\ &= \sum_{n=0}^{\infty}\frac{\Gamma(\upsilon+q+n)\,\Gamma\left(\upsilon+\frac{1}{2}\right)}{\Gamma\left(2\upsilon+q+n+\frac{1}{2}\right)}\frac{(2x)^n}{n!}.\end{aligned}$$

Using the Legendre's duplication formula, we get

$$\begin{aligned}\Gamma(\upsilon+q+n) &= \Gamma\left(\upsilon+q+n-\frac{1}{2}+\frac{1}{2}\right)\\ &= \frac{\sqrt{\pi}}{2^{2\upsilon+2q+2n-2}}\frac{\Gamma(2\upsilon+2q+2n-1)}{\Gamma\left(\upsilon+q+n-\frac{1}{2}\right)} \qquad (2.10)\\ &= \frac{\sqrt{\pi}}{2^{2\upsilon+2q+2n-1}}\frac{\Gamma(2\upsilon+2q+2n)}{\Gamma\left(\upsilon+q+n+\frac{1}{2}\right)}.\end{aligned}$$

Substituting equations (2.9) and (2.10) into equation (2.7), we obtain

$$I_\upsilon(q;x) = \frac{\left(\frac{x}{2}\right)^\upsilon}{2^{q+\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{\Gamma(2\upsilon+2q+2n)}{\Gamma(\upsilon+q+n+\frac{1}{2})\,\Gamma(2\upsilon+q+n+\frac{1}{2})\,n!}\left(\frac{x}{2}\right)^n. \qquad (2.11)$$

□

Theorem 2.4. *The relation between the extended modified Bessel function $I_\upsilon(q;x)$ and the confluent hypergeometric function ${}_1F_1(\alpha,\beta,x)$ is*

$$I_\upsilon(q;x) = \frac{\left(\frac{x}{2}\right)^\upsilon\ 2^{2\upsilon+q-\frac{1}{2}}\ \Gamma(\upsilon+q)}{\sqrt{\pi}\,\Gamma(2\upsilon+q+\frac{1}{2})}\,{}_1F_1\left(\upsilon+q,2\upsilon+q+\frac{1}{2},2x\right). \qquad (2.12)$$

Proof. Recall that

$$I_\upsilon(q;x) = \frac{\left(\frac{x}{2}\right)^\upsilon 2^{2\upsilon+q-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\upsilon+\frac{1}{2}\right)}\int_0^1 t^{\upsilon+q-1}\,(1-t)^{\upsilon-\frac{1}{2}}\exp(2xt)\,dt. \qquad (2.13)$$

Consider the representation of the function ${}_1F_1(\alpha,\beta,x)$ as

$${}_1F_1(\alpha,\beta,2x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}\,(1-t)^{\beta-\alpha-1}\exp(2xt)\,dt. \qquad (2.14)$$

Hence, the special cases $\alpha=\upsilon+q$ and $\beta=2\upsilon+q+\frac{1}{2}$ give us a new relation between the extended modified Bessel function and the confluent hypergeometric function as follows:

$$I_\upsilon(q;x) = \frac{\left(\frac{x}{2}\right)^\upsilon\ 2^{2\upsilon+q-\frac{3}{2}}\ \Gamma(\upsilon+q)}{\sqrt{\pi}\,\Gamma\left(2\upsilon+q+\frac{1}{2}\right)}\,{}_1F_1\left(\upsilon+q,2\upsilon+q+\frac{1}{2},2x\right) \qquad (2.15)$$

which proves the theorem.

The relation (2.12) provides a wide range of applications of the function (2.7). Since $I_\upsilon(q;x)$ represents both modified Bessel and confluent hypergeometric functions, the special function $I_\upsilon(q;x)$ can effectively used to generalize many special functions.

Example 2.5. *The generalized function(2.6) can be reduced to many well known special function by choosing the values of υ and q together. Some relations can be given as follows*

$$I_{1/2}(1/2;x) = e^x I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} e^x \sinh x.$$

$$I_0(1/2;x) = e^x J_0(x), \quad I_1(1/2;x) = e^x J_1(x), \quad I_0(3/2;x) = e^x (J_0(x) + J_1(x)).$$

$$I_\upsilon(1/2;x) = e^x J_\upsilon(x), \quad Re(\upsilon) > -\frac{1}{2}.$$

$$I_0(q;x) = \frac{2^{q-\frac{1}{2}}}{\sqrt{\pi}}\, {}_1F_1(q, q+\frac{1}{2}, 2x), \quad Re(q) > 0.$$

$$I_{1/2}(q;x) = (-x)^{-q-\frac{1}{2}} \sqrt{\frac{x}{2\pi}} \left(\Gamma(q+\frac{1}{2}) - \Gamma(q+\frac{1}{2}, -2x) \right), \quad Re(q) > -\frac{1}{2}.$$

The next theorem deals with an asymptotic formula of extended modified Bessel function (2.6).

Theorem 2.6. *The special function $I_\upsilon(q;x)$ as $x \to \infty$ approaches to*

$$I_\upsilon(q;x) \sim \frac{2^{q-1} e^{2x}}{\sqrt{\pi x}}.$$

Proof. Consider the integral representation of the $I_\upsilon(q;x)$ as

$$I_\upsilon(q;x) = \frac{\left(\frac{x}{2}\right)^\upsilon 2^{2\upsilon+q-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\upsilon+\frac{1}{2}\right)} \int_0^1 t^{\upsilon+q-1} (1-t)^{\upsilon-\frac{1}{2}} \exp(2xt)\, dt.$$

