Punjab University Journal of Mathematics (2025), 57(11), 1210-1221
https://doi.org/10.52280/pujm.2025.57(11)03

Fixed Points of q-Rung Orthopair Fuzzy Contractive Mappings in q-ROF Metric Spaces with an Application to Boundary Value Problems

Received
22 August, 2025

Revised
30 January, 2026

Accepted
01 March, 2026

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Abstract. Owing to their enhanced expressive capability in handling uncertainty, q-rung orthopair fuzzy (q-ROF) metric spaces provide a flexible framework for analyzing nonlinear problems under vagueness. In this article, we introduce a new class of q-ROF fuzzy contractive mappings and establish a corresponding fixed point theorem in complete q-ROF metric spaces. The uniqueness and existence of the fixed point are proved under suitable contractive conditions. To demonstrate the practical relevance of the theoretical results, an application to a second-order nonlinear boundary value problem with homogeneous boundary conditions is presented. By transforming the boundary value problem into an equivalent integral equation and employing the proposed fixed point theorem, the existence of a unique continuous solution is obtained. Illustrative examples are also included to validate the applicability and effectiveness of the proposed approach.

AMS (MOS) Subject Classification Codes: 47H10; 54H25
Key Words: Fuzzy q-ROF metric space; fuzzy contractive condition; fixed point result; boundary value problem

1. INTRODUCTION

The emergence of imprecision and vagueness in handling data sparked the introduction of fuzzy set theory by Zadeh in 1965 [25]. Fuzzy set theory faced many challenges over the years, but it has greatly progressed since its inception and has discovered applications that provide useful tools for modeling uncertainty across a wide number of fields. Works labeled in the literature that require systematic treatment combined with optimal computation show that fuzzy sets are positioned at the core of mathematics not only as domains of logic and set theory but also in approximate reasoning. A complete synthesis of practical applications and recent modifications of fuzzy sets is given in [8]. An understanding of the architecture and properties of fuzzy sets is important in appreciating because of the broad framework of fuzzy logic and its integration in applied sciences.

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In 1975, Kramosil and Michalek [12] integrated fuzzy sets with metric spaces, coining the term fuzzy metric spaces. This initial construction was improved upon by Veeramani and George [7], where they applied a continuous t-norm and created a Haudorff topology for these spaces. Such developments have opened up new areas of research, especially in image analysis, machine learning and data clustering [16]. In the scope of Fixed point theory, fuzzy contractive mappings were introduced by Sapena and Gregori [9].

In 1986, Atanassov presented the concept of intuitionistic fuzzy sets, which expand on conventional fuzzy sets by incorporating degrees of membership, non-membership, and a corresponding hesitation margin, in order to understand an extra layer of uncertainty [2]. Numerous operators and extensions of IFSs have been developed as a result of this method's success in real-world decision-making situations [3, 4]. By adding geometric representations of intuitionistic fuzzy sets and distance measures, Kacprzyk and szmidt[23] enhanced the theory even more.

In 2004, Park [15] generalized the notion of fuzzy metric space into the framework of intuitionistic fuzzy metric spaces (IFMS), out stretching the Hausdorff topology and demonstrating that every classical metric naturally induces an intuitionistic fuzzy metric. Ongoing research continues to explore the algebraic structure, topological properties, and practical implications of IFMS in various complex systems [1, 10, 13, 18, 19, 20]. Some fixed point results were explored for the class of intuitionistic fuzzy mappings [5, 6, 11, 17].

Although intuitionistic fuzzy sets and their associated metric spaces have significantly extended the classical fuzzy framework by incorporating degrees of non-membership, their applicability remains constrained when modelling higher levels of uncertainty. In particular, the restriction imposed by the condition that the sum of membership and non-membership degrees must not exceed unity limits their expressive capability in complex systems. This limitation becomes more pronounced in metric and fixed point settings, where contractive conditions must simultaneously regulate the behaviour of both membership and non-membership functions.

The introduction of q-rung orthopair fuzzy (q-ROF) sets [24] provides a natural and powerful generalisation by allowing the sum of the q-th powers of membership and non-membership degrees to be bounded by one. This additional flexibility enables a more refined representation of uncertainty and hesitation, especially in problems involving non-linear dynamics. However, despite recent advances in the study of q-ROF metric spaces, existing fixed point results largely rely on classical fuzzy or intuitionistic fuzzy contraction mappings, which do not fully reflect the intrinsic structure of orthopair fuzzy metrics [22].

Motivated by this theoretical gap, there is a clear need to develop contractive conditions that are inherently compatible with the q-ROF framework. In particular, a contraction mapping must account for the simultaneous influence of q-powered membership and non-membership functions while preserving the topological completeness of the underlying space. This observation forms the basis for introducing a new class of q-ROF fuzzy contractive mappings in the present work, along with corresponding fixed point results and applications.

