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Picard-Tikhonov-Mann (PTM) iteration process with applications

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Abstract. In mathematics, a fixed point of a function is the process of finding a solution to an equation that can be written as $\Upsilon x = x$ for a suitable function Υ. Fixed point theory has many applications. For instance, in compilers (computer programs), fixed point computations are used for program analysis. An example of this is the data-flow analysis that is often required to optimize code. The vector of PageRank values of web pages is the fixed point of a derived linear transformation of the world wide web connectivity structure. Our main focus in this paper is to consider a novel algorithm for finding the fixed point in some abstract spaces. Some applications are presented regarding split feasibility/constrained minimization problems/signal enhancement. Furthermore, we perform the numerical illustrations to investigate the basic techniques using Matlab R2016a. The proposed novel algorithm demonstrates strong convergence to fixed points of the considered mapping, with practical applications in solving split feasibility and constrained minimization problems. Numerical results confirm the effectiveness of the method in signal enhancement tasks. This approach offers promising potential for further research in abstract metric spaces and iterative solution techniques.

AMS (MOS) Subject Classification Codes: 47H09; 47H10
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1. Introduction

Let $\Phi \neq \emptyset$ be a convex subset of a normed space Ψ, and $\Upsilon : \Phi \to \Phi$ be a map. Next, we express the set of all fixed points (FPs) of Υ by F. Υ is said $L-$Lipschitzian if there is a constant $L > 0$ with $\|\Upsilon\kappa - \Upsilon\tau\| \leq L\,\|\kappa - \tau\|$ for each $\kappa, \tau \in \Phi$. A $L-$Lipschitzian is nonexpansive if $L = 1$, and contraction if $L \in (0, 1)$.

FP theory is an interdisciplinary field that combines notions from geometry, topology, applied and pure analysis. Nonlinear analysis has proven to be an inestimable tool in biology, economics, engineering, game theory, and so forth. One of the most significant contributions of FP theory is its ability to figure out all sorts of mathematical problems, such as integral/differential equations, and variational inequalities and essentially to reveal the existence/uniqueness of the solutions to these problems. As early as 1965, Browder [9] was the former researcher who acquired a basic existence theorem of FPs for nonexpansive operators in the closed convex bounded sets on abstract spaces, and later various papers have explored FP theory for uniformly convex Banach spaces, as documented in references ([8], [18], [24]). The simplest and most fundamental iteration method in the FP theory is defined by Picard [30]. This method can used for computing FPs of contraction-type operators, but it is unapplicable for nonexpansive operators. The FP problem for nonexpansive operators stands for a crucial and natural extension of the class of operators beyond contraction mappings. Therefore, when an operator is nonexpansive, we try to apply the iteration [28], which is more general than the stated [30] method.

In 1967, Halpern [19] introduced that the sequence $\{x_n\}$ constructed hereinbelow:

$$\omega, x_1 \in \Phi, \;\; x_{n+1} = \alpha_n \omega + (1-\alpha_n)\,\Upsilon x_n, n \geq 0, \tag{1. 1}$$

here $\{\alpha_n\} \subset (0,1]$ and Φ is a closed convex subset of a Hilbert space.

Various researchers have examined and improved the (1. 1) technic to certain Banach spaces and $\{\alpha_n\} \subset (0,1]$ providing the parameters given below:

(S_1) $\lim_{n\to\infty} \alpha_n = 0$;
(S_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(S_3) ([45]) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(S_4) ([48]) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Suzuki [36], in 2009, discussed several sufficient terms on $\{\alpha_n\} \subset (0,1]$ that assure the convergence to a FP of nonexpansive operators. Conversely, the authors [16] and [35] dividedly found out that together just parameters $(S_1)\&(S_2)$ are sufficient for the strong convergence.

Cheval&Leuştean [14], in 2022, considered subsequently generalisations of the notable Halpern and Mann methods acquired by uniting them with the so-called the Tikhonov–Mann (TM) iteration as follows:

$$\begin{aligned} \omega, x_1 &\in \Phi, \\ z_n &= (1-\alpha_n)\,\omega + \alpha_n x_n, \\ x_{n+1} &= (1-\lambda_n)\,z_n + \lambda_n \Upsilon z_n, n \geq 0, \end{aligned} \tag{1. 2}$$

where $\{\alpha_n\}, \{\gamma_n\} \subset (0,1]$. If additional articles on the TM iteration are needed in the literature, various papers are available ([13], [15], [17], [27]).

Motivated by all the above-mentioned advancements, we establish a strong connection between the Picard and the TM iteration schemes in a general nonlinear setting. This algorithm (1. 3) is defined as indicated below:

Algorithm 1.1. *Let $\Phi \neq \emptyset$ be a closed convex subset of a Banach space Ψ and $\Upsilon : \Phi \to \Phi$ be a map. Let $\{x_n\} \subseteq \Phi$ is defined hereinbelow:*

$$\begin{aligned} \omega, x_1 &\in \Phi, \\ x_{n+1} &= \Upsilon y_n, \\ y_n &= \lambda_n z_n + (1-\lambda_n)\Upsilon z_n \\ z_n &= (1-\alpha_n) x_n + \alpha_n \omega,\ n \geq 0, \end{aligned} \tag{1. 3}$$

here $\{\alpha_n\}, \{\lambda_n\} \subset (0,1]$ satisfying (S_1) and (S_2) condition.

The iterative sequence $\{x_n\}$ identified by (1. 3) is said Picard-Tikhonov-Mann (PTM) iteration process. The proposed iteration (1. 3) can be easily reduced to Tikhonov-Mann and Picard iterations.

This research first presents a novel algorithm that converges strongly to a FP of Υ. Our conclusions are implemented to solve solutions of split feasibility/constrained minimization problems. Secondly, a convergence theorem is obtained for the proposed iteration for $G-$nonexpansive maps on Hilbert space via a digraph. Further, we perform the numerical illustrations to investigate the basic techniques by using Matlab R2016a.