Let $I = \int_0^1 t^{\upsilon+q-1} (1-t)^{\upsilon-\frac{1}{2}} \exp(2xt)\, dt$. By using the substitution $t = 1 - \frac{u}{u+x}$, the integral I will be

$$\begin{aligned} I &= \int_0^\infty \frac{x^{\upsilon+q-1}}{(u+x)^{\upsilon+q-1}} \left(\frac{u}{u+x}\right)^{\upsilon-\frac{1}{2}} \exp(2x(1-\frac{u}{u+x})) \left(\frac{x}{(u+x)^2}\right) du \\ &= \exp(2x) \cdot \int_0^\infty \frac{u^{\upsilon-\frac{1}{2}} x^{\upsilon+q}}{(u+x)^{2\upsilon+q+\frac{1}{2}}} \exp(\frac{-2xu}{u+x})\, du \\ &= \frac{\exp(2x)}{x^{\upsilon+\frac{1}{2}}} \int_0^\infty u^{\upsilon-\frac{1}{2}} \exp(-2u) du \end{aligned}$$

where $\frac{u}{x} \to 0$ and $\frac{2xu}{u-x} \to -2u$ for large number x. Since $\int_0^\infty u^{\upsilon-\frac{1}{2}} \exp(-2u) du = 2^{-\upsilon-\frac{1}{2}} \Gamma\left(\upsilon+\frac{1}{2}\right)$, we have

$$I_\upsilon(q;x) = \frac{\left(\frac{x}{2}\right)^\upsilon 2^{2\upsilon+q-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\upsilon+\frac{1}{2}\right)} \frac{\exp(2x) 2^{-\upsilon-\frac{1}{2}} \Gamma\left(\upsilon+\frac{1}{2}\right)}{x^{\upsilon+\frac{1}{2}}},$$

which proves the theorem. $\square$

Remark 2.7. *If we consider the relation*

$$I_v(\frac{1}{2};x) = e^x I_v(x),$$

the corresponding asymptotic formula of modified Bessel function of first kind can easily be derived as

$$I_v(x) \to \frac{e^x}{\sqrt{2\pi x}},\ x \to \infty.$$

Theorem 2.8. *For $\left|\frac{2}{z}\right| < 1$, the following generating function holds true:*

$$\sum_{n=-\infty}^{\infty} I_{n+\frac{1}{2}}(-n+\frac{1}{2};x)z^n = \sqrt{\frac{2}{\pi x}}\left(\frac{ze^{xz}}{z-2}\right).$$

Proof. By using Legendre's duplication formula, the series representation of $I_\upsilon(q;x)$ can be given as

$$I_\upsilon(q;x) = \sum_{n=0}^{\infty} \frac{2^{\upsilon+q+n-\frac{1}{2}}\Gamma(\upsilon+q+n)}{\sqrt{\pi}\Gamma(2\upsilon+q+n+\frac{1}{2})\, n!} x^{n+\upsilon}.$$

Consequently,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} I_{n+\frac{1}{2}}(-n+\frac{1}{2};x)z^n &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{\infty} \frac{\Gamma(k+1)\,2^{k+\frac{1}{2}}}{\sqrt{\pi}k!\Gamma(n+k+2)} x^{n+k+\frac{1}{2}}\right) z^n \\
&= \sum_{n=-\infty}^{\infty}\left(\sum_{m=n+k+1}^{\infty} \frac{2^{k+\frac{1}{2}}}{\sqrt{\pi}\Gamma(m+1)} x^{m-\frac{1}{2}}\right) z^n \\
&= \sqrt{\frac{2}{\pi x}}\left(\sum_{m=0}^{\infty}\frac{(xz)^m}{m!}.\sum_{k=0}^{\infty} 2^k z^{-k}\right).
\end{aligned}$$

Since $\left|\frac{2}{z}\right| < 1$, the geometric series

$$\sum_{k=0}^{\infty} 2^k z^{-k} = \frac{1}{1-\frac{2}{z}}.$$

Hence,

$$\sum_{n=-\infty}^{\infty} I_{n+\frac{1}{2}}(-n+\frac{1}{2};x)z^n = \sqrt{\frac{2}{\pi x}} e^{xz}\frac{z}{z-2}.$$

□

Next, we attempt to find generating functions involving the special function $I_v(q;x)$,mainly motivated by the paper of Agarwal et al. ([4]).

Theorem 2.9. *For $\upsilon, q \in \mathbb{C}$, the following generating function holds true:*

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} I_{\upsilon-k}(q+2k;x)t^k = \left(1-\frac{2t}{x}\right)^{-\upsilon-q} I_\upsilon(q;\frac{x^2}{x-2t}). \tag{2.16}$$

Proof. By using Legendre's duplication formula, the series representation of $I_v(q;x)$ can be given as

$$I_v(q;x)=\sum_{n=0}^{\infty}\frac{2^{v+q+n-\frac{1}{2}}\Gamma(v+q+n)}{\sqrt{\pi}\Gamma(2v+q+n+\frac{1}{2})\,n!}x^{n+v}.$$

Consequently, by a little simplifications,

$$\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(k+1\right)}I_{v-k}(q+2k;x)t^k=\sum_{n=0}^{\infty}\frac{2^{v+q+n-\frac{1}{2}}x^{n+v}}{\sqrt{\pi}\Gamma(2v+q+n+\frac{1}{2})\,n!}\cdot\sum_{k=0}^{\infty}\frac{\Gamma(v+q+n+k)2^k t^k}{k!x^k}. \tag{2.17}$$

Since $\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+k)t^k}{k!\Gamma(\lambda)}=(1-t)^{\lambda}$ for $\lambda\in\mathbb{C}$,

$$\sum_{k=0}^{\infty}\frac{\Gamma(v+q+n+k)2^k t^k}{k!x^k}=\Gamma(v+q+n)\sum_{k=0}^{\infty}\frac{\Gamma(v+q+n+k)\left(\frac{2t}{x}\right)^k}{\Gamma(v+q+n)\;\;k!}.$$

Therefore the infinite sum (2.17) becomes

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(k+1\right)}I_{v-k}(q+2k;x)t^k &= \sum_{n=0}^{\infty}\frac{2^{v+q+n-\frac{1}{2}}x^{n+v}\Gamma(v+q+n)}{\sqrt{\pi}\Gamma(2v+q+n+\frac{1}{2})\,n!}\cdot\left[\left(1-\frac{2t}{x}\right)^{-(v+q+n)}\right]\\ &= \sum_{n=0}^{\infty}\frac{2^{v+q+n-\frac{1}{2}}x^{v}\left(\frac{x}{1-\frac{2t}{x}}\right)^n\Gamma(v+q+n)}{\sqrt{\pi}\Gamma(2v+q+n+\frac{1}{2})\,n!}\cdot\left[\left(1-\frac{2t}{x}\right)^{-(v+q)}\right],\end{aligned}$$

which gives the generating function given in (2.16). □

We aim to continue to generate a new generating function involving confluent hypergeometric function ${}_1F_1(\alpha,\beta,x)$ via generating function(2.16).

Theorem 2.10. *For $\alpha,\beta\in\mathbb{C}$, the following generating function holds true:*

$$\sum_{k=0}^{\infty}\frac{\Gamma\left(k+\alpha\right)}{\Gamma\left(k+1\right)}{}_1F_1\left(\alpha+k,\beta,x\right)z^k=\left(1-z\right)^{-\alpha}\Gamma\left(\alpha\right){}_1F_1\left(\alpha,\beta,\frac{x}{1-z}\right). \tag{2.18}$$

Proof. Use the generating function (2.16) together with relation (2.12) and set $\alpha\to v+q,\beta\to 2v+q+\frac{1}{2},z\to\frac{2t}{x}$. □

For one interesting reference from generating functions of special functions, we refer the paper Cohl. et al. ([9]).

2.2. **Extended Hypergeometric, Confluent Hypergeometric and Beta Functions via extended modified Bessel Function.** In this subsection, newly introduced function (2.6) will be used for introducing another extended special function to generalize the hypergeometric, confluent hypergeometric and extended beta functions. The following definition consists a new and generalized special function based on the function (2.6).

Definition 2.11. *The extended beta-hypergeometric function*

$$B_{v,q}^{(\mu,\sigma)}(x,y;p)=\sqrt{\frac{2}{\pi}}\int_0^1 t^x(1-t)^y I_{v+\frac{1}{2}}(q;\frac{-p}{t^\mu\,(1-t)^\sigma})dt, \tag{2.19}$$
$$(Re(p)>0,\ \mu,\sigma\geq 0,\min\{Re\left(v+q+\frac{1}{2}\right),Re\left(2v+q+\frac{3}{2}\right)\}>0,$$
$$Re(x+\mu q)>-1,Re(y+\sigma q)>-1).$$

is defined.

Remark 2.12. *The necessary conditions for existence of integral given in (2.19) can also be derived by using the relation (2.12) and the paper (*[14]*, pp. 633, theorem 2.1).*

It is clear that the new extension (2.19) reduces to many defined special functions as

Case 2.13. *function (1.1) in the paper when* $v=0,q=\frac{1}{2}$ *and* $\mu,\sigma=1$*, precisely,* $B_{0,\frac{1}{2}}^{(1,1)}(x,y;p)=B_p(x,y)$,

Case 2.14. *function (1.3) when* $v=0,q=\frac{1}{2}$ *and* $\mu,\sigma=m$*,precisely,* $B_{0,\frac{1}{2}}^{(m,m)}(x,y;p)=B_p^{(\alpha,\beta;m)}(x,y)$,

Case 2.15. *function (1.4) when* $q=2\alpha-\beta+\frac{1}{2},v=\beta-\alpha-\frac{1}{2}$ *and* $\mu=\rho;\sigma=\lambda$,

Case 2.16. *function (1.5) in the paper by using the relation* $K_v(x)=\frac{\pi\exp(-x)}{2\sin v\pi}\left(I_{-v}(\frac{1}{2};x)-I_v(\frac{1}{2};x)\right)$ *where* $v\notin\mathbb{Z}$.

Consequently, we use the function (2.19) to extend the hypergeometric functions and beta functions as follows:

$$F_{v,q;p}^{(\mu,\sigma)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{v,q}^{(\mu,\sigma)}(b+n,c-b;p)}{B(b,c-b)}\frac{z^n}{n!}, \tag{2.20}$$
$$(Re(p)>0,\ \ |z|<1,\ \ \min\{Re\left(v+q+\frac{1}{2}\right),Re\left(2v+q+\frac{3}{2}\right)\},$$
$$Re(\mu),Re(\sigma)\geq 0,Re(c)>Re(b)>0,Re(a)>0),$$

and

$$\Phi_{v,q;p}^{(\mu,\sigma)}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_{v,q}^{(\mu,\sigma))}(b+n,c-b;p)}{B(b,c-b)}\frac{z^n}{n!}, \tag{2.21}$$
$$(Re(p)>0,\ \ \min\{Re\left(v+q+\frac{1}{2}\right),Re\left(2v+q+\frac{3}{2}\right)\},$$
$$Re(\mu),Re(\sigma)\geq 0,Re(c)>Re(b)>0,Re(a)>0).$$

Theorem 2.17. *The special functions (2.20) and (2.21) , respectively, has the following integral representation*

$$\begin{aligned}
F_{v,q;p}^{(\mu,\sigma)}(a,b;c;z) &= \frac{\sqrt{\frac{2}{\pi}}}{B(b,c-b)}\int_0^1 t^b(1-t)^{c-b}(1-zt)^{-a}I_{v+\frac{1}{2}}(q;\frac{-p}{t^\mu(1-t)^\sigma})dt,\\
(Re(p) &> 0,\ |\arg(1-z)|<\pi,\ Re(\mu),Re(\sigma)\geq 0, Re(c)>Re(b)>0,\\
Re(v+q) &> 0, Re(v)>\frac{-1}{2}, Re(a)>0),
\end{aligned}$$

and

$$\Phi_{v,q;p}^{(\mu,\sigma)}(b;c;z) = \frac{\sqrt{\frac{2}{\pi}}}{B(b,c-b)}\int_0^1 t^b(1-t)^{c-b}e^{zt}I_{v+\frac{1}{2}}(q;\frac{-p}{t^\mu(1-t)^\sigma})dt, \tag{2.22}$$
$$(Re(p)>0,\ Re(\mu),Re(\sigma)\geq 0, Re(c)>Re(b)>0,$$
$$Re(v+q)>0, Re(v)>\frac{-1}{2}, Re(a)>0).$$