2. PRELIMINARIES

[q-ROF set [24]] A *q-rung orthopair fuzzy subset* Λ of $\mathfrak{I}$, denoted as a q-ROF set, is an ordered pair

$$\Lambda = \langle \mu_\Lambda, \eta_\Lambda \rangle_q,$$

where $\mu_\Lambda, \eta_\Lambda : \mathfrak{I} \to I = [0,1]$ denote the membership and non-membership degrees, respectively, satisfying for all $\xi \in \mathfrak{I}$:

(1) $q \geq 1$,
(2) $(\mu_\Lambda(\xi))^q + (\eta_\Lambda(\xi))^q \leq 1$.

[Continuous t-norm [21]] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if it satisfies:

(C1) $*$ is continuous, associative and commutative,
(C2) $\phi * 1 = \phi$ for all $\phi \in [0,1]$,
(C3) $\phi * \omega \leq \zeta * \rho$ whenever $\phi \leq \zeta$ and $\omega \leq \rho$.

[Continuous t-conorm [15]] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-conorm* if it satisfies:

(D1) $\diamond$ is continuous, associative and commutative,
(D2) $\phi \diamond 0 = \phi$ for all $\phi \in [0,1]$,
(D3) $\phi \diamond \omega \leq \zeta \diamond \rho$ whenever $\phi \leq \zeta$ and $\omega \leq \rho$.

[q-ROF metric space [22]] Let $\mathfrak{I}$ be a non-empty set, $*$ a continuous t-norm, $\diamond$ a continuous t-conorm, and let $\Im, \aleph : \mathfrak{I}^2 \times (0,\infty) \to [0,1]$ be fuzzy sets representing the membership and non-membership grades, respectively. The 5-tuple $(\mathfrak{I}, \Im, \aleph, *, \diamond)$ is called a *q-ROF metric space* if for all $\xi, \rho, \nu \in \mathfrak{I}$ and all $s, \phi > 0$ the following axioms hold:

(1) $(\Im(\xi,\rho,\phi))^q + (\aleph(\xi,\rho,\phi))^q \leq 1$,
(2) $\Im(\xi,\rho,\phi) > 0$,
(3) $\Im(\xi,\rho,\phi) = 1$ if and only if $\xi = \rho$,
(4) $\Im(\xi,\rho,\phi) = \Im(\rho,\xi,\phi)$,
(5) $(\Im(\xi,\rho,\phi))^q * (\Im(\rho,\nu,s))^q \leq (\Im(\xi,\nu,\phi+s))^q$,
(6) $\Im(\xi,\rho,\cdot) : (0,\infty) \to (0,1]$ is continuous,
(7) $\aleph(\xi,\rho,\phi) > 0$,
(8) $\aleph(\xi,\rho,\phi) = 0$ if and only if $\xi = \rho$,
(9) $\aleph(\xi,\rho,\phi) = \aleph(\rho,\xi,\phi)$,
(10) $(\aleph(\xi,\rho,\phi))^q \diamond (\aleph(\rho,\nu,s))^q \geq (\aleph(\xi,\nu,\phi+s))^q$,
(11) $\aleph(\xi,\rho,\cdot) : (0,\infty) \to (0,1]$ is continuous.

The choice of the continuous t-norm $*$ and t-conorm $\diamond$ is not arbitrary; they must be compatible with the triangular-inequality axioms (5) and (10) above, which are essential for generating a well-defined topology. In particular, the continuity of $*$ and $\diamond$ guarantees that the induced topology is Hausdorff and metrizable. The family

$$\{B^q(\xi,\zeta,\phi) : \xi \in \mathfrak{I},\ \zeta \in (0,1),\ \phi > 0\},$$

where

$$B^q(\xi,\zeta,\phi) = \{\rho \in \mathfrak{I} : \Im^q(\xi,\rho,\phi) > 1-\zeta,\ \aleph^q(\xi,\rho,\phi) < \zeta\},$$

forms a base for the topology $\tau_{(\Im,\aleph)}$ on $\mathfrak{I}$. This topology is completely determined by the q-ROF metric structure and ensures that sequential convergence agrees with the topological convergence described below. [Convergence and Cauchy sequences] Let $(\mathfrak{I}, \Im, \aleph, *, \diamond)$ be a q-ROF metric space.