2. PRELIMINARIES

We apply S_Ψ to express the unit sphere $S_\Psi = \{a \in \Psi : \|a\| = 1\}$ in Banach space Ψ. If $a, b \in S_\Psi$ via $b \neq a \Rightarrow \|(1-\eta) a + \eta b\| < 1$ for $\forall \eta \in (0,1)$, then Ψ is called to be strictly convex. In a strictly convex Banach space Ψ we obtain that if $\|b\| = \|a\| = \|(1-\zeta) b + \zeta a\|$ for $a, b \in \Psi$ and $\zeta \in (0,1)$, then $b = a$. A Banach space Ψ is called to be smooth if

$$\lim_{t \to 0^+} \frac{\|a + tb\| - \|a\|}{t} \tag{2. 4}$$

exists for $\forall a, b \in S_\Psi$. Then, the norm of Ψ is called to be $G\hat{a}teaux$ differentiable. It is called to be uniformly $G\hat{a}teaux$ differentiable if for $\forall b \in S_\Psi$, (2. 4) is obtained uniformly for $a \in S_\Psi$. It is worth noting that, every uniformly smooth space holds uniformly $G\hat{a}teaux$ differentiable norm. Let Ψ be an arbitrary real normed space via dual space Ψ^*. We state by $J : \Psi \to 2^{\Psi^*}$ is the normalized duality map identified by

$$J(\tau) := \left\{ g^* \in \Psi^* : \langle \tau, g^* \rangle = \|g^*\|^2 = \|\tau\|^2 \right\}, \tau \in \Psi,$$

where $\langle ., . \rangle$ states the generalized duality pairing. In this case, there exists $j(\tau + \kappa) \in J(\tau + \kappa)$ with

$$\|\tau + \kappa\|^2 \leq \|\tau\|^2 + 2 \langle \kappa, j(\tau + \kappa) \rangle \text{ for } \forall \kappa, \tau \in \Psi.$$

It is worthy of noting that, Ψ is smooth iff J is single-valued. A Banach space $\Phi \subseteq \Psi$ is called to a retract of Ψ if there is a continuous $P : \Psi \to \Phi$ with $Pa = a$ for $\forall a \in \Phi$. We say such P is a retraction of Ψ onto Φ. Herefrom, if a map P is a retraction, then $Pb = b$ for each b in the range of P. A retraction P is called to be sunny if $P(Pa + t(a - Pa)) = Pa$ for all $a \in \Psi$ and $0 \leq t$. Further, Φ is called to be a sunny nonexpansive retract of Ψ, if a sunny retraction P is nonexpansive ([1]).

Lemma 2.1. [48] *Let* $\{\kappa_n\}$ *be a sequence of nonnegative real numbers providing* $\kappa_{n+1} \leq (1-\sigma_n)\kappa_n + \sigma_n\omega_n$, $n \geq 1$, *here* $\{\omega_n\}$ *and* $\{\sigma_n\}$ *are sequences of real numbers which provide the terms:*
(i) $\sum_{n=1}^{\infty}\sigma_n = \infty$ *and* $\{\sigma_n\} \subset [0,1]$,
(ii) $\limsup_{n\to\infty}\omega_n = 0$.
Then $\lim\limits_{n\to\infty}\kappa_n = 0$.

Lemma 2.2. [47] *Let* $m > 1$ *and* $Z > 0$ *be two constant numbers and* Ψ *a Banach space. Then* Ψ *is uniformly convex iff there is a convex, strictly increasing, and continuous function* $\varphi : [0,\infty) \to [0,\infty)$ *via* $\varphi(0) = 0$ *such that*

$$\|\xi a + (1-\xi)b\|^m \leq \xi\|a\|^m + (1-\xi)\|b\|^m - \Gamma_m(\xi)\varphi(\|a-b\|)$$

for $\forall a, b \in B_Z(0) = \{a \in \Psi : \|a\| \leq Z\}$ *and* $\xi \in [0,1]$, *here* $\Gamma_m(\xi) = \xi(1-\xi)^m + \xi^m(1-\xi)$.

Henceforward, we show the weak and strong convergence of $\{x_n\}$ to a point $\varsigma \in \Phi$ by $x_n \rightharpoonup \varsigma$ and $x_n \to \varsigma$, resp.

Lemma 2.3. [40] *Let* Ψ *be a reflexive Banach space whose norm is uniformly* $G\widehat{a}teaux$ *differentiable,* $\Phi \neq \emptyset$ *be a closed convex subset of* Ψ *and* $\Upsilon : \Phi \to \Phi$ *be a nonexpansive map via* $F \neq \emptyset$. *Given that every closed convex bounded subset of* Φ *holds FP property for nonexpansive maps. Then* F *is the sunny nonexpansive retract of* Φ. *Furthermore, if* $\omega \in \Phi$ *and* z_t *be the unique point in* Φ *identified by* $z_t = t\omega + (1-t)\Upsilon z_t$ *for* $t \in (0,1)$, *then* $\{z_t\} \to R_F(\omega)$ *when* $t \to 0^+$, *here* $R_F : \Phi \to F$ *is the sunny nonexpansive retraction.*

Lemma 2.4. [46] *Let* Ψ *be a Banach space involving a uniformly* $G\widehat{a}teaux$ *differentiable norm,* $\Phi \neq \emptyset$ *a closed convex subset of* Ψ, $h : \Phi \to \Phi$ *a continuous operator,* $\Upsilon : \Phi \to \Phi$ *be a nonexpansive map and* $\{x_n\} \subseteq \Phi$ *such that* $\lim_{n\to\infty}\|\Upsilon x_n - x_n\| = 0$. *Assume* $\{z_t\} \subseteq \Phi$ *is a path identified by* $z_t = thz_t + (1-t)\Upsilon z_t$ *for* $t \in (0,1)$, *such that* $z_t \to e^*$ *when* $t \to 0^+$. *Then* $\limsup_{n\to\infty}\langle he^* - e^*, J(x_n - e^*)\rangle \leq 0$.

3. Main Results

Proposition 3.1. *Let* $\Phi \neq \emptyset$ *be a closed convex subset of a Banach space* Ψ *and* $\Upsilon : \Phi \to \Phi$ *be a nonexpansive map via* $F \neq \emptyset$. *For* $\omega, x_1 \in \Phi$, *a sequence* $\{x_n\} \subseteq \Phi$ *is identified by (1. 3). Then we hold the below terms:*
(i) $\{x_n\}$ *is bounded,*
(ii) $\lim_{n\to\infty}\|\Upsilon z_n - z_n\| = \lim_{n\to\infty}\|\Upsilon y_n - y_n\| = \lim_{n\to\infty}\|\Upsilon x_n - x_n\| = 0$.

