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Complex Picture Fuzzy Subgroup and its Application in Modeling of a Parallel Robotic Grippers

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Abstract. Complex fuzzy sets and picture fuzzy sets portray powerful enhancements of fuzzy set theory that led the formulation of complex fuzzy subgroups and picture fuzzy subgroups. Their combined structure, namely complex picture fuzzy sets found numerous applications in the recent years. However it has not been applied to group theory. Consequently, this work initiates complex picture fuzzy subgroups (CPFSGs) and studies their properties. Important notions including level complex picture fuzzy sets, CPF support, CPF core, CPF conjugacy, CPF normalizer, CPF order and CPF index are defined and their mutual connections are discussed. A complex picture fuzzy form of Lagrange's theorem is presented. In addition, complex picture fuzzy normal subgroups are studied in terms of their CPF cosets, normalizer and conjugate CPFSGs. The CPFSGs are discussed under group homomorphism. Finally, a novel group theoretic idea of CPFSG is used to model a two-finger robotic gripper. It assesses grip success, uncertainty, and instability using the CPFSG of $SE(3)$.

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Key Words: Complex picture fuzzy subgroup, complex fuzzy subgroup, picture fuzzy subgroup, homomorphism, complex picture fuzzy set, robotic grippers.

1. INTRODUCTION

Research Background and Related Work. Revolutionizing the set theoretical concepts, Zadeh [40] in 1965 introduced a set based on a membership function $\chi : \Phi \to [0, 1]$ which mapped each element from the crisp set Φ to some number from 0 to 1. By doing so, this new set called the fuzzy set (FS) depicted the partial satisfaction of an element towards a criteria, hence allowing for depiction of scenarios impossible under the binary membership degrees 0 and 1 of a crisp set. Considering the impact of this novel structure, Rosenfeld [29] defined fuzzy subgroups (FSGs) and elucidated the algebraic properties and restrictions of these new structures. Following this, Das [12] further studied FSGs and presented level FSGs based on fuzzy thresholds. Wu [37] studied normality in the fuzzy sense and presented his results on normal FSGs. In 1986, considering the limitation of fuzzy sets in

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depicting dissatisfaction efficiently. Exploring algebraic structures is a significant research direction in modern mathematics and has received considerable attention from many researchers; for instance, see [30, 31]. Atanassov [8] developed the notion of intuitionistic fuzzy sets (IFSs). These new sets considered two fuzzy memberships separately depicting the intensity of satisfaction and dissatisfaction, though constrained by their sum not exceeding unity. A few years later, Biswas [10] extended the fuzzy subgroups to intuitionistic fuzzy subgroups (IFSGs). This new group theoretical extension of FSs laid foundation to an ever expending study on fuzzy set based extensions and their merger with the group theory. Recently, Rasuli [27] discussed the norms over IFSGs for studying direct product of groups. The IFSGs were applied as a foundation to define a binary two fold algebraic structure by Yasar and Hatip [39]. Vijayabalaji *et al.* [36] extended the IFSGs to intuitionistic fuzzy soft expert groups, hence accomodating the opinions of multiple experts in their structure.

In 2002, Ramot *et al.* [26] proposed a significant contribution to the FS theoretical advancements, namely complex fuzzy sets (CFSs). These CFSs mapped the elements to a unit modulus circle in the complex domain unlike the real unit interval, hence expanding the range of depiction. Following the same, Ramot *et al.* [**?**] discussed complex fuzzy (CF) logic with distinct fuzzy memberships for amplitude and phase terms in the CF depiction. [25] developed confidence level aggregation operators for bipolar CFSs. The concept of CF subgroups (CFSGs) was brought forward by Alsarahead and Ahmad [6]. In 2021, another attempt was made by Abuhijleh *et al.* [1] based on approach of Rosenfeld. Extending it further, Alolaiyan *et al.* [5] introduced (α, β)-CFSGs and studied their properties. Alsarahead and Al-Husban [7] combined CFSGs with multi-fuzzy sets introducing complex multi-fuzzy subgroups. Yang *et al.* [38] initiated the notion of bipolar CFSGs. Rasuli [28] defined CFSGs and normal CFSGs by means of T-norms. Recently, Imtiaz *et al.* [17] applied conjunctive complex fuzzy subgroups to sylow theory. Ali *et al.* [4] proposed the (ϵ, δ)-complex anti FSGs.

In 2014, Cuong and Kreinovich [11] introduced picture fuzzy sets (PFSs) by considering three fuzzy memberships that portray satisfaction, neutrality and dissatisfaction. This enhanced approach attracted numerous researchers leading to multiple extensions and applications. For instance, Ali *et al.* [3] combined bipolar soft sets with PFSs allowing for uncertainty handling under bipolar parameters. Verma and Rohtagi [35] applied their newly developed picture fuzzy measures to medical diagnostics. In 2023, He and Wang [16] evaluated energy vehicles by analyzing online reviews with the help of PFSs. Utilizing the sentiment analytic capabilities of PFSs, Bani-Doumi *et al.* [9] applied PFS based algorithm on patient feedbacks to customize hospital recommendations. Recently, Kahraman [19] introduced proportional picture fuzzy sets. In 2023, Dogra and Pal [14] considered the need for picture fuzzy extension of fuzzy subgroups. The algebraic study of different extensions of fuzzy sets has attracted considerable attention in the literature; see, for instance, [15,20]. This led to the formation of picture fuzzy subgroups (PFSGs).

Taking into account the powerful capabilities of CFSs and PFSs, complex picture fuzzy sets (CPFSs) were developed with their novel aggregation operators (Akram *et al.* [2]).

Shoaib *et al.* [33] utilized CPF numbers in graphs and developed certain operations on these CPF graphs. Jan *et al.* [18] utilized CPFSs in analyzing generative adversarial networks. Recently, Dhumras *et al.* [13] and Khan *et al.* [21] discussed the solutions to pattern recognition problems using CPFS similarity and distance measures, respectively. Tanoli *et al.* [34] brought forward the concept of complex cubic PFSs and applied it to solve decision-problems. Petchimuthu *et al.* [24] developed aggregation operators for their novel complex q-rung PFSs and utilized these operators in pharmacological analysis.

Motivation and Research Gaps. There has been much research on fuzzy subgroup theory in classical fuzzy, intuitionistic fuzzy, picture fuzzy and complex fuzzy scenarios, but there are still some significant theoretical and practical issues that are not addressed in the literature. Specifically, a comprehensive algebraic structure that combines the complex picture fuzzy set theory and classical group theory is not available. The fuzzy subgroup models that exist are either abstinence-aware uncertainty models or phase-based uncertainty models, but not both at once. The uncertainty of multidimensional information, such as positive, neutral and negative information, and amplitude and phase characteristics therefore have not been explored much. Moreover, there is not any systematic theory for Complex Picture Fuzzy Subgroups (CPFSGs) that can be found. The level complex picture fuzzy sets, CPF support, CPF core, CPF conjugacy, CPF normalizer, CPF order and CPF index are new concepts and have not been previously defined and studied in a unified (complex picture fuzzy) framework setting. In addition, there is no general extension of some of the significant classical group theoretic results in the complex picture fuzzy environment. Especially, there has been no previous results on the complex picture fuzzy analogue of Lagrange's theorem. Also, the field of Complex Picture Fuzzy Normal Subgroups is not well developed. These relations between the CPF normality, CPF cosets, conjugate CPFSGs and CPF normalizers are rarely explored in existing works.

Another noteworthy discrepancy relates to the behaviour of the CPFSGs under group homomorphisms. In classical algebraic systems, homomorphic mappings are the main tools, and in generalized fuzzy algebraic systems, they are used to preserve and transform certain properties of algebraic systems. In the case of preserving and transforming certain properties of the subgroup in picture fuzzy algebraic systems, the analysis of such homomorphic mappings has not been thoroughly studied. Furthermore, there is also significant lack of real world implementations of generalized fuzzy subgroup theory for applications. Specifically, the utilization of complex picture fuzzy subgroup structures in robotics, intelligent control systems, motion analysis and engineering uncertainty modelling has not been addressed at all. Past robotic grasping models typically lack the ability to represent simultaneously the stability of the grasp, the uncertainty of grasping, the abstinence of grasping, and the motion behaviour of the grasp, in the form of algebraic uncertainty frameworks. It is obvious from these restrictions that there is a need for a single and mathematically sound theory of Complex Picture Fuzzy Subgroups to extend the classical subgroup analysis in a multidimensional uncertainty environment with phase-sensitive information and abstinence.

Novelties of the Proposed Work. This study proposes a new algebraic structure by combining the CPFS and classical group theory. The present work begins the theory of Complex Picture Fuzzy Subgroups (CPFSGs) and systematically explores the algebraic structure and structural properties of CPFSGs. In this paper, several new concepts are introduced as CPF conjugacy, CPF normalizer, CPF order and CPF index. Rigorous relationships between the above concepts are carefully set up, greatly enriching the classical and generalized fuzzy subgroup theory into a complex picture fuzzy setting. These ideas lead to a better understanding of the internal algebraic properties of CPFSGs and open the door for further studies of fuzzy algebra and abstract group theory. The other novelty is development of a complex picture fuzzy analogue of Lagrange's Theorem.

The study is an extension of one of the basics of classical group theory into the CPF environment, showing that much of what is known about the algebra can be retained when considering multidimensional and phase uncertainty. This extension adds to the mathematical maturity of CPFSGs and emphasises their mathematical consistency with classical algebraic structures.

The work also presents and investigates in detail the concept of Complex Picture Fuzzy Normal Subgroups. They are studied in term of CPF cosets, CPF normalizers, and conjugate CPFSGs. These investigations extend the classical ideas of normal and conjugacy in a fuzzy system to a complex picture fuzzy system, thus offering a more flexible mechanism of analysing an uncertain algebraic system. The proposed theory also investigates the properties of CPFSGs with respect to group homomorphisms, which enable the transfer and modification of CPFG subgroup properties between algebraic structures. This helps to establish a more complete and unified theory of complex picture fuzzy algebra. Apart from the theoretical advancements, the proposed work will be a novel interdisciplinary application of CPFSGs in robotics and intelligent systems. In particular, a two finger robotic gripper is modeled with the CPFSG of the special special Euclidean group. The proposed model uses the three criteria of grip success, uncertainty, instability and motion reliability to assess grasp under uncertain grasping conditions. The proposed CPFSGs are used in the context of robotic motion analysis, introducing phase-sensitive and abstinence-aware uncertainty, thus showing the applicability of the proposed approach to real-world engineering systems. It is one of the first attempts to integrate advanced fuzzy algebraic subgroup theory to robotic grasping and intelligent control mechanisms. In general, the work proposed is a contribution both theoretically and practically for the development of Complex fuzzy algebra. The comparison and summary of various fuzzy algebraic structures with CPFG are given in the following Table 1.

Paper Layout. In the rest of this paper, we describe the implementation of the technique and present examples of its application. In Section 2 the key existing concepts and results, and basic concepts of mathematical modeling are revisited to establish a mathematical foundation for the proposed study. In Section 3, the notion of Complex Picture Fuzzy Subgroups (CPFSGs) is introduced and some basic algebraic properties and some of the important theorems are derived. Theory of level complex picture fuzzy sets, the concept of CPF support and CPF core, and an investigation of the structural properties and interrelationships were presented in Section 4. In Section 5 the conjugacy in complex picture fuzzy subgroups is studied in detail. In this section, some important concepts like the complex

TABLE 1. Comparative Summary of Generalized CPFG

Characteristics	**Fuzzy Subgroup**	**Complex Fuzzy Subgroup**	**Complex Intuitionistic Fuzzy Subgroup**	**Complex Picture Fuzzy Subgroup**
Membership Degree	✓	✓	✓	✓
Non-membership Degree	×	×	✓	✓
Neutral / Abstinence Degree	×	×	×	✓
Phase Information	×	✓	✓	✓
Oscillatory / Cyclic Uncertainty Modeling	×	✓	✓	✓
Multidimensional Uncertainty Handling	×	Partial	✓	✓
Advanced Algebraic Structures	Basic	Moderate	Advanced	Highly Advanced
Homomorphism Structures	Basic	Partial	Moderate	Advanced
Suitability for Intelligent Systems	Limited	Moderate	Moderate	Highly Suitable
Applications in Robotics and Signal Processing	Rare	Possible	Limited	Highly Suitable
Phase-sensitive Decision Modeling	×	✓	✓	✓
Research Maturity	Mature	Developing	Developing	Emerging

picture fuzzy order and complex picture fuzzy index are introduced and analyzed. Moreover, a complex picture fuzzy analogue of Lagrange's Theorem is defined. In section 6 two new notions of complex picture fuzzy normal subgroups are introduced and studied their algebraic properties associated to CPF cosets, conjugate CPFSGs and normalizers. In Section 7 the behaviour of complex picture fuzzy subgroups under group homomorphisms is studied by considering homomorphic images and preimages in CPF. The proposed theory is illustrated in Section 8 by considering a robotic-gripper modeling problem based on the CPF subgroup structure of $SE(3)$. Finally, in Section 9 the key results obtained are summarized, the importance of the introduced framework is emphasized and some areas of future research are discussed.

2. PRELIMINARIES

Here are revisited the key ideas and results that build foundation for the upcoming work.

Definition 2.1. [40] A fuzzy set (FS) $\mathcal{F}$ on a set Ξ is defined by the mapping $\chi_{\mathcal{F}} : \Xi \to [0,1]$, such that,

$$\mathcal{F} = \{(\varsigma, \chi_{\mathcal{F}}(\varsigma)) \mid \varsigma \in \Xi\}.$$

The above notion represents a powerful approach towards uncertainties, where partial truth is depicted by memberships in the closed unit interval, unlike the rigid binary evaluation in crisp sets. Motivated by the concept, fuzzy theory was merged with group theory to introduce fuzzy subgroups as recalled below.

Definition 2.2. [29] A FS $\mathcal{F}$ on a group $(\Xi, \cdot)$ is called a fuzzy subgroup (FSG) of Ξ if

(1) $\chi_{\mathcal{F}}(\varsigma \cdot \upsilon) \geq \min\{\chi_{\mathcal{F}}(\varsigma), \chi_{\mathcal{F}}(\upsilon)\},\ \forall \varsigma, \upsilon \in \Xi.$
(2) $\chi_{\mathcal{F}}(\varsigma^{-1}) \geq \chi_{\mathcal{F}}(\varsigma),\ \forall \varsigma \in \Xi.$

Here ς^{-1} portrays the inverse of ς in Ξ.

Proposition 2.3. [29] If $\mathcal{F}$ is a FSG of Ξ, then

$$\chi_{\mathcal{F}}(\varsigma^{-1}) = \chi_{\mathcal{F}}(\varsigma),\ \forall \varsigma \in \Xi.$$

The following gives the necessary and sufficient condition for FSGs.

Proposition 2.4. [29] A FS $\mathcal{F}$ is called a FSG of Ξ if and only if $\forall \varsigma, \upsilon \in \Xi$,

$$\chi_{\mathcal{F}}(\varsigma \cdot \upsilon^{-1}) \geq \min\{\chi_{\mathcal{F}}(\varsigma), \chi_{\mathcal{F}}(\upsilon)\}.$$

For FSGs defined on groups in a group homomorphism (homomorphism simply for convenience), the following notions present the corresponding fuzzy images and fuzzy pre-images.

Definition 2.5. [29] Consider a homomorphism $\Upsilon : \Xi \to \Theta$. Let $\mathcal{F}$ be the FSG of Ξ. Then, the image of $\mathcal{F}$ under Υ given by $\Upsilon(\mathcal{F})$ is a FS of Θ, such that

$$\chi_{\Upsilon(\mathcal{F})}(\mathfrak{p}) = \max_{\varsigma \in \Upsilon^{-1}(\mathfrak{p})} \{\chi_{\mathcal{F}}(\varsigma)\},\ \forall \mathfrak{p} \in \Theta.$$

If nothing maps from Ξ to $\mathfrak{p}$, then $\chi_{\Upsilon(\Phi)}(\mathfrak{p}) = 0$.

Definition 2.6. [29] Consider a homomorphism $\Upsilon : \Xi \to \Theta$. Let $\mathcal{K}$ be the FSG of Θ. Then, the pre-image of $\mathcal{K}$ given by $\Upsilon^{-1}(\Psi)$ is a FS of Ξ, such that

$$\chi_{\Upsilon^{-1}(\mathcal{K})}(\varsigma) = \chi_{\mathcal{K}}(\Upsilon(\varsigma)),\ \forall \varsigma \in \Xi.$$

Next theorem elucidates the preservation of CPFSGs by homomorphism.

Theorem 2.7. [29] Consider two groups $(\Xi, \cdot)$ and $(\Theta, \circ)$ with a homomorphism $\Upsilon : \Xi \to \Theta$. Suppose Φ is a CPFSG of Ξ and Ψ is a CPFSG of Θ, then the CPF image $\Upsilon(\Phi)$ and the CPF inverse image $\Upsilon^{-1}(\Psi)$ are CPFSGs of Θ and Ξ, respectively.

An important and extensively studied extension of FS theory is picture fuzzy set theory, which considers three fuzzy memberships for representing an object, restricted by unit sum. The notion of this extension is provided in the coming definition.

Definition 2.8. [11] A picture fuzzy set (PFS) $\mathcal{P}$ on a set Ξ is defined by the mappings $\mathfrak{r}_{\mathcal{P}}, \mathfrak{s}_{\mathcal{P}}, \mathfrak{t}_{\mathcal{P}} : \Xi \to [0, 1]$, such that,

$$\mathcal{P} = \{(\varsigma, \mathfrak{r}_{\mathcal{P}}(\varsigma), \mathfrak{s}_{\mathcal{P}}(\varsigma), \mathfrak{t}_{\mathcal{P}}(\varsigma)) \mid \varsigma \in \Xi\}$$

Here, $\mathfrak{r}_{\mathcal{P}}(\varsigma)$, $\mathfrak{s}_{\mathcal{P}}(\varsigma)$, and $\mathfrak{t}_{\mathcal{P}}(\varsigma)$ portray the degrees of satisfaction, neutrality and dissatisfaction of the element ς in $\mathcal{P}$, while constrained by the condition

$$0 \le \mathfrak{r}_{\mathcal{P}}(\varsigma) + \mathfrak{s}_{\mathcal{P}}(\varsigma) + \mathfrak{t}_{\mathcal{P}}(\varsigma) \le 1, \forall \varsigma \in \Xi.$$

Following are recalled some operations of PFSs.

Definition 2.9. [11] Consider two PFSs $\mathcal{P}$ and $\mathcal{Q}$ defined on the set Ξ, then $\forall \varsigma \in \Xi$,

- $\mathcal{P} \subseteq \mathcal{Q} \iff \mathfrak{r}_{\mathcal{P}}(\varsigma) \le \mathfrak{r}_{\mathcal{Q}}(\varsigma),\ \mathfrak{s}_{\mathcal{P}}(\varsigma) \le \mathfrak{s}_{\mathcal{Q}}(\varsigma)$ and $\mathfrak{t}_{\mathcal{P}}(\varsigma) \ge \mathfrak{t}_{\mathcal{Q}}(\varsigma)$
- $\mathcal{P} = \mathcal{Q} \iff \mathcal{P} \subseteq \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{P}$
- $\mathcal{P}^C = \{(\varsigma, \mathfrak{r}^C_{\mathcal{P}}(\varsigma), \mathfrak{s}^C_{\mathcal{P}}(\varsigma), \mathfrak{t}^C_{\mathcal{P}}(\varsigma)) \mid \mathfrak{r}^C_{\mathcal{P}}(\varsigma) = \mathfrak{t}_{\mathcal{P}}(\varsigma), \mathfrak{s}^C_{\mathcal{P}}(\varsigma) = \mathfrak{s}_{\mathcal{P}}(\varsigma), \mathfrak{t}^C_{\mathcal{P}}(\varsigma) = \mathfrak{r}_{\mathcal{P}}(\varsigma)\}$
- $\mathcal{P} \cap \mathcal{Q} = \{(\varsigma, \min\{\mathfrak{r}_{\mathcal{P}}(\varsigma), \mathfrak{r}_{\mathcal{Q}}(\varsigma)\}, \min\{\mathfrak{s}_{\mathcal{P}}(\varsigma), \mathfrak{s}_{\mathcal{Q}}(\varsigma)\}, \max\{\mathfrak{t}_{\mathcal{P}}(\varsigma), \mathfrak{t}_{\mathcal{Q}}(\varsigma)\})\}$
- $\mathcal{P} \cup \mathcal{Q} = \{(\varsigma, \max\{\mathfrak{r}_{\mathcal{P}}(\varsigma), \mathfrak{r}_{\mathcal{Q}}(\varsigma)\}, \min\{\mathfrak{s}_{\mathcal{P}}(\varsigma), \mathfrak{s}_{\mathcal{Q}}(\varsigma)\}, \min\{\mathfrak{t}_{\mathcal{P}}(\varsigma), \mathfrak{t}_{\mathcal{Q}}(\varsigma)\})\}$

Extending the enhanced structure of PFSs to group theory, picture fuzzy subgroups came into existence as defined below.

Definition 2.10. [14] A PFS $\mathcal{P}$ of a group Ξ is called a picture fuzzy subgroup (PFSG) of Ξ, if the following satisfy:

(1) $\begin{cases} \mathfrak{r}_{\mathcal{P}}(\varsigma \cdot \upsilon) \ge \min\{\mathfrak{r}_{\mathcal{P}}(\varsigma), \mathfrak{r}_{\mathcal{P}}(\upsilon)\} \\ \mathfrak{s}_{\mathcal{P}}(\varsigma \cdot \upsilon) \ge \min\{\mathfrak{s}_{\mathcal{P}}(\varsigma), \mathfrak{s}_{\mathcal{P}}(\upsilon)\} \\ \mathfrak{t}_{\mathcal{P}}(\varsigma \cdot \upsilon) \le \max\{\mathfrak{t}_{\mathcal{P}}(\varsigma), \mathfrak{t}_{\mathcal{P}}(\upsilon)\} \end{cases}, \forall \varsigma, \upsilon \in \Xi.$

(2) $\begin{cases} \mathfrak{r}_{\mathcal{P}}(\varsigma^{-1}) \ge \mathfrak{r}_{\mathcal{P}}(\varsigma) \\ \mathfrak{s}_{\mathcal{P}}(\varsigma^{-1}) \ge \mathfrak{s}_{\mathcal{P}}(\varsigma) \\ \mathfrak{t}_{\mathcal{P}}(\varsigma^{-1}) \le \mathfrak{t}_{\mathcal{P}}(\varsigma) \end{cases}, \forall \varsigma, \varsigma^{-1} \in \Xi$, such that ς^{-1} is the inverse element of ς in Ξ.

Following proposition elucidates the necessary and sufficient condition conforming a PFS as a PFSG.

Proposition 2.11. [14] Consider $\mathcal{P}$ represents a PFS of a group Ξ. Then, $\mathcal{P}$ is called a PFSG of Ξ if and only if $\forall \varsigma, \upsilon \in \Xi$:

$$\begin{cases} \mathfrak{r}_{\mathcal{P}}(\varsigma \cdot \upsilon^{-1}) \ge \min\{\mathfrak{r}_{\mathcal{P}}(\varsigma), \mathfrak{r}_{\mathcal{P}}(\upsilon)\} \\ \mathfrak{s}_{\mathcal{P}}(\varsigma \cdot \upsilon^{-1}) \ge \min\{\mathfrak{s}_{\mathcal{P}}(\varsigma), \mathfrak{s}_{\mathcal{P}}(\upsilon)\} \\ \mathfrak{t}_{\mathcal{P}}(\varsigma \cdot \upsilon^{-1}) \le \max\{\mathfrak{t}_{\mathcal{P}}(\varsigma), \mathfrak{t}_{\mathcal{P}}(\upsilon)\} \end{cases}$$

The complex fuzzy sets make another powerful extension of the FSs that allow depiction of uncertainty on a unit circle in the complex plane. By utilizing the exponential form of complex numbers, the complex fuzzy membership depicts uncertainty in terms of both amplitude and phase terms as defined below.

Definition 2.12. [26] A complex fuzzy set (CFS) $\mathcal{C}$ on a set Ξ is defined as

$$\mathcal{C} = \{(\varsigma, \mathfrak{r}_{\mathcal{C}}(\varsigma)e^{ix_{\mathcal{C}}(\varsigma)}) \mid i = \sqrt{-1}, \varsigma \in \Xi\}$$

where $\mathfrak{r}_{\mathcal{C}} : \Xi \to [0,1]$ and $x_{\mathcal{C}} : \Xi \to [0, 2\pi]$ represent the phase and amplitude terms confining the complex fuzzy memberships to a complex plane's unit circle.

The basic operations on CFSs are recalled below.

Definition 2.13. [26] Consider two CFSs $\mathcal{C}$ and $\mathcal{D}$ defined on the set Ξ, then $\forall \varsigma \in \Xi$,

- $\mathcal{C} \subseteq \mathcal{D} \iff \mathfrak{r}_{\mathcal{C}}(\varsigma) \leq \mathfrak{r}_{\mathcal{D}}(\varsigma)$ and $x_{\mathcal{C}}(\varsigma) \leq x_{\mathcal{D}}(\varsigma)$
- $\mathcal{C} = \mathcal{D} \iff \mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{C}$
- $\mathcal{C} \cap \mathcal{D} = \{(\varsigma, \min\{\mathfrak{r}_{\mathcal{C}}(\varsigma), \mathfrak{r}_{\mathcal{D}}(\varsigma)\}e^{i\min\{x_{\mathcal{C}}(\varsigma), x_{\mathcal{D}}(\varsigma)\}})\}$
- $\mathcal{C} \cup \mathcal{D} = \{(\varsigma, \max\{\mathfrak{r}_{\mathcal{C}}(\varsigma), \mathfrak{r}_{\mathcal{D}}(\varsigma)\}e^{i\max\{x_{\mathcal{C}}(\varsigma), x_{\mathcal{D}}(\varsigma)\}})\}$

The idea of π-FSs is provided in the upcoming definition.

Definition 2.14. [6] Consider a FS $\Phi = \{(\varsigma, \chi_{\Phi}(\varsigma)) \mid \varsigma \in \Xi\}$ on the set Ξ. Then the corresponding π-fuzzy set (π-FS) Φ_π is defined as

$$\Phi_\pi = \{(\varsigma, \chi_{\Phi_\pi}(\varsigma)) \mid \chi_{\Phi_\pi} \in [0, 2\pi],\ \varsigma \in \Xi\}$$

such that $\forall \varsigma \in \Xi,\ \chi_{\Phi_\pi}(\varsigma) = 2\pi\chi_{\Phi}(\varsigma)$.

Following the above definition, a π-FSG is defined as follows.

Definition 2.15. [6] A π-FS Φ_π on a group $(\Xi, \cdot)$ is called a π-fuzzy subgroup (π-FSG) of Ξ if

(1) $\chi_{\Phi_\pi}(\varsigma \cdot \upsilon) \geq \min\{\chi_{\Phi_\pi}(\varsigma), \chi_{\Phi_\pi}(\upsilon)\},\ \forall \varsigma, \upsilon \in \Xi$.
(2) $\chi_{\Phi_\pi}(\varsigma^{-1}) \geq \chi_{\Phi_\pi}(\varsigma),\ \forall \varsigma \in \Xi$.

The coming definition recalls the notion of homogeneity in case of CFSs.

Definition 2.16. [6] Consider two CFSs $\mathcal{C}$ and $\mathcal{D}$ on a set Ξ with memberships $\mathfrak{r}_{\mathcal{C}}(\varsigma)e^{ix_{\mathcal{C}}(\varsigma)}$ and $\mathfrak{r}_{\mathcal{D}}(\varsigma)e^{ix_{\mathcal{D}}(\varsigma)}$, respectively. Then $\forall \varsigma, \upsilon \in \Xi$,

(1) The CFS $\mathcal{C}$ is called a homogeneous CFS, if

$$\mathfrak{r}_{\Phi}(\varsigma) \leq \mathfrak{r}_{\Phi}(\upsilon) \iff x_{\Phi}(\varsigma) \leq x_{\Phi}(\upsilon)$$

(2) The CFS $\mathcal{C}$ is homogeneous with the CFS $\mathcal{D}$, if

$$\mathfrak{r}_{\Phi}(\varsigma) \leq \mathfrak{r}_{\Psi}(\upsilon) \iff x_{\Phi}(\varsigma) \leq x_{\Psi}(\upsilon)$$

By utilizing this homogeneity, a complex fuzzy subgroup is defined as follows.

Definition 2.17. [6] Consider a group Ξ. A homogeneous CFS $\mathcal{C}$ of Ξ with membership function $\varphi_{\mathcal{C}}(\varsigma) = \mathfrak{r}_{\mathcal{C}}(\varsigma)e^{ix_{\mathcal{C}}(\varsigma)}$ is called a complex fuzzy subgroup (CFSG) of Ξ, if the following axioms satisfy:

(1) $\varphi_{\mathcal{C}}(\varsigma \cdot \upsilon) \geq \min\{\varphi_{\mathcal{C}}(\varsigma), \varphi_{\mathcal{C}}(\upsilon)\},\ \forall \varsigma, \upsilon \in \Xi$
(2) $\varphi_{\mathcal{C}}(\varsigma^{-1}) \geq \varphi_{\mathcal{C}}(\varsigma),\ \forall \varsigma \in \Xi$, where ς^{-1} is the inverse of ς in Ξ.

The upcoming theorem relates a CFSG to its corresponding FSG and π-FSG.

Theorem 2.18. [6] Consider $\mathcal{C} = \{(\varsigma, \mathfrak{r}_{\mathcal{C}}(\varsigma)e^{ix_{\mathcal{C}}(\varsigma)}) \mid i = \sqrt{-1}, \varsigma \in \Xi\}$ is a homogeneous CFS of group Ξ. Then $\mathcal{C}$ is called a CFSG of Ξ if and only if:

(1) The FS $\hat{\mathcal{C}} = \{(\varsigma, \mathfrak{r}_{\mathcal{C}}(\varsigma)) \mid \varsigma \in \Xi)\}$ is a FSG of Ξ.
(2) The π-FS $\breve{\mathcal{C}} = \{(\varsigma, x_{\mathcal{C}}(\varsigma)) \mid \varsigma \in \Xi\}$ is a π-FSG of Ξ.

Finally, the next definition recalls the combination of CFSs and PFSs, which considers three complex fuzzy memberships for depicting satisfaction, neutrality and dissatisfaction.

Definition 2.19. [2] A complex picture fuzzy set (CPFS) Φ over a universe Ξ is defined as

$$\begin{aligned}\Phi &= \{(\varsigma, \varphi_\Phi(\varsigma), \psi_\Phi(\varsigma), \xi_\Phi(\varsigma)) \mid \varsigma \in \Xi\}\\ &= \{(\varsigma, \mathfrak{r}_\Phi(\varsigma)e^{ix_\Phi(\varsigma)}, \mathfrak{s}_\Phi(\varsigma)e^{iy_\Phi(\varsigma)}, \mathfrak{t}_\Phi(\varsigma)e^{iz_\Phi(\varsigma)}) \mid \varsigma \in \Xi\}.\end{aligned}$$

For each element $\varsigma \in \Xi$,

$$\varphi_\Phi(\varsigma) = \mathfrak{r}_\Phi(\varsigma)e^{ix_\Phi(\varsigma)}, \quad \psi_\Phi(\varsigma) = \mathfrak{s}_\Phi(\varsigma)e^{iy_\Phi(\varsigma)},$$
$$\xi_\Phi(\varsigma) = \mathfrak{t}_\Phi(\varsigma)e^{iz_\Phi(\varsigma)},$$

where $\varphi_\Phi(\varsigma)$, $\psi_\Phi(\varsigma)$, and $\xi_\Phi(\varsigma)$ denote the satisfaction, neutrality, and dissatisfaction degrees of ς in Φ, respectively.

Here, $\mathfrak{r}_\Phi(\varsigma)$, $\mathfrak{s}_\Phi(\varsigma)$, and $\mathfrak{t}_\Phi(\varsigma)$ are the amplitude terms satisfying

$$0 \le \mathfrak{r}_\Phi(\varsigma) + \mathfrak{s}_\Phi(\varsigma) + \mathfrak{t}_\Phi(\varsigma) \le 1, \qquad \forall\varsigma \in \Xi.$$

Similarly, $x_\Phi(\varsigma)$, $y_\Phi(\varsigma)$, and $z_\Phi(\varsigma)$ are the phase terms satisfying

$$0 \le x_\Phi(\varsigma) + y_\Phi(\varsigma) + z_\Phi(\varsigma) \le 2\pi, \qquad \forall\varsigma \in \Xi.$$

From now on, a CPFS will simply be denoted by

$$\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi).$$

3. Complex Picture Fuzzy Subgroups

This section defines complex picture fuzzy subgroups by extending CFSGs and PFSGs. The following definitions extend PFSs and PFSGs to π-PFSs and π-PFSGs for phase terms in the CPFSs, respectively.

Definition 3.1. Consider a PFS $\mathcal{D} = \{(\varsigma, \mathfrak{r}_{\mathcal{D}}(\varsigma), \mathfrak{s}_{\mathcal{D}}(\varsigma), \mathfrak{t}_{\mathcal{D}}(\varsigma)) \mid \varsigma \in \Xi\}$ on the set Ξ. The corresponding π-PFS $\mathcal{D}_\pi$ is defined as

$$\mathcal{D}_\pi = \{(\varsigma, \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma)) \mid \mathfrak{r}_{\mathcal{D}_\pi}, \mathfrak{s}_{\mathcal{D}_\pi}, \mathfrak{t}_{\mathcal{D}_\pi} \in [0, 2\pi],\ \varsigma \in \Xi\}$$

such that $\forall\varsigma \in \Xi,\ \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma) = 2\pi\mathfrak{r}_{\mathcal{D}}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma) = 2\pi\mathfrak{s}_{\mathcal{D}}(\varsigma)$, and $\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma) = 2\pi\mathfrak{t}_{\mathcal{D}}(\varsigma)$, with the condition

$$0 \le \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma) + \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma) + \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma) \le 1.$$

Definition 3.2. A π-PFS $\mathcal{D}_\pi$ of a group Ξ is called π-PFSG, if the following satisfy:

(1) $\begin{cases}\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon) \ge \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon)\}\\ \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon) \ge \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon)\}\\ \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon) \le \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon)\}\end{cases}$, $\forall\varsigma, \upsilon \in \Xi$.

(2) $\begin{cases}\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma^{-1}) \ge \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma)\\ \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma^{-1}) \ge \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma)\\ \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma^{-1}) \le \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma)\end{cases}$, $\forall\varsigma, \varsigma^{-1} \in \Xi$, such that ς^{-1} is the inverse element of ς in Ξ.

The next proposition relates PFSGs to their corresponding π-PFSGs.

Proposition 3.3. A π-PFS $\mathcal{D}_\pi$ is called a π-PFSG of Ξ if and only if $\mathcal{D}$ is a PFSG of Ξ.

Proof. Suppose $\mathcal{D}$ is a PFSG of Ξ. Since $\mathcal{D}_\pi$ is obtained by scaling the memberships in $\mathcal{D}$ by a positive scale factor 2π, therefore, the conditions in Definition 3.2 satisfy directly from those of the PFSG. The converse can be proved similarly. □

Proposition 3.4 discusses the necessary and sufficient condition for a π-PFS to be a π-PFSG.

Proposition 3.4. Consider $\mathcal{D}_\pi$ represents a π-PFS of a group Ξ. Then, $\mathcal{D}_\pi$ is called a π-PFSG of Ξ if and only if $\forall \varsigma, \upsilon \in \Xi$:

$$\begin{cases} \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon^{-1}) \geq \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon)\} \\ \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon^{-1}) \geq \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon)\} \\ \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon^{-1}) \leq \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon)\} \end{cases}$$

Proof. Let $\mathcal{D}_\pi$ be a π-PFSG of Ξ. Then $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{cases} \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon^{-1}) \geq \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon^{-1})\} \geq \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon)\} \\ \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon^{-1}) \geq \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon^{-1})\} \geq \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon)\} \\ \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon^{-1}) \leq \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon^{-1})\} \leq \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon)\} \end{cases}$$

Conversely, let the given conditions satisfy. Suppose that $\forall \varsigma \in \Xi$, $\mathfrak{e} = \varsigma\cdot\varsigma^{-1}$ represents the identity in Ξ. Then,

$$\begin{cases} \mathfrak{r}_{\mathcal{D}_\pi}(\mathfrak{e}) = \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma\cdot\varsigma^{-1}) \geq \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma)\} = \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma) \\ \mathfrak{s}_{\mathcal{D}_\pi}(\mathfrak{e}) = \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma\cdot\varsigma^{-1}) \geq \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma)\} = \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma) \\ \mathfrak{t}_{\mathcal{D}_\pi}(\mathfrak{e}) = \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma\cdot\varsigma^{-1}) \leq \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma)\} = \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma) \end{cases}$$

This implies that $\forall \upsilon \in \Xi$,

$$\begin{cases} \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon^{-1}) = \mathfrak{r}_{\mathcal{D}_\pi}(\mathfrak{e}\cdot\upsilon^{-1}) \geq \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\mathfrak{e}), \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon)\} = \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon) \\ \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon^{-1}) = \mathfrak{s}_{\mathcal{D}_\pi}(\mathfrak{e}\cdot\upsilon^{-1}) \geq \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\mathfrak{e}), \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon)\} = \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon) \\ \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon^{-1}) = \mathfrak{t}_{\mathcal{D}_\pi}(\mathfrak{e}\cdot\upsilon^{-1}) \leq \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\mathfrak{e}), \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon)\} = \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon) \end{cases} \quad \text{(A)}$$

The combination of results in (A) with the given conditions lead to the following:

$$\begin{cases} \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon) = \mathfrak{r}_{\mathcal{D}_\pi}(\varsigma\cdot(\upsilon^{-1})^{-1}) \geq \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon^{-1})\} \geq \min\{\mathfrak{r}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{r}_{\mathcal{D}_\pi}(\upsilon)\} \\ \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon) = \mathfrak{s}_{\mathcal{D}_\pi}(\varsigma\cdot(\upsilon^{-1})^{-1}) \geq \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon^{-1})\} \geq \min\{\mathfrak{s}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{s}_{\mathcal{D}_\pi}(\upsilon)\} \\ \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma\cdot\upsilon) = \mathfrak{t}_{\mathcal{D}_\pi}(\varsigma\cdot(\upsilon^{-1})^{-1}) \leq \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon^{-1})\} \leq \max\{\mathfrak{t}_{\mathcal{D}_\pi}(\varsigma), \mathfrak{t}_{\mathcal{D}_\pi}(\upsilon)\} \end{cases} \quad \text{(B)}$$

Finally, from (A) and (B), it proves that $\mathcal{D}_\pi$ is a π-PFSG of Ξ. □

Definition 3.5 imposes complex fuzzy homogeneity on CPFSs to introduce homogeneous CPFSs, which will act as a foundation for the complex picture fuzzy subgroups.

Definition 3.5. Consider two CPFSs $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ and $\Psi = (\varphi_\Psi, \psi_\Psi, \xi_\Psi)$ with memberships $\varphi_\Phi(\varsigma) = \mathfrak{r}_\Phi(\varsigma)e^{ix_\Phi(\varsigma)}$, $\psi_\Phi(\varsigma) = \mathfrak{s}_\Phi(\varsigma)e^{iy_\Phi(\varsigma)}$, $\xi_\Phi(\varsigma) = \mathfrak{t}_\Phi(\varsigma)e^{iz_\Phi(\varsigma)}$, $\varphi_\Psi(\varsigma) = \mathfrak{r}_\Psi(\varsigma)e^{ix_\Psi(\varsigma)}$, $\psi_\Psi(\varsigma) = \mathfrak{s}_\Psi(\varsigma)e^{iy_\Psi(\varsigma)}$, and $\xi_\Psi(\varsigma) = \mathfrak{t}_\Psi(\varsigma)e^{iz_\Psi(\varsigma)}$, $\forall \varsigma \in \Xi$. Then

(1) The CPFS Φ is called a homogeneous CPFS, if

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma) \leq \mathfrak{r}_\Phi(\upsilon) \iff x_\Phi(\varsigma) \leq x_\Phi(\upsilon) \\ \mathfrak{s}_\Phi(\varsigma) \leq \mathfrak{s}_\Phi(\upsilon) \iff y_\Phi(\varsigma) \leq y_\Phi(\upsilon) \\ \mathfrak{t}_\Phi(\varsigma) \leq \mathfrak{t}_\Phi(\upsilon) \iff z_\Phi(\varsigma) \leq z_\Phi(\upsilon) \end{cases}, \ \forall \varsigma, \upsilon \in \Xi.$$

(2) The CPFS Φ is homogeneous with the CPFS Ψ, if

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma) \leq \mathfrak{r}_\Psi(\upsilon) \iff x_\Phi(\varsigma) \leq x_\Psi(\upsilon) \\ \mathfrak{s}_\Phi(\varsigma) \leq \mathfrak{s}_\Psi(\upsilon) \iff y_\Phi(\varsigma) \leq y_\Psi(\upsilon) \\ \mathfrak{t}_\Phi(\varsigma) \leq \mathfrak{t}_\Psi(\upsilon) \iff z_\Phi(\varsigma) \leq z_\Psi(\upsilon) \end{cases}, \ \forall \varsigma, \upsilon \in \Xi.$$

Non-homogeneous complex fuzzy sets are usually avoided in fuzzy subgroup theory because arbitrary complex phases destroy the natural ordering and algebraic compatibility needed for subgroup axioms. All the coming developments in this work will consider a CPFS as a homogeneous CPFS.

Proposition 3.6. A CPFS Φ is homogeneous if and only if its CPFS complement Φ^C is homogeneous.

Proof. Straightforward. □

The upcoming definition introduces the complex picture fuzzy subgroups as an extension of both CFSGs and PFSGs.

Definition 3.7. Consider a group Ξ. A homogeneous CPFS $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ of Ξ is called a complex picture fuzzy subgroup (CPFSG) of Ξ, if the following axioms satisfy:

(1) $\begin{cases} \varphi_\Phi(\varsigma \cdot \upsilon) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Phi(\upsilon)\} \\ \psi_\Phi(\varsigma \cdot \upsilon) \geq \min\{\psi_\Phi(\varsigma), \psi_\Phi(\upsilon)\} \\ \xi_\Phi(\varsigma \cdot \upsilon) \leq \max\{\xi_\Phi(\varsigma), \xi_\Phi(\upsilon)\} \end{cases}, \ \forall \varsigma, \upsilon \in \Xi$

(2) $\begin{cases} \varphi_\Phi(\varsigma^{-1}) \geq \varphi_\Phi(\varsigma) \\ \psi_\Phi(\varsigma^{-1}) \geq \psi_\Phi(\varsigma) \\ \xi_\Phi(\varsigma^{-1}) \leq \xi_\Phi(\varsigma) \end{cases}, \ \forall \varsigma, \varsigma^{-1} \in \Xi$ such that ς^{-1} is the inverse element of $\varsigma \in \Xi$.

The existence of a CPFSG is linked with the existence of corresponding PFSG (amplitude terms) and π-PFSG (phase terms) in Theorem 3.8.

Theorem 3.8. Consider $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ is a homogeneous CPFS of group Ξ. Then Φ is called a CPFSG of Ξ if and only if:

(1) The PFS $\hat{\Phi} = \{(\varsigma, \mathfrak{r}_\Phi(\varsigma), \mathfrak{s}_\Phi(\varsigma), \mathfrak{t}_\Phi(\varsigma) \mid \varsigma \in \Xi)\}$ is a PFSG of Ξ.
(2) The π-PFS $\breve{\Phi} = \{(\varsigma, x_\Phi(\varsigma), y_\Phi(\varsigma), z_\Phi(\varsigma)) \mid \varsigma \in \Xi\}$ is a π-PFSG of Ξ.

Proof. Suppose Φ is a CPFSG of Ξ. Then $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{aligned} & \varphi_\Phi(\varsigma \cdot \upsilon) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Phi(\upsilon)\} \\ \implies \quad & \mathfrak{r}_\Phi(\varsigma \cdot \upsilon) e^{i x_\Phi(\varsigma \cdot \upsilon)} \geq \min\{\mathfrak{r}_\Phi(\varsigma) e^{i x_\Phi(\varsigma)}, \mathfrak{r}_\Phi(\upsilon) e^{i x_\Phi(\upsilon)}\} \\ & = \min\{\mathfrak{r}_\Phi(\varsigma), \mathfrak{r}_\Phi(\upsilon)\} e^{i \min\{x_\Phi(\varsigma), x_\Phi(\upsilon)\}} \text{ (By homogeneity)} \end{aligned}$$

This implies that

$$\mathfrak{r}_\Phi(\varsigma\cdot\upsilon)\geq\min\{\mathfrak{r}_\Phi(\varsigma),\mathfrak{r}_\Phi(\upsilon)\}\quad\text{and}\quad x_\Phi(\varsigma\cdot\upsilon)\geq\min\{x_\Phi(\varsigma),x_\Phi(\upsilon)\}.$$

Similarly, the following result from ψ_Φ and ξ_Φ:

$$\mathfrak{s}_\Phi(\varsigma\cdot\upsilon)\geq\min\{\mathfrak{s}_\Phi(\varsigma),\mathfrak{s}_\Phi(\upsilon)\}\quad\text{and}\quad y_\Phi(\varsigma\cdot\upsilon)\geq\min\{y_\Phi(\varsigma),y_\Phi(\upsilon)\},$$

$$\mathfrak{t}_\Phi(\varsigma\cdot\upsilon)\leq\max\{\mathfrak{t}_\Phi(\varsigma),\mathfrak{t}_\Phi(\upsilon)\}\quad\text{and}\quad z_\Phi(\varsigma\cdot\upsilon)\leq\max\{z_\Phi(\varsigma),z_\Phi(\upsilon)\}.$$

Consider ς^{-1} is the inverse of ς. Then $\forall\varsigma,\varsigma^{-1}\in\Xi$,

$$\varphi_\Phi(\varsigma^{-1})\geq\varphi_\Phi(\varsigma)\implies\mathfrak{r}_\Phi(\varsigma^{-1})e^{ix_\Phi(\varsigma^{-1})}\geq\mathfrak{r}_\Phi(\varsigma)e^{ix_\Phi(\varsigma)}$$

This implies that $\mathfrak{r}_\Phi(\varsigma^{-1})\geq\mathfrak{r}_\Phi(\varsigma)$ and $x_\Phi(\varsigma^{-1})\geq x_\Phi(\varsigma)$. Similarly, from ψ_Φ and ξ_Φ,

$$\mathfrak{s}_\Phi(\varsigma^{-1})\geq\mathfrak{s}_\Phi(\varsigma),\ y_\Phi(\varsigma^{-1})\geq y_\Phi(\varsigma),\ \mathfrak{t}_\Phi(\varsigma^{-1})\leq\mathfrak{t}_\Phi(\varsigma),\ \text{and } z_\Phi(\varsigma^{-1})\geq z_\Phi(\varsigma).$$

Hence, $\hat{\Phi}$ and $\breve{\Phi}$ are PFSG and π-PFSG of Ξ, respectively.

Conversely, suppose $\hat{\Phi}$ and $\breve{\Phi}$ are PFSG and π-PFSG of Ξ, respectively. Then $\forall\varsigma,\upsilon\in\Xi$,

$$\begin{cases}\mathfrak{r}_\Phi(\varsigma\cdot\upsilon)\geq\min\{\mathfrak{r}_\Phi(\varsigma),\mathfrak{r}_\Phi(\upsilon)\}\\ \mathfrak{s}_\Phi(\varsigma\cdot\upsilon)\geq\min\{\mathfrak{s}_\Phi(\varsigma),\mathfrak{s}_\Phi(\upsilon)\}\\ \mathfrak{t}_\Phi(\varsigma\cdot\upsilon)\leq\max\{\mathfrak{t}_\Phi(\varsigma),\mathfrak{t}_\Phi(\upsilon)\}\end{cases}\text{and}\ \begin{cases}x_\Phi(\varsigma\cdot\upsilon)\geq\min\{x_\Phi(\varsigma),x_\Phi(\upsilon)\}\\ y_\Phi(\varsigma\cdot\upsilon)\geq\min\{y_\Phi(\varsigma),y_\Phi(\upsilon)\}\\ z_\Phi(\varsigma\cdot\upsilon)\leq\max\{z_\Phi(\varsigma),z_\Phi(\upsilon)\}\end{cases},$$

and $\forall\varsigma,\varsigma^{-1}\in\Xi$,

$$\begin{cases}\mathfrak{r}_\Phi(\varsