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Local convergence analysis of a three-step iterative scheme with Lagrange interpolation and basin of attraction

Maira Khalid
Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan.
Times University Multan, Pakistan

*Saima Akram
Department of Mathematics, Government College Women University Faisalabad, 38000, Faisalabad, Pakistan
Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan

Muhammad Ibrahim
Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan

*Corresponding Author
saimaakram@gcwuf.edu.pk

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Abstract. Developing efficient and robust iterative methods for solving non-linear equations is a critical task in various scientific and engineering fields. In this study, we presented a three-step eighth-order derivative-free iterative scheme based on Lagrange interpolation. The method involves four parameters and one variable weight function, and it is specifically designed to avoid the computation of higher-order derivatives. A detailed local convergence analysis is carried out under the assumption that the method relies only on the first-order derivative, satisfying a Lipschitz condition. This analysis establishes the convergence radius, provides error estimates, and confirms the existence and uniqueness of the solution. These results support the effective selection of a suitable initial guess based on the computed convergence region. Numerical experiments are conducted to validate the theoretical findings and demonstrate that the presented method provides a larger radius of convergence compared to existing methods of the respective domain. Furthermore, the dynamic behavior of the method is examined using basin of attraction, which illustrates improved stability and reduced chaotic behavior when applied to transcendental equations.

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1. Introduction

Numerical analysis is a fundamental discipline in applied mathematics, providing efficient techniques for approximating solutions to mathematical problems that are difficult or impossible to solve analytically. Among these, non-linear equations are of particular importance, as they frequently arise in various scientific and engineering fields, including fluid dynamics, control theory, signal processing, and machine learning. Non-linear problems also appear in the formulation of differential and integral equations, which are central to the modeling of many physical and industrial systems. In many cases, exact analytical solutions to such problems are either unknown or too complex to derive, which significantly limits the applicability of traditional analytical methods. As a result, the development and analysis of numerical methods, especially iterative schemes, have become essential for the accurate solution of non-linear problems.
To solve non-linear equations numerically, iterative methods are commonly employed to produce successive approximations that converge to the true solution. Among the classical methods, the Newton-Raphson method is widely used for its simplicity and quadratic rate of convergence. However, it requires the computation of derivatives at each iteration, which can be computationally difficult for certain functions. These limitations have motivated the development of alternative approaches, including higher-order and derivative-free iterative methods. Such methods aim to improve convergence behavior while reducing computational cost, especially in cases where the derivative is unavailable or highly sensitive to numerical errors. As a result, derivative-free iterative methods have become an important area of numerical analysis, particularly for solving non-linear equations arising in practical applications. Therefore, many researchers have focused on iterative methods [18]. The goal is to approximate a unique solution, denoted as Φ, and the equation of the form

$$P(q) = 0. \tag{1. 1}$$

Here, P is defined on an open convex set $\Theta \subset \Lambda$ with values in Λ, where Λ is either real or complex numbers.
In evaluate the performance of an iterative method in solving equation(1. 1), it becomes crucial to analyze its convergence properties. It provides the theoretical foundation to ensure that the generated sequence remains stable and approaches the desired solution under specific conditions. Moreover, convergence analysis is generally carried out using two approaches: local and semi-local convergence. In the case of local (the ball of convergence is centered at the solution), whereas semi-local (the ball of convergence is centered at the initial guess q_0).
However, in many practical applications, the exact solution is not known in advance, making the selection of an appropriate initial guess a critical and often difficult task. Although a method may demonstrate strong local stability, its convergence largely depends on the

proximity of the initial guess to the actual solution. Semilocal convergence analysis plays a vital role in addressing this issue by formulating convergence criteria centered at the initial guess rather than the solution itself.

In local convergence analysis [3, 10], the method's behavior is studied under the assumption that the initial guess lies sufficiently close to the exact solution. This analysis examines the convergence radius to determine the region near the root where the method is guaranteed to converge. It is widely used to evaluate the accuracy, stability, and convergence rate of the iterative schemes when proximity to the root can be assumed. In this study, well-defined conditions for convergence are established to provide meaningful guidance on the selection of initial approximations, thereby enhancing the practical utility and robustness of the proposed method.

Many researchers have investigated local convergence under Lipschitz conditions [14]. For instance, Devi et al. [5], in 2024, proposed a three-step sixth-order iterative method. They analyzed its convergence domain and evaluated the performance and efficiency of the scheme. In 2023, Devi et al. [6] studied the local convergence of a parameter-based, derivative-free continuation method, exploring the existence and uniqueness of the developed scheme.

In this article, our primary focus is on a modified version of Kung and Traub's iterative scheme [13]. In 1974, they developed an optimal two iterative multi-step families of arbitrary order. Their study is divided into two parts: the first is a scheme without memory, and the second is a with-memory iterative scheme. These schemes are based on inverse interpolating polynomials. Here, we emphasize the local convergence analysis of a three-step, iterative scheme [1], defined for each $\tau = 0, 1, 2, \dots$ by

$$\begin{aligned} h_\tau &= q_\tau + \eta_1 P(q_\tau), \\ d_\tau &= q_\tau - \frac{P(q_\tau)}{P[h_\tau, q_\tau] + \eta_2 P(h_\tau)}, \\ b_\tau &= d_\tau - S(A_\tau) \times B_\tau \times C_\tau \times P(d_\tau), \\ q_{\tau+1} &= b_\tau - \frac{P(b_\tau)}{G_\tau}, \end{aligned} \tag{1. 2}$$

where,

$$\begin{aligned} &A_\tau = \frac{P(d_\tau)}{P(q_\tau)}, \quad B_\tau = \frac{1}{P[d_\tau, h_\tau] + \eta_2 P(h_\tau) + \eta_3 (d_{\theta a u} - h_\tau)(d_\tau - q_\tau)}, \\ &C_\tau = \frac{P(q_\tau) + \beta P(d_\tau)}{P(q_\tau) + (\beta - 2) P(d_\tau)}, \quad G_\tau = T_1 + T_2 + T_3 + T_4 + T_5, \\ &T_1 = P[b_\tau, h_\tau] + P[b_\tau, q_\tau] + P[b_\tau, d_\tau] - P[q_\tau, d_\tau] - P[h_\tau, d_\tau] - P[q_\tau, h_\tau] \\ &T_2 = \frac{(q_\tau - b_\tau) P(q_\tau)}{(q_\tau - h_\tau)(q_\tau - d_\tau)}, \quad T_3 = \frac{(h_\tau - b_\tau) P(h_\tau)}{(h_\tau - q_\tau)(h_\tau - d_\tau)} \\ &T_4 = \frac{(d_\tau - b_\tau) P(d_\tau)}{(d_\tau - h_\tau)(d_\tau - q_\tau)}, \quad T_5 = \eta_4 (b_\tau - h_\tau)(b_\tau - d_\tau)(b_\tau - q_\tau), \end{aligned}$$

and $S(A_\tau)$ is a real-valued weight function, defined for the special case as

$$S(A_\tau) = \frac{1}{1 + A_\tau - 2A_\tau^2}.$$

The three-step iterative scheme (1. 2), expressed as Method MSI, is of eighth order, derivative-free, and without memory. It requires only four function evaluations per cycle and has an efficiency index of $8^{\frac{1}{4}} \approx 1.68179$. As a conventional approach, the iterative scheme (1. 2) is evaluated by expanding the Taylor series, which involves a set of hypotheses that the non-linear function is up to the eighth-order on the interval Θ. Such hypotheses certainly restrict the applicability of the technique, as scheme (1. 2) involves only up to the first-order derivative. Now, we consider some motivational applications to find the radius of the convergence of the iterative scheme (1. 2):

(i) **Motivational example** Let us consider a real-valued function $P : T \subseteq \mathbb{R} \to \mathbb{R}$, $\Omega = [-0.5, 1.5]$, which is given by

$$P(q) = \begin{cases} q^5 - q^4 + q^3 \ln(q^2) & \text{if } q \neq 0, \\ 0 & \text{if } q = 0. \end{cases}$$

The corresponding derivatives are

$$P'(q) = 5q^4 - 4q^3 + 3q^2 \ln(q^2) + 2q^2,$$

$$P''(q) = 20q^3 - 12q^2 + 6q \ln(q^2) + 10q,$$

$$P'''(q) = 60q^2 - 24q + 6 \ln(q^2) + 22.$$

It is evident that P''' is unbounded over the interval I, and thus, the assumptions required for convergence are not satisfied. However, both P' and P'' remain bounded on I, as

$$\lim_{q \to 0} q^2 \ln(q^2) = \lim_{q \to 0} q \ln(q^2) = 0.$$

Hence, the findings in [1] cannot be utilized to determine the convergence scheme (1. 2), as they require hypotheses concerning the eighth derivative of the function P or higher.

(ii) Initial approximation selection is often uncertain because the radius of convergence is not identified. Moreover, the radius is also valuable even when the solution is unknown.

(iii) The study in [1] does not provide error estimates between two successive iterations, making it difficult to predict how many iterations are needed to achieve a certain level of accuracy using method (1. 2).

(iv) Local convergence results for the presented three-step eighth-order iterative scheme is highly valuable because it addresses the challenge of selecting the initial guess and ensures the convergence of the multi-point method.

(v) Additional, the uniqueness of the solution Φ is not addressed in [1], which leaves the theoretical justification for uniqueness unresolved.

The article is organized as follows: In Section 2, we present the basic terminology, establish local convergence conditions, and state a convergence theorem for the derivative-free iterative scheme (1. 2). In Section 3, we discussed the existing three-step iterative schemes and used differential and integral equation to determine the radius of convergence. In Section 4, the basin of attraction is examined using transcendental equations to illustrate the behavior of iterative methods. The conclusion is provided in Section 5.