Proof. (i) Let $\varsigma \in F$. Invoking (1. 3), we get

$$\|z_n - \varsigma\| \leq \alpha_n\|\omega - \varsigma\| + (1-\alpha_n)\|x_n - \varsigma\|, \tag{3. 5}$$
$$\|y_n - \varsigma\| \leq \lambda_n\|z_n - \varsigma\| + (1-\lambda_n)\|\Upsilon z_n - \varsigma\| \leq \|z_n - \varsigma\|. \tag{3. 6}$$

Due to (3. 5) and (3. 6), we attain

$$\begin{aligned} \|x_{n+1}-\varsigma\| &= \|\Upsilon y_n-\varsigma\| \\ &\leq \|z_n-\varsigma\| \\ &\leq \alpha_n\|\omega-\varsigma\|+(1-\alpha_n)\|x_n-\varsigma\| \\ &\leq \max\{\|x_n-\varsigma\|,\|\omega-\varsigma\|\} \\ &\vdots \\ &\leq \max\{\|x_1-\varsigma\|,\|\omega-\varsigma\|\}. \end{aligned}$$

Hence $\{x_n\}$ is bounded, due to (3. 5) and (3. 6), then $\{y_n\}$ and $\{z_n\}$ are bounded.

(ii) By (1. 3) and (S_1) condition, we have

$$\begin{aligned} \|z_n-x_n\| &= \alpha_n\|x_n-\omega\| \\ &\to 0 \text{ as } n\to\infty, \end{aligned} \tag{3. 7}$$

and using (3. 7), we get

$$\begin{aligned} \|\Upsilon z_n-z_n\| &\leq \alpha_n\|x_n-\omega\|+\|z_n-x_n\| \\ &\to 0 \text{ as } n\to\infty. \end{aligned} \tag{3. 8}$$

Due to (3. 8), we obtain

$$\begin{aligned} \|y_n-z_n\| &\leq (1-\lambda_n)\|\Upsilon z_n-z_n\| \\ &\to 0 \text{ when } n\to\infty. \end{aligned} \tag{3. 9}$$

Further,

$$\begin{aligned} \|x_{n+1}-\Upsilon z_n\| &\leq \|z_n-y_n\| \\ &\to 0 \text{ when } n\to\infty, \end{aligned} \tag{3. 10}$$

and

$$\begin{aligned} \|y_n-x_n\| &\leq \|y_n-z_n\|+\|z_n-x_n\| \\ &\to 0 \text{ when } n\to\infty. \end{aligned} \tag{3. 11}$$

Owing to (3. 8) and (3. 9), we know that

$$\begin{aligned} \|x_{n+1}-z_n\| &\leq \|\Upsilon z_n-\Upsilon y_n\|+\|\Upsilon z_n-z_n\| \\ &\to 0 \text{ when } n\to\infty, \end{aligned} \tag{3. 12}$$

and

$$\begin{aligned} \|\Upsilon y_n-y_n\| &\leq \|\lambda_n z_n+(1-\lambda_n)\Upsilon z_n-\Upsilon y_n\| \\ &\leq \lambda_n\|\Upsilon z_n-z_n\|+(1-\lambda_n)\|\Upsilon z_n-\Upsilon y_n\| \\ &\to 0 \text{ as } n\to\infty. \end{aligned} \tag{3. 13}$$

Using (3. 7), (3. 8) and (3. 14), we attain

$$\begin{aligned} \|\Upsilon z_n-x_n\| &\leq \|x_n-z_n\|+\|\Upsilon z_n-z_n\| \\ &\to 0 \text{ when } n\to\infty. \end{aligned} \tag{3. 14}$$

and

$$\begin{aligned}\|x_n - \Upsilon x_n\| &\leq \|x_n - \Upsilon z_n\| + \|\Upsilon z_n - \Upsilon x_n\| \\ &\to 0 \text{ when } n \to \infty.\end{aligned} \qquad (3.\ 15)$$

□

Theorem 3.2. *Let Ψ be a uniformly convex Banach space involving a uniformly $G\widehat{a}teaux$ differentiable norm, $\Phi \neq \emptyset$ be a closed convex subset of Ψ, and $\Upsilon : \Phi \to \Phi$ be a nonexpansive map via $F \neq \emptyset$. Let a sequence $\{x_n\} \subseteq \Phi$ is constructed by (1. 3) for $\omega, x_1 \in \Phi$. Then $\{x_n\} \to R_F(\omega)$, where $R_F : \Phi \to F$ is the sunny nonexpansive retraction.*

Proof. By Lemma 2.3, we see that the path $\{z_t\} \to R_F(\omega)$ for $t \in (0,1)$. Let $e^* := R_F(\omega) = \lim_{t\to 0^+} z_t$. Invoking Lemma 2.4, we get $\limsup_{n\to\infty} \langle \omega - e^*, J(z_n - e^*)\rangle \leq 0$.

Invoking (1. 3), we have

$$\begin{aligned}\|z_n - e^*\|^2 &= \|\alpha_n(\omega - e^*) + (1-\alpha_n)(x_n - e^*)\|^2 \\ &\leq (1-\alpha_n)\|x_n - e^*\|^2 + 2\alpha_n \langle \omega - e^*, J(z_n - e^*)\rangle .\end{aligned} \qquad (3.\ 16)$$

Due to Lemma 2.2 and (3. 16), we attain

$$\begin{aligned}\|x_{n+1} - e^*\|^2 &= \|\Upsilon y_n - e^*\|^2 \\ &\leq \|y_n - e^*\|^2 \\ &= \|\lambda_n(z_n - e^*) + (1-\lambda_n)(\Upsilon z_n - e^*)\|^2 \\ &\leq \lambda_n\|z_n - e^*\|^2 + (1-\lambda_n)\|\Upsilon z_n - e^*\|^2 \\ &\leq \|z_n - e^*\|^2 \\ &\leq (1-\alpha_n)\|x_n - e^*\|^2 + 2\alpha_n\langle \omega - e^*, J(z_n - e^*)\rangle \\ &\leq (1-\alpha_n)\|x_n - e^*\|^2 + \alpha_n \xi_n,\end{aligned}$$

here $\xi_n := 2\langle \omega - e^*, J(z_n - e^*)\rangle$. Since (S_2) condition, $\limsup_{n\to\infty} \xi_n \leq 0$ and by Lemma 2.1 we deduce that $\{x_n\} \to e^*$. □

Remark 3.3. *(a) Letting $\lambda_n \equiv 1$ in Theorem 3.2, we obtain Theorem 5.6 in* [34].

(b) Taking $\alpha_n \equiv 0$ in Theorem 3.2, we get the result for the normal $S-$iteration process in [34].

Rhoades [33] proposed a framework for comparing the rates of convergence between two iterative algorithms, as outlined below.

