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Solutions of Two Diophantine Equations

$$\mathcal{P}_l = \mathcal{J}_r + \mathcal{J}_s \text{ and } \mathcal{R}_l = \mathcal{J}_r + \mathcal{J}_s$$

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Abstract. Suppose that the sequences $\{\mathcal{P}_i\}_{i\geq 0}$, $\{\mathcal{R}_i\}_{i\geq 0}$, and $\{\mathcal{J}_i\}_{i\geq 0}$ correspond to the Padovan numbers, Perrin numbers, and Jacobsthal numbers, respectively. In this paper, we determine all Padovan and Perrin numbers that can be written as the sum of two Jacobsthal numbers by applying Baker's method for linear forms in logarithms, combined with the reduction technique of Dujella and Pethő.

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1. Introduction

The Padovan sequence $\{\mathcal{P}_i\}_{i\geq 0}$ is a recursively defined sequence given by $\mathcal{P}_i = \mathcal{P}_{i-2} + \mathcal{P}_{i-3}$ for $i \geq 3$, with the initial terms $\mathcal{P}_0 = \mathcal{P}_1 = \mathcal{P}_2 = 1$. R. Padovan introduced it as a sequence connected to the plastic number previously defined by H. V. D. Loan. Numerous mathematicians have explored Diophantine equations related to the Padovan sequence. See [6], [16], [17], [19] and [9]. The sequence of Perrin numbers begins with $\mathcal{R}_0 = 3$, $\mathcal{R}_1 = 0$, $\mathcal{R}_2 = 2$ and obeys the relation $\mathcal{R}_i = \mathcal{R}_{i-2} + \mathcal{R}_{i-3}$ for $i \geq 3$. For some recent works concerning Diophantine equations with Perrin numbers, see [1] and [5]. Perrin and Padovan numbers exhibit many similar properties since they share the same Binet formula. We mention some recent works concerning the Diophantine equations related to Padovan and Perrin numbers. In [14], N. Irmak solved the two equations $\mathcal{P}_n =$

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$m!$ and $\mathcal{R}_n = m!$. In [20], S. E. Rihane *et al.* identified which Padovan and Perrin numbers are Fermat numbers. In [2], K. N. Adédji *et al.* they found all Padovan and Perrin numbers that can be represented as the product of two repdigits in base b for $2 \leq b \leq 10$. In [22], S. E. Rihane and A. Togbé investigated which product of two repdigits is the Padovan or Perrin numbers. Also, in [21], they solved the Diophantine equations $\mathcal{F}_n^{(k)} = \mathcal{R}_m$ where $\mathcal{F}_n^{(k)}$ is the n^{th} $k-$generalized Fibonacci number. The sequence of Jacobsthal numbers is defined recursively as $\mathcal{J}_0 = 0$, $\mathcal{J}_1 = 1$ and $\mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2}$ for $n \geq 2$. This sequence appeared in [11, 12]. In [13], F. Horadarm provided a detailed study of this sequence. Basic properties and generalizations of the Jacobsthal sequence can be found in [3, 4, 7, 15, 21]. In [10], F. Erduran and R. Keskin solved the Diophantine equations $\mathcal{F}_k = \mathcal{J}_m\mathcal{J}_n$ and $\mathcal{J}_k = \mathcal{F}_m\mathcal{F}_n$. This paper focuses on investigating two Diophantine equations.

$$\mathcal{P}_l = \mathcal{J}_r + \mathcal{J}_s, \tag{1. 1}$$

and

$$\mathcal{R}_l = \mathcal{J}_r + \mathcal{J}_s. \tag{1. 2}$$

We apply linear forms in logarithms and reduction techniques. The full set of solutions is given in the theorems

Theorem 1.1. *The set contains all non-negative triples (l, r, s) that satisfy Eq. (1. 1) while ensuring $r \geq s$ is*

$$\left\{ \begin{array}{ccccc} (0,1,0), & (0,2,0), & (1,1,0), & (1,2,0), & (2,1,0) \\ (2,2,0), & (3,1,1), & (3,2,1), & (3,2,2), & (4,1,1) \\ (4,2,1), & (4,2,2) & (5,3,0), & (6,3,1), & (6,3,2) \\ (7,4,0), & (10,5,1) & (10,5,2), & (11,5,4), & (12,6,0) \\ & (17,8,1), & (17,8,2), & (17,7,7), & \end{array} \right\}$$

and

Theorem 1.2. *The all non-negative solutions (l, r, s) that satisfy Eq. (1. 2) with $r \geq s$ belong to the set.*

$$\left\{ \begin{array}{ccccc} (0,3,0), & (1,0,0), & (2,1,1), & (2,2,1), & (2,2,2) \\ (3,3,0), & (4,1,1), & (4,2,1), & (4,2,2), & (5,4,0) \\ (6,4,0), & (8,4,4), & (9,5,1), & (9,5,2), & (11,6,1) \\ & (11,6,2), & (11,5,5), & (16,8,4) & \end{array} \right\}.$$