2. LOCAL CONVERGENCE ANALYSIS

A definition related to how functions behave monotonically is necessary:

Definition 2.1 (Monotonicity)**.** *Let us suppose that* $P : \mathbb{D} \subseteq \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a function and* P *is non-decreasing on* $\mathbb{D}$*, if for each* $(s_1, s_2), (s_3, s_4) \in \mathbb{D}$ *with* $s_1 \leq s_3, s_2 \leq s_4$*, and*

$$P(s_1, s_2) \leq P(s_3, s_4). \tag{2. 3}$$

Furthermore, P *is increasing function on* $\mathbb{D}$*, if* $s_1 \leq s_3$ *and* $s_2 < s_4$*, or* $s_1 < s_3$ *and* $s_2 \leq s_4$ *or* $s_1 < s_3$*, and* $s_2 < s_4$ *implies that*

$$P(s_1, s_2) < P(s_3, s_4). \tag{2. 4}$$

In this section, we introduce the functions and parameters to develop the analysis of ball convergence. Assume that $\eta_1, \eta_2, \eta_3, \eta_4 \in \mathbb{R}, \beta \neq 2, \varpi \geq 0$, and set the domain $V = [0, +\infty)$.

Suppose that:

(i) Let $\zeta_0 : V \times V \to V, \lambda : V \to V$ be continuous and non-decreasing functions such that $\zeta_0(0, 0) = 0$, and $\lambda(0) = 0$. Let us define the parameter ξ_0 is given by

$$\xi_0 = sup\{\rho \geq 0 : \zeta_0(0, 0) < 1\}, \tag{2. 5}$$

so that the equation is defined as

$$a_1(\rho) - 1 = 0,$$

where, $a_1(\rho) = \zeta_0(\varpi\varrho, \rho) + |\eta_2|\lambda(\varpi\rho)\varpi\rho$ has a minimal zeros $\xi_0 \in V - \{0\}$, respectively. Let $V_0 = [0, \xi_0)$.

(ii) Let $\zeta_1 : V_0 \times V_0 \to V$ be a continuous and non-decreasing function. Consider the function a_3 and Υ_1 defined on the interval $[0, \xi_0)$ by

$$a_3(\rho) = \frac{\zeta_1(a_2(\rho), \rho) + \eta_2\lambda(\varpi\rho)\varpi\rho}{1 - a_1(\rho)},$$

and

$$\Upsilon_1(\rho) = a_3(\rho) - 1.$$

Suppose that

$$\zeta_1(0, 0) < 0.$$

Then

$$\Upsilon_1(0) = \frac{\zeta_1(0, 0)}{1 - a_1(0)} - 1 < 0, \tag{2. 6}$$

and

$$\Upsilon_1(\rho) \to \quad a \quad positive \quad number \quad or \quad +\infty \quad as \quad \rho \to \xi_0^-. \tag{2. 7}$$

Using equations (2. 6) and (2. 7) and the intermediate value theorem, we can conclude that there exists at least one root in the interval $(0, \xi_0)$. Let us suppose that the equation $\Upsilon_1(\rho) = 0$ has its smallest root ω_1.

(iii) There exist continuous and non-decreasing function $\zeta : V_0 \to V_0$. Define another equation

$$a_4(\rho) - 1 = 0, \quad a_7(\rho) - 1 = 0, \quad a_8(\rho) - 1 = 0,$$

where,

$$a_4(\rho) = (1 + \lambda(\rho))\rho,$$
$$a_7 = \xi_0(a_3\rho, \varpi\rho) + |\eta_2|\lambda(\varpi\rho)\varpi\rho + |\eta_3|a_6(\rho),$$
$$a_8(\rho) = \zeta(\rho) + |\beta - 2|\lambda(a_3(\rho)\rho)a_3(\rho),$$

have minimal zeros $\xi_1, \xi_2, \xi_3 \in V_0 - \{0\}$. Let us suppose that $\xi_4 = min\{\xi_1, \xi_2, \xi_3\}$ and $V_1 = [0, \xi_4)$.

(iv) Assume that a_{11} and Υ_2 are two functions defined on the interval $[0, \xi_4)$ by

$$a_{11} = (1 + a_{10}(\rho)\lambda(a_3(\rho)))a_3(\rho),$$
$$a_{11} = \left(1 + \frac{S(a_5(\rho))}{1 - a_7(\rho)} \times a_9(\rho)\lambda(a_3(\rho))\right) a_3(\rho),$$

and

$$\Upsilon_2 = a_{11} - 1.$$

Suppose that

$$\left(1 + \frac{S(0)}{1 - a_7(0)} \times a_9(0)\lambda(0)\right) \zeta_1(0, 0) < 1. \tag{2. 8}$$

and

$$\Upsilon_2(\rho) \to \quad a \quad positive \quad number \quad or \quad +\infty \quad as \quad \rho \to \xi_4^-. \tag{2. 9}$$

By using (2. 8), we can say that $\Upsilon_2(0) < 0$. Therefore, by the intermediate value theorem equation $\Upsilon_2(\rho) = 0$ has at-least one root in the interval $(0, \xi_4)$. Denote the smallest root ω_2.

(v) There exist continuous and non-decreasing function $\lambda_0 : V_0 \to V_0$. Define equation

$$a_{17}(\rho) - 1 = 0,$$

where,

$$\begin{aligned} a_{17}(\rho) = {} & \xi_0(a_{11}(\rho)\rho, \varpi\rho) + \lambda(a_{11}(\rho)\rho + \rho) + \lambda(a_{11}(\rho)\rho + a_3(\rho)\rho) \\ & + \lambda(a_3(\rho)\rho + \rho) + \lambda(\varpi\rho + a_3(\rho)\rho) + \lambda(\varpi\rho + \rho) \\ & + a_{13}(\rho) + a_{14}(\rho) + a_{15}(\rho) + a_{16}(\rho), \end{aligned}$$

has a minimal zero $\xi_5 \in V_0 - \{0\}$. Let $V_2 = [0, \xi_5)$.

(vi) Suppose that a_{18} and Υ_3 are the functions on the interval $[0, \xi_5)$ by

$$a_{18} = \left(1 + \frac{\lambda(a_{11}(\rho)\rho)}{1 - a_{17}(\rho)}\right) a_{11}(\rho),$$

and

$$\Upsilon_3 = a_{18} - 1.$$

Suppose that

$$\left(1 + \frac{\lambda(0)}{1 - a_{17}(0)}\right) (1 + S(0)a_9(0)\lambda(0))\, \zeta_1(0, 0) < 1, \tag{2. 10}$$

$$\Upsilon_2(\rho) \to \quad a \quad positive \quad number \quad or \quad +\infty \quad as \quad \rho \to \xi_5^-. \tag{2. 11}$$

By employing, we get $\Upsilon_3(0) < 0$. So, by intermediate value theorem, there exists at least one root in the interval $(0, \xi_5)$. Let ω_3 denote the smallest root of the equation $\Upsilon_3(\rho) = 0$.

Consider the radius of convergence ω for the iterative scheme (1. 2) is given below

$$\omega = min\{\omega_j\}, \quad j = 1, 2, 3. \tag{2. 12}$$

Then, set $V_3 = [0, \omega)$.
These definitions follow by the parameter ω, we have

$$0 \leq a_1(\rho) < 1, \tag{2. 13}$$
$$0 \leq a_4(\rho) < 1, \tag{2. 14}$$
$$0 \leq a_7(\rho) < 1, \tag{2. 15}$$
$$0 \leq a_8(\rho) < 1, \tag{2. 16}$$
$$0 \leq a_{17}(\rho) < 1, \tag{2. 17}$$
$$0 \leq \Upsilon_j(\rho) < 1, \tag{2. 18}$$

hold for all $\rho \in V_3$, and $j = 1, 2, 3$. Let us suppose that a continuous differentiable function $P : \Theta \subset \Lambda \to \Lambda$ and $[., ., P] : \Lambda \times \Lambda \to \mathbb{R}$ be a first divided difference of the function P. Also, suppose that there exists $\Phi \in \Theta$, and $\zeta_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous and non-decreasing function with $\zeta_0(0, 0) = 0$ such that for each $q, w \in \Theta$

$$P(\Phi) = 0, \quad P'(\Phi) \in \mathbb{R} \tag{2. 19}$$
$$|P'(\Phi)^{-1}([q, w; P] - P'(\Phi))| \leq \zeta_0(|q - \Phi|, |w - \Phi|), \tag{2. 20}$$
$$|P'(\Phi)^{-1}([q, \Phi; P] - P'(\Phi))| \leq \zeta(|q - \Phi|). \tag{2. 21}$$

Assume that $\Theta_0 = \Theta \cap M(\Phi, r_0)$. and the parameters $\eta_1, \eta_2, \eta_3, \eta_4 \in \mathbb{R}, \beta \neq 2, \varpi \geq 0$, functions $\lambda_0, \lambda, \zeta_2 : [0, \omega_0) \to [0, +\infty), \zeta_1, \zeta_3 : [0, \omega_0) \times [0, \omega_0) \to [0, +\infty)$ such that for each $q, w, t \in \Theta_0$

$$|I + \eta_1[q, \Phi; P]| \leq \varpi, \tag{2. 22}$$
$$|[q, \Phi; P]| \leq \lambda_0(|q - \Phi|), \tag{2. 23}$$
$$|P'(\Phi)^{-1}[q, \Phi; P]| \leq \lambda(|q - \Phi|), \tag{2. 24}$$
$$|P'(\Phi)^{-1}([q, w; P] - [w, \Phi; P])| \leq \zeta_1(|q - w||, ||w - \Phi|), \tag{2. 25}$$
$$|P'(\Phi)^{-1}([q, w; P] - [t, w; P])| \leq \zeta_2(|q - t|), \tag{2. 26}$$
$$|P'(\Phi)^{-1}([q, w; P] - [t, q; P])| \leq \zeta_3(|q - t|, |w - q|), \tag{2. 27}$$
$$|P'(\Phi)^{-1}| \leq \mu, \tag{2. 28}$$
$$\overline{M}(\Phi, R^*) \subseteq \Theta. \tag{2. 29}$$

Then, the sequence q_τ starting from initial guess $q_0 \in M(\Phi, \omega) - \{\Phi\}$ and generated by three-step derivative-free scheme (1. 2), is well-defined in $\overline{M}(\Phi, R^*)$. The solution converges to the approximate root Φ, where Υ_3 is a function with a smallest radius ω_3. Additionally, the following error bounds are valid for $\tau \geq 0$

$$|d_\tau - \Phi| \leq a_3(|q_\tau - \Phi|) < |q_\tau - \Phi| < \omega, \tag{2. 30}$$
$$|b_\tau - \Phi| \leq a_{11}(|q_\tau - \Phi|) < |q_\tau - \Phi| < \omega, \tag{2. 31}$$
$$|d_{\tau+1} - \Phi|| \leq a_{18}(|q_\tau - \Phi|) < |q_\tau - \Phi| < \omega. \tag{2. 32}$$

Here, a_m are the functions that defined previously. Moreover, if $R_1 \geq \omega$ then $\lim_{x\to\infty} P(q) = \Phi$ is the only limit point of the non-linear equation $P(q) = 0$ in $\Theta_1 = \Theta \cap M(\Phi, R_1)$.