Definition 3.4. [33] *Let $\Phi \neq \emptyset$ be a closed convex subset of a Banach space Ψ and $\Upsilon : \Phi \to \Phi$ be a map. Suppose that $\{\varrho_n\}$ and $\{v_n\}$ are two iterative sequences which converge to a fixed point f in Φ. The sequence $\{\varrho_n\}$ is said to converge to f faster than $\{v_n\}$ if the following inequality holds*

$$\|\varrho_n - f\| \leq \|v_n - f\| \text{ for every } n \geq 1.$$

Example 3.5. *Let Ψ be equipped with the standard norm, let Φ be all real numbers from zero to infinity including zero. Suppose that $\Upsilon x = \sin x$ for $x \in \Phi$. Then Υ is a self-map. Observe that $F = \{0\}$. Let*

$$\alpha_n = \lambda_n = 3/7 \textit{ for } n \geq 1 \textit{ and } \omega = 0.5.$$

Table 1 provides a comparative analysis of the convergence rates to the fixed point achieved by the Halpern iteration, the (TM) iteration process, and the (PTM) iteration process using Matlab R2016a.

Table 1. Convergence Performance of Iteration Schemes

n	*Iteration (1. 1)*	*Iteration (1. 2)*	*Iteration (1. 3)*
1	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$
2	0.7092	0.7070	0.6913
3	0.5864	0.5744	0.5548
4	0.5305	0.5213	0.4945
5	0.5034	0.4998	0.4665
6	0.4899	0.4911	0.4532
7	0.4832	0.4876	0.4469
8	0.4798	0.4861	0.4439
9	0.4780	0.4855	0.4424
10	0.4772	0.4853	0.4417

An analysis of the data presented in Table 1 reveals that the Picard–Tikhonov–Mann iteration approach achieves faster convergence when compared to the previously outlined iterations.

4. Application

Some applications are given this part. Let Ξ be a real Hilbert space via $\langle ., . \rangle$. Let $\Phi \neq \emptyset$ be a closed convex subset of Ξ and $\Upsilon : \Phi \to \Xi$ a nonlinear mapping. Υ is called to be:

- monotone if $\langle \Upsilon\kappa - \Upsilon\tau, \kappa - \tau \rangle$ for $\forall \kappa, \tau \in \Phi$,
- $\lambda-$strongly monotone $(\lambda - sm)$ if there is a constant $0 < \lambda$ with $\langle \Upsilon\kappa - \Upsilon\tau, \kappa - \tau \rangle \geq \lambda \left\| \kappa - \tau \right\|^2, \forall \kappa, \tau \in \Phi$,
- $\nu-$inverse strongly monotone $(\nu - ism)$ if there is a constant $0 < \nu$ with $\langle \Upsilon\kappa - \Upsilon\tau, \kappa - \tau \rangle \geq \nu \left\| \Upsilon\kappa - \Upsilon\tau \right\|^2, \forall \kappa, \tau \in \Phi$.

The variational inequality problem identified by Φ and Υ will be stated by $VI(\Phi, \Upsilon)$ ([23]). The $VI(\Phi, \Upsilon)$ is the problem of obtaining a vector α in Φ with $\langle \Upsilon\alpha, \alpha - l \rangle \geq 0$, $\forall l \in \Phi$. The set of all these vectors that solve the $VI(\Phi, \Upsilon)$ problem is stated by $\Omega(\Phi, \Upsilon)$. The $VI(\Phi, \Upsilon)$ is related to various types of problems such as the complementarity problem, the convex minimization problem, and such-like. In examining variational inequalities, the approximation and existence of solutions are substantial issues. The $VI(\Phi, \Upsilon)$ is equivalent to the FP problem, viz

$$\text{to compute } \vartheta^* \in \Phi \text{ such that } \vartheta^* = F_\mu^* \vartheta = P_\Phi \left(I - \mu\Upsilon \right) \vartheta^*$$

here $P_\Phi : \Xi \to \Phi$ is the metric projection. The operator $F_\mu := P_\Phi \left(I - \mu\Upsilon \right)$ is a contraction on Φ with $2\lambda/L^2 > \mu > 0$, if Υ is $\lambda - sm$ and $L-$Lipschitzian. Then, an application of the Banach contraction principle means that $\Omega(\Phi,\Upsilon)=\{\vartheta^*\}$ and the sequence of the [30] method, defined by

$$x_{n+1}=F_\mu x_n,\; n\geq 1,$$

where $\{x_n\}\to\vartheta^*$.

Construction of FPs of nonexpansive maps is a significant matter in the theory of nonexpansive maps and holds applications in many areas such as signal processing and image recovery ([10], [25], [49]). Thus, the split feasibility problem (SFP) of Φ and T is

$$\text{to compute a point } \vartheta\subseteq\Phi \text{ with } \Upsilon\vartheta\in Q, \tag{4. 17}$$

herefrom $\Upsilon:\Xi_1\to\Xi_2$ is a bounded linear map, where Φ is a closed convex subset of Hilbert spaces Ξ_1 and Ξ_2. The SFP is called to be consistent if (4. 17) holds a solution. It is obvious that SFP is consistent iff the following FP problem has a solution:

$$\text{to compute } \vartheta\in\Phi \text{ with } \vartheta=P_\Phi\left(I-\gamma\Upsilon^*\left(I-P_Q\right)\Upsilon\right)\vartheta \text{ for } 0<\gamma, \tag{4. 18}$$

where Υ^* is the adjoint of Υ, P_Φ and P_Q are the orthogonal projections onto Φ and Q, resp., $P_\Phi\left(I-\gamma\Upsilon^*\left(I-P_Q\right)\Upsilon\right)$ is nonexpansive.

4.1. **Application to constrained optimization problems.** Let Φ be a closed convex subset of a Hilbert space Ξ, $P_\Phi:\Xi\to\Phi$ be the metric projection and $\Upsilon:\Phi\to\Xi$ be a $\nu-ism$. Note that $P_\Phi\left(I-\mu\Upsilon\right)$ is a nonexpansive map providing $\mu\in(0,2\nu)$. The methods for signal/image processing are usually iterative constrained optimization processes constituted to minimize a convex differentiable function Υ over a closed convex set $\Phi\subseteq\Xi$. It is worth noting that, every $L-$Lipschitzian map is $2/L-ism$. Hence, we hold the next theorem which forms the sequence of vectors in the constrained or feasible set Φ which converges weakly to the optimal solution which minimizes Υ.

Theorem 4.2. *Let Φ be a closed convex subset of a Hilbert space Ξ, and Υ a differentiable and convex function on an open set N including Φ. Supposing $\nabla\Upsilon$ is an $L-$Lipschitz map on N, $\mu\in(0,2/L)$ and minimizers of Υ concerning Φ exist. Let a sequence $\{x_n\}\subseteq\Phi$ is constructed by*

$$\begin{aligned}\omega,x_1 &\in \Phi,\\ x_{n+1} &= P_\Phi\left(I-\mu\nabla\Upsilon\right)y_n,\\ y_n &= \lambda_n z_n+\left(1-\lambda_n\right)P_\Phi\left(I-\mu\nabla\Upsilon\right)z_n,\\ z_n &= \alpha_n\omega+\left(1-\alpha_n\right)x_n,\; n\geq 0,\end{aligned}$$

here $\{\alpha_n\}$, $\{\lambda_n\}\subseteq[\epsilon,1-\epsilon]$ for some $\epsilon\in(0,1)$. Then $\{x_n\}$ converges weakly to a minimizer of Υ.

4.3. **Application to $SFPs$.** An operator Υ in a Hilbert space Ξ is called to be averaged if Υ can be expressed as $(1-\sigma)I+\sigma S$ for $\sigma\in(0,1)$, where S is a nonexpansive operator on Ξ. Let $q(x):\frac{1}{2}\left\|\left(\Upsilon-P_Q\Upsilon\right)x\right\|$, $x\in\Phi$. Take into consideration the minimization problem

$$\text{compute } \min_{x\in\Phi} q\left(x\right).$$

The gradient of q is $\nabla q=\Upsilon^*\left(I-P_Q\right)\Upsilon$, here Υ^* is the adjoint of Υ. ∇q is $L-$Lipschitzian via $\|\Upsilon\|^2=L$, as $I-P_Q$ is nonexpansive. Thus, ∇q is $1/L-ism$ and $I-\mu\nabla q$

is averaged for $0 < \mu < 2/L$. So, the composition $P_\Phi(I - \mu\nabla q)$ is averaged. Let $\Upsilon := P_\Phi(I - \mu\nabla q)$. Consider the solution set of SFP is F ([2]).

Next, we establish an iteration algorithm that can be applied to compute solutions of SFP.

Theorem 4.4. *Suppose that SFP is consistent. Assume $\{\alpha_n\}$, $\{\lambda_n\} \subseteq [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Let $\{x_n\} \subseteq \Phi$ is constructed by*

$$\begin{aligned} \omega, x_1 &\in \Phi, \\ x_{n+1} &= P_\Phi(I - \mu\nabla q)\, y_n, \\ y_n &= \lambda_n z_n + (1-\lambda_n) P_\Phi(I - \mu\nabla q)\, z_n, \\ z_n &= \alpha_n \omega + (1-\alpha_n)\, x_n,\ n \geq 0, \end{aligned}$$

here $0 < \mu < 2/\|\Upsilon\|^2$. Then $\{x_n\}$ converges weakly to a solution of SFP.

Remark 4.5. *Since the PTM iteration process reduces to the normal $S-$iteration process, Theorem 4.4 extends Corollary 6.7 of Sahu* [34].

Example 4.6. *Take $\Psi = R$, $\Phi = [-1, 1]$ and $\omega = 1$. Describe an operator $\Upsilon : \Phi \to \Phi$ by $\Upsilon x = -x$ for $x \in \Phi$. Consider $\alpha_n = \frac{1}{13+n}$ and $\lambda_n = \frac{9+n}{7n+11}$ for $n \geq 0$. Then Υ is a nonexpansive operator. We also get $F = \{0\} \neq \emptyset$. All numerical calculations were performed using Matlab R2016a. The first twenty values for the initial point $x_0 = 0.50000$ are as follows: $x_1 = 0.0059524$, $x_2 = -0.0014667$, $x_3 = -0.0012187$, $x_4 = -0.0015784$, $x_5 = -0.0015906$, $x_6 = -0.0016301$, $x_7 = -0.0016107$, $x_8 = -0.0015899$, $x_9 = -0.0015548$, $x_{10} = -0.0015186$, $x_{11} = -0.0014789$, $x_{12} = -0.0014395$, $x_{13} = -0.0014000$, $x_{14} = -0.0013615$, $x_{15} = -0.0013240$, $x_{16} = -0.0012878$, $x_{17} = -0.0012531$, $x_{18} = -0.0012196$, $x_{19} = -0.0011877$, $x_{20} = -0.011570$.*

5. THE PTM ITERATION PROCESS FOR G-NONEXPANSIVE MAPPINGS

Let G be a digraph involving the set of vertices V and the set of edges E includes overall the loops, namely $(\kappa, \kappa) \in E$ for $\forall \kappa \in V$. Furthermore, supposing G holds no parallel edges, and we may hereby define $G = (V, E)$. A digraph G is called to be *transitive* if, (κ, τ), $(\tau, \varsigma) \in E$ for $\forall \kappa, \tau, \varsigma \in V$, we acquire $(\kappa, \varsigma) \in E$ ([21]).

Let $\Phi \neq \emptyset$ be a convex subset of a Banach space, G be a digraph, here $V = \Phi$ and $\Upsilon : \Phi \to \Phi$. In this case, Υ is called to be $G-$nonexpansive if the below terms have:

- Υ is edge-preserving, viz, for each $\kappa, \tau \in \Phi$ such that $(\kappa, \tau) \in E \Rightarrow (\Upsilon\kappa, \Upsilon\tau) \in E$;
- $\|\Upsilon\kappa - \Upsilon\tau\| \leq \|\kappa - \tau\|$, whenever $(\kappa, \tau) \in E$ for each $\kappa, \tau \in \Phi$ ([42]).

Definition 5.1. [41] *Let $\Phi \neq \emptyset$ be a subset of a normed space Ψ and let G be a digraph with $V = \Phi$. Then Φ is called to hold Property G if every sequence $\{x_n\} \rightharpoonup \kappa \in \Phi$, there is a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ with $(x_{n_l}, \kappa) \in E$ for each $k \in N$.*

Definition 5.2. *[*[3], [29]*] Let G be a digraph. $V \supseteq \Psi$ is called a dominating set if each $a \in V \setminus \Psi$ there is $x \in \Psi$ with $(x, a) \in E$ and we call that x dominates a or a is dominated by x. Let $a \in V$, $V \supseteq \Psi$ is dominated by a if $(a, x) \in E$ for each $x \in \Psi$ and we call that Ψ dominates a if $(x, a) \in E$ for each $x \in \Psi$.*

Let $\Phi \neq \emptyset$ be a closed convex subset of a real Hilbert space Ξ. There is a unique closest point in Φ, indicated by $P_\Phi \kappa$, with $\|\kappa - P_\Phi \kappa\| \leq \|\kappa - \tau\|$ for each $\tau \in \Phi$, $\forall \kappa \in \Xi$. $P_\Phi : \Xi \to \Phi$ is said the metric projection.