2. Preliminary results

2.1. **Padovan sequence.** We will discuss several key properties of the Padovan sequence $\{\mathcal{P}_r\}_{r\geq 0}$. The characteristic polynomial

$$P(x) := x^3 - x - 1 = 0,$$

has zeros α, β, and $\gamma = \bar{\beta}$, where γ is the complex conjugate of β. where

$$\alpha = \frac{r_1 + r_2}{6} \quad \text{and} \quad \beta = \frac{-(r_1 + r_2) + \sqrt{-3}\,(r_1 - r_2)}{12}, \tag{2. 1}$$

with

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}. \tag{2. 2}$$

Additionally, the Binet formula is expressed as

$$\mathcal{P}_l = a\alpha^l + b\beta^l + c\gamma^l \quad \text{for all} \quad l \geq 0, \tag{2. 3}$$

where

$$a = \frac{\alpha + 1 + \beta\gamma}{(\alpha - \beta)(\alpha - \gamma)}, \quad b = \frac{\beta + 1 + \alpha\gamma}{(\beta - \alpha)(\beta - \gamma)}, \quad c = \frac{\gamma + 1 + \alpha\beta}{(\gamma - \alpha)(\gamma - \beta)} = \bar{b}. \tag{2. 4}$$

The minimum polynomial of a with a positive leading coefficient over the integers is

$$23x^3 - 23x^2 + 6x - 1,$$

which zeros a, b, c with $|a|$, $|b|$, $|c| < 1$.

The following approximations are valid:

$$\begin{aligned} 1.32 < \alpha < 1.33; \\ 0.86 < |\beta| = |\gamma| = \alpha^{\frac{-1}{2}} < 0.87; \\ 0.72 < a < 0.73; \\ 0.24 < |b| = |c| < 0.25. \end{aligned} \tag{2. 5}$$

Using Eqs.(2. 1), (2. 2) and (2. 5), we can prove that

$$\left|\mathcal{P}_l - a\alpha^l\right| = \left|b\beta^l + c\gamma^l\right| < \frac{1}{\alpha^{\frac{l}{2}}} \quad \forall l \geq 1. \tag{2. 6}$$

Furthermore, by induction, we can prove that

$$\alpha^{l-3} \leq \mathcal{P}_l \leq \alpha^{l-1} \quad \forall l \geq 1. \tag{2. 7}$$

2.2. **Perrin sequence.** The Perrin sequence's Binet formula can be written as follows:

$$\mathcal{R}_l = \alpha^l + \beta^l + \gamma^l \quad \forall l \geq 0.$$

This formula provides a way to calculate the l-th term of the Perrin sequence using the values α, β, and γ, raised to the power of l.

Induction allows us to prove that

$$\alpha^{l-2} \leq \mathcal{R}_l \leq \alpha^{l+1} \quad \forall l \geq 2. \tag{2. 8}$$

2.3. **Jacobsthal sequence.** The Jacobsthal sequence has the characteristic polynomial

$$P(x) = x^2 - x - 2.$$

The Binet formula is provided by

$$\mathcal{J}_r = \frac{2^r - (-1)^r}{3}. \tag{2. 9}$$

It is easy to prove that

$$2^{r-2} \leq \mathcal{J}_r \leq 2^{r-1} \quad \text{for all} \quad r \geq 1. \tag{2. 10}$$