Proof. Substituting the function (2.19) with $x\to b+n, y\to c-b$ into function (2.20), we have after interchanging the order of summation and integration which is guaranteed

$$\begin{aligned}
F_{v,q;p}^{(\mu,\sigma)}(a,b;c;z) &= \frac{\sqrt{\frac{2}{\pi}}}{B(b,c-b)}\int_0^1 t^b(1-t)^{c-b}I_{v+\frac{1}{2}}(q;\frac{-p}{t^\mu(1-t)^\sigma})\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}dt\\
&= \frac{\sqrt{\frac{2}{\pi}}}{B(b,c-b)}\int_0^1 t^b(1-t)^{c-b}(1-zt)^{-a}I_{v+\frac{1}{2}}(q;\frac{-p}{t^\mu(1-t)^\sigma})dt,
\end{aligned}$$

where $(1-zt)^{-a} = \sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}, \forall\ |zt|<1$. Similarly, from the definitions of the functions (2.19) and (2.21), we can derive the integral representation (2.22) with $\exp(zt) = \sum_{n=0}^{\infty}\frac{(zt)^n}{n!}$. □

Also, it can be easily seen that the new extensions (2.20) and (2.22) reduce to the following special functions as

Case 2.18. *extended Gauss hypergeometric function and extended confluent hypergeometric function in Lee et al. (*[13]*, pp. 189, Equations. (1.11) and (1.12)) respectively when* $\upsilon = 0$, $q=\frac{1}{2}$ *and* $\mu=\sigma=1$. *The extended Gauss hypergeometric function is defined as*

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\exp\left[\frac{-p}{t(1-t)}\right]dt$$

and extended confluent hypergeometric function

$$\Phi_p(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp\left[zt-\frac{p}{t(1-t)}\right]dt.$$

Case 2.19. *new generalized beta function in Özergin et al. (*[17]*; pp. 4607, Equations. (11)) when $q = 2\alpha - \beta + \frac{1}{2}, v = \beta - \alpha - \frac{1}{2}$ and $\mu = \sigma = 1$. Precisely, the new generalized beta function is defined as*

$$ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp[zt]\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt. $$

Remark 2.20. *An interesting generalization of extension of gamma function and generalized gamma function given together in the paper (*[17]*) can be considered as*

$$ \begin{aligned} \Gamma_p^{v,q}(x) \quad &: \quad = \int_0^1 t^{x-1} I_{v+\frac{1}{2}}(q; -(t+\frac{p}{t})dt, \\ (Re(p) \quad &> \quad 0, Re(x) > 0, Re(v+q) > -\frac{1}{2}, Re(2v+q) > -\frac{3}{2}). \end{aligned} $$

2.3. **The Mellin and Laplace Transforms.** In this subsection, we derive the Mellin and Laplace transforms into extended modified Bessel and extended beta-hypergeometric functions. The necessary conditions for their existences can be followed through existences of the special functions appearing in their respective formulae.

Theorem 2.21. *The Mellin transform of*

$$ \begin{aligned} M[I_v(q;x);s] \quad &: \quad = \int_0^\infty x^{s-1} I_v(q;x)dx \\ &: \quad = \frac{(-1)^{v+s-1}\, 2^{q-s-\frac{1}{2}}\Gamma(v+s)\Gamma(q-s)}{\sqrt{\pi}\Gamma\left(q+v-s+\frac{1}{2}\right)} \end{aligned} $$

whenever integral exists.

Proof. Assume that the Mellin transform of $I_v(q;x)$ exists. Then,

$$ \begin{aligned} M[I_v(q;x);s] \quad &: \quad = \int_0^\infty x^{s-1} I_v(q;x)dx \\ &: \quad = \frac{2^{v+q-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)} \int_0^\infty x^{s-1} \int_0^1 x^v t^{v+q-1}(1-t)^{v-\frac{1}{2}} \exp(2xt)\, dtdx. \end{aligned} $$

By using uniform convergency of the integration with substitutions $\sigma = -2xt$ and $\lambda = t$, we have

$$ \int_0^\infty x^{s-1} I_v(q;x)dx = \frac{2^{q-s-\frac{1}{2}}(-1)^{v+s-1}}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)} \int_0^1 \lambda^{q-s-1}(1-t)^{v-\frac{1}{2}} d\lambda \int_0^\infty \sigma^{v+s-1}\exp(-\sigma)\, d\sigma. $$

Hence,

$$ \begin{aligned} \int_0^\infty x^{s-1} I_v(q;x)dx \quad &= \quad \frac{2^{q-s-\frac{1}{2}}(-1)^{v+s-1}\Gamma(v+s)}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)} B\left(q-s, v+\frac{1}{2}\right) \\ &= \quad \frac{2^{q-s-\frac{1}{2}}(-1)^{v+s-1}\Gamma(v+s)\Gamma(q-s)}{\sqrt{\pi}\Gamma\left(q+v-s+\frac{1}{2}\right)}. \end{aligned} $$

□

In the paper ([3],pp. 410, equation (3.3)), the Mellin transform of the hypergeometric function ${}_1F_1\left(\alpha,\beta,-b\right)$ is used as

$$\int_0^\infty b^{s-1}\ {}_1F_1\left(\alpha,\beta,-b\right)db=\frac{\Gamma\left(\alpha-s\right)\Gamma\left(\beta\right)\Gamma\left(s\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-s\right)}. \tag{2.23}$$

By using Mellin transform of $I_v(q;x)$, the Mellin transform of ${}_1F_1$ can easily be showed by the following corollary.

Corollary 2.22. *From the Mellin transform of $I_v(q;x)$, we can easily derive the Mellin transform of ${}_1F_1\left(\alpha,\beta,-p\right)$ as*

$$\int_0^\infty p^{s-1}\ {}_1F_1\left(\alpha,\beta,-p\right)dp=\frac{\Gamma\left(\alpha-s\right)\Gamma\left(\beta\right)\Gamma\left(s\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-s\right)}.$$

Proof. Considering the Mellin transform of $I_v(q;x)$ with relation (2.12), we have

$$\int_0^\infty p^{s-1}\left[\frac{(p)^v\ 2^{v+q-\frac{1}{2}}\ \Gamma(v+q)}{\sqrt{\pi}\ \Gamma(2v+q+\frac{1}{2})}{}_1F_1\left(v+q,2v+q+\frac{1}{2},2p\right)\right]dp=\frac{2^{q-s-\frac{1}{2}}\left(-1\right)^{v+s-1}\Gamma\left(v+s\right)\Gamma\left(q-s\right)}{\sqrt{\pi}\Gamma\left(q+v-s+\frac{1}{2}\right)}.$$

