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Characterization of Amenability in Non-Expansive Dynamical Systems by the Schauder Fixed Point Property

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Abstract. In the following manuscript, we are addressing the question:"Which amenability property of dynamical system is characterized by the Schauder fixed point property (Sfpp)?". We have first introduced the property F_C and then we provided suitable answers to this problem for the respective class $\mathcal{ELA}$ (extremely left amenable) of semitopological semigroups by developing a relation between properties F_C, F_E and Sfpp.

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Key Words: Dynamical system, Semitopological semigroups, Schauder fixed point property, Amenability.

1. Introduction

A well-known finding by Brouwer[1] is that any self-mapping that is continuously defined for a unit closed ball in $\Re^n$ attains a fixed point, served as the catalyst for the examination of the presence of a fixed point regarding nonlinear maps as well. The following more general fact is also produced by this result: any continuous map defined on a non-empty compact convex subset of a Euclidean space surely attains an invariant point. Later, Schauder [15] enhanced Brouwer's findings by demonstrating the details that, in reality, Brouwer's theorem holds true even when $\Re^n$ is substituted with a normed space. Then we have a Sfpp which states whenever (S, C, σ) is a jointly continuous map with a non-empty compact convex set of a separated locally convex topological vector space then a common fixed point for S in C exists. Related to this, we have another property F_E which states

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that whenever (S, C, σ) is a jointly continuous map with a compact Hausdorff space, then there exists a common fixed point for S in C. Mitchell [11] then proved that $\mathcal{ELA}$ of a semitopological semigroup is described by the property F_E.
In this article, we will prove that $F_E \Rightarrow F_S$ then it is obviously natural to ask that "Is there any amenability property of a semitopological semigroup that is categorized by Sfpp F_S?" or "Does $F_S \Rightarrow F_E$?" We will attempt to address the above question by introducing an additional property F_C, followed by a promising approach to provide a concise solution to the open problem published by Lau in [7].

2. Notations and Preliminaries

Let us recall some fundamental concepts which will be used in our work and which can be found in [1, 15, 10, 13, 2].

Fixed Point. A point $v \in V$ is known as a FP of mapping $\eta : V \to V$ if $v = \eta v$. In other words , it is the point which remains invariant or unchanged under given transformation.

Example 1. Let η be a self mapping on $\Re$ defined by $\eta(x) = x^2$ and $\eta(x) = -x$, which implies $x = 0, 1$ and x=0 are the FPs of the given functions respectively.

FP Property. Consider a Banach space S and a non-empty set N of S where N is closed, bounded and contained within S. If a FP for any non-expansive mapping $\eta : N \to N$ are in N, then it possesses the FP property.

Weak FP Property. Let S be a Banach space and any weakly compact convex subset N of S has the FP property then N has weak FP property.

Left and Right Translation. For a semigroup σ , let $l^\infty(\sigma)$ be the C^*-algebra of complex-valued functions which are also bounded on σ having the supremum norm and multiplication which is point-wise. For every element $s \in \sigma$, $k \in l^\infty(\sigma)$, if

$$l_s k(t) = k(st) \; where \; t \in \sigma$$

then $l_s k$ is called the left translate of k by s. Similarly if

$$\varrho_s k(t) = k(ts) \; where \; s \in \sigma$$

then $\varrho_s k$ is said to be the right translate of k by s.

Mean. Let $\mathbb{T}$ be a closed subspace of $l^\infty(\sigma)$ containing constants and invariant under translations. Then a linear functional $w \in \mathbb{T}^*$ is called a mean if

$$\|w\| = w(e) = 1,$$

h is referred to as a Left invariant imply if $w(l_t h) = w(h)$ for all $t \in \sigma, h \in \mathbb{T}$.Similarly w is called Right invariant mean if $w(\varrho_a h) = w(h)$ for all $a \in \sigma$, $h \in \mathbb{T}$.

Now, we are going to list some spaces as follows:
$\mathcal{C}_\mathrm{b}(\sigma)$ denotes the space of all bounded continuous complex-valued functions defined on σ.