2.4. **Linear Forms in Logarithms.** Consider an algebraic number ζ with a minimal polynomial of degree d over $\mathbb{Z}$, given by $a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(x - \zeta^{(i)})$, where a_0 is the positive leading coefficient, and $\zeta^{(i)}$ represents the conjugates of ζ. The logarithmic height of ζ is defined as $h(\zeta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\max\left\{|\zeta^{(i)}|, 1\right\}\right)$. The logarithmic height function $h(.)$ satisfies The following properties:

$$\begin{aligned} h(\zeta_1 \pm \zeta_2) &\le h(\zeta_1) + h(\zeta_2) + \log 2; \\ h\left(\zeta_1\zeta_2^{\pm 1}\right) &\le h(\zeta_1) + h(\zeta_1); \\ h(\zeta^s) &= |s|h(\zeta) \quad (s \in \mathbb{Z}). \end{aligned} \tag{2. 11}$$

The theorem in [18], proposed by Matveev, is a fundamental result in the theory of linear forms in logarithms and serves as a crucial component in our proofs.

Theorem 2.5. *(Matveev) Consider positive real algebraic numbers $\zeta_1, \ldots, \zeta_k$ in a real algebraic number field $\mathbb{L} \subset \mathbb{R}$ of degree D. Let $b_1, \ldots, b_k$ be nonzero integers such that*

$$\Lambda := \zeta_1^{b_1} \cdots \zeta_k^{b_k} - 1 \neq 0.$$

Then, it holds that

$$\log|\Lambda| > -1.4 \times 30^{l+3} \times k^{4.5} \times D^2 \times (1 + \log D) \times (1 + \log B) \times C_1 \cdots C_k,$$

where

$$B \ge \max\{|b_1|, \ldots, |b_k|\}$$

and

$$C_i \ge \max\{Dh(\zeta_i), |\log\zeta_i|, 0.16\}, \quad \textit{for each } i = 1, \ldots, k.$$

2.6. **Reduction process.** To improve the upper bounds of our variables using Theorem 2.5, we refer to the following lemma by Dujella and Pethő [8]. For any real number X, we define $\|X\|$ as the shortest distance from X to the nearest integer, i.e.,

$$\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}.$$

Lemma 2.7. *Let M be a positive integer. Consider the real numbers λ, η, A, B with $A > 0$ and $B > 1$. Let $\frac{p}{q}$ be a convergent of λ with $q > 6M$ and define $\epsilon := \|\eta q\| - M\|\lambda q\| > 0$. If (r, s, ω) is a positive integer solution to the inequality*

$$0 < |r\lambda - s + \eta| < \frac{A}{B^\omega},$$

with $r \le M$, then it follows that

$$\omega < \frac{\log\left(\frac{Aq}{\epsilon}\right)}{\log B}.$$

Theorem 2.8. [6] *(Legendre Theorem of Continued Fractions) Let x be a real number, and $p, q \in \mathbb{Z}$ such that $x = [a_0, a_1, \ldots]$. If the following inequality holds:*

$$\left|\frac{p}{q} - x\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ *is a convergent of the continued fraction representation of* x*. Furthermore, let* M *and* n *be non-negative integers with* $q_n > M$*. Define* $b = \max\{a_i : 0 \leq i \leq n\}$*. It follows that for all pairs of integers* (r, s) *where* $0 < s < M$*:*

$$\frac{1}{(b+2)s^2} < \left|\frac{r}{s} - x\right|.$$

3. Proof of Theorem 1.1

3.1. **Bounding s, r, and l.** Assume that $s \leq r$. Then, by applying inequalities (2. 10) and (2. 7), we can derive,

$$\alpha^{l-3} \leq \mathcal{P}_l \leq 2^r \quad \text{and} \quad 2^{r-2} \leq \mathcal{P}_l \leq \alpha^{l-1}.$$

This leads to the conclusion that:

$$(r-2)\frac{\log 2}{\log \alpha} + 1 \leq l \leq r\frac{\log 2}{\log \alpha} + 3.$$

Using Sage, the collection of all non-negative solutions (l, r, s) of Eq.(1.1) with $s \leq r \leq 130$ and $l \leq 400$ corresponds to the solution set outlined in Theorem 1.1. We will now consider the case where $l > 400$, $r \geq s$, and $r \geq 130$. We can assume that $l < 3r$. Utilizing Eqs. (2. 3) and (2. 9), we can express Eq. (1. 1) as follows:

$$\left|a\alpha^l - \frac{2^r}{3}\right| = \left|\frac{2^s}{3} - \frac{((-1)^r + (-1)^s)}{3} - b\beta^l - c\gamma^l\right|.$$