Lemma 5.3. [39] *Let Φ be a convex subset of a Hilbert space Ξ and let $\kappa \in \Xi$, $\tau \in \Phi$. Then the undermentioned are equivalent:*
(a) $\|\kappa - \tau\| = d(\kappa, \Phi)$;
(b) $\langle \kappa - \tau, \tau - z \rangle \geq 0$ *for every* $z \in \Phi$.

Lemma 5.4. [39] *Let Ξ be a Hilbert space. Let $\{x_n\} \subseteq \Xi$ with $x_n \rightharpoonup \kappa$. If $\kappa \neq \tau$, then* $\liminf_{n\to\infty} \|x_n - \kappa\| < \liminf_{n\to\infty} \|x_n - \tau\|$.

Proposition 5.5. *Let Φ be a convex subset of a vector space Ψ and G is a digraph and transitive involving $V = \Phi$ and E is convex. Let $\Upsilon : \Phi \to \Phi$ be edge-preserving. Let $\{x_n\}$ be constructed by (1. 3), here $\omega = x_0$. Let $c^* \in F$ be such that $(x_0, c^*), (c^*, x_0) \in E$. If $\{x_n\}$ dominates x_0, then (x_0, x_n), $(x_n, \Upsilon x_n)$, $(x_n, x_{n+1}) \in E$ for $\forall n \in N$.*

Theorem 5.6. *Let $\Phi \neq \emptyset$ be a closed convex subset of a Hilbert space Ξ and G be a digraph and transitive involving $V = \Phi$ and E be convex. Assume Φ owns Property G. Let $\Upsilon : \Phi \to \Phi$ be a $G-$nonexpansive map. Supposing $F \neq \emptyset$ and $E \supseteq F \times F$. Given $\omega, x_1 \in \Phi$, a sequence $\{x_n\}$ in Φ is constructed by (1. 3) for $\omega = x_0$. Let $c^* \in F$ be such that $(x_0, c^*), (c^*, x_0) \in E$. If $\{x_n\}$ dominates x_0, then (x_0, x_n), $(x_n, \Upsilon x_n)$, $(x_n, x_{n+1}) \in E$ for $\forall n \in N$. If $\{x_n\}$ is dominated by Px_0 and $\{x_n\}$ dominates x_0, then $\{x_n\} \to Px_0$, here P is the metric projection on F.*

Proof. Let $a_0 = Px_0$. Using the same proof as in Proposition 3.1, by Proposition 5.5, we can deduce that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|\Upsilon x_n - x_n\| = 0$. Now, we show that $\limsup_{n\to\infty} \langle x_n - a_0, x_0 - a_0 \rangle \leq 0$. In fact, consider a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - a_0, x_0 - a_0 \rangle = \limsup_{l\to\infty} \langle x_{n_l} - a_0, x_0 - a_0 \rangle.$$

Since all the x_{n_l} lie in the weakly compact set Φ and Φ holds Property G, without losing of generality we can suppose that $x_{n_l} \rightharpoonup y$ for some $y \in \Phi$ and $(x_{n_l}, y) \in E$. Assume $y \neq \Upsilon y$. Owing to Lemma 5.4, $\lim_{n\to\infty} \|\Upsilon x_n - x_n\| = 0$, we attain

$$\begin{aligned} \liminf_{l\to\infty} \|x_{n_l} - y\| &< \liminf_{l\to\infty} \|x_{n_l} - \Upsilon y\| \\ &\leq \liminf_{l\to\infty} [\|x_{n_l} - \Upsilon x_{n_l}\| + \|\Upsilon x_{n_l} - \Upsilon y\|] \\ &= \liminf_{l\to\infty} \|\Upsilon x_{n_l} - \Upsilon y\| \leq \liminf_{l\to\infty} \|x_{n_l} - y\|, \text{ (By } G-\text{nonexpansiveness of } \Upsilon) \end{aligned}$$

which is a contradiction. Thus $y = \Upsilon y$. So, from Lemma 5.3, we obtain

$$\limsup_{l\to\infty} \langle x_{n_l} - a_0, x_0 - a_0 \rangle = \langle y - a_0, x_0 - a_0 \rangle \leq 0.$$

Hence, $\limsup_{n\to\infty} \langle x_n - a_0, x_0 - a_0 \rangle \leq 0$. Invoking the analogous proof as in Theorem 3.2, we can deduce that $\lim_{n\to\infty} \|x_n - a_0\|^2 = 0$. Hence $\{x_n\} \to x_0 = a_0$. □

Remark 5.7. *Since the PTM iteration process reduces to the Halpern iteration process, Theorem 5.6 improves Theorem 4.5 of Tiammee et al.* [41].

Motivated by [22] and [41] we offer the below instances to demonstrate the convergence of the algorithms given in this writing.

Example 5.8. *Let* $\Xi = R$ *and* $\Phi = \left[0, \frac{1}{2}\right]$ *via* $\|x-y\| = |x-y|$ *and let* G *such that*

$$V = \Phi \text{ and } E = \left\{(x,y) : x, y \in \left[0, \frac{3}{8}\right] \text{ such that } |x-y| \leq 8^{-1}\right\}.$$

Describe $\Upsilon : \Phi \to \Phi$ *by*

$$\Upsilon x = \begin{cases} \frac{4x^2}{3} \text{ if } x \in \left[0, \frac{1}{2}\right) \\ \left(\frac{5}{8}\right)^2 \text{ if } x = \frac{1}{2} \end{cases},$$

for each $x \in \Phi$. *Let* $\alpha_n = \frac{1}{n+15}$, $\lambda_n = \frac{3n+5}{16n+7}$ *and* $\omega = \frac{1}{2}$. *It is straightforward to show that* Υ *is* $G-$*nonexpansive mapping, but is not nonexpansive. We also have* $F = \{0.00000\}$. *Then* F *is* $\omega_n = \frac{-\sqrt{3}\sqrt{3n^2+28n+67}+3n+15}{8n+32}$. *Hence, it can be easily seen that* $\omega_n \leq 8^{-1}$ *for each* $n \in N$. *Therefore* $(\omega_n, Px_0) = (\omega_n, 0) \in E$, *scilicet,* Px_0 *is dominated by* $\{\omega_n\}$ *and* $\omega_n \to 0 = Px_0$ *when* $n \to \infty$.