$\mathrm{LUC}(\sigma)$ is the space of all left uniformly continuous functions on σ, i.e. all $g \in \mathcal{C}_{\mathrm{b}}(\sigma)$ in this manner the mapping $t \mapsto l_t g : \sigma \mapsto \mathcal{C}_{\mathrm{b}}(\sigma)$.
$\mathrm{AP}(\sigma)$ is the space of all $g \in \mathcal{C}_{\mathrm{b}}(\sigma)$ such that $\mathcal{LO}(g) = \{l_s(g) : s \in \sigma\}$ is relatively compact with respect to the norm topology of $\mathcal{C}_{\mathrm{b}}(\sigma)$.
$\mathrm{WAP}(\sigma)$ is said to be space of all $g \in \mathcal{C}_{\mathrm{b}}(\sigma)$ such that $\mathcal{LO}(g) = \{l_s(g) : s \in \sigma\}$ is relatively compact in the WT of $\mathcal{C}_{\mathrm{b}}(\sigma)$.
$\mathrm{m}(\sigma)$ denotes the space of all bounded real-valued functions on σ.
$\beta(\sigma)$ denotes the space of all multiplicative means on $\mathrm{m}(\sigma)$.

Semitopological Semigroup. If σ is a semigroup with the H.S topology is said to be a semitopological semigroup if the mapping for every $s \in \sigma$ $v \to uv$ and $v \to vu$ from σ into itself are continuous.

Left Amenable.
The semitopological semigroup S is said to be left amenable if L.U.C(S) has a LIM.
Observe that if S is discrete then

$$L.U.C(S) = \mathcal{C}_b(S) = l^{\infty}(S)$$

So both the definitions are same when S is discrete.

Left Reversible. If any two closed right ideals of the semitopological semigroup S have a non-void intersection, then S is said to be left reversible. , i.e

$$\overline{aS} \bigcap \overline{dS} \ for\ whatever\ a,\ d\ \in\ S.$$

Dynamical System. A dynamical system consists of a pair (S, ϱ), where S is a semitop-logical semigroup and ϱ is topological space together with separately continuous action of S on ϱ. If the action is continuous then the dynamical system is said to be continuous.

Q-Nonexpansive Dynamical System. Dynamical system is Q-Nonexpansive Dynamical System where Q is a family of seminorms if

$$q(sv - sw) \leq (v - w)\ for\ every\ q \in Q, s \in S\ and\ v, w \in \varrho \tag{2. 1}$$

Convex Hull. Let V be a vector space and G be a subset of V . The convex hull of G is defined as the set

$$CH(G) = \{\sum_{j=1}^{n} t_j x_j, x_j \in G; t_j \geq 0; \sum_{j=1}^{n}; n \geq 1\}$$

Left Thick. Let $\sigma^{'}$ be a subset of a semigroup σ is called left thick in σ if for subset $\sigma^{''} \subseteq \sigma$ that is finite, there is some $s^{''} \in \sigma$ so that $\sigma^{''} s^{''} \subseteq \sigma^{'}$.

Example 2. Let $\sigma^{'}$ be the set of all integers greater than or equal to 100.Then

$$\sigma^{'} = \{100, 101, ...\}$$

Take a finite subset

$$\sigma^{''} = \{-3, 7, 20\}$$

We want $s^* \in \mathbb{Z}$ such that

$$\sigma^{''} + s^* \subseteq \sigma^{'}$$

We need

$$-3 + s^* \geq 100, \quad 7 + s^* \geq 100, \quad 20 + s^* \geq 100$$

For $s^* = 103$

$$\{-3 + 103,\ 7 + 103, 0 + 103\} = \{100, 110, 123\} \subseteq \sigma^{'}$$

The set $\{100, 101, ...\}$ is left thick in semigroup $\mathbb{Z}$.

Jointly Continuous Mapping. Let X, Y, and Z be topological spaces, and let

$$f : X \times Y \to Z$$

be a function. We say that f is *jointly continuous*(JC) if it is a continuous function from the product space $X \times Y$ (equipped with the product topology) to the space Z.