Dividing by $\frac{2^r}{3}$ and using Eq.(2. 6), we get

$$\left|\frac{3a\alpha^l}{2^r} - 1\right| < \frac{4}{2^{r-s}}. \tag{3. 1}$$

Define

$$\Lambda_1 = 3a\alpha^l 2^{-r} - 1,$$

Initially, we prove that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, then $3a\alpha^l = 2^r$. Let σ be the automorphism defined by $\sigma(a\alpha) = b\beta$. From this, we have $\left|3b\beta^l\right| = 2^r$. However, since $\left|3b\beta^l\right| < 3$, it follows that $2^r < 3$. This leads to a contradiction because we have $r \geq 10$. Using Eq.(3. 1) we find,

$$\log|\Lambda_1| < \log 4 + (s-r)\log 2. \tag{3. 2}$$

If we choose

$$\begin{aligned} &\zeta_1 = 3a, && \zeta_2 = \alpha, && \zeta_3 = 2, \\ &b_1 = 1, && b_2 = l && b_3 = -r, \\ &k = 3. \end{aligned}$$

Take $\mathbb{L} = \mathbb{Q}(\alpha)$, therefore $D = 3$.
Using Eq.(2. 11) we get,

$$h(\zeta_1) \leq \log 3 + \frac{1}{3}\log 23;$$
$$h(\zeta_2) = \frac{1}{3}\log \alpha;$$
$$h(\zeta_3) = \log 2.$$

We take

$$C_1 = 6.5, \;\; C_2 = \log \alpha, \;\; \text{and} \;\; C_3 = 3\log 2, B = 3r.$$

Therefore, Theorem 2.5 implies

$$\log|\Lambda_1| > -744017400000\sqrt{3} \times (1+\log 3)(1+\log 3r)(\log 621)(\log 8 \log \alpha),$$

which can be simplified to

$$\log|\Lambda_1| > -4 \times 10^{14}(1+\log 3r). \tag{3. 3}$$

By analyzing the inequalities in Eqs.(3. 2) and (3. 3), we derive:

$$(r-s)\log 2 - \log 4 < 4 \times 10^{14}(1+\log 3r). \tag{3. 4}$$

Then

$$s > r - \frac{4 \times 10^{14}(1+\log 3r)}{\log 2} - 2. \tag{3. 5}$$

Now, we rewrite Eq.(1. 1) as

$$a\alpha^l - \frac{2^r(1+2^{s-r})}{3} = -b\beta^l - c\gamma^l - \frac{(-1)^r - (-1)^s}{3}, \tag{3. 6}$$

It can be expressed as

$$\left|\frac{3a\alpha^l 2^{-r}}{1+2^{s-r}} - 1\right| = \left|\frac{3\cdot 2^{-r}}{1+2^{s-r}}\left(-b\beta^l - c\gamma^l - \frac{(-1)^r-(-1)^s}{3}\right)\right|. \tag{3. 7}$$

By applying Eq. (2. 6) and utilizing the fact that

$$\left|\frac{(-1)^r-(-1)^s}{3}\right| \leq \frac{2}{3}$$

we obtain the following inequality:

$$\left|\frac{3a\alpha^l 2^{-r}}{1+2^{s-r}} - 1\right| < \frac{5}{2^r}. \tag{3. 8}$$

Define $\Lambda_2 = \frac{3a}{1+2^{r-s}}\alpha^l 2^{-r} - 1$.

In a manner very similar to the case of Λ_1, we also find that $\Lambda_2 \neq 0$.
Therefore,

$$\log|\Lambda_2| < \log 5 - r\log 2. \tag{3. 9}$$

Now, let

$$\begin{aligned} &\zeta_1 = \frac{3a}{1+2^{s-r}}, && \zeta_2 = \alpha, && \zeta_3 = 2, \\ &b_1 = 1, && b_2 = l, && b_3 = -r, \\ &k = 3. \end{aligned}$$

Take $\mathbb{L} = \mathbb{Q}(\alpha)$, thus $D = 3$.

$$\begin{aligned} h(\zeta_1) &\leq \log 3 + \frac{1}{3}\log 23 + (r-s)\log 2 + \log 2; \\ h(\zeta_2) &= \frac{1}{3}\log \alpha; \\ h(\zeta_3) &= \log 2. \end{aligned}$$

We choose,

$$C_1 = 3\log 3 + \log 23 + 3(r-s)\log 2 + 3\log 2, \quad C_2 = \log \alpha, \quad \text{and} \quad C_3 = 3\log 2.$$