- A sequence $\{\xi_\nu\}$ in $\mathfrak{I}$ *converges* to $\xi \in \mathfrak{I}$ (denoted $\xi_\nu \to \xi$) if and only if for every $\phi > 0$

  $$\lim_{\nu\to\infty} \Im^q(\xi_\nu, \xi, \phi) = 1 \quad \text{and} \quad \lim_{\nu\to\infty} \aleph^q(\xi_\nu, \xi, \phi) = 0.$$

  This notion of convergence is exactly the convergence in the topology $\tau_{(\Im,\aleph)}$.
- $\{\xi_\nu\}$ is called a *Cauchy sequence* if for every $\phi > 0$ and every $\epsilon > 0$ there exists $\nu_0 \in \mathbb{N}$ such that

  $$\Im^q(\xi_\nu, \xi_m, \phi) > 1 - \epsilon \quad \text{and} \quad \aleph^q(\xi_\nu, \xi_m, \phi) < \epsilon \qquad \text{for all } \nu, m \geq \nu_0.$$

  Equivalently,

  $$\lim_{\nu,m\to\infty} \Im^q(\xi_\nu, \xi_m, \phi) = 1 \quad \text{and} \quad \lim_{\nu,m\to\infty} \aleph^q(\xi_\nu, \xi_m, \phi) = 0 \quad \text{for all } \phi > 0.$$
- The space is *complete* if every Cauchy sequence converges (with respect to $\tau_{(\Im,\aleph)}$) to a point of $\mathfrak{I}$.

The definitions of convergence and Cauchy sequences are fully compatible with the topological structure induced by the q-ROF metric. The triangular inequalities (5) and (10), together with the continuity of $*$ and $\diamond$, guarantee that the limit of a convergent sequence is unique and that Cauchy sequences behave as expected in a metric-like topology. Consequently, all standard topological arguments (e.g., closure, completeness, continuity) can be carried out within this framework.

[22] Consider $\gimel = \mathbb{N}$. Define $\vartheta * \omega = \vartheta\omega$ and $\vartheta \Diamond \omega = \vartheta + \omega - \vartheta\omega$, for all $\vartheta, \omega \in [0,1]$ and let $\Im$ and $\aleph$ be defined as:

$$\Im(\xi, \varrho, \wp) = \left(\frac{1}{\exp\left(\frac{|\xi-\varrho|}{\wp}\right)} \right), \quad \aleph(\xi, \varrho, \wp) = \left(1 - \frac{1}{\exp \frac{|\xi-\varrho|}{\wp}} \right)^{\frac{1}{q}}$$

for all $\xi, \varrho \in \gimel$ and $\wp > 0$. Then $(\gimel, \Im, \aleph, *, \Diamond)$ is a q-ROF metric space. [9] Let$(\gimel, \Im, *)$ be a fuzzy metric space. A mapping $T : \gimel \longrightarrow \gimel$ is termed as a fuzzy contractive mapping if there exists $\lambda \in (0,1)$ such that $\forall \xi, \varrho \in \gimel, \forall \wp > 0$

$$\frac{1}{\Im(T\xi, T\varrho, \wp)} - 1 \leq \lambda \left[\frac{1}{\Im(\xi, \varrho, \wp)} - 1 \right]$$

In 2023, Onbaşıoğlu and Pazar [14] extended the idea of fuzzy contractive mapping in intuitionistic fuzzy metric-like spaces and also provided some fixed point results. [14] Let $(\gimel, \Im, \aleph, *, \Diamond)$ denotes an intuitionistic fuzzy metric-like space. A mapping $T : \gimel \longrightarrow \gimel$ is called intuitionistic fuzzy contractive if there exists $\lambda \in (0,1)$ such that $\forall \xi, \varrho, \in \gimel, \forall \wp > 0$

$$\frac{1}{\Im(T\xi, T\varrho, \wp)} - 1 \leq \lambda \left[\frac{1}{\Im(\xi, \varrho, \wp)} - 1 \right] \text{ and } \aleph(T\xi, T\varrho, \wp) \leq \lambda \aleph(\xi, \varrho, \wp)$$

where, λ is called the intuitionistic fuzzy constant of the mapping T.

Novelty and Contribution of This Work. Although recent research has explored topological and geometric properties of q-ROF metric spaces [22], the present manuscript introduces several distinct advancements that extend the existing theory in meaningful ways:

(1) **New Class of Contractions:** We introduce a *q-ROF fuzzy contraction* (Definition 10) that simultaneously generalizes both the classical fuzzy contraction of Sapena and Gregori [9] and the intuitionistic fuzzy contraction of Onbaşoğlu and Pazar [14]. Unlike earlier intuitionistic fuzzy contractions, which only impose conditions of the form

$$\frac{1}{\Im(T\xi, T\rho, \phi)} - 1 \leq \lambda\Big[\frac{1}{\Im(\xi, \rho, \phi)} - 1\Big], \qquad \aleph(T\xi, T\rho, \phi) \leq \lambda\, \aleph(\xi, \rho, \phi),$$

our q-ROF contraction employs the q-th powers of the membership and non-membership functions:

$$\frac{1}{\Im^q(T\xi, T\rho, \phi)} - 1 \leq \lambda\Big[\frac{1}{\Im^q(\xi, \rho, \phi)} - 1\Big], \qquad \aleph^q(T\xi, T\rho, \phi) \leq \lambda\, \aleph^q(\xi, \rho, \phi).$$