Proof. We shall verify the inequalities given in (2. 30)-(2. 32) using mathematical induction on τ. First, by applying (2. 19), (2. 22) and the hypothesis $\Phi \in M(\Phi,\omega) - \{\Phi\}$ for $\tau = 0$, we get

$$\begin{aligned} |h_0 - \Phi| &= |q_0 - \Phi + \eta_1(P(q_0) - P(\Phi))|, \\ &= |q_0 - \Phi + \eta_1[q_0 - \Phi; P](q_0 - \Phi)|, \\ &= |(I + \eta_1[q_0 - \Phi; P](q_0 - \Phi)|, \\ &\leq |I + \eta_1[q_0 - \Phi; P]||q_0 - \Phi|, \\ &\leq \varpi|q_0 - \Phi|. \end{aligned} \tag{2. 33}$$

Now, we take the first sub-step of the derivative-free iterative scheme in (1. 2) for $\tau = 0$, to establish the error bound of given in (2. 30), as follows

$$\begin{aligned} d_0 - \Phi &= q_0 - \Phi - P(q_0)(P[h_0, q_0] + \eta_2 P(h_0))^{-1}, \\ &= -(P[h_0, q_0] + \eta_2 P(h_0))^{-1}\big(P(q_0) - P(\Phi) \\ &\quad - (P[h_0, q_0] + \eta_2 P(h_0))(q_0 - \Phi)\big), \\ |d_0 - \Phi| &\leq |P'(\Phi)(P[h_0, q_0] + \eta_2 P(h_0))^{-1}| \\ &\quad \times |P'(\Phi)^{-1}([q_0, \Phi; P] - (P[h_0, q_0] + \eta_2 P(h_0)))| \times |q_0 - \Phi|. \end{aligned} \tag{2. 34}$$

In order to prove the invertibility, we must show that $(P[h_0, q_0] + \eta_2 P(h_0))$ exist. By employing equations (2. 20) and (2. 24), we get the following expression

$$\begin{aligned} &|P'(\Phi)^{-1}(P[h_0, q_0] + \eta_2 P(h_0) - P'(\Phi))| \\ &\qquad = |P'(\Phi)^{-1}([h_0, q_0; P] - P'(\Phi) + \eta_2[h_0, \Phi; P](h_0 - \Phi))| \\ &\qquad \leq \zeta_0(|h_0 - \Phi|, |q_0 - \Phi|) + |\eta_2|\lambda(|h_0 - \Phi|)|h_0 - \Phi|, \\ &\qquad \leq \zeta_0(\varpi|q_0 - \Phi|, |q_0 - \Phi|) + |\eta_2|\lambda(\varpi|q_0 - \Phi|)\varpi|q_0 - \Phi|, \\ &\qquad = a_1(|q_0 - \Phi|) < 1. \end{aligned} \tag{2. 35}$$

From the condition (2. 35) and Banach lemma on invertible [15] operators, we have

$$|P'(\Phi)(P[h_0, q_0] + \eta_2 P(h_0))^{-1}| < \frac{1}{1 - a_1(|q_0 - \Phi|)} \tag{2. 36}$$

By (2. 24) and (2. 25), we get that

$$\begin{aligned} &|P'(\Phi)^{-1}([q_0, \Phi; P] - (P[h_0, q_0] + \eta_2 P(h_0)))| \\ &= |P'(\Phi)^{-1}([q_0, \Phi; P] - [h_0, q_0; P] - \eta_2[h_0, \Phi, P](h_0 - \Phi))|, \\ &\leq \zeta_1(|h_0 - q_0|, |q_0 - \Phi|) + \eta_2\lambda(|h_0 - \Phi|)|h_0 - \Phi|. \end{aligned} \tag{2. 37}$$

Inserting equations (2. 36) and (2. 37) in (2. 34) to obtained the first sub-step of the iterative scheme (1. 2)

$$|d_0 - \Phi| \leq \frac{\zeta_1(|h_0 - q_0|, |q_0 - \Phi|) + \eta_2\lambda(|h_0 - \Phi|)|h_0 - \Phi|}{1 - a_1(|q_0 - \Phi|)}|q_0 - \Phi|. \tag{2. 38}$$

Here,

$$\begin{aligned}|h_0 - q_0| &= |\eta_1(P(q_0) - P(\Phi))|,\\ &= |\eta_1[q_0, \Phi; P](q_0 - \Phi)|,\\ &\leq |\eta_1|\lambda_0(|q_0 - \Phi|)|q_0 - \Phi|, \quad by \quad (2.23)\\ &= a_2(|q_0 - \Phi|). \end{aligned} \tag{2. 39}$$

Thus, assertion (2. 38) becomes as

$$\begin{aligned}|d_0 - \Phi| &\leq \frac{\zeta_1(a_2(|q_0 - \Phi|), |q_0 - \Phi|) + |\eta_2|\lambda(\varpi|q_0 - \Phi|)\varpi|q_0 - \Phi|}{1 - a_1(|q_0 - \Phi|)}|q_0 - \Phi|,\\ &= a_3(|q_0 - \Phi|)|q_0 - \Phi| < |q_0 - \Phi| < \omega. \end{aligned} \tag{2. 40}$$

Thus, d_0 is well-defined and the error bound (2. 30) hold for $\tau = 0$. Now, we take the second sub-step of the iterative scheme (1. 2) for $\tau = 0$

$$|b_0 - \Phi| \leq |d_0 - \Phi| + |S(A_0)| \times |B_0| \times |C_0|. \tag{2. 41}$$

Here,

$$|A_0| \leq |P'(\Phi)^{-1}P(d_0)||P'(\Phi)P(q_0)^{-1}|, \tag{2. 42}$$

$P(q_0)$ can be written as

$$P(q_0) = P(q_0) - P(\Phi) = [q_0, \Phi; P](q_0 - \Phi).$$

To prove $P(q_0)$ is invertible. Using the estimates (2. 24), we get that

$$|P'(\Phi)^{-1}P(q_0)| \leq (1 + \lambda(|q_0 - \Phi|))|q_0 - \Phi| = a_4(|q_0 - \Phi|) < 1. \tag{2. 43}$$

Therefore, by Banach lemma on invertible operators $P(q_0)^{-1}$ exist, and

$$|P'(\Phi)P(q_0)^{-1}| \leq \frac{1}{1 - a_4(|q_0 - \Phi|)}. \tag{2. 44}$$

Thus, inserting (2. 24) and (2. 44)in (2. 42), we obtain

$$\begin{aligned}|A_0| \leq \frac{\lambda(|d_0 - \Phi|)|d_0 - \Phi|}{1 - a_4(|q_0 - \Phi|)} &\leq \frac{\lambda(a_3(|q_0 - \Phi|)|q_0 - \Phi|)a_3(|q_0 - \Phi|)|q_0 - \Phi|}{1 - a_4(|q_0 - \Phi|)}\\ &= a_5(|q_0 - \Phi|). \end{aligned} \tag{2. 45}$$

By applying the first sub-step of the derivative-free iterative scheme (1. 2), we get the following product

$$(d_0 - h_0)(d_0 - q_0) = \frac{\eta_1 P(q_0)^2}{P[h_0, q_0] + \eta_2 P(h_0)} + \left(\frac{P(q_0)}{P[h_0, q_0] + \eta_2 P(h_0)}\right)^2. \tag{2. 46}$$

Putting conditions (2. 24), (2. 28), and (2. 36) in (2. 46) and taking the norm on both sides, and resulting expression is as follows

$$\begin{aligned}|d_0 - h_0| \times |d_0 - q_0| &\leq \frac{|\eta_1|\lambda(|q_0 - \Phi|)^2|q_0 - \Phi|^2}{1 - a_1(|q_0 - \Phi|)} + \frac{\mu\lambda(|q_0 - \Phi|)^2|q_0 - \Phi|^2}{(1 - a_1(|q_0 - \Phi|))^2},\\ &= a_6(|q_0 - \Phi|). \end{aligned} \tag{2. 47}$$