Example 5.9. *Let* $\Phi := \left\{\mathbf{x} = (x_1, x_2, \ldots) \in l_2 : \|\mathbf{x}\|_{l_2} \leq 1 \text{ and } x_i \in [0,1] \text{ for } i \in N\right\}$ *with* $\|\mathbf{x}\|_{l_2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$ *and* $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{\infty} x_i y_i$ *for* $y = (x_1, x_2, \ldots) \in l_2 : \|x\|_{l_2} \leq 1$ *and* $y_i \in [0,1]$. *Let* $G = (V,E)$ *such that*

$$V = \Phi \text{ and } E = \left\{(x,y) : x, y \in \left[0, \frac{1}{3}\right] \text{ with } \|\mathbf{x} - \mathbf{y}\|_{l_2} \leq 5^{-1} \text{ for } i \in N\right\}.$$

Identified $\Upsilon : \Phi \to \Phi$ *by* $\Upsilon \mathbf{x} = \left(0.5x_1^2, 0.375x_2^2, 0, 0, \ldots\right)$ *for* $\forall \mathbf{x} \in \Phi$. *It is simple to show that* Υ *is* $G-$*nonexpansive map, but is not nonexpansive. We also have* $F = \{\mathbf{0}\}$ *where* $\mathbf{0} = (0, 0, \ldots)$ *is the null vector on* l_2.

Assuming the sequence $\{x_n\}$ *is constructed by*

$$\begin{aligned} x_{n+1} &= \Upsilon y_n, \\ y_n &= \left(\frac{3n+5}{16n+7}\right) z_n + \left(1 - \frac{3n+5}{16n+7}\right) \Upsilon z_n, \\ z_n &= \left(\frac{1}{n+15}\right) \omega + \left(1 - \frac{1}{n+15}\right) x_n, n \geq 0, \end{aligned}$$

where $\omega = \mathbf{x}_0 = \left(\frac{1}{6}, \frac{1}{8}, 0, 0, \ldots\right)$. *Due to definition of* $\{x_n\}$ *and* $Px_0 = \{\mathbf{0}\}$, *we own* $\|x_n - \mathbf{0}\|_{l_2} \leq 0.2$ *It follows that* $(x_n, Px_0) \in E$. *Then* Px_0 *is dominated by* $\{x_n\}$. *It is clear that* $\{x_n\}$ *dominates* x_0 *and also* Px_0 *is dominated by* $\{x_0\}$. *From Theorem 5.6, the sequence* $\{x_n\} \to Px_0 = \{\mathbf{0}\}$. *The first five values for the initial point* $x_0 = \left(\frac{1}{6}, \frac{1}{8}, 0, 0, \ldots\right)$ *are as follows:* $x_1 = (7.5E-3, 0.6E-2, 0, 0, \ldots)$, $x_2 = (1.323E-5, 6.18E-6, 0, 0, \ldots)$, $x_3 = (3.84E-6, 1.5E-6, 0, 0, \ldots)$, $x_4 = (2.04E-6, 8.6E-7, 0, 0, \ldots)$ *and* $x_5 = (1.887E-7, 7.9072E-8, 0, 0, \ldots)$.

6. Signal Enhancement

Signal enhancement is a process for reducing noise leading to an improved signal-to-noise ratio [43]. Signal enhancement is a crucial topic in various engineering fields, particularly in communications, image processing, medicine, and audio engineering [44]. Iterative algorithms are frequently used to solve complex signal enhancement problems [10]. These algorithms are essential for optimization and improving the accuracy of the solution in signal enhancement [11].

Motivated these facts, we write code which generates a noisy signal from a sine wave, applies an iterative signal improvement algorithm, and visualizes the original, noisy, and improved signals on Matlab R2016a. Below is a step-by-step explanation of the Pseudocode:

Step1: Creating the Original Signal (the clean sine wave) and Noisy Signal (the sine wave with added Gaussian noise)

```
t = 0:0.01:2*pi; % Time vector
original_signal = sin(t); % Original sine wave
noise = 0.3 * randn(size(t)); % Gaussian noise
noisy_signal = original_signal + noise; % Noisy signal
```

Initialize (PTM) iteration parameters

```
x_k = noisy_signal; % Initial guess
alpha = 0.5; % Step size
num_iterations = 50; % Number of iterations
% Define smoothing filter: 5-point moving average
filter_func = @(x) filter(ones(1,5)/5, 1, x);
```

Step 2: Applying a Novel Iterative Signal Enhancement Algorithm

Step 3: Visualize the Results

```
figure;
subplot(3,1,1);
plot(t, original_signal, 'k', 'LineWidth', 1.5);
title('Original Signal');
xlabel('Time'); ylabel('Amplitude');
subplot(3,1,2);
plot(t, noisy_signal, 'r', 'LineWidth', 1.5);
title('Noisy Signal');
xlabel('Time'); ylabel('Amplitude');
subplot(3,1,3);
plot(t, x_k, 'b', 'LineWidth', 1.5);
title('Improved signal');
xlabel('Time'); ylabel('Amplitude').
```

Here, x_n is initial signal, and α_n is the learning rate controlling the influence of the previous iteration versus the new signal. This rate controls the balance between the current and previous signals. If the learning rate is too high, the algorithm may become too aggressive and overshoot (Figure 3). If it is too low, the denoising might be slow or ineffective (Figure 1). The algorithm updates the signal in a balanced way by giving equal weight to the previous signal and the current signal, ensuring a smoother and more gradual improvement

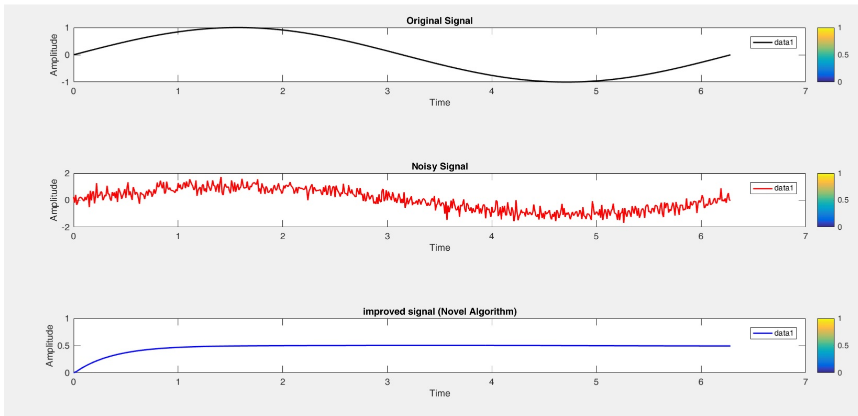

FIGURE 1. we see Original/Noisy/Improved Signals when $\alpha_n = \lambda_n = 0.1$ and $\omega = 0.5$