This not only extends the intuitionistic fuzzy case (which corresponds to $q = 1$) but also provides a more flexible framework for modeling uncertainty when $q > 1$, because the q-ROF model allows a wider range of membership/non-membership pairs satisfying $(\mu^q + \eta^q) \leq 1$.

(2) **Fixed-Point Theorem in Complete q-ROF Metric Spaces:** We prove the existence and uniqueness of fixed points for the proposed contraction in complete q-ROF metric spaces (Theorem 1). The proof explicitly utilizes the completeness of the space and the specific topological structure induced by the q-ROF metric, thereby illustrating how the theoretical foundation laid in [22] can be applied to obtain concrete analytical results.

(3) **Non-Trivial Examples and an Application:** We provide explicit examples (Example 2) that illustrate situations where our q-ROF contraction works while classical intuitionistic fuzzy contractions fail (e.g., when $q = 4$ and axiom (a) of Definition 4 is violated). Moreover, we present an application to a boundary-value problem (Section 3), demonstrating how the new contraction can be used to guarantee the existence of solutions in function spaces endowed with a q-ROF metric structure. This practical application underscores the utility of the theory beyond purely abstract settings.

(4) **Clarification of Topological Consistency:** In contrast to earlier works that often assume generic t-norms, we carefully discuss the role of continuous t-norms and t-conorms in generating a Hausdorff topology (see Remark after Definition 4). We also verify that the definitions of Cauchy sequences and convergence are fully compatible with this topology, thereby ensuring that all subsequent fixed-point arguments are topologically sound.

In summary, this article advances the theory of q-ROF metric spaces by introducing a new, more general class of contractive mappings, establishing a corresponding fixed-point theorem, and illustrating its applicability both through counter-examples and a concrete differential-equations application. The results strictly extend earlier intuitionistic fuzzy fixed-point theorems and provide a robust analytical tool for problems involving higher-rung orthopair fuzzy uncertainty.

3. MAIN RESULTS

According to Definition 2, we are considering the following contractive mapping for the class of q-ROF metric spaces. Let $(\gimel, \Im, \aleph, *, \Diamond)$ be a complete q-ROF metric space. Self mapping T defined on $\gimel$ is called q-ROF fuzzy contractive if there exists $\lambda \in (0,1)$ such that for all $\xi, \varrho, \in \gimel$ and for all $\wp > 0$

$$\frac{1}{\Im^q(T\xi, T\varrho, \wp)} - 1 \leq \lambda \left[\frac{1}{\Im^q(\xi, \varrho, \wp)} - 1 \right] \text{ and } \aleph^q(T\xi, T\varrho, \wp) \leq \lambda \aleph^q(\xi, \varrho, \wp)$$

where, λ represents the q-ROF constant of the mapping T. The contraction condition in Definition 8 is consistent with the q-ROF metric axioms. In particular, using the triangular inequalities

$$(\Im(\xi,\rho,\phi))^q * (\Im(\rho,\nu,s))^q \leq (\Im(\xi,\nu,\phi+s))^q, \quad (\aleph(\xi,\rho,\phi))^q \Diamond (\aleph(\rho,\nu,s))^q \geq (\aleph(\xi,\nu,\phi+s))^q,$$

one can verify that if T satisfies Definition 8, then the iterated images $T^n\xi$ form a Cauchy sequence in the q-ROF sense. This compatibility ensures that the contraction operates harmoniously within the q-ROF topological structure. Let $(\gimel, \Im, \aleph, *, \Diamond)$ be a q-ROF metric space and $T : \gimel \longrightarrow \gimel$ be a q-ROF fuzzy contractive mapping. Then T has a unique fixed point $w \in \gimel$ such that $\Im(w, w, \wp) = 1$ and $\aleph(w, w, \wp) = 0$, for all $\wp > 0$. **Proof.** Let ξ_0 be an arbitrary element of $\mathfrak{I}$. Define a sequence $\{\xi_v\}$ in $\mathfrak{I}$ such that $\xi_v = T(\xi_{v-1})$ for all $v \in \mathbb{N}$. If $\xi_v = \xi_{v-1}$ for some $v \in \mathbb{N}$, then ξ_v is a fixed point of T.