Now, prove the invertibility of the denominator B_0. Using inequalities (2. 20), (2. 24), and (2. 26), we have

$$\begin{aligned}
&|P'(\Phi)^{-1}(P[d_0, h_0] - P'(\Phi) + \eta_2 P(h_0) + \eta_3(d_0 - h_0)(d_0 - q_0))|,\\
&= |P'(\Phi)^{-1}([d_0, h_0; P] - P'(\Phi) + \eta_2[h_0, \Phi, P](h_0 - \Phi) + \eta_3(d_0 - h_0)(d_0 - q_0))|,\\
&\leq \zeta_0(|d_0 - \Phi|, |h_0 - \Phi|) + |\eta_2|\lambda(|h_0 - \Phi|)|h_0 - \Phi| + |\eta_3|a_6(|q_0 - \Phi|),\\
&\leq \zeta_0(a_3(|q_0 - \Phi|), \varpi|q_0 - \Phi|) + |\eta_2|\lambda(\varpi|q_0 - \Phi|)\varpi|q_0 - \Phi| + |\eta_3|a_6(|q_0 - \Phi|),\\
&\leq a_7(|q_0 - \Phi|) < 1. \qquad (2.\ 48)
\end{aligned}$$

Hence, by the Banach lemma $(P[d_0, h_0] - P'(\Phi) + \eta_2 P(h_0) + \eta_3(d_0 - h_0)(d_0 - q_0))^{-1}$ exist and

$$|P'(\Phi)(P[d_0, h_0] - P'(\Phi) + \eta_2 P(h_0) + \eta_3(d_0 - h_0)(d_0 - q_0))^{-1}| \leq \frac{1}{1 - a_7(|q_0 - \Phi|)}. \qquad (2.\ 49)$$

Next, we check the invertibility of $P(q_0) + (\beta - 2)P(d_0)$ By employing the assertions (2. 21) and (2. 24), we have

$$\begin{aligned}
&|P'(\Phi)^{-1}(q_0 - \Phi)^{-1}(P(q_0) + (\beta - 2)P(d_0) - P'(\Phi)(q_0 - \Phi))|,\\
&= |(q_0 - \Phi)^{-1}(P'(\Phi)^{-1}(([q_0 - \Phi, P] - P'(\Phi))(q_0 - \Phi) + (\beta - 2)[d_0, \Phi, P](d_0 - \Phi)))|,\\
&\leq |q_0 - \Phi|^{-1}(\zeta(|q_0 - \Phi|)|q_0 - \Phi| + |\beta - 2|\lambda(|d_0 - \Phi|)|d_0 - \Phi|),\\
&\leq |q_0 - \Phi|^{-1}(\zeta(|q_0 - \Phi|)|q_0 - \Phi| + |\beta - 2|\lambda(a_3(|q_0 - \Phi|)|q_0 - \Phi|)a_3(|q_0 - \Phi|)|q_0 - \Phi|),\\
&= \zeta(|q_0 - \Phi|) + |\beta - 2|\lambda(a_3(|q_0 - \Phi|)|q_0 - \Phi|)a_3(|q_0 - \Phi|),\\
&\leq a_8(|q_0 - \Phi|) < 1. \qquad (2.\ 50)
\end{aligned}$$

Thus, we accomplish that $(P(q_0) + (\beta - 2)P(d_0))^{-1}$ exists and the lemma of perturbation with Banach inverses, we obtain the following expression

$$|P'(\Phi)(P(q_0) + (\beta - 2)P(d_0))^{-1})| \leq \frac{1}{|q_0 - \Phi|(1 - a_8(|q_0 - \Phi|))}, \qquad (2.\ 51)$$

and using equations (2. 24) and (2. 51), we get the following inequality for $|C_0|$

$$\begin{aligned}
|C_0| &\leq \frac{\lambda(|q_0 - \Phi|)|q_0 - \Phi| + |\beta|\lambda(|d_0 - \Phi|)|q_0 - \Phi|}{|q_0 - \Phi|(1 - a_8(|d_0 - \Phi|))},\\
|C_0| &\leq \frac{\lambda(|q_0 - \Phi|) + |\beta|\lambda(a_3(|q_0 - \Phi|)|q_0 - \Phi|)a_3(|q_0 - \Phi|)}{1 - a_8(|q_0 - \Phi|)},\\
|C_0| &= a_9(|q_0 - \Phi|). \qquad (2.\ 52)
\end{aligned}$$

Using estimates (2. 45),(2. 49) and (2. 52), we get

$$|S(A_0)||B_0||C_0| = \frac{S(a_5(|q_0 - \Phi|))}{1 - a_7(|q_0 - \Phi|)} \times a_9(|q_0 - \Phi|) = a_{10}(|q_0 - \Phi|). \qquad (2.\ 53)$$

Inserting (2. 24), and (2. 53) in (2. 41), we obtain the second sub-step of derivative-free iterative scheme (1. 2):

$$\begin{aligned} |b_0-\Phi| &\le |d_0-\Phi| + a_{10}(|q_0-\Phi|)\lambda(|d_0-\Phi|)|d_0-\Phi|, \\ &\le (1+a_{10}(|q_0-\Phi|)\lambda(|d_0-\Phi|))\,|d_0-\Phi|, \\ &\le (1+a_{10}(|q_0-\Phi|)\lambda(a_3(|q_0-\Phi|)))\,a_3(|q_0-\Phi|)|q_0-\Phi|, \\ |b_0-\Phi| &= a_{11}(|q_0-\Phi|)|q_0-\Phi| < |q_0-\Phi| < \omega. \end{aligned} \tag{2. 54}$$

Hence, the error bound (2. 31) and the iteration for $d_0 \in M(\Phi,\omega)$ is validated for $\tau = 0$. Now, we take the third sub-step of the iterative scheme (1. 2) for $\tau = 0$

$$|q_1-\Phi| \le |b_0-\Phi| + |P'(\Phi)^{-1}P(b_0)||P'(\Phi)G_0^{-1}|. \tag{2. 55}$$

Now, simplifying the denominator of the third sub-step of the iterative scheme (1. 2)

$$(h_0-d_0) = \eta_1 P(q_0) + \frac{P(q_0)}{P[h_0,q_0]+\eta_2 P(h_0)}. \tag{2. 56}$$

To prove the invertibility of (2. 56), and get the following expression

$$\begin{aligned} &\left|(\eta_1 P'(\Phi)|q_0-\Phi|)^{-1}\left(\eta_1(P(q_0)-P'(\Phi)(q_1-\Phi)) + \frac{P(q_0)}{P[h_0,q_0]+\eta_2 P(h_0)}\right)\right| \\ &\le \frac{1}{|\eta_1|}|q_0-\Phi|^{-1}\left(|\eta_1|\zeta(|q_0-\Phi|)|q_0-\Phi| + \frac{\lambda(|q_0-\Phi|)|q_0-\Phi|}{1-a_1(|q_0-\Phi|)}\right), \\ &= \frac{1}{|\eta_1|}\left(|\eta_1|\zeta(|q_0-\Phi|) + \frac{\lambda(|q_0-\Phi|)}{1-a_1(|q_0-\Phi|)}\right), \\ &= a_{12}(|q_0-\Phi|) < 1. \end{aligned} \tag{2. 57}$$

By the Banach lemma, $\left(\eta_1 P(q_0) + \frac{P(q_0)}{P[h_0,q_0]+\eta_2 P(h_0)}\right)^{-1}$ exists, and

$$\left|P'(\Phi)\left(\eta_1 P(q_0) + \frac{P(q_0)}{P[h_0,q_0]+\eta_2 P(h_0)}\right)^{-1}\right| \le \frac{1}{\eta_1|q_0-\Phi|(1-a_{12}(|q_0-\Phi|))}. \tag{2. 58}$$

T_2, T_3, T_4 and T_5 can be expressed as

$$\begin{aligned} T_2 &= -\frac{1}{\eta_1} + S(A_0)\times B_0\times C_0\times\left(\frac{P[h_0,q_0]+\eta_2(Ph_0)}{\eta_1 P(q_0)}\right), \\ |T_2| &\le \frac{\mu}{|\eta_1|} + a_{10}(|q_0-\Phi|)\times\left(\frac{\lambda(|h_0-q_0|)+|\eta_2|\lambda(|h_0-\Phi|)|h_0-\Phi|}{|\eta_1|(1-a_4(|q_0-\Phi|))}\right), \\ &\le \frac{\mu}{|\eta_1|} + a_{10}(|q_0-\Phi|)\times\left(\frac{\lambda(a_1(|q_0-\Phi|))+|\eta_2|\lambda(\varpi|q_0-\Phi|)\varpi|q_0-\Phi|}{|\eta_1|(1-a_4(|q_0-\Phi|))}\right), \\ &= a_{13}(|q_0-\Phi|). \end{aligned} \tag{2. 59}$$

$$T_3 = \frac{\left(\eta_1 P(q_0) + \frac{P(q_0)}{P[h_0,q_0]+\eta_2(Ph_0)} + S(A_0)\times B_0 \times C_0\right) P(h_0)}{\eta_1 P(q_0)\left(\eta_1 P(q_0) + \frac{P(q_0)}{P[h_0,q_0]+\eta_2(Ph_0)}\right)},$$

$$|T_3| \leq \frac{\left(|\eta_1|\lambda_0(|q_0-\Phi|)|q_0-\Phi| + \frac{\lambda(|q_0-\Phi|)|q_0-\Phi|}{1-a_1(|q_0-\Phi|)} + a_{10}(|q_0-\Phi|)\right)\lambda(|h_0-\Phi|)|h_0-\Phi|}{(|\eta_1\mu|(1-a_4(|q_0-\Phi|)))(|\eta_1||q_0-\Phi|(1-a_{12}(|q_0-\Phi|)))},$$

$$\leq \frac{\left(|\eta_1\lambda_0(|q_0-\Phi|)|q_0-\Phi| + \frac{\lambda(|q_0-\Phi|)|q_0-\Phi|}{1-a_1(|q_0-\Phi|)} + a_{10}(|q_0-\Phi|)\right)\lambda(\varpi|q_0-\Phi|)\varpi|q_0-\Phi|}{(|\eta_1|\mu(1-a_4(|q_0-\Phi|)))(|\eta_1||q_0-\Phi|(1-a_{12}(|q_0-\Phi|)))},$$