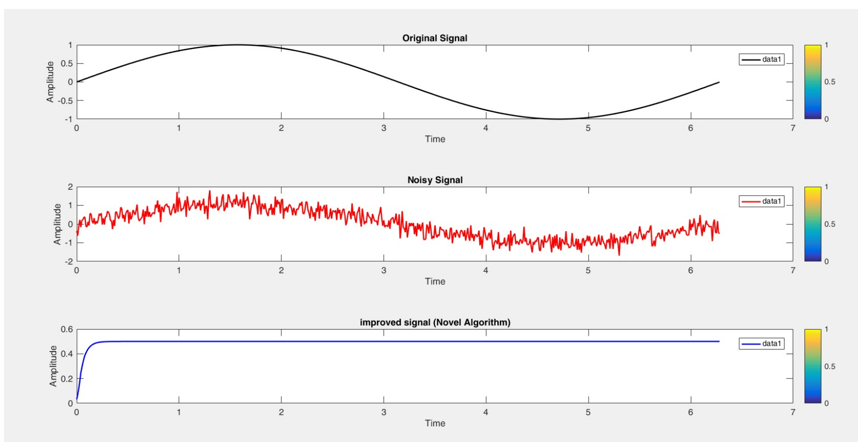

FIGURE 2. we see Original/Noisy/Improved Signals when $\alpha_n = \lambda_n = \omega = 0.5$

(Figure 2) ([20], [31]). The steady progress of the improved signal shows how effective and stable the algorithm's noise reduction process is, the signal becomes cleaner as it improves, and eventually the signal becomes stable and close to the original. This indicates that the algorithm is working correctly and is only reducing the noise without making unnecessary smoothing ([20], [31]).

Remark 6.1. *(Boundary value problems (BVPs) via the proposed algorithm) BVPs constitute fundamental mathematical models that involve finding solutions to differential equations subject to specific boundary conditions. These problems have extensive applications across applied sciences, physics and engineering. Analytical (or numerical) solutions to BVPs are often obtained through techniques rooted in functional analysis (*[4], [5]*). In this*

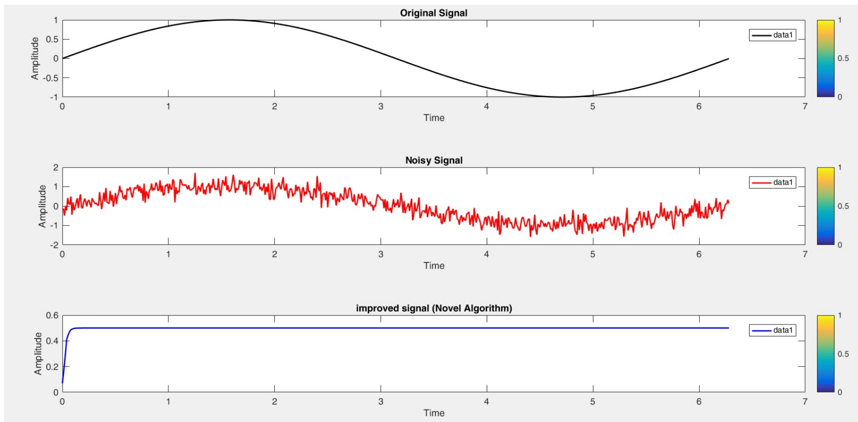

FIGURE 3. we see Original/Noisy/Improved Signals when $\alpha_n = \lambda_n = 0.8$ and $\omega = 0.5$

context, fixed point theory serves as a powerful tool, providing an effective framework for investigating the existence and uniqueness of solutions to BVPs, particularly in the study of nonlinear differential equations ([37], [38]*). The question of the existence of solutions to differential equations using the proposed algorithm remains an open problem. While the algorithm shows promising convergence behavior and potential applicability to a variety of linear and nonlinear problems, rigorous proofs establishing existence and convergence have yet to be fully developed. Further theoretical investigation is needed to confirm these aspects and to better understand the scope and limitations of the method in addressing boundary value problems and nonlinearities.*

7. CONCLUSION

We have proposed a novel algorithm that converges strongly to a FP of Υ. Our conclusions are applied to solve solutions of split feasibility/constrained minimization problems. Secondly, a convergence theorem is obtained for the proposed iteration for $G-$nonexpansive maps on a Hilbert space via a digraph. Furthermore, we provide the numerical illustrations to investigate the basic techniques using Matlab R2016a. Within the future scope of this idea, readers may explore quadratic rates of asymptotic regularity for the PTM method on $W-$hyperbolic spaces (for the definition of this notion, see [26] *and* [32]). Fixed point theory forms the basis of a number of iterative algorithms used particularly in computer science and engineering. Iterations are used to converge to fixed points in areas such as numerical analysis, optimization problems, physical simulations, data mining, and machine learning. These methods provide effective solutions in a wide range of applications. Signal recovery is a field concerned with recovering lost or corrupted signals. Image deblurring is the process of converting a blurred image back to its original, clear state. This process is usually performed to remove blurs that occur in images and is important in many areas such as image processing, space imaging, medical imaging, and computer vision. Our main result can be applied to signal processing and image deblurring (see, [12], [49], [50]).

Signal enhancement and iterative algorithms play a crucial role in modern engineering applications [7]. Bayesian enhancement, adaptive methods, filtering, and iterative algorithms are used across a wide range of problems to improve signal quality ([6], [20], [31]*)*. These techniques are particularly important in preventing noise and distortions and in improving the accuracy of solutions [44]. An interesting direction for future research would be to explore the applicability of the proposed PTM iteration process to fixed point problems in more general and complex settings, such as hyperbolic spaces and CAT(0) spaces, as well as to investigate its potential extension to stochastic environments. In light of the findings presented in this work, we suggest that the proposed iteration process may serve as a useful and efficient tool for researchers studying fixed point problems in various abstract spaces. Readers are encouraged to further explore its potential in broader mathematical models, including those arising in optimization, nonlinear analysis, and applied mathematics.

CONFLICT OF INTEREST

Author declares no conflict of interest.

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