Assume now that $\xi_v \neq \xi_{v-1}$ for all $v \in \mathbb{N}$. From Definition 10 we obtain

$$\begin{aligned}
\frac{1}{\Im^q(\xi_v, \xi_{v+1}, \phi)} - 1 &= \frac{1}{\Im^q(T\xi_{v-1}, T\xi_v, \phi)} - 1 \\
&\leq \lambda \Big[\frac{1}{\Im^q(\xi_{v-1}, \xi_v, \phi)} - 1 \Big] \\
&\leq \lambda^2 \Big[\frac{1}{\Im^q(\xi_{v-2}, \xi_{v-1}, \phi)} - 1 \Big] \\
&\vdots \\
&\leq \lambda^v \Big[\frac{1}{\Im^q(\xi_0, \xi_1, \phi)} - 1 \Big] \longrightarrow 0, \qquad v \to \infty,
\end{aligned}$$

hence

$$\lim_{v\to\infty} \Big[\frac{1}{\Im^q(\xi_v, \xi_{v+1}, \phi)} - 1 \Big] = 0,$$

which implies

$$\lim_{v\to\infty} \Im^q(\xi_v, \xi_{v+1}, \phi) = 1.$$

Similarly,

$$\begin{aligned}\aleph^q(\xi_v,\xi_{v+1},\phi) &= \aleph^q(T\xi_{v-1},T\xi_v,\phi)\\ &\le \lambda\,\aleph^q(\xi_{v-1},\xi_v,\phi)\\ &\le \lambda^2\aleph^q(\xi_{v-2},\xi_{v-1},\phi)\\ &\vdots\\ &\le \lambda^v\aleph^q(\xi_0,\xi_1,\phi)\longrightarrow 0, \qquad v\to\infty,\end{aligned}$$

so that

$$\lim_{v\to\infty}\aleph^q(\xi_v,\xi_{v+1},\phi)=0.$$

Now, for any $l\in\mathbb{N}$,

$$\begin{aligned}\Im^q(\xi_v,\xi_{v+l},\phi) &\ge \Im^q\Big(\xi_v,\xi_{v+1},\frac{\phi}{l}\Big)*\Im^q\Big(\xi_{v+1},\xi_{v+2},\frac{\phi}{l}\Big)*\cdots*\Im^q\Big(\xi_{v+l-1},\xi_{v+l},\frac{\phi}{l}\Big)\\ &\longrightarrow 1*1*\cdots*1=1 \quad \text{as } v\to\infty,\end{aligned}$$

and

$$\begin{aligned}\aleph^q(\xi_v,\xi_{v+l},\phi) &\le \aleph^q\Big(\xi_v,\xi_{v+1},\frac{\phi}{l}\Big)\diamond\aleph^q\Big(\xi_{v+1},\xi_{v+2},\frac{\phi}{l}\Big)\diamond\cdots\diamond\aleph^q\Big(\xi_{v+l-1},\xi_{v+l},\frac{\phi}{l}\Big)\\ &\longrightarrow 0\diamond 0\diamond\cdots\diamond 0=0 \quad \text{as } v\to\infty.\end{aligned}$$

Thus $\{\xi_v\}$ is a Cauchy sequence in $(\mathfrak{I},\Im,\aleph,*,\diamond)$.

Since the space $(\mathfrak{I},\Im,\aleph,*,\diamond)$ is **complete**, every Cauchy sequence converges. Consequently, there exists an element $\xi\in\mathfrak{I}$ such that

$$\lim_{v\to\infty}\Im^q(\xi_v,\xi,\phi)=1 \qquad \text{and} \qquad \lim_{v\to\infty}\aleph^q(\xi_v,\xi,\phi)=0 \quad \text{for all } \phi>0.$$

We now show that ξ is a fixed point of T. Applying the q-ROF contractive condition,

$$\frac{1}{\Im^q(T\xi_v,T\xi,\phi)}-1\le\lambda\Big[\frac{1}{\Im^q(\xi_v,\xi,\phi)}-1\Big]\longrightarrow 0\ (v\to\infty),$$

hence $\lim\limits_{v\to\infty}\Im^q(T\xi_v,T\xi,\phi)=1$. Likewise,

$$\aleph^q(T\xi_v,T\xi,\phi)\le\lambda\,\aleph^q(\xi_v,\xi,\phi)\longrightarrow 0\ (v\to\infty),$$

so $\lim\limits_{v\to\infty}\aleph^q(T\xi_v,T\xi,\phi)=0$. Therefore $\lim_{v\to\infty}T\xi_v=T\xi$. Because $\xi_{v+1}=T\xi_v$, we obtain

$$\xi=\lim_{v\to\infty}\xi_{v+1}=\lim_{v\to\infty}T\xi_v=T\xi,$$

which proves that ξ is a fixed point of T.