Hausdorff Space. A topological space $(\mathcal{R}, \varrho)$ is called a Hausdorff space (HS) if for any pair of distinct points $s, t \in \mathcal{R}$, there exist two open sets P_1 and P_2 such that

$$s \in P_1, \quad t \in P_2, \quad \text{and} \quad P_1 \cap P_2 = \varnothing.$$

Topological Vector Space. Consider a vector space V over the field F and let ϱ be a topology on V. Then (V, ϱ) is said to be a topological vector space(TVS) if it satisfies the following conditions:

(1) The operation of addition is jointly continuous; that is, given any $w, z \in V$ and any ϱ-neighbourhood B of $w + z$ in V, there exist ϱ-neighbourhoods C of w and D of z in V such that

$$C + D \subseteq B.$$

(2) The operation of scalar multiplication

$$F \times R \to R$$

is jointly continuous; i.e., given any $w \in R$ and $k \in F$ and any ϱ-neighbourhood B of kw in R, there exist a ϱ-neighbourhood C of w in V and a neighbourhood X of k in F such that

$$XC \subseteq B.$$

Locally Convex Space. A TVS (V, ϱ) is called a LCS if it has a base of neighborhoods of 0 consisting of convex sets.

3. Some Aspects of Schauder Fixed Point Property.

Within this section, first we intend to study Sfpp then we will prove it's relation with another fixed point property. To further extent we will discuss some examples related to these fpp.

(F_E) : Every JC mapping of S on a non-empty Hausdorff space that is compact attains a fixed point that is common[11].

(F_S) : Every JC mapping of S on a non-empty compact convex set ω in a separated locally convex topological vector space has a common fixed point [8].

Above given property (F_S) is called Sfpp.

A semigroup S is Extremely Left Amenable if and only if every extreme point of the compact, convex set of left-invariant means is a multiplicative left-invariant mean[13]. Mitchell proved in [11] semitopological semigroup is referred as $\mathcal{ELA}$ if it has the FPP F_E.

The first examples of extremely amenable groups were provided by Herer and Christensen [5]. They showed that certain abelian Polish groups of the form $L_0(\mu, \mathbb{R})$, consisting of all μ-measurable real-valued functions (where μ is a pathological submeasure), are extremely amenable. Later, more general groups of the form $L_0(\mu, G)$, with G a topological group, were also shown to be extremely amenable. The first examples of extremely amenable groups were provided by Herer and Christensen [5]. They showed that certain abelian Polish groups of the form $L_0(\mu, \mathbb{R})$, consisting of all μ-measurable real-valued functions (where μ is a pathological submeasure), are extremely amenable. Later, more general groups of the form $L_0(\mu, G)$, with G a topological group, were also shown to be extremely amenable.

Furthermore, Furstenberg–Weiss and independently Glasner [4] proved that the group of measurable functions from the unit interval I, equipped with the Lebesgue measure λ, to the unit circle $\mathbb{T}$ is extremely amenable. Farah and Solecki [3] extended this result by demonstrating that $L_0(\mu, G)$ is extremely amenable whenever G is a compact solvable topological group and μ is a diffused submeasure. Later, Sabok [14] generalized these results to all solvable topological groups G together with diffused submeasures μ.

Theorem 1. If the semigroup is $\mathcal{ELA}$ then it will possess property F_S.

Proof. Suppose $\{J_s : s \in S\}$ be a JC representation of σ on a non-empty ω of a separated LCS which is compact and convex. Then we have a strong version of Tychnoff's Theorem [12] that implies for every representation $\{J_s : s \in S\}$ we have at least one fixed point. Here we are considering the induced topology on K this implies K is a subspace(a subspace of a topological space X is contained in X which is equipped with a topology induced from that of X called subspace toplogy or relative topology) of a LC Hausdorff TVS and a subspace of Hausdorff is Hausdorff. This implies that K is a Hausdorff subspace of LC Hausdorff TVS which is compact. Now by F_E for every jointly continuous representation $\{J_s : s \in S\}$ on K we have a common fixed point.