Finally, by setting $B = 3r$, Theorem 2.5 yields the following result.

$$\log|\Lambda_2| > -744017400000\sqrt{3}\times(1+\log 3)(1+\log 3r)(\log(4968)+3(r-s)\log 2)(\log 8\log \alpha),$$

By employing Eqs. (3. 4), (3. 5), and (3. 9), along with some computations, we obtain the following result.

$$r\log 2 < 4.2\times 10^{14}(1+\log 3r) + 2\times 10^{27}(1+\log 3r)^2 + 3. \tag{3. 10}$$

Using Sage, we find that

$$r < 2\times 10^{31}. \tag{3. 11}$$

3.2. **Lowering the bound on r.** We will decrease the upper bound on r. Assume that $r - s \geq 20$ and define

$$\Gamma_1 = \log(3a) + l\log\alpha - r\log 2.$$

Using Eq.(3. 1), we have

$$\left|e^{\Gamma_1} - 1\right| < \frac{4}{2^{s-r}} < \frac{1}{4}, \tag{3. 12}$$

which implies that

$$|\Gamma_1| < \frac{1}{2}. \tag{3. 13}$$

Then $|\Gamma_1| < 2|e^{\Gamma_1} - 1|$. Thus,

$$|\Gamma_1| < \frac{8}{2^{s-r}}. \tag{3. 14}$$

We get $\Gamma_1 \neq 0$ since $\Lambda_1 \neq 0$. Then

$$0 < \left|\frac{\log 3a}{\log 2} - r + l\left(\frac{\log\alpha}{\log 2}\right)\right| < \frac{8}{2^{s-r}\log 2} < \frac{12}{2^{s-r}}. \tag{3. 15}$$

Select $M = 6\times 10^{31}$ and utilize Lemma 2.7 with the following data:

$$\lambda = \frac{\log \alpha}{\log 2}, \qquad \eta = \frac{\log 3a}{\log 2},$$
$$A = 12, \qquad B = 2.$$

Expressing λ as a continued fraction $[a_0, a_1, \ldots]$, we find that

$$q_{86} = 692660461516288491499622810754244220739 7 > 2 \times 10^{32} > 6M,$$

where q_{86} denotes the denominator of the 86th convergent in the continued fraction expansion of γ.

We utilize Lemma 2.7 with $M = 6 \times 10^{31}$ (where $M > 3r > l$), $\gamma = \frac{\log \alpha}{\log 2}$, $\eta = \frac{\log 3a}{\log 2}$, $C = 12$, and $D = 2$. We compute:

$$\epsilon = \|\eta q_{86}\| - M \|\gamma q_{86}\| > 0.3.$$

Hence, applying Lemma 2.7, we obtain $r - s \leq 137$ unconditionally.

We define:

$$\Gamma_2 = \log\left(\frac{3a}{1 + 2^{s-r}}\right) + l \log \alpha - r \log 2.$$

From Eq. (3. 8), we can derive further results.

$$\left|e^{\Gamma_2} - 1\right| < \frac{5}{2^r} < \frac{1}{4}, \tag{3. 16}$$

which implies that

$$|\Gamma_2| < \frac{1}{2}. \tag{3. 17}$$

Consequently, we have $|\Gamma_2| < 2|e^{\Gamma_2} - 1|$. Thus, we obtain

$$|\Gamma_2| < \frac{10}{2^r}. \tag{3. 18}$$

We note that $\Gamma_2 \neq 0$ because $\Lambda_2 \neq 0$. Then

$$0 < \left|\frac{\log\left(\frac{3a}{1+2^{s-r}}\right)}{\log 2} - r + l\left(\frac{\log \alpha}{\log 2}\right)\right| < \frac{15}{2^r}. \tag{3. 19}$$

Choose $M = 6 \times 10^{31} (M > 3r > l)$ and we utilize Lemma 2.7 with the subsequent data:

$$\lambda = \frac{\log \alpha}{\log 2}, \qquad \eta = \frac{\log\left(\frac{3a}{1+2^{s-r}}\right)}{\log 2}$$
$$A = 15, \qquad B = 2.$$

Expressing λ as a continued fraction $[a_0, a_1, \ldots]$, we find that

$$q_{86} = 675779972932909731435512028489410 06 > 2 \times 10^{32} > 6M,$$

where q_{86} represents the denominator of the 86^{th} convergent of the continued fraction of λ. We evaluate ϵ under the condition $r - s \leq 137$:

$$\epsilon = \|\eta q_{86}\| - M \|\gamma q_{86}\| > 0.003.$$

Consequently, using Lemma 2.7, we conclude that $r < 130$, leading to $s < 130$ and $l < 400$. Using Sage, we solved Eq. (1. 1) with the constraints $s < 130$, resulting in $r < 130$ and $l < 400$. We derived the solutions presented in Theorem 1.1, thereby completing the proof.