$$= a_{14}(|q_0-\Phi|). \qquad (2.\ 60)$$

$$T_4 = \frac{(S(A_0)\times B_0\times C_0)\times P(d_0)\times\left(\frac{P[h_0,q_0]+\eta_2 P(h_0)}{P(q_0)}\right)}{\eta_1 P(q_0) + \frac{P(q_0)}{P[h_0,q_0]+\eta_2 P(h_0)}},$$

$$|T_4| \leq \frac{a_{10}(|q_0-\Phi|)\lambda(|d_0-\Phi|)|d_0-\Phi| \times (\lambda(|h_0-q_0|) + |\eta_2|\lambda(|h_0-\Phi|)|h_0-\Phi|)}{(1-a_{12}(|q_0-\Phi|))(1-a_4(|q_0-\Phi|))},$$

$$\leq \frac{a_{10}(|q_0-\Phi|)\lambda(|d_0-\Phi|)|d_0-\Phi| \times (\lambda(a_2|q_0-\Phi|) + |\eta_2|\lambda(\varpi|q_0-\Phi|)\varpi|q_0-\Phi|)}{(1-a_{12}(|q_0-\Phi|))(1-a_4(|q_0-\Phi|))},$$

$$= a_{14}(|q_0-\Phi|). \qquad (2.\ 61)$$

$$|T_5| \leq |\eta_4|\mu\left(|\eta_1|\lambda_0(|q_0-\Phi|)|q_0-\Phi| + \frac{\lambda(|q_0-\Phi|)|q_0-\Phi|}{1-a_1(|q_0-\Phi|)} + a_{10}(|q_0-\Phi|)\right)$$
$$\times a_{10}(|q_0-\Phi|)\times\left(\frac{\lambda(|q_0-\Phi|)|q_0-\Phi|}{1-a_1(|q_0-\Phi|)} + a_{10}(|q_0-\Phi|)\right),$$
$$|T_5| = a_{16}(|q_0-\Phi|). \qquad (2.\ 62)$$

Since $G_0 \in \Theta$, using (2. 20), (2. 24),(2. 59), (2. 60 , (2. 61), and (2. 62), we obtain

$$\begin{aligned}|P'(\Phi)^{-1}(G_0 - P'(\Phi))| &= |P'(\Phi)^{-1}([b_0,h_0;P] - P'(\Phi) + [b_0,q_0;P] + [b_0,d_0;P] - [q_0,d_0;P] \\ &\quad - [h_0,d_0;P] - [q_0,h_0;P] + T_2 + T_3 + T_4 + T_5)|, \\ &\leq \zeta_0(|b_0-\Phi|,|h_0-\Phi|)| + \lambda(|b_0-q_0|) + \lambda(|b_0-d_0|) + \lambda(|q_0-d_0|) \\ &\quad + \lambda(|h_0-d_0|) + \lambda(|q_0-h_0|) + a_{13}(|q_0-\Phi|) + a_{14}(|q_0-\Phi|) \\ &\quad + a_{15}(|q_0-\Phi|) + a_{16}(|q_0-\Phi|), \\ &= \zeta_0(|b_0-\Phi|,|h_0-\Phi|)| + \lambda(|b_0-\Phi| + |q_0-\Phi|) + \lambda(|b_0-\Phi| \\ &\quad + |d_0-\Phi|) + \lambda(|d_0-\Phi| + |q_0-\Phi|) + \lambda(|h_0-\Phi| + |d_0-\Phi|) \\ &\quad + \lambda(|h_0-\Phi| + |q_0-\Phi|) + a_{13}(|q_0-\Phi|) + a_{14}(|q_0-\Phi|) \\ &\quad + a_{15}(|q_0-\Phi|) + a_{16}(|q_0-\Phi|), \\ &= a_{17}(|q_0-\Phi|) < 1. \qquad (2.\ 63)\end{aligned}$$

Therefore, by Banach lemma on invertible operators G_0^{-1} exist and

$$|P'(\Phi)G_0^{-1}| < \frac{1}{1-a_{17}(|q_0-\Phi|)}. \qquad (2.\ 64)$$

By using assertions (2. 24) and (2. 64), we get the third sub-step of iterative scheme (1. 2) for $\tau = 1$

$$\begin{aligned} |q_1 - \Phi| &\leq |b_0 - \Phi| + \frac{\lambda(|b_0 - \Phi|)|b_0 - \Phi|}{1 - a_{17}(|q_0 - \Phi|)}, \\ &\leq \left(1 + \frac{\lambda(|b_0 - \Phi|)}{1 - a_{17}(|q_0 - \Phi|)}\right)|b_0 - \Phi|, \\ &\leq \left(1 + \frac{\lambda(a_{11}(|q_0 - \Phi|)|q_0 - \Phi|)}{1 - a_{17}(|q_0 - \Phi|)}\right)a_{11}(|q_0 - \Phi|)|q_0 - \Phi|, \\ &= a_{18}(|q_0 - \Phi|)|q_0 - \Phi| < |q_0 - \Phi| < \omega. \end{aligned} \tag{2. 65}$$

So, by the definition of P and item (2. 32) hold for $\tau = 0$ and $q_1 \in M(\Phi, \omega)$. By replacing q_0, h_0, d_0, b_0, q_1 with $q_\tau, h_\tau, d_\tau, b_\tau, q_{\tau+1}$ respectively in the previous calculation, we easily get the assertion(2. 30)-(2. 32). Using the inequality

$$|q_{\tau+1} - \Phi| \leq k|q_\tau - \Phi| < \omega,$$

where $c = a_{18}(|q_0 - \Phi|)$. We conclude that $\lim_{\tau\to\infty} x_\tau = \Phi$ and $q_{\tau+1} \in M(\Phi, \omega)$. □

Now, we prove that Φ is the unique solution of the equation $P(q) = 0$ in a particular set Let us suppose that there exists a solution $y_* \in M(\Phi, \omega_4)$ of the equation $P(q) = 0$ for some $\omega_4 > 0$. The inequality given in (2. 20) is verified on $\omega_5 \leq \omega_4$ and the open ball $M(\Phi, \omega_4)$ so that

$$\zeta(\omega_5) < 1.$$

Assume that another ball $M_1 = \Theta \cap M[\Phi, \omega_5]$. Hence, there is only one unique solution Φ in the set M_1 of the equation $P(q) = 0$.

Proof. By the hypothesis and $U = [\Phi, y_*]$, we have

$$\begin{aligned} |P'(\Phi)^{-1}(U - P'(\Phi))| &\leq \zeta(|y^* - \Phi|), \\ &\leq \zeta(\omega_5) < 1, \end{aligned}$$

and by using the following approximation

$$y^* - \Phi = U^{-1}(P(y^* - P(\Phi))) = U^{-1}(0) = 0.$$

Hence, we conclude that $y^* = \Phi$.
We can easily select $\omega_4 = \Phi$. □

From assertions (2. 20) and (2. 22), we derive the general condition

$$\zeta_0 \leq \zeta_1 \tag{2. 66}$$

The ratio $\frac{\zeta_1}{\zeta_0}$ can, in fact, be arbitrarily large. Notably, while condition (2. 20) could be removed, applying both conditions (2. 20) and (2. 22) yields a broader domain for the method (1. 2) than using only assertion (2. 22).
In 2000, Weerakoon and Fernando [19] introduced the concept of the computational order of convergence (COC) to estimate error bounds. The COC is defined by the expression

$$COC = \frac{ln\frac{|q_{\tau+2} - \Phi|}{|q_{\tau+1} - \Phi|}}{ln\frac{|q_{\tau+1} - \Phi|}{|q_\tau - \Phi|}}, \quad \forall \quad \tau = 0, 1, 2, ...$$

where q_τ, $q_{\tau+1}$, and $q_{\tau+2}$, are consecutive approximations of the root Φ. This formula calculates the convergence rate by analyzing the error ratios across successive iterations. The approximate computational order of convergence (ACOC) [8] is expressed as

$$ACOC = \frac{ln\frac{|q_{\tau+2}-q_{\tau+1}|}{|q_{\tau+1}-q_\tau|}}{ln\frac{|q_{\tau+1}-q_\tau|}{|q_\tau-q_{\tau-1}|}}, \quad \forall \quad \tau = 0, 1, 2, ...$$

.