Uniqueness. Suppose $\eta\in\mathfrak{I}$ is another fixed point of T, i.e. $T\eta=\eta$. For any $\phi>0$,

$$\frac{1}{\Im^q(\eta,\xi,\phi)}-1=\frac{1}{\Im^q(T\eta,T\xi,\phi)}-1\le\lambda\Big[\frac{1}{\Im^q(\eta,\xi,\phi)}-1\Big].$$

Rearranging gives

$$(1-\lambda)\Big[\frac{1}{\Im^q(\eta,\xi,\phi)}-1\Big]\le 0.$$

Since $\lambda \in (0,1)$, we have $1-\lambda > 0$, which forces

$$\frac{1}{\Im^q(\eta,\xi,\phi)} - 1 \leq 0,$$

i.e. $\Im^q(\eta,\xi,\phi) \geq 1$. By the definition of a q-ROF metric, $\Im^q(\eta,\xi,\phi) \leq 1$, hence $\Im^q(\eta,\xi,\phi) = 1$ for all $\phi > 0$.

Now consider the non-membership part:

$$\aleph^q(\eta,\xi,\phi) = \aleph^q(T\eta,T\xi,\phi) \leq \lambda\,\aleph^q(\eta,\xi,\phi).$$

If $\aleph^q(\eta,\xi,\phi) > 0$, dividing by it yields $1 \leq \lambda < 1$, a contradiction. Therefore $\aleph^q(\eta,\xi,\phi) = 0$ for all $\phi > 0$.

Having $\Im^q(\eta,\xi,\phi) = 1$ and $\aleph^q(\eta,\xi,\phi) = 0$ for every $\phi > 0$ implies, by the axioms of a q-ROF metric, that $\eta = \xi$. Hence the fixed point is unique.

Let $\daleth = \mathbb{N}$ and define the continuous t-norm and t-conorm by

$$\alpha * \beta = \alpha\beta, \qquad \alpha \diamond \beta = \alpha + \beta - \alpha\beta, \quad \forall\, \alpha,\beta \in [0,1].$$

Define the membership and non-membership functions $\Im, \aleph : \daleth \times \daleth \times (0,\infty) \to (0,1]$ by

$$\Im(\xi,\varrho,\wp) = \begin{cases} \left(\dfrac{\xi+\wp}{\varrho+\wp}\right)^{\frac{1}{q}}, & \text{if } \xi \leq \varrho, \\ \left(\dfrac{\varrho+\wp}{\xi+\wp}\right)^{\frac{1}{q}}, & \text{if } \varrho \leq \xi, \end{cases}$$

$$\aleph(\xi,\varrho,\wp) = \begin{cases} \left(\dfrac{\varrho-\xi}{\varrho+\wp}\right)^{\frac{1}{q}}, & \text{if } \xi \leq \varrho, \\ \left(\dfrac{\xi-\varrho}{\xi+\wp}\right)^{\frac{1}{q}}, & \text{if } \varrho \leq \xi, \end{cases}$$

for all $\xi, \varrho \in \daleth$ and $\wp > 0$.

We now verify that $(\daleth, \Im, \aleph, *, \diamond)$ is a q-rung orthopair fuzzy metric space.

(i) *Boundedness:* For all $\xi, \varrho \in \daleth$ and $\wp > 0$,

$$\Im^q(\xi,\varrho,\wp) + \aleph^q(\xi,\varrho,\wp) = \frac{\min\{\xi,\varrho\} + \wp + |\xi-\varrho|}{\max\{\xi,\varrho\} + \wp} \leq 1.$$

(ii) *Identity and positivity:* $\Im(\xi,\varrho,\wp) = 1$ and $\aleph(\xi,\varrho,\wp) = 0$ if and only if $\xi = \varrho$. Moreover, $\Im(\xi,\varrho,\wp) > 0$ and $\aleph(\xi,\varrho,\wp) \geq 0$ for all $\xi \neq \varrho$.

(iii) *Symmetry:* It follows directly from the definitions that

$$\Im(\xi,\varrho,\wp) = \Im(\varrho,\xi,\wp), \quad \aleph(\xi,\varrho,\wp) = \aleph(\varrho,\xi,\wp).$$

(iv) *Triangular inequalities:* For all $\xi, \varrho, \nu \in \daleth$ and $\wp, \sigma > 0$,

$$\Im^q(\xi,\varrho,\wp) * \Im^q(\varrho,\nu,\sigma) \leq \Im^q(\xi,\nu,\wp+\sigma),$$

and

$$\aleph^q(\xi,\varrho,\wp) \diamond \aleph^q(\varrho,\nu,\sigma) \geq \aleph^q(\xi,\nu,\wp+\sigma),$$

which follow from the monotonicity of the rational expressions and properties of $*$ and $\diamond$.