If we consider the substitutions $p\to-\frac{p}{2},\alpha=v+q,\beta=2v+q+\frac{1}{2}$ for above integration, we have

$$\int_0^\infty\frac{p^{(s+v)-1}\Gamma\left(\alpha\right)2^{q-s-\frac{1}{2}}\left(-1\right)^{v+s-1}}{\sqrt{\pi}\Gamma\left(\beta\right)}\ {}_1F_1\left(\alpha,\beta,-p\right)dp=\frac{2^{q-s-\frac{1}{2}}\left(-1\right)^{v+s-1}\Gamma\left(v+s\right)\Gamma\left(q-s\right)}{\sqrt{\pi}\Gamma\left(q+v-s+\frac{1}{2}\right)}$$

which gives Mellin transform (2.23). $\square$

Theorem 2.23. *The Mellin transform of*

$$\begin{aligned}M\left[B_{v,q}^{(\mu,\sigma)}(x,y;p);s\right]\ &:=\int_0^\infty p^{s-1}B_{v,q}^{(\mu,\sigma)}(x,y;p)dp\\ &:=\frac{2^{q-s}}{\pi}\frac{\left(-1\right)^v\Gamma\left(v+s\right)\Gamma\left(q-s\right)}{\Gamma\left(q+v-s+\frac{1}{2}\right)}B\left(x+\mu s+1,y+\sigma s+1\right)\end{aligned}$$

whenever integral exists.

Proof. In the light of Mellin transform of $I_v(q;x)$, the Mellin transform of $B_{v,q}^{(\mu,\sigma)}(x,y;p)$ can be represented as

$$M\left[B_{v,q}^{(\mu,\sigma)}(x,y;p);s\right]=\sqrt{\frac{2}{\pi}}\int_0^1 t^x\left(1-t\right)^y\int_0^\infty p^{s-1}\ I_{v+\frac{1}{2}}(q;-\frac{p}{t^\mu\left(1-t\right)^\sigma})dpdt.$$

Let $p=\Theta\left(t^\mu\left(1-t\right)^\sigma\right)$ and $\lambda=t\ \left(dp=d\Theta\left(t^\mu\left(1-t\right)^\sigma\right)\text{ and }d\lambda=dt\right)$. Then,

$$\begin{aligned}M\left[B_{v,q}^{(\mu,\sigma)}(x,y;p);s\right]&=\sqrt{\frac{2}{\pi}}\int_0^1\lambda^{x+\mu s}\left(1-\lambda\right)^{y+\sigma s}d\lambda\left[\frac{2^{q-s-\frac{1}{2}}\left(-1\right)^v\Gamma\left(v+s\right)\Gamma\left(q-s\right)}{\sqrt{\pi}\Gamma\left(q+v-s+\frac{1}{2}\right)}\right]\\ &=\sqrt{\frac{2}{\pi}}\left[\frac{2^{q-s-\frac{1}{2}}\left(-1\right)^v\Gamma\left(v+s\right)\Gamma\left(q-s\right)}{\sqrt{\pi}\Gamma\left(q+v-s+\frac{1}{2}\right)}\right]\int_0^1\lambda^{x+\mu s}\left(1-\lambda\right)^{y+\sigma s}d\lambda\\ &=\frac{\sqrt{2}}{\pi}\frac{2^{q-s-\frac{1}{2}}\left(-1\right)^v\Gamma\left(v+s\right)\Gamma\left(q-s\right)}{\Gamma\left(q+v-s+\frac{1}{2}\right)}B\left(x+\mu s+1,y+\sigma s+1\right).\end{aligned}$$

□

Remark 2.24. *Since the special function (2.19) is extensions of some recently introduced special functions, the Mellin transforms of these covered functions can be derived.*

Theorem 2.25. *The Laplace transform, if exists, of extended modified Bessel function is*

$$L\{I_v(q;x);s\} := \int_0^\infty e^{-sx} I_v(q;x)dx \tag{2.24}$$
$$:= \frac{2^{q+v-\frac{1}{2}}\Gamma(v+q)\Gamma(v+1)}{\sqrt{\pi}s^{v+1}\Gamma\left(q+2v+\frac{1}{2}\right)} F\left(v+1, v+q; 2v+q+\frac{1}{2}; \frac{2}{s}\right).$$

where $F(a,b;c;z)$ is Gauss hypergeometric function (see ([15]*,pp.11, equation (2)).*

Proof. Consider the Laplace transform of $I_v(q;x)$

$$L\{I_v(q;x);s\} = \int_0^\infty e^{-sx} I_v(q;x)dx = \int_0^\infty e^{-sx} \frac{\left(\frac{x}{2}\right)^v 2^{2v+q-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)} \int_0^1 t^{v+q-1}(1-t)^{v-\frac{1}{2}} \exp(2xt)\, dtdx.$$

By using uniform convergency of the integration, we have

$$\int_0^\infty e^{-sx} I_v(q;x)dx = \frac{2^{v+q-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)} \int_0^1 t^{v+q-1}(1-t)^{v-\frac{1}{2}}\, dt \int_0^\infty (x)^v e^{x(-s+2t)}\, dx.(s>2t).$$

By using the substitutions $x \to \frac{\sigma}{s-2t}$ and $\lambda = t$, we have

$$\begin{aligned}\int_0^\infty e^{-sx} I_v(q;x)dx &= \frac{2^{v+q-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)} \int_0^1 \lambda^{v+q-1}(1-\lambda)^{v-\frac{1}{2}}(s-2\lambda)^{-v-1} d\lambda \int_0^\infty (\sigma)^v e^{-\sigma}\, d\sigma \\ &= \frac{2^{v+q-\frac{1}{2}}\Gamma(v+1)}{\sqrt{\pi}\, s^{v+1}\Gamma\left(v+\frac{1}{2}\right)} \int_0^1 \lambda^{v+q-1}(1-\lambda)^{v-\frac{1}{2}}\left(1-\frac{2}{s}\lambda\right)^{-v-1} d\lambda.\end{aligned}$$