3. Numerical experiments

In this section, we determine the domain of the three-step, derivative-free eighth-order iterative scheme and compare its performance with existing schemes within the same domain. Furthermore, we assess the efficiency of the optimal iterative scheme (1. 2) across various values of the parameter β, demonstrating its enhanced performance relative to known schemes. The following established schemes are presented for comparison: Kumar et al. [12] considered the three-step derivative-free iterative scheme and discussed the local convergence involving first-order derivative. The scheme is represented by Method KSS and is described as follows

$$\begin{aligned}
a_\tau &= q_\tau + P(q_\tau), \quad b_\tau = q_\tau - P(q_\tau), \quad E_\tau = [a_\tau, b_\tau; P],\\
d_\tau &= q_\tau - E_\tau^{-1}P(q_\tau),\\
u_\tau &= d_\tau - (3I - 2E_\tau^{-1}[d_\tau, q_\tau; P])E_\tau^{-1}P(d_\tau),\\
q_{\tau+1} &= u_\tau - \left(\frac{13}{4}I - E_\tau^{-1}[u_\tau, d_\tau; P]\left(\frac{7}{2}I - \frac{5}{4}E_\tau^{-1}[u_\tau, d_\tau; P]\right)\right)E_\tau^{-1}P(u_\tau). \quad (3.\ 67)
\end{aligned}$$

In 2017, Argyros et al. [2] analyzed the local convergence of a three-step, sixth-order iterative scheme in Banach spaces. The scheme, originally developed by Grau et al. [9], is a derivative-free method represented as Method A17, is presented as follows

$$\begin{aligned}
d_\tau &= q_\tau - T_\tau^{-1}P(q_\tau),\\
u_\tau &= d_\tau - K_\tau^{-1}P(d_\tau),\\
q_{\tau+1} &= u_\tau - K_\tau^{-1}P(u_\tau), \qquad (3.\ 68)
\end{aligned}$$

where,

$$\begin{aligned}
T_\tau = [a_\tau, b_\tau; P], K_\tau = 2[d_\tau, q_\tau; P] - [a_\tau, b_\tau; P],\\
a_\tau = q_\tau + P(q_\tau), \quad b_\tau = q_\tau - P(q_\tau).
\end{aligned}$$

In 2018, George et al. [7] presented the local convergence by using first-order derivative. The seventh-order derivative Chebyshev-type iterative scheme [16] referred to as Method A18, is given as follows

$$\begin{aligned}
d_\tau &= q_\tau - T_\tau^{-1}P(q_\tau),\\
u_\tau &= d_\tau - K_\tau P(d_\tau),\\
q_{\tau+1} &= u_\tau - S_\tau P(u_\tau), \qquad (3.\ 69)
\end{aligned}$$

where,

$$\begin{aligned}
T_\tau &= [a_\tau, q_\tau; P],\\
K_\tau &= (3I - T_\tau^{-1}([d_\tau, q_\tau; P] + [d_\tau, a_\tau; P]))T_\tau^{-1},
\end{aligned}$$

$$S_\tau = [u_\tau, q_\tau; P]^{-1}([a_\tau, q_\tau; P] + [d_\tau, q_\tau; P] - [u_\tau, q_\tau; P])T_\tau^{-1},$$
$$a_\tau = q_\tau + \delta P(q_\tau), \quad \delta \in \mathbb{R}.$$

Singh et al. [17] analyzed the local convergence of a three-step, fifth-order iterative method. The derivative-free method, represented as Method SSE, is given below

$$\begin{aligned} d_\tau &= q_\tau - [q_\tau, q_\tau + P(q_\tau); P]^{-1}P(q_\tau), \\ u_\tau &= q_\tau - 2([q_\tau, q_\tau + P(q_\tau); P] + [d_\tau, d_\tau + P(d_\tau); P])^{-1}P(q_\tau), \\ q_{\tau+1} &= u_\tau - [d_\tau, d_\tau + P(d_\tau); P]^{-1}P(u_\tau). \end{aligned} \tag{3. 70}$$

Example 3.1. *Let us suppose that the motion in* 3*-dimensional object is governed by a system of differential equations*

$$p_1^{'}(\theta_1) - p_1(\theta_1) - 1 = 0, p_2^{'}(\theta_2) - (e-1)\theta_2 - 1 = 0, p_3^{'}(\theta_3) - 1 = 0$$

with $M(q, Phi) = \overline{M}(q, \Phi) = \mathbb{R}^3$, $I = Q(0,1)$, *and* $p_1(0) = p_2(0) = p_3(0) = 0$. *For the solution of the system is given by*

$$\Phi = (\theta_1, \theta_2, \theta_3)^T \in \Theta.$$

Define function $P = (h_1, h_2, h_3) : \Theta = \mathbb{R}^3$ *which is given as*

$$P_1(\theta) = (e^{\theta_1} - 1, \frac{e-1}{2}\theta_2^2 + \theta_2, \theta_3)^T.$$

Then, the derivative of P_1 *is*

$$\begin{bmatrix} e^{\theta_1} & 0 & 0 \\ 0 & (e-1)\theta_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $\Phi = (0,0,0)^\Theta, P'(\Phi) = P'(\Phi)^{-1} = diag\{1,1,1\}$, *Using the conditions (2. 19)-(2. 29), we get the following expressions*

$$\zeta_0(\alpha_1, \alpha_2) = \frac{S_0}{2}(\alpha_1 + \alpha_2),$$
$$\zeta_1(\alpha_1, \alpha_2) = \frac{S\alpha_1 + S_0\alpha_2}{2},$$
$$\zeta_2(\alpha) = \frac{1}{2}e^{\frac{1}{S_0}}\alpha,$$
$$\zeta_3(\alpha_1, \alpha_2) = \frac{S}{2}(\alpha_1 + \alpha_2),$$
$$\lambda_0(\alpha) = \lambda_\alpha = \frac{1}{2}(1 + e^{\frac{1}{S_0}}),$$
$$\zeta(\alpha) = \frac{1}{2}(e-1)\alpha,$$
$$\varpi = 1 + \frac{1}{2}|\beta_1|(1 + e^{\frac{1}{S_0}}), S_0 = e - 1, S = e.$$

By applying the definition of the function "ω*" and the three-step iterative scheme (1. 2) for different values of the parameter* β*, we get the results presented in Table (1).*

Table (1) clearly demonstrates that our method MSI attains a significantly larger radius *of convergence when compared to the existing methods KSS, A17, A18, and SSE. This improvement not only reflects enhanced convergence speed but also greater numerical stability. Furthermore, the expanded convergence radius implies that MSI is more robust and capable of converging effectively from a wider variety of initial guesses, making it more reliable for practical applications.*

TABLE 1. Comparison of radius of convergence for different values of β and already existing schemes

Radius of convergence of iterative Method MSI for different values of β				
β	ω_1	ω_2	ω_3	ω
0.5	*0.379265*	*0.139088*	*0.139085*	*0.139085*
0.75	*0.379265*	*0.142188*	*0.142187*	*0.142187*
1	*0.379265*	*0.145169*	*0.145168*	*0.145168*
Method	*Radius of convergence for different methods*			
Method KSS	*0.204156*	*0.131098*	*0.119606*	*0.119606*
Method A17	*0.152436*	*0.074998*	*0.057836*	*0.057836*
Method A18	*0.201018*	*0.083025*	*0.063907*	*0.063907*
Method SSE	*0.054900*	*0.050916*	*0.050171*	*0.050171*

Example 3.2. *Assume that* $\mathbb{Z}$ *is the continuous function on the complex space* $\mathbb{C}[0,1]$ *with closed ball* $\overline{M}(\Phi,1)$. *Let us suppose that the mixed Hammerstein-type integral equation* [11] *by*

$$\mathrm{X}(y) = \int_0^1 U(y,z)\left(\mathrm{X}(z)^{\frac{3}{2}} + \frac{\mathrm{X}(z)^2}{2}\right) dz \tag{3. 71}$$

where,

$$U(y,z) = \begin{cases} (1-y)z, & \textit{if } z \le y \\ y(1-z), & \textit{if } y \le z \end{cases}$$

Observe that $\Phi(y) = 0$. *Define mapping* $P : \Theta \subseteq [0,1] \to \mathbb{C}[0,1]$, *and function can be expressed as*

$$P(\mathrm{X})(y) = \mathrm{X}(y) - \int_0^1 U(y,z)\left(\mathrm{X}(z)^{\frac{3}{2}} + \frac{\mathrm{X}(z)^2}{2}\right) dz,$$

The derivative of P' *is given below*

$$P'(\mathrm{X})w(y) = w(y) - \int_0^1 U(y,z)\left(\frac{3}{2}\mathrm{X}(z)^{\frac{1}{2}} + \mathrm{X}(z)\right) dz.$$

if $P'(\Phi(y)) = 1$, *then, we get the following values*

$$\zeta_0(\alpha_1,\alpha_2) = \zeta_1(\alpha_1,\alpha_2) = \zeta_2(\alpha_1,\alpha_2) = \zeta_3(\alpha_1,\alpha_2) = \alpha_1 + \alpha_2,$$

$$\zeta(\alpha) = t, \lambda(\alpha) = \lambda_0(\alpha) = \frac{9}{16},$$

$$\varpi = 1 + |\beta_1|\frac{9}{16}.$$

By employing Maple 18 for numerical computations, it is clear from Table (2) that method (1. 2) exhibits a larger radius of convergence. These results highlight the efficiency and robustness of the proposed method. The increased radius of convergence indicates that the method can be applied to a broader range of initial approximations.

TABLE 2. Comparison of radius of convergence for different values of β and existing schemes

Radius of convergence of iterative Method MSI for different values of β				
β	ω_1	ω_2	ω_3	ω
0.5	*0.496124*	*0.293345*	*0.248475*	*0.248475*
0.75	*0.496124*	*0.297446*	*0.292920*	*0.292920*
1	*0.496124*	*0.301204*	*0.300345*	*0.300345*
Method	*Radius of convergence for different methods*			
Method KSS	*0.175392*	*0.026781*	*0.009082*	*0.009082*
Method A17	*0.612432*	*0.189863*	*0.121415*	*0.121415*
Method A18	*0.580509*	*0.262323*	*0.146373*	*0.146373*
Method SSE	*0.290249*	*0.270018*	*0.266803*	*0.266803*

4. Basin of attraction

Polynomiography is derived from two words, ”Polynomial” and ”Graphy.” It is defined as ”Visualization of art and science.” This technique involves approximating the roots of complex polynomials and generating the fractal and non-fractal images based on the mathematical convergence characteristics of iterative functions. Using algorithms and thousands of pixels on a computer monitor, polynomiographs are developed to beautifully blend mathematics and artistic visualization.