Example 3. The bicylic semigroup represented by $S_1 = \langle a, b |\, ab = e\rangle$ is a semigroup produced by identity e and some other elements a and b such that $ab = e$ has the semigroup property F_S but is not extremely left amenable(see[7]).

Proof. Elements of set S_1 are of the type

$$s = b^c a^d \tag{3. 2}$$

for certain integer $c \geq 0$ and $d \geq 0$. If

$$bs = s. \tag{3. 3}$$

Now from (3. 2) put $s = b^c a^d$ in (3. 3),we get

$$b^{c+1}a^d \quad = \quad b^c a^d \tag{3. 4}$$

Now multiplying a^c and b^d towards the left and the right, respectively, in the above equation,we get

$$b \quad = \quad e,$$

Which is a not true as given that $S_1 = \langle a, b : ab = e\rangle$ if $b = e$ then $a.e = e$ which is not possible. Hence S_1 does not contains right zero element. This implies that S_1 is not extremely left amenable. Let σ denote a continuous representation of S_1 over a non empty set C which is convex. From Sfpp, say z is a fixed point of C.Then

$$\begin{aligned} b(ez) &= (be)z = (eb)z = e(bz) = ez \\ a(ez) &= a(bez) = (ab)ez = e.ez = e^2 z = ez. \end{aligned}$$

Therefore, ez has become an invariant point in C that is common for every generator including a, b and e. This presents that ez is a fixed point for S_1 that is commom but S_1 is not $\mathcal{ELA}$. This implies that F_E does not holds in S_1.

Example 4. The free commutative cyclic semigroup has the fpp F_S (see[7]).

Proof. Let Σ^* be the free commutative semigroup on one generator $\Sigma^* = \{\nu\}$.Then

$$\Sigma^* = \{\nu, \nu\nu, \nu\nu\nu, \nu\nu\nu\nu...\}$$

Let σ be the continuous representation of Σ^* on non-empty set K which is compact and convex.Then by Sfpp K contains fixed point for ν. i.e

$$\begin{aligned}
\nu z &= z \\
\nu(\nu z) &= \nu(\nu(z)) = \nu z = z \\
\nu\nu\nu(z) &= \nu\nu(\nu(z)) = \nu\nu(z) = \nu(\nu\nu(z)) = \nu z = z, \\
&\vdots
\end{aligned}$$

In this way we will get the common fixed point in K for all elements of Σ^*. Hence z is a common fixed point for Σ^*. This implies that Σ^* has F_S property.

Example 5. A group that is generated by one element has the fpp F_S (see[7]).

Proof. Let B be a group generated by one element that is $\exists$ some $g \epsilon B$ s.t. $B = \langle g \rangle$. This implies that g is the generator. Now, let σ be the continuous depiction of B on a non-empty compact and convex set, then by Schauder fixed point theorem $H(g) \neq \emptyset$. Let $z \epsilon H(g)$, i.e

$$g.z = z \tag{3. 5}$$

For $g \in B$, then for some $i \in I$, $g^{-1} \in B$ such that

$$\begin{aligned}
g^{-1} &= g^i \\
g^{-1}.z &= g^i.z
\end{aligned}$$

From the above, we get $g^{-1}.z = z$.This implies that $z \in H(g^{-1}) \Rightarrow H(g) \subseteq H(g^{-1})$. Similarly, $H(g^{-1}) \subseteq H(g)$ Hence, a group (finite or infinite) that is generated by one element has the property F_S.

Example 6. The semigroup $(Q, +)$ has the Schauder fixed point property , where Q is the set of all the rational numbers and Q_+ is the set of all positive rational numbers (see[7]).

Proof. If Q_+ acts on a set C which is compact and convex then the fixed point set $H(s)$ contains some element for every $r \in Q_+$. It is clear that if $r = zt$ for some integer z, then $H(t) \subset H(s)$. Let $r_1, r_2, \ldots, r_n$ be finite element of Q_+. It can be written $r_j = \frac{l_j}{z_j}(j = 1, 2, \ldots n)$, where z_j and l_j are integers. Let $r_j = \frac{1}{z_1, z_2, \ldots, z_k}$ then all r_j are multiples of r. So $G(r) \subset G(r_j)$ for all j. This implies that the collection

$\gamma = \{G(r) : r \in Q_+\}$ that contains subsets of C which are compact has the finite intersection property. Therefore $\bigcap_{r \in Q_+} H(r) \neq \phi$ i.e Q_+ has a fixed point that is common in C.