Since

$$F(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-z\lambda)^{-a}\, dt \text{ with } |\arg(1-z)| < \pi,$$

then

$$\begin{aligned}\int_0^\infty e^{-sx} I_v(q;x)dx &= \frac{2^{v+q-\frac{1}{2}}\Gamma(v+1)}{\sqrt{\pi}s^{v+1}\,\Gamma\left(v+\frac{1}{2}\right)} F(v+1, v+q; 2v+q+\frac{1}{2}; \frac{2}{s}) \cdot B(v+q, v+\frac{1}{2}) \\ &= \frac{2^{v+q-\frac{1}{2}}\Gamma(v+1)}{\sqrt{\pi}\, s^{v+1}\Gamma\left(v+\frac{1}{2}\right)} F(v+1, v+q; 2v+q+\frac{1}{2}; \frac{2}{s}) \frac{\Gamma(v+q)\Gamma\left(v+\frac{1}{2}\right)}{\Gamma\left(2v+q+\frac{1}{2}\right)},\end{aligned}$$

which gives the formula (2.24). □

As an particular case, the Laplace transform of $I_v(q;x)$ (2.24) when $v=0$ and $q=\frac{1}{2}$ gives

$$L\{I_0(\frac{1}{2};x);s\} = \frac{1}{\sqrt{s^2-2s}}.$$

Consequently, Laplace transform of modified Bessel function of the first kind for $v = 0$ via Laplace transform of $I_v(q; x)$ can easily be obtained

$$L\{I_0\left(x\right);s\} = L\{e^{-x}I_0(\frac{1}{2};x);s\} = \frac{1}{\sqrt{\left(s+1\right)^2 - 2\left(s+1\right)}} = \frac{1}{\sqrt{s^2-1}}.$$

Corollary 2.26. *The Laplace transform of modified Bessel function is*

$$L\{I_v\left(x\right);s\} = \frac{1}{2^v\left(s+1\right)^{v+1}\Gamma\left(q+2v+\frac{1}{2}\right)}F\left(v+1, v+\frac{1}{2}; 2v+1; \frac{2}{s+1}\right).$$

Proof. Assume that Laplace transform of $I_v(q; x)$ exists and equals to $F\left(s\right)$. Consequently,

$$L\{I_v\left(x\right);s\} = L\{e^{-x}I_v\left(\frac{1}{2},x\right);s\} = F\left(s+1\right).$$

By using formula (**??**) together with Legendre's duplication formula, we derive the corresponding formula. □

3. 3. GENERALIZATION OF FRACTIONAL DERIVATIVES

3.1. **Extended Fractional Derivative via Extended Modified Bessel Function.** In this subsection, we introduce an interesting extended fractional derivative which can be generalization of a large set of fractional derivatives. Let $z > 0$ then the new extension of Riemann-Liouville fractional derivative ${}_{\mu,\sigma}D^{\alpha,\eta,p}_{v,q;z}f(z)$ is defined as follows:

$$ {}_{\mu,\sigma}D^{\alpha,\eta,p}_{v,q;z}\left(f(z)\right) := \frac{\sqrt{\frac{2}{\pi}}}{\Gamma(\alpha)}\int_0^z f(t)(z-t)^{\alpha-1}t^{\eta}I_{v+\frac{1}{2}}(q;\frac{-pz^{\mu+\sigma}}{t^{\mu}\left(z-t\right)^{\sigma}})dt, \tag{3.25}$$

$(\min\{Re(\alpha) > 0, Re(p), Re(\eta) > 0, Re(v+q+\frac{1}{2}) > 0, Re(2v+q+\frac{3}{2}) > 0),\ \mu,\sigma \geq 0),$

where $n-1 < Re(\alpha) < n\ (n = 1, 2, 3, ...)$.

Now, we start with the extended fractional derivative of elementary function $f(z) = z^{\lambda}$.

Corollary 3.1. *Let* $Re(\eta + \lambda + \mu q) > -1$. *Then*

$$ {}_{\mu,\sigma}D^{\alpha,\eta,p}_{v,q;z}(z^{\lambda}) = \frac{z^{\eta+\lambda+\alpha}}{\Gamma(\alpha)}B^{(\mu,\sigma)}_{v,q}(\eta+\lambda, \alpha-1; p)$$

whenever the function $B^{(\mu,\sigma)}_{v,q}$ *exists.*

Proof. Consider the fractional derivative (3.25), we get

$$ {}_{\mu,\sigma}D^{\alpha,\eta,p}_{v,q;z}(z^{\lambda}) = \frac{\sqrt{\frac{2}{\pi}}}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}t^{\eta+\lambda}I_{v+\frac{1}{2}}(q;\frac{-pz^{\mu+\sigma}}{t^{\mu}\left(z-t\right)^{\sigma}})dt.$$

Taking $t = zu$, after a little simplification, gives

$$\begin{aligned} {}_{\mu,\sigma}D^{\alpha,\eta,p}_{v,q;z}(z^{\lambda}) &= \frac{\sqrt{\frac{2}{\pi}}z^{\eta+\lambda+\alpha}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{\eta+\lambda} I_{v+\frac{1}{2}}(q; \frac{-p}{u^{\mu}\left(1-u\right)^{\sigma}})du \\ &= \frac{z^{\eta+\lambda+\alpha}}{\Gamma(\alpha)} B^{(\mu,\sigma)}_{v,q}(\eta+\lambda, \alpha-1; p). \end{aligned}$$

□

Corollary 3.2. *Let $\xi \neq 0$ and $\xi \in \mathbb{C}$. Then*

$$ {}_{\mu,\sigma}D^{\alpha,\eta,p}_{v,q;z}((z-\xi)^r) := \frac{(-\xi)^r B(\eta, \alpha-1) z^{\eta+\alpha}}{\Gamma(\alpha)} F^{(\mu,\sigma)}_{v,q;p}(-r, \eta; \eta+\alpha-1; \frac{z}{\xi}), \quad (3.26)$$

whenever the function $F^{(\mu,\sigma)}_{v,q;p}$ exists.