The term ”fractal,” used in the context of polynomiography, was introduced by the renowned research scientist Benoit Mandelbrot. It describes sets or geometric figures that exhibit self-similarity and are scale-invariant. This means that no matter how much one zooms in, new details continuously emerge, displaying complexity at every level of magnification. Interestingly, some fractal patterns can be generated through simple iterative processes, resulting in sets like the Julia set and the well-known Mandelbrot set. The simplicity of these

iterative methods, which may or may not serve a specific purpose, has led to the creation of numerous websites where both enthusiasts and experts share their fractal designs. Many of these images are connected to the famous Mandelbrot set.
Now, we will examine some foundational concepts that help us to visualize the behavior of iterative methods [4]. Assume a rational operator $\mathbb{R} : \widetilde{C} \to \widetilde{C}$, where $\widetilde{C}$ be the Riemann sphere. A point $\breve{z}_0 \in \widetilde{C}$ is said to be an orbit, which can be written as the following sequence:

$$orb(\breve{z}_0) = \{\breve{z}_0, \breve{z}_1 = \mathbb{R}(\breve{z}_0), \breve{z}_2 = \mathbb{R}^2(\breve{z}_0), ..., \breve{z}_t = \mathbb{R}^t(\breve{z}_0)\}.$$

Moreover, a fixed point $\breve{z}_0$ can be divided into following categories:

(1) **Periodic point:** Suppose that n is the period of the periodic point $\breve{z}_0$ i.e., $\mathbb{R}^n(\breve{z}_0) = \breve{z}_0$.
(2) **Critical point:** $\breve{z}_0$ is called a critical point if the derivative of the rational operator at $\breve{z}_0$ is zero, that is, $\mathbb{R}'(\breve{z}_0) = 0$.
(3) **Strange fixed point:** If the rational operator $\mathbb{R}$ that are not related to the roots of the non-linear equation $P(q) = 0$, is known as strange fixed point.
(4) **Fixed point:** If $n = 1$, then $\breve{z}_0$ is the fixed point, which can be further categorized as:
   - Attractor: If $|\mathbb{R}^{'}(\breve{z}_0)| < 1$.
   - Repulsor: If $|\mathbb{R}'(\breve{z}_0)| > 1$.
   - Super attractor: If $|\mathbb{R}'(\breve{z}_0)| = 0$.
   - Parabolic: if $|\mathbb{R}'(\breve{z}_0)| = 1$.

Now, we define the basin of attraction for an attractor η, as the set

$$\chi(\eta) = \{\breve{z}_0 \in \mathcal{C} : \mathbb{R}^m(\breve{z}_0) \to \eta, m \to \infty\}. \tag{4. 72}$$

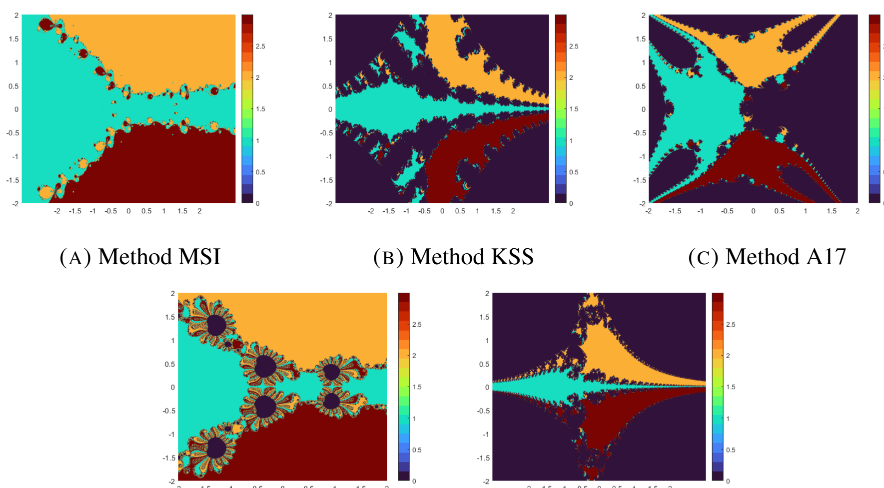
This set contains all points whose iterative sequences converge to η. The Fatou set $\digamma(\mathbb{R})$ includes points where the orbits behave regularly, converging to an attractor such as periodic orbits, fixed points, or infinity. The complement of this set in $\widetilde{C}$ is called the Julia set, denoted $\Im(\mathbb{R})$, which often indicates the boundary between stable and chaotic dynamics.
In this section, we examine the behavior of the iterative schemes using the concept of the basin of attraction. We take some higher-degree non-linear complex polynomials to compare the iterative scheme MSI with the already known schemes of the same domain, namely KSS, A17, A18, and SSE. The basins of attraction are created using the MATLAB R2023a computer algebra system that is highly effective for analyzing the stability of iterative schemes.
For n^{th}-degree complex polynomial, the roots partitions the complex plane into n distinct basins. Each basin corresponds to a specific root and is assigned a unique color for visualization. From the dynamical perspective, we consider a square region $\Delta = [-2, 2] \times [-2, 2]$ in the complex plane, with 1000×1000 mesh grid. The iterative scheme is deemed to converge if the root of the complex polynomial lies in the basin of attraction, with a maximum of 50 iterations and stopping criterion $|P(q_\tau)| < 10^{-6}$.
In this visualization, darker shades represent faster convergence, requiring fewer iterations to reach the root. Conversely, lighter shades indicate that more iterations are needed for convergence. The intense purple color indicates that the root does not converge within 50 iterations, implying that the scheme has diverged for the respective root. This pattern

gives a valuable insight for evaluating the stability, performance, efficiency, and the convergence region of the multi-step iterative scheme across the different starting points. Through this dynamical analysis, we can effectively compare the performance of various iterative schemes. Now, we consider some complex polynomials which is given below:

Example 4.1. *Let us suppose that the third-degree complex polynomial*

$$P_3(\breve{z}) = (\breve{z}_2 + 1)(\breve{z} + 1).$$

The approximate roots of this polynomial are 1 *and* $\pm\iota$. *The basin of attraction for the iterative schemes MSI, KSS, A17, A18, and SSE are shown in figure (1).*
For comparison, the complex plane is represented within the square region $\Delta = [-2, 2] \times [-2, 2]$. *Each colored region corresponds to initial guesses converging to a specific root, while the purple region indicates divergence. Among the five schemes, MSI exhibits the most stable performance, with large, well-defined basins and smoother boundaries, reflecting reduced sensitivity to initial conditions. In contrast, KSS, A17, and SSE demonstrate the highest degree of chaotic behavior, with irregular boundaries and extensive non-convergence regions, while A18 shows fragmented basins and larger divergence zones, indicating that these methods require more careful selection of initial guesses.*



(A) Method MSI (B) Method KSS (C) Method A17

(D) Method A18 (E) Method SSE

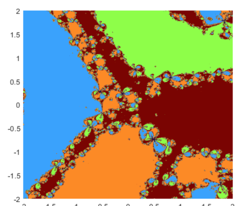
FIGURE 1. Visualization of basin of attraction for the function $P_3(\breve{z}) = (\breve{z}_2 + 1)(\breve{z} + 1)$.

Example 4.2. *Let us assume that a trigonometric polynomial*

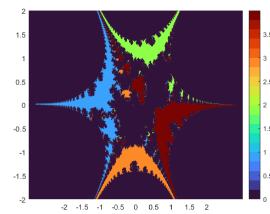
$$P_4(\breve{z}) = \breve{z}^4 + sin(\breve{z}) + icos(\breve{z}) - 1.$$

The computed approximate roots are $-1.1883 + 0.0590i$, $0.5696 + 1.1625i$, $-0.0875 - 1.0185i$, *and* $0.8726 - 0.2100i$. *The roots are colored in dark red, green, blue, and orange.*

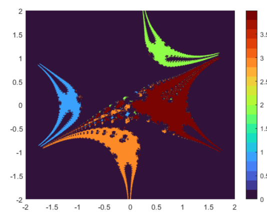
The basins of attraction for the iterative methods MSI, KSS, A17, A18, and SSE are shown in Figures (2a), (2b), (2c), (2d), and (2e) generated using MATLAB R2023a. Each color in the figure represents the set of initial points converging to its corresponding root. The MSI scheme exhibits broad, continuous regions of attraction with smooth boundaries, reflecting greater stability. In contrast, the methods KSS, A17, A18, and SSE yield intricate, fractal-type boundaries and larger divergence areas, suggesting reduced robustness. These visual patterns highlight how the geometry of the basin of attraction can reveal the efficiency and reliability of iterative schemes.



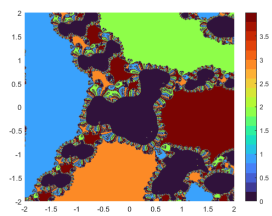
(A) Method MSI



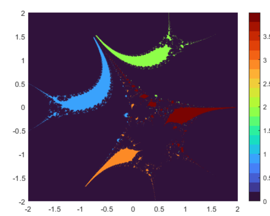
(B) Method KSS



(C) Method A17



(D) Method A18



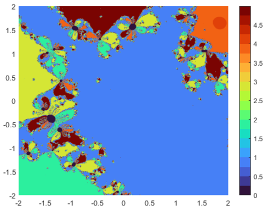
(E) Method SSE

FIGURE 2. Visualization of basin of attraction for the function $P_4(\breve{z}) = \breve{z}^4 + sin(\breve{z}) + icos(\breve{z}) - 1$.