4. Proof of Theorem 1.2

4.1. **Establishing bounds for s, r, and l.** Assume $s \leq r$, then we use the inequalities (2. 8) and (2. 10) to get a relation between l and r. Thus we have

$$\alpha^{l-2} \leq \mathcal{R}_l \leq 2^r \text{ and } 2^{r-2} \leq \mathcal{R}_l \leq \alpha^{l+1}. \tag{4. 1}$$

These imply that

$$(r-2)\frac{\log 2}{\log \alpha} - 1 \leq l \leq r\frac{\log 2}{\log \alpha} + 2. \tag{4. 2}$$

Using Sage computations for all $l \leq 400$, we obtained only the specified solutions. We now consider the case $l > 400$ with $r \geq 130$. Because of the symmetry in Eq. (1. 2), we may also assume that $r \geq s$. In particular, we focus on the range $l < 3r$. In this context, we substitute the Perrin and Jacobsthal sequences in Eq. (1. 2) with their respective Binet formulas as follows:

$$\alpha^l + \beta^l + \gamma^l = \frac{2^r - (-1)^r}{3} + \frac{2^s - (-1)^s}{3}. \tag{4. 3}$$

Then

$$\left|\alpha^l - \frac{2^r}{3}\right| = \left|\frac{2^s}{3} - \frac{((-1)^r + (-1)^s)}{3} - \beta^l - \gamma^l\right|. \tag{4. 4}$$

Using Eq.(2. 5), we get

$$\begin{aligned}\left|\alpha^l - \frac{2^r}{3}\right| &\leq \frac{2^s}{3} + \frac{2}{3} + 2\left|\beta^l\right| \\ &< \frac{2^s}{3} + \frac{2^s}{3} + \frac{4 \cdot 2^s}{3}.\end{aligned}$$

Then

$$\left|\alpha^l - \frac{2^r}{3}\right| < \frac{6 \cdot 2^s}{3} \tag{4. 5}$$

Thus

$$\left|\frac{3\alpha^l}{2^r} - 1\right| < \frac{6}{2^{r-s}}. \tag{4. 6}$$

Define

$$\Lambda_3 = 3\alpha^l 2^{-r} - 1,$$

We now utilize Theorem 2.5 with the following data:

$$\begin{aligned} \zeta_1 &= 3, & \zeta_2 &= \alpha, & \zeta_3 &= 2, \\ b_1 &= 1, & b_2 &= l, & b_3 &= -r, \\ k &= 3. \end{aligned}$$

Similar to the previous cases, we can demonstrate that $\Lambda_3 \neq 0$.
Taking $\mathbb{L} = \mathbb{Q}(\alpha)$, using Eq.(2. 11), we get

$$\begin{aligned} h(\zeta_1) &= \log 3; \\ h(\zeta_2) &= \frac{1}{3}\log \alpha; \\ h(\zeta_3) &= \log 2. \end{aligned}$$

We take

$$C_1 = \log 27, \;\; C_2 = \log \alpha, \;\; \text{and} \;\; C_3 = \log 8.$$

We choose $B = 3r$. Therefore Theorem 2.5 provides

$$\log|\Lambda_3| > -744017400000\sqrt{3} \times (1+\log 3)(1+\log 3r)(\log 27)(\log 8 \log \alpha),$$

which can be simplified to

$$\log|\Lambda_3| > -9 \times 10^{12}(1+\log 3r). \tag{4. 7}$$

Thus, from Eq. (4. 6), we obtain

$$\log|\Lambda_3| < 2\log 3 + (s-r)\log 2. \tag{4. 8}$$

Thus, by analyzing (4. 7) and (4. 8), we derive the following conclusion

$$(r-s)\log 2 < 8 \times 10^{12}(1+\log 3r) + \log 6. \tag{4. 9}$$

Hence,

$$s > r - \frac{8 \times 10^{12}(1+\log 3r)}{\log 2} - \frac{\log 6}{\log 2}. \tag{4. 10}$$