Proof. Consider the fractional derivative (3.25), we get

$$\begin{aligned} &D^{\mu,\eta,p}_z((z-\xi)^r) \\ &= \frac{\sqrt{\frac{2}{\pi}}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1}(t-\xi)^r t^{\eta} I_{v+\frac{1}{2}}(q; \frac{-pz^{\mu+\sigma}}{t^{\mu}\left(z-t\right)^{\sigma}})dt \\ &= \frac{\sqrt{\frac{2}{\pi}}(-\xi)^r z^{\eta+\alpha}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1}(1-\frac{z}{\xi}u)^r u^{\eta} I_{v+\frac{1}{2}}(q; \frac{-p}{u^{\mu}\left(1-u\right)^{\sigma}})du \ (t = uz) \\ &= \frac{(-\xi)^r B(\eta, \alpha-1) z^{\eta+\alpha}}{\Gamma(\alpha)} F^{(\mu,\sigma)}_{v,q;p}(-r, \eta; \eta+\alpha-1; \frac{z}{\xi}). \end{aligned}$$

□

The special case of new extension (3.25) with $p \to 2p, \mu = \sigma = 1; v = 0, q = \frac{1}{2}; \alpha = -\mu - \frac{1}{2}, \eta = \frac{-1}{2}$ reduces the generalized Riemann-Liouville fractional derivative which is defined by Özarslan et al ([16]) as

$$\begin{aligned} &D^{\mu,\eta,p}_z f(z) := \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1} exp(\frac{-pz^2}{t(z-t)})dt, \\ &(Re(\mu) < 0, Re(p) > 0). \end{aligned}$$

Also, the particular case $\mu = \sigma = 0; v = 0, q = \frac{1}{2}; \alpha = -\mu, \eta = 0$ for extended fractional derivative (3.25) reduces the Riemann-Liouville fractional derivative

$$\begin{aligned} &D^{\mu}_z f(z) := \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1}dt, \\ &(Re(\mu) < 0). \end{aligned}$$

It is also important to note that the extended fractional derivative (3.25) reduces to extended fractional derivative

$$I_z^{\mu,b}\{f(z)\} := \frac{1}{\Gamma(\mu)}\int_0^z f(t)(z-t)^{\mu-1}{}_1F_1\left(\gamma,\beta,-\frac{bz^{\rho+\lambda}}{t^\rho(z-t)^\lambda}\right)dt,$$
$$(\rho>0,\lambda>0,\min\{Re(\gamma),Re(\beta),Re(\mu),Re(b)\}>0),$$

defined in ([14],pp.647) when $p\to\frac{b}{2},\mu=\rho,\sigma=\lambda;v=0,q=2\gamma-\beta+\frac{1}{2},v=\beta-\gamma-\frac{1}{2};\alpha=\mu+\beta\lambda-\gamma\lambda-\frac{\lambda}{2},\eta=\beta\rho-\gamma\rho-\frac{\rho}{2}$.

Finally, Katugampola in the paper ([12]*)* introduced a new fractional integral operator given by,

$$\left({}^\rho I_{a+}^\alpha f\right)(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{\tau^{\rho-1}f(\tau)}{(x^\rho-\tau^\rho)^{1-\alpha}}d\tau, \tag{3.27}$$

which is generalization of the Riemann-Liouville and the Hadamard fractional integrals. The extended fractional derivative (3.25) reduces to the fractional derivative (3.27) when $z\to x^\rho-a^\rho, f(z)\to f\left((z+a^\rho)^{\frac{1}{\rho}}\right);\alpha\to\alpha,\eta=0,v=0,q=\frac{1}{2}$ and $\mu=\sigma=0$.

In the light of these reductions, we can easily understand that the extended fractional derivative (3.25) is generalization of many defined fractional derivatives.

3.2. **Fractional Derivative of Rational Functions.** In this subsection, we will derive the extended fractional derivative of arbitrary rational functions. Consequently, the general representation of fractional derivatives of many defined fractional derivatives of arbitrary rational functions can firstly be derived.

Assume that $P(z)$ and $Q(z)$ are polynomials such that $\deg(P)<\deg(Q)$. In this case, the real partial fraction decomposition of the rational function $\frac{P(z)}{Q(z)}$ can be represented as

$$\frac{P(z)}{Q(z)}=\sum_{i=1}^{p}\sum_{r=1}^{k_i}\frac{a_{ir}}{(z-z_i)^r}+\sum_{j=1}^{q}\sum_{s=1}^{l_j}\frac{\beta_{js}z+\gamma_{js}}{\left(z^2-2Re(z_j)z+|z_j|^2\right)^s} \tag{3.28}$$

where $a_{ir},\beta_{js},\gamma_{js}\in\mathbb{R}$. In the representation (3.28), the inverse of quadratic functions can not be worked well in many calculations. Because of this quadratic functions of the denominators, for example, we can not derive the fractional derivatives of rational functions. In this paper, we will use complex partial fraction decomposition method together with formula (3.26) to derive extended fractional derivatives of rational functions. In the paper ([18]), the complex partial fraction decomposition of arbitrary rational function was derived by the following theorem:

Theorem 3.3. *Let $x_1,\dots,x_p$ be pairwise different real numbers and $z_1,\dots,z_q\in\mathbb{C}\backslash\mathbb{R}$ be also pairwise different. If $P(x)$ is a polynomial with real coefficients whose degree satisfies the inequality* $\deg\big(P(x)\big)<p+2(l_1+\cdots+l_q)$*, then there exists $a_{ir},\beta_{js},\gamma_{js}\in\mathbb{R}$ and*

$b_{js} \in \mathbb{C}$ *such that*

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \sum_{i=1}^{p}\sum_{r=1}^{k_i}\frac{a_{ir}}{(x-x_i)^r}+\sum_{j=1}^{q}\sum_{s=1}^{l_j}\frac{\beta_{js}x+\gamma_{js}}{\left(x^2-2Re(z_j)x+|z_j|^2\right)^s}\\ &= \sum_{i=1}^{p}\sum_{r=1}^{k_i}\frac{a_{ir}}{(x-x_i)^r}+\sum_{j=1}^{q}\sum_{s=1}^{l_j}\Big(\frac{b_{js}}{\left(x-z_j\right)^s}+\frac{\bar{b}_{js}}{\left(x-\bar{z}_j\right)^s}\Big),\end{aligned}$$

where

$$Q(x)=(x-x_1)^{k_1}\dots(x-x_p)^{k_p}\left(x^2-2Re(z_1)x+|z_1|^2\right)^{l_1}\dots\left(x^2-2Re(z_q)x+|z_q|^2\right)^{l_q}.$$