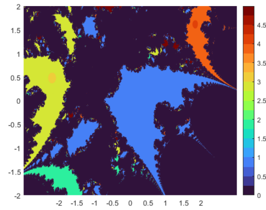
Example 4.3. *Suppose that the transcendental equation*

$$P_5(\breve{z}) = e^{\breve{z}}(cos(\breve{z}) + isin\breve{z}) - 1.$$

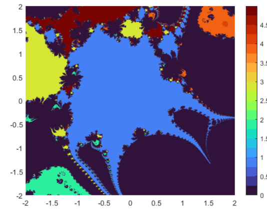
The basin of attraction of the three-step derivative-free iterative methods MSI, KSS, A17, A18, and SSE are illustrated in figure (3). The approximate roots of the given polynomial are $0, -1.6280 - 1.8124i, -2.1942 + 0.4902i, 1.8124 + 1.6280i, -0.4902 + 2.1942i$ *depicted in maroon, orange, yellow, green, light blue, respectively. Each colored region in the figure corresponds to the set of initial points that converge to the associated root under the given iterative scheme. It is evident from the graphical patterns that the MSI method forms large, cohesive regions leading to the roots, indicating a relatively more stable convergence behavior compared to KSS, A17, A18, and SSE, which display highly fragmented and chaotic boundaries. The comparative analysis also shows that divergence occurs for all methods, as certain initial points fail to converge within the maximum number of iterations, highlighting the need for more iterations to ensure convergence for such points.*



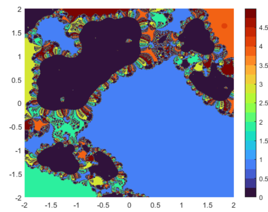
(A) Method MSI



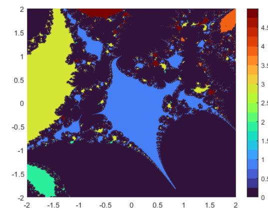
(B) Method KSS



(C) Method A17



(D) Method A18



(E) Method SSE

FIGURE 3. Visualization of basin of attraction for the function $P_5(\breve{z}) = e^{\breve{z}}(cos(\breve{z}) + isin\breve{z}) - 1$.

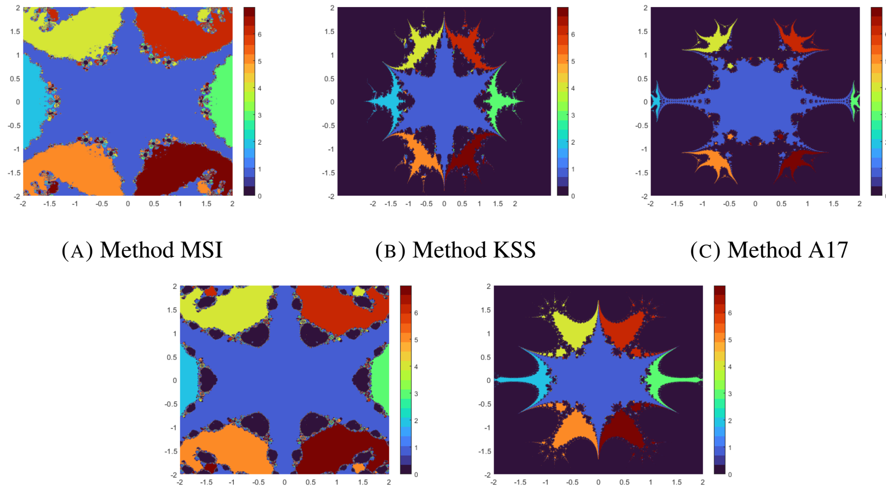
Example 4.4. *Assume that the non-linear polynomial*

$$P_4(\breve{z}) = tan^{-1}(\breve{z})cosh\breve{z}.$$

The dynamical analysis of the iterative schemes MSI, KSS, A17, A18, and SSE are shown in 4a, 4b, 4c, 4d, and 4e. The polynomial under consideration has roots $0, -1.8954, 1.8954, -0.7495 + 1.2964i, -0.7495 - 1.2964i, 0.7495 + 1.2964i, 0.7495 - 1.2964i$. *which are depicted in the basin of attraction plots using the colors blue, cyan, yellow, red, orange, green, and maroon. The analysis is conducted over the complex plane in the square region* $\Delta = [-2, 2] \times [-2, 2]$.
From these visualizations, it is evident that MSI method yields smoother and more distinct attraction regions, showing chaotic patterns than A18. In the MSI results, the regions corresponding to different roots are more evenly distributed, which minimizes overlap and reduces uncertainty in the convergence paths. In contrast The boundaries between different regions are more defined, suggesting enhanced stability in convergence. The figures clearly show that the presented three-step eighth-order iterative scheme MS1 attains a broader and more evenly distributed region of convergence compared to the competing schemes KSS, A17, A18, and SSE, enabling it to handle a wider range of initial guesses. Furthermore, the boundaries between different converging regions are smoother and less fragmented, reflecting a lower degree of chaotic behavior.

Example 4.5. *Let us consider the one-dimensional heat equation*

$$\frac{\partial P(q,t)}{\partial q} = \lambda \frac{\partial^2 P(q,t)}{\partial t^2}, \tag{4. 73}$$



(A) Method MSI (B) Method KSS (C) Method A17

(D) Method A18 (E) Method SSE

FIGURE 4. Visualization of basin of attraction for the function $P_4(\breve{z}) = tan^{-1}(\breve{z})cosh\breve{z}$.

where $P(q,t)$ represents the temperature distribution in a rod of length d, and $\lambda > 0$ is the thermal diffusivity of the material.
Boundary conditions: *The rod is held at zero temperature at both ends for all $t > 0$*

$$P(0,t) = 0 \quad P(d,t) = 0, \quad t > 0.$$

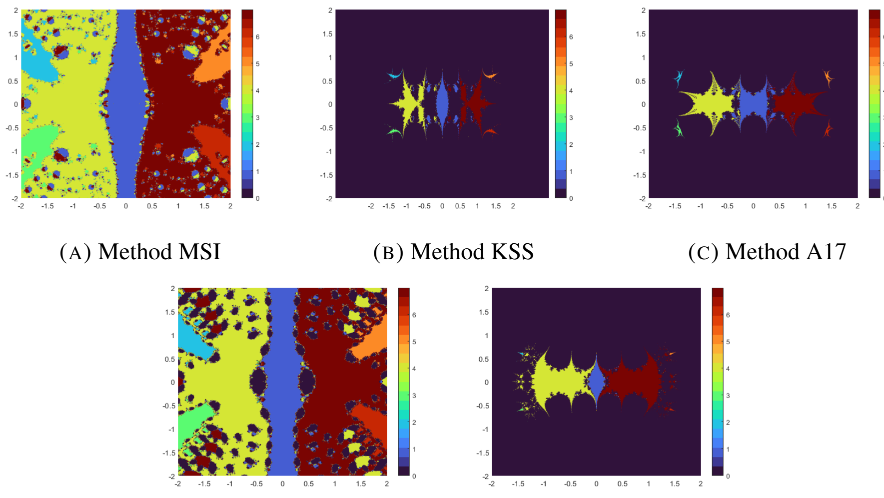
Initial conditions: *The initial temperature distribution along the rod is given by*

$$P(q,0) = g(q), \quad 0 \leq q \leq d.$$

Here, $g(q)$ is a prescribed function defining the temperature at $t = 0$. The solution of the above equation (4. 73) is given below

$$P_7(\breve{z}) = M_\tau sin(\frac{\tau\pi}{d})\breve{z}.$$

We consider the parameter $\tau = d = 1$ and $M_1 = 1$. The approximations of the nonlinear equation are $0, -1.4114+0.5869i, -1.4114-0.5869i, -0.9800+0.0000i, 1.4114+0.5869i, 1.4114-0.5869i, 0.98$. The roots are depicted in blue, cyan, yellow, red, orange, green, and maroon, respectively, while the region of divergence is depicted in purple color. The dynamical analysis of the methods MSI, KSS, A17, A18, and SSE are illustrated in figure (5). The basin of attraction is plotted in the complex plane within the square region $\Delta = [-2,2] \times [-2,2]$. These basins help us understand the sensitivity and stability of iterative methods under different initial guesses. From the figures, it is clear that the method MSI exhibits less chaotic behavior when compared to the existing schemes, indicating more stable convergence characteristics. The comparative analysis reveals that divergence occurs primarily in regions where the iterative methods require a higher number of iterations *to reach convergence. This behavior is indicative of the sensitivity of the method to initial conditions and highlights the importance of choosing efficient root-finding techniques.*



(A) Method MSI (B) Method KSS (C) Method A17

(D) Method A18 (E) Method SSE

FIGURE 5. Visualization of basin of attraction for the function $P_7(\breve{z}) = M_\tau sin(\frac{\tau\pi}{d})\breve{z}$.

5. CONCLUSION

The convergence analysis of the three-step derivative-free iterative scheme (1. 2) was previously addressed in [1], where Taylor series expansions and derivatives up to the eighth order were employed to establish the eighth-order convergence. However, the need for high-order derivatives limits the practical applicability of the scheme. The major challenges identified in [1] include selecting an appropriate initial approximation, determining the domain of convergence, and ensuring the uniqueness of the solution. To overcome these issues, we analyzed the convergence behavior using only the first-order derivative under a Lipschitz condition. Additionally, we have constructed the local convergence theorem to guarantee the existence and uniqueness of the solution, along with an associated error bound.

To verify the theoretical results, we considered examples including a 3-dimensional differential equation and Hammerstein-type integral equation to determine the radius of convergence. Numerical experiments were performed using the computer algebra system Maple 18, showing that the three-step iterative scheme (1. 2) has a larger convergence radius compared to existing methods. Additionally, a dynamical analysis has also been carried out using various transcendental and complex polynomials. The basin of attraction highlights reduced chaotic behavior and a small divergence region. In conclusion, both numerical and dynamical analysis confirm that the iterative scheme MSI demonstrates strong stability, efficiency, and robustness.

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AUTHOR CONTRIBUTION

Methodology, Software, Writing original draft and review (Maira Khalid, Saima Akram); Formal analysis, and Investigation, (Maira Khalid, Saima Akram, Muhammad Ibrahim); Supervision (Saima Akram, Muhammad Ibrahim). All authors have read and agreed to the published version of the manuscript.

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DATA AVAILABILITY

No datasets were generated or analyzed during the current study.

COMPETING INTERESTS

The authors declare no competing interests.

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