(v) *Continuity:* For fixed $\xi, \varrho \in \gimel$, both $\Im(\xi, \varrho, \cdot)$ and $\aleph(\xi, \varrho, \cdot)$ are continuous functions on $(0, \infty)$.

Hence, $(\gimel, \Im, \aleph, *, \diamond)$ is a complete q-rung orthopair fuzzy metric space.

Now define a self-mapping $T : \gimel \to \gimel$ by

$$T(\xi) = \eta,$$

where $\eta \in \mathbb{N}$ is a fixed constant.

Then, for all $\xi, \varrho \in \gimel$ and $\wp > 0$,

$$\Im^q(T\xi, T\varrho, \wp) = 1, \qquad \aleph^q(T\xi, T\varrho, \wp) = 0.$$

Consequently,

$$\frac{1}{\Im^q(T\xi, T\varrho, \wp)} - 1 = 0 \leq \lambda \left(\frac{1}{\Im^q(\xi, \varrho, \wp)} - 1 \right),$$

and

$$\aleph^q(T\xi, T\varrho, \wp) \leq \lambda \aleph^q(\xi, \varrho, \wp),$$

for any $\lambda \in (0, 1)$.

Therefore, T is a q-ROF fuzzy contractive mapping and admits a unique fixed point $\eta \in \gimel$.

This q-ROF metric space is not an intuitionistic fuzzy metric space for $q = 4$. In order to verify take $\xi = 4$, $\varrho = 9$ and $\wp = 0.5$ then axioms of the intuitionistic fuzzy metric space do not hold. Let $\mathfrak{I} = [0, 1]$ and define for $\xi, \rho \in \mathfrak{I}$, $\phi > 0$:

$$\Im(\xi, \rho, \phi) = \exp\left(-\frac{|\xi - \rho|^q}{\phi}\right), \qquad \aleph(\xi, \rho, \phi) = 1 - \exp\left(-\frac{|\xi - \rho|^q}{\phi}\right),$$

with $* = \min$, $\diamond = \max$, and $q = 3$. Then $(\mathfrak{I}, \Im, \aleph, *, \diamond)$ is a complete q-ROF metric space. Define $T\xi = \frac{\xi}{2}$. One can check that for $\lambda = \frac{1}{2}$,

$$\frac{1}{\Im^3(T\xi, T\rho, \phi)} - 1 \leq \lambda \left[\frac{1}{\Im^3(\xi, \rho, \phi)} - 1 \right], \qquad \aleph^3(T\xi, T\rho, \phi) \leq \lambda \aleph^3(\xi, \rho, \phi).$$

Thus T is a q-ROF contraction with a unique fixed point $\xi = 0$. For $q = 1$ (intuitionistic fuzzy case), however, the condition $\mu + \eta \leq 1$ fails for some ξ, ρ, showing that this mapping is not an intuitionistic fuzzy contraction. Hence the example works in the q-ROF setting but not in the intuitionistic one.

4. AN APPLICATION

To demonstrate the practical utility of the theoretical framework developed in the preceding sections, we now apply the q-ROF fixed-point theorem to a classical boundary value problem. The problem under consideration involves a second-order differential equation with Dirichlet boundary conditions—a structure frequently encountered in physics, engineering, and applied mathematics. By reformulating the problem as an integral equation and endowing the relevant function space with a suitable q-ROF metric, we can interpret the solution operator as a self-mapping satisfying the q-ROF fuzzy contraction defined in Section 3. The completeness of the constructed q-ROF metric space, together with Theorem 1, then guarantees the existence and uniqueness of a solution. This application not only illustrates the applicability of our abstract results but also highlights how q-ROF metric spaces can provide a flexible and powerful tool for analyzing problems in which uncertainty or imprecision is modeled through higher-rung orthopair fuzzy sets. Consider the following boundary value problem:

$$\begin{cases} \frac{d}{du}\left(e^{u}\frac{dy}{du}\right) = g(u, y(u)),\ u \in [0,1] \\ \qquad y(0) = 0,\ y(1) = 0. \end{cases} \tag{4. 1}$$

We first justify the choice of the Green function associated with the given boundary value problem (4. 1).