Example 7. A discrete group on $z \geq 2$ does not have fixed point property (F_S)(see[7].

Proof. It is enough to show the case $z = 2$. As for $z = 2$ the semigroup is isomorphic to $Z_\circ \times Z_\circ$ where $Z_\circ$ is the semigroup of non negative integers with respect to addition $\{0, 1, 2 \ldots\}$(which is the discrete cyclic semigroup). Notably, two continuous functions g and l exist, mapping the unit $[0, 1]$ into itself and satisfying the commutative property under composition function but do not have any common fixed point in [0,1] . Let the representation of $Z_\circ \times Z_\circ$ on [0,1] defined by $\urcorner_{(0,0)} = id[0, 1]$, $\urcorner_{(1,0)} = g$, $\urcorner_{(0,1)} = l$ then $Z_\circ \times Z_\circ$ has no fixed point in [0,1] that is common although $Z_\circ$ has the fpp (F_S).

4. Amenabilty of Dynamical System

In this section we will first introduce the property F_C and then develop the relation between the property F_s and F_c which will be helpful in proving our main result.

(F_C) : Every JC representation of semitoplogical semigroup on a convex hull of non-empty compact Hausdorff space has a fixed point that is common.

Now we will prove that F_S implies F_C.

Theorem 2. If a semitoplogical semigroup $\mathbb{S}$ has the Schauder fpp F_s then it has the fpp F_c.

Proof. Let $\mathcal{C}$ be a HS and $\{G_s : s \in S\}$ be a JC representation of σ. Also let $\mathcal{K}$ be a separated L.C.S and C be a subspace of $\mathcal{K}$. $CH(\mathcal{C})$ is the convex hull of $\mathcal{C}$. As $\mathcal{C}$ is compact so $CH(\mathcal{C})$ is also compact. This implies $CH(\mathcal{C})$ is a compact set of a separated LCS. If $\{G_s : s \in S\}$ is JC for $CH(\mathcal{C})$ then by F_s for every JC representation $\{G_s : s \in S\}$ on $CH(\mathcal{C})$ we have a common fixed point.

Theorem 3. Let σ be a semigroup then the following are

(1) Left invariant multiplicative mean exists in σ .

(2) For each pair of subsets $\sigma_1, \sigma_2 \subseteq \sigma$ such that $\sigma_1 \cup \sigma_2$ is left thick in σ, then atleast one subset σ_1 or σ_2 must be a left thick in σ.

(3) For every finite family of subsets $\sigma_i \subseteq \sigma$ so that $\sigma = \bigcup_{i=1}^{n} \sigma_i$, it implies that at least one of the subset σ_i is left thick in σ.

(4) Every JC representation of σ on a convex hull of non-empty compact HS has a fixed point that is common for σ.

Proof. $(1) \Longrightarrow (2)$

Let $\sigma^{'}$ be left thick in σ, where $\sigma^{'} = \sigma_1 \bigcup \sigma_2$. Let $\{\sigma_\Upsilon\}$ be the finite family of subsets of σ, for each finite subset $\sigma_\Upsilon \in \sigma$ there exists $s_\Upsilon \in \sigma$ such that $\sigma_\Upsilon s_\Upsilon \subseteq \sigma^{'}$. As $\kappa(\sigma)$ is w^* compactness, the net $\{\Omega s_\Upsilon\}$ has a subnet $\{\Omega s_\delta\}$ which is w^*-convergent to some $\nu^{'} \in \kappa(\sigma)$[16].