Now we write Eq.(4. 3) as

$$\alpha^l - \frac{2^r(1+2^{s-r})}{3} = -\beta^l - \gamma^l - \frac{(-1)^r - (-1)^s}{3}, \tag{4. 11}$$

From this, we find that

$$\left|\frac{3\alpha^l 2^{-r}}{1+2^{s-r}} - 1\right| = \left|\frac{3 \cdot 2^{-r}}{1+2^{s-r}}\left(-\beta^l - \gamma^l - \frac{(-1)^r - (-1)^s}{3}\right)\right|. \tag{4. 12}$$

Using Eq.(2. 5) and $\left|\frac{(-1)^r-(-1)^s}{3}\right| \leq \frac{2}{3}$ and $s \leq r$, we obtain

$$\left|\frac{3\alpha^l 2^{-r}}{1+2^{s-r}} - 1\right| < \frac{8}{2^r}. \tag{4. 13}$$

Define $\Lambda_4 = \frac{3}{1+2^{s-r}}\alpha^l 2^{-r} - 1$. As before, we can show that $\Lambda_4 \neq 0$. Therefore,

$$\log|\Lambda_4| < \log 8 - r\log 2. \tag{4. 14}$$

Let

$$\begin{aligned} \zeta_1 &= \frac{3}{1+2^{s-r}}, & \zeta_2 &= \alpha, & \zeta_3 &= 2, \\ b_1 &= 1, & b_2 &= l, & b_3 &= -r, \\ k &= 3. \end{aligned}$$

Then we take $\mathbb{L} = \mathbb{Q}(\alpha)$, the logarithmic heights are,

$$\begin{aligned} h(\zeta_1) &\leq \log 3 + (r-s)\log 2 + \log 2; \\ h(\zeta_2) &= \frac{1}{3}\log \alpha; \\ h(\zeta_3) &= \log 2. \end{aligned}$$

We choose

$$C_1 = 3\log 3 + 3(r-s)\log 2 + 3\log 2, \;\; C_2 = \log \alpha, \;\; \text{and} \;\; C_3 = \log 8.$$

We take $B = 3r$. Therefore Theorem 2.5 implies

$$\log|\Lambda_4| > -744017400000\sqrt{3}\times(1+\log 3)(1+\log 3r)(\log 216+3(r-s)\log 2)(3\log 2\log \alpha),$$

Then, by applying Eqs. (4. 9), (4. 10), and (4. 14) along with some calculations, we obtain the following results.

$$r\log 2 < 2.3\times 10^{13}(1+\log 3r) + 6\times 10^{25}(1+\log 3r)^2 + 4. \tag{4. 15}$$

Using Sage, we find that

$$r < 5\times 10^{29}. \tag{4. 16}$$

4.2. **Lowering bound on r.** We will decrease the upper limit on r. Let us assume that $r - s \geq 20$ and put

$$\Gamma_3 = \log(3) + l\log\alpha - r\log 2.$$

Then we have from Eq.(4. 6) that

$$\left|e^{\Gamma_3} - 1\right| < \frac{6}{2^{s-r}} < \frac{1}{4}, \tag{4. 17}$$

This implies that

$$|\Gamma_3| < \frac{1}{2}. \tag{4. 18}$$

Thus, we have $|\Gamma_3| < 2\left|e^{\Gamma_3} - 1\right|$. Therefore, it follows that

$$|\Gamma_3| < \frac{12}{2^{r-s}}. \tag{4. 19}$$

We note that $\Gamma_3 \neq 0$ since $\Lambda_3 \neq 0$. Consequently, we find that

$$0 < \left|\frac{\log 3}{\log 2} - r + l\left(\frac{\log\alpha}{\log 2}\right)\right| < \frac{12}{2^{r-s}\log 2} < \frac{18}{2^{r-s}}. \tag{4. 20}$$

Choose $M = 15 \times 10^{29} (M > 3r > l)$ and we apply Lemma 2.7 with the subsequent data:

$$\lambda = \frac{\log \alpha}{\log 2}, \qquad \eta = \frac{\log 3}{\log 2},$$
$$A = 18, \qquad B = 2.$$

Expressing λ as a continued fraction $[a_0, a_1, ...]$, we find that
$q_{86} = 692660461516288491499622810754244220739 7 > 9 \times 10^{30} > 6M$.
Where q_{86} represents the denominator of the 86^{th} convergent of the continued fraction of λ. We compute

$$\epsilon = \|\eta q_{86}\| - M \|\lambda q_{86}\| > 0.03.$$

Thus by Lemma 2.7, we obtain $r - s \leq 142$ unconditionally.
Now we put

$$\Gamma_4 = \log\left(\frac{3}{1+2^{s-r}}\right) + l \log \alpha - r \log 2.$$

Using Eq.(4. 13), we obtain

$$\left|e^{\Gamma_4} - 1\right| < \frac{8}{2^r} < \frac{1}{4}, \tag{4. 21}$$

which implies that

$$|\Gamma_4| < \frac{1}{2}. \tag{4. 22}$$

Then $|\Gamma_4| < 2\left|e^{\Gamma_4} - 1\right|$. Thus,

$$|\Gamma_4| < \frac{16}{2^r}. \tag{4. 23}$$

We note that $\Gamma_4 \neq 0$ since $\Lambda_4 \neq 0$. Then

$$0 < \left|\frac{\log\left(\frac{3}{1+2^{s-r}}\right)}{\log 2} - r + l\left(\frac{\log \alpha}{\log 2}\right)\right| < \frac{24}{2^r}. \tag{4. 24}$$

Choose $M = 15 \times 10^{29}$ $(M > 3r > l)$ and we apply Lemma 2.7 with the subsequent data:

$$\lambda = \frac{\log \alpha}{\log 2}, \qquad \eta = \frac{\log\left(\frac{3}{1+2^{s-r}}\right)}{\log 2},$$
$$A = 24, \qquad B = 2.$$

When we represent λ as a continued fraction $[a_0, a_1, \ldots]$, we obtain

$$q_{86} = 675779972932909731435512028489410 06 > 9 \times 10^{30} > 6M.$$

Where q_{86} is the denominator of the 86^{th} convergent of the continued fraction of τ. We evaluate every ϵ such that $r - s \leq 142$, except in the case where $r - s = 1$:

$$\epsilon = \|\eta q_{86}\| - M \|\lambda q_{86}\| > 0.003.$$

Thus, by applying Lemma 2.7, we conclude that $r < 130$, which implies $s < 130$ and $l < 400$. We used Sage to solve Eq. (1. 2) under the conditions $s < 130$, $r < 130$, and $l < 400$, obtaining the specified solutions in Theorem 1.2.

If $r - s = 1$, we find that ϵ is always negative. Therefore, we solve Eq. (1. 2) for the case where $r - s = 1$. In this scenario, Eq. (1. 2) can be expressed as $R_l = J_s + J_{s+1}$ and can be simplified to

$$\mathcal{R}_l = 2^s. \tag{4. 25}$$

This leads to the conclusion that $l < 3s$, and from Eq. (4. 16), we find that $s < 5 \times 10^{29}$. Employing the same way of proving $\Lambda_3 < \frac{6}{2^{s-r}}$, we get

$$\left|\alpha^l 2^{-s} - 1\right| < \frac{6}{2^s},$$

Thus

$$\left| l\frac{\log \alpha}{\log 2} - s \right| < \frac{18}{2^s}.$$

If $s > 10$ using $l < 3s$ then $\frac{18}{2^s} < \frac{1}{2l^2}$. From Legendre's Theorem, this implies that $\frac{s}{l}$ is a convergent of $\frac{\log \alpha}{\log 2}$. Since $l < M$, computations performed with Sage indicate that

$$q_{70} \leq M < q_{71} \quad \text{and} \quad b := \max\{a_i : i = 0, 2, ..., 71\} = 80.$$

Therefore, based on the properties of continued fractions, we have:

$$\frac{1}{(b+2)l} < \left| l\frac{\log \alpha}{\log 2} - s \right| < \frac{18}{2^r}.$$

Thus, it follows that:

$$2^s < 18 \cdot 82 \cdot M.$$

This leads to the conclusion that $s \leq 110$. We utilized Sage to solve Eq. (4. 25) under the condition $s < 299$ and obtained the solutions indicated in Theorem 1.2. Therefore, Theorem 1.2 is proven.

5. Conflict of interest

The authors declare no conflict of interest.

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