The relations between the coefficients of the real partial fraction decomposition and the coefficients of the complex partial fraction decomposition are

$$\begin{aligned} b_{j1} &= \sum_{s=2}^{l}\beta_{js}\omega_j^2|\omega|^{2(s-2)}C_{2s-3}^{s-2}+\sum_{s=1}^{l}(\beta_{js}\omega_j z+\omega_j\gamma_{js})|\omega_j|^{2(s-1)}C_{2(s-1)}^{s-1},\\ b_{j2} &= \sum_{s=3}^{l}\beta_{js}\omega_j^3|\omega_j|^{2(s-3)}C_{2s-4}^{s-3}+\sum_{s=2}^{l}(\beta_{js}\omega^2 z+\omega_j^2\gamma_{js})|\omega_j|^{2(s-2)}C_{2s-3}^{s-2},\\ &\vdots\\ b_{jl_j-1} &= \beta_{jl_j}\omega^l+\beta_{j\,l_j-1}\omega_j^{l_j-1}z+\omega_j^{l_j-1}\gamma_{j\,l_j-1}+(\beta_{jl_j}\omega_j^{l_j-1}z+\omega_j^{l_j-1}\gamma_{jl_j})|\omega_j|^2C_{l_j}^1\\ b_{jl_j} &= \beta_{jl_j}\omega_j^l z+\omega_j^l\gamma_{jl_j},\end{aligned} \tag{3.29}$$

where $\omega_j=\frac{1}{2iIm(z_j)}$.

Theorem 3.4. *Let $Re(\eta)>0$ and $Re(\alpha)>0$. The extended fractional derivative of arbitrary rational function satisfying previous theorem is*

$$\begin{aligned}&{}_{\mu,\sigma}D_{v,q;z}^{\alpha,\eta,p}\left(\frac{P(z)}{Q(z)}\right)=\\ &=\sum_{i=1}^{p}\sum_{r=1}^{k_i}\frac{a_{ir}B(\eta,\alpha-1)z^{\eta+\alpha}}{(-x_i)^r\Gamma(\alpha)}F_{v,q;p}^{(\mu,\sigma)}(r,\eta;\eta+\alpha-1;\frac{z}{x_i})\\ &+\sum_{j=1}^{q}\sum_{s=1}^{l_j}\Big(b_{js}\frac{B(\eta,\alpha-1)z^{\eta+\alpha}}{(-z_j)^s\Gamma(\alpha)}F_{v,q;p}^{(\mu,\sigma)}(s,\eta;\eta+\alpha-1;\frac{z}{z_j})\\ &+\bar{b}_{js}\frac{B(\eta,\alpha-1)z^{\eta+\alpha}}{(-\bar{z}_j)^s\Gamma(\alpha)}F_{v,q;p}^{(\mu,\sigma)}(s,\eta;\eta+\alpha-1;\frac{z}{\bar{z}_j})\Big)\end{aligned}$$

whenever the extended hypergeometric functions $F_{v,q;p}^{(\mu,\sigma)}$ exist.

Proof. Considering the formula (3.26) and complex partial fraction decomposition, the proof of the theorem can easily be done. □

A numerical example of extended fractional derivative of the rational function given in ([18]) will be derived by the following example.

Example 3.5. *Consider the rational function given in (*[18]*)*

$$f(z) = \frac{2z+1}{\left(z^2+6z+10\right)^3}. \tag{3.30}$$

The complex partial fraction decomposition of function (2.20) can be given as

$$f(z) = \sum_{s=1}^{3}\left(\frac{z_s}{\left(x-(-3+i)\right)^s} + \frac{\bar{z}_s}{\left(x-(-3-i)\right)^s}\right), \tag{3.31}$$

where $z_1 = \frac{15i}{16}, z_2 = \frac{15}{16} - \frac{i}{8}$ *and* $z_3 = \frac{-1}{4} - \frac{5i}{8}$. *By using the decomposition (3.31) , the extended fractional derivative of rational function (3.30) can be given as*

$$\begin{aligned}{}_{\mu,\sigma}D^{\alpha,\eta,p}_{v,q;z}(f(z)) &= \sum_{s=1}^{3}(z_s\frac{B(\eta,\alpha-1)z^{\eta+\alpha}}{(3-i)))^s\Gamma(\alpha)}F^{(\mu,\sigma)}_{v,q;p}(s,\eta;\eta+\alpha-1;\frac{z}{i-3})+ \\ &+ \bar{z}_s\frac{B(\eta,\alpha-1)z^{\eta+\alpha}}{(3+i)))^s\Gamma(\alpha)}F^{(\mu,\sigma)}_{v,q;p}(s,\eta;\eta+\alpha-1;\frac{z}{-i-3})).\end{aligned}$$

4. CONCLUSION

Recently, the investigation into extensions of some special functions has become important. Thus, many extensions of special functions have been obtained by the authors of different studies. From this point of view, we present extended modified Bessel function $I_v(q;x)$ which generalizes the Bessel and modified Bessel functions, by using an additional parameter in the integral representation. An extension of the well-known functions of the literature such as the hypergeometric function, the confluent hypergeometric function and the extended beta functions are also given via extended modified Bessel function. A necessary relation between extended modified Bessel function $I_v(q;x)$ and the confluent hypergeometric function ${}_1F_1(\alpha,\beta,x)$ is easily given. Moreover, Mellin and Laplace transforms for some newly derived special functions are obtained as a common coverage. We derive asymptotic formulae and the generating functions of the extended modified Bessel function. Hence, a lot of relations with respect to this new function can be proved by using its generating functions. The fractional derivative of rational functions is explicitly found by using the new definition of fractional derivative and complex partial fraction decomposition. It can be easily seen that the results obtained in this paper are new and effective mathematical tools and, also extensions of many results of the literature.

5. REFERENCES

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