The corresponding homogeneous problem admits a unique solution under the given boundary conditions, and the Green function $\mathcal{P}(v, u)$ is obtained using standard variational methods for second-order linear differential operators with variable coefficients. The function

$$\mathcal{P}(v, u) = \begin{cases} -e^{-u}, & 0 \le v \le u \le 1, \\ -e^{-v}, & 0 \le u \le v \le 1, \end{cases}$$

is continuous on $[0, 1] \times [0, 1]$ and satisfies the imposed boundary conditions, ensuring that the solution of the boundary value problem can be represented in integral form.

Using this Green function, the boundary value problem is equivalent to the integral equation

$$y(u) = \int_0^1 \mathcal{P}(v, u)\, g(v, y(v))\, dv,$$

which motivates the definition of the operator $\sigma : Y \to Y$ by

$$\sigma(y)(u) = \int_0^1 \mathcal{P}(v, u)\, g(v, y(v))\, dv,$$

where $Y := C_2\left([0, 1], \mathbb{R}\right)$ represent the space of continuous function with continuous derivatives upto the second order. Suppose

$$\mathfrak{D}(y, z) = \sup_{u\in[0,1]} |y(u) - z(u)|\,, \text{ and } \mathfrak{D}_{\mathfrak{g}}(y, z) = \sup_{u\in[0,1]} |g(u, y(u)) - g(u, z(u))|\,.$$

Define $\alpha * \beta = \min\{\alpha, \beta\}$, $\alpha \diamondsuit \beta = \max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, 1]$ and for $0 < \lambda \le r < 1$

$$\Psi(y, z, r) = \left(\frac{r}{r + \mathfrak{D}(y, z)}\right)^{\frac{1}{q}} \text{ and } \aleph(y, z, r) = \left(\frac{\mathfrak{D}(y, z)}{r + \mathfrak{D}(y, z)}\right)^{\frac{1}{q}}$$

Then $(Y, \Im, \aleph, *, \diamondsuit)$ is a complete q-ROF metric space.

We now derive the contractive estimates in detail. For any $y, z \in Y$ and $r > 0$, we have

$$\begin{aligned} \sup_{u\in[0,1]} |\sigma(y)(u) - \sigma(z)(u)| &= \sup_{u\in[0,1]} \left| \int_0^1 \mathcal{P}(v, u)\big(g(v, y(v)) - g(v, z(v))\big)\, dv \right| \\ &\le \sup_{u\in[0,1]} \int_0^1 |\mathcal{P}(v, u)|\, |g(v, y(v)) - g(v, z(v))|\, dv \\ &\le D_g(y, z) \int_0^1 |\mathcal{P}(v, u)|\, dv. \end{aligned}$$

By assumption, the Green function satisfies

$$\int_0^1 |\mathcal{P}(v,u)|\, dv \leq \lambda < 1,$$

uniformly for all $u \in [0,1]$. Hence,

$$\sup_{u\in[0,1]} |\sigma(y)(u) - \sigma(z)(u)| \leq \lambda D_g(y,z).$$

Using the condition $D_g(y,z) \leq D(y,z)$, we obtain

$$\sup_{u\in[0,1]} |\sigma(y)(u) - \sigma(z)(u)| \leq \lambda D(y,z).$$

Consequently, the membership function satisfies

$$\begin{aligned}\frac{1}{\Im^q(\sigma(y),\sigma(z),r)} - 1 &= \frac{\sup_{u\in[0,1]} |\sigma(y)(u) - \sigma(z)(u)|}{r} \\ &\leq \lambda \frac{D(y,z)}{r} = \lambda \left(\frac{1}{\Im^q(y,z,r)} - 1 \right),\end{aligned}$$

and similarly,

$$\aleph^q(\sigma(y),\sigma(z),r) = \frac{\sup_{u\in[0,1]} |\sigma(y)(u) - \sigma(z)(u)|}{r + \sup_{u\in[0,1]} |\sigma(y)(u) - \sigma(z)(u)|} \leq \lambda \aleph^q(y,z,r).$$

Thus, the operator σ satisfies the q-rung orthopair fuzzy contractive condition defined in Theorem 1. By completeness of the space $(Y, \Psi, \mathcal{N}, *, \diamond)$, σ admits a unique fixed point, which corresponds to a unique continuous solution of the boundary value problem.

5. CONCLUSION

In this study, we have explored the potential of q-ROF metric spaces by introducing a novel category of fuzzy-type contractive mappings. The fixed-point theorem established within this framework broadens the scope of classical fixed-point results under uncertainty. The provided examples and applications not only validate the theoretical findings, but also highlight the practical significance of the proposed approach in real-world scenarios. These results pave the way for further research in fuzzy analysis, particularly in areas involving complex decision-making and vague data structures.

DECLARATIONS

Conflict of Interest: The author declares no conflicts of interest.

Funding: There is no specific funding for this research article. This work was supported and funded by the author herself.

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