For any $s \in \sigma$

$$l_s^* \nu^{'} = l_s^*(w^* lim_\delta(\Omega s_\delta)) = w^* lim_\delta(l_s^*(\Omega s_\delta))$$

As l_s^* is w^*-continuous so we can follow the second equality. Let $h_\circ \in n(\sigma)$ be the characteristic function of $\sigma^{'}$. Then

$$\begin{aligned}(l_s^* \nu^{'}, h_\circ) &= lim_\delta(l_s^*(\Omega s_\delta), h_\circ) = lim_\delta(\Omega s_\epsilon, l_s h_\circ) \\ &= lim_\delta(l_s h_\circ) s_\delta) = lim_\delta h_\circ(s s_\delta) = 1\end{aligned}$$

for each $s \in \sigma$. By (1), σ has a multiplicative left invariant mean ν. Suppose $\nu^{''} \in \kappa(\sigma)$ be stated by $\nu^{''} = \nu \bigodot \nu^{'}$. Then $\nu^{''}$ is a left invariant mean by [[2], Corollary 2, p.529]. Furthermore, for each $s \in \sigma$

$$\nu_l^* h^\circ(s) = (\nu^{'}, l_s h_\circ) = (l_s^* \nu^{'}, h_\circ) = 1$$

hence $\nu^{'} h_\circ = e$. Thus

$$\nu^{''} h_\circ = (\nu, \nu_l^{'} h_\circ) = (\nu, e) = 1$$

Let h_1 and h_2 be the characteristic functions of σ_1 and σ_2 . As $\nu^{''}$ is multiplicative, it will give zero on characteristic function. As $\nu^{''} h_\circ = 1$, it is not possible that $\nu^{''} h_1 = 0$ and $\nu^{''} h_2 = 0$. So there must be at least one of the function h_j where $j = 1, 2, \ldots$ satisfies $\nu^{''} h_i = 1$. But if K is any set that is contained in left amenable semigroup σ, in this case K is left thick in σ iff there exists left invariant mean on σ that assign the value 1 to the characteristic function K [[10], Theorem 7]. So, there must exist at least one of subset σ_j that is left thick in σ.

$(2) \Longrightarrow (3)$

σ is left thick in itself, this implies that $\bigcup_{i=1}^{n} \sigma_i$ is left thick.Then by (b) there exist at least one of the subset σ_i is left thick in σ.

$(3) \Longrightarrow (4)$

Let $\phi : \sigma \to \pi$ be a JC representation of σ onto π, a semigroup of continuous map of $CH(Z)$ into $CH(Z)$, where Z is a compact HS. This implies that $CH(Z)$ is also compact. Let ϱ be a topology on $CH(Z)$, such that $CH(Z)$ is Hausdorff. This implies $CH(Z)$ is also compact Hausdorff space. Let z be point of $CH(Z)$.Now at first we will show that their

exist $y_o \in CH(Z)$ such that for every open neighbourhood W of y_o where $W \subseteq CH(Z)$, the set $\{s \in \sigma; (\phi s)y \in W\}$ is a left thick in σ.
Suppose that $y_o \notin CH(Z)$. Then there exist an open neighbourhood of z for each $z \in CH(Z)$, the set $W_z \subseteq CH(Z)$ such that the set σ_z where $\sigma_z = \{s \in \sigma; \phi(s)y \in \sigma_z\}$ does not satisfy the condition to be left thick in σ. However, the family $\{W_z; z \in CH(Z)\}$ is open covering of $CH(Z)$, $CH(Z)$ is compact so $\exists$ a finite number of element $z(j) \in CH(Z)$ such that $CH(Z) = \bigcup_{j=1}^{n} \sigma_{z(j)}$ which is contradicting (3). Hence $y_o \in CH(Z)$.
Let ϱ be the family of finite subsets of σ and χ the family of all open neighbourhoods of y_o. Let $\sigma_\kappa^{''}$ be subset of σ that is finite and letY_ρ be an open neighborhood of y_o Further $\{\sigma_\rho^{'}\}$ will designate the net left thick of σ defined as $\sigma_\rho^{'} = \{s \in \sigma; \phi(s)y \in Y_\rho\}$. Let $\Psi = \varrho \times \chi$ the direct product set of ϱ and χ. For every $\psi = (\kappa, \rho) \in \Psi$, there exist $s_\psi \in \sigma$ such that $\sigma_\kappa^{''} s_\psi \subseteq \sigma_\rho$ by left thickness of $\sigma_\rho^{'}$ in σ. For any $s \in \sigma$, the net$\{ss_\psi\}$ is finally in all $\sigma_\rho^{'}$, which means that the net $\{\phi(ss_\psi)y\}$ is finally in all open neighborhood Y_ρ of y_π. Therefore $y_o = lim_\psi(\phi(ss_\psi)y)$ for all $s \in \sigma$. Let s_o be a particular element of σ. Then for any $s \in \sigma$,

$$\begin{aligned}\phi(s)y_o = \phi(s)lim_\psi(\phi(s_os_\psi)y) &= lim_\psi(\phi(s)\phi(s_os_\psi)y) \\ &= lim_\psi(\phi(s_os_\psi)y) = y_o\end{aligned}$$

we can follow the second equality as $\phi(s)$ is continuous and similarly as $\phi(s)$ is JC so we can follow the third inequality. Hence y_o is the required common point of family π that is fixed.
$(4) \implies (1)$
Let $\mathfrak{C}$ be the convex hull of $\alpha(\sigma)$, where $\alpha(\sigma)$ is compact HS with respect to w^* topology. For each $s \in \sigma$, define a map $\mathbb{G}_s : \alpha(\sigma) \to \alpha(\sigma)$ by $\mathbb{G}_s\nu = l_s^*\nu$ for $\nu \in \alpha(\sigma)$. The set $\Im = \{\mathbb{G}_s : s \in \sigma\}$ forms a semigroup, jointly continuous to σ of $\alpha(\sigma)$ into itself. Let the set $\Im = \{\mathbb{G}_s : s \in \sigma\}$ be such that it forms a jointly continuous representation to $\mathfrak{C}$ of $\mathfrak{C}$ into itself. As $\alpha(\sigma)$ is compact HS then $\mathfrak{C}$ is also compact and let τ be discrete topology on $\mathfrak{C}$ then $\mathfrak{C}$ is Hausdorff w.r.t τ.Now by (4), there exist $\nu_o \in \mathfrak{C}$ such that $\nu_o = \mathbb{G}_s\nu_o = l_s^*\nu_o$ for all $s \in \mathbb{S}$. i.e ν_o is a fixed point that is common in $\mathfrak{C}$. Now as $\alpha(\sigma) \subseteq \mathfrak{C}$, so we will show that $\nu_o \in \alpha(\sigma)$. Let $\nu_o \notin \alpha(\sigma)$, as $\alpha(\sigma)$ is a compact HS then by property F_E $\exists$ some fixed point in $\alpha(\sigma)$ for $\Im = \{\mathbb{G}_s : s \in \sigma\}$ that is common, this implies that $\nu_o \in \alpha(\sigma) \subseteq \mathfrak{C}$. This shows that ν_o is a multiplicative invariant mean.

5. Conclusion

In this work, we examined the limitation that the fixed point property F_S does not directly lead to amenability or to the property F_E that characterizes extreme left amenability.We have shown in examples that F_S alone is insuffcient. To overcome this issue, we introduced an intermediate property F_C to strengthen the connection between F_S and F_E. Establishing the implications

$F_S \implies F_C \implies F_E \implies$ Extremely Left Amenable

We obtain that F_S implies F_E indirectly via F_C. Since F_E already characterizes extremely left amenable dynamical systems, provides a full dynamical characterization of extreme left amenability

Property	Implies	Reason
F_S	F_C	Established in this work
F_C	F_E	Introduced to connect $F_S$and F_E
F_E	Extremely Left Amenable	Classical implication
F_S	Extremely Left Amenable	Follows indirectly via F_C and F_E

TABLE 1. Summary of implications between fixed point properties and extreme left amenability

Credit authorship contribution statement:

Amna Kalsoom: Conceptualization, formal analysis, validation. **Sana Siddique:** Formal analysis, investigation, writing original draft, methodology. **Maliha Rashid:** Conceptualization, formal analysis, supervision, validation.

Declaration of competing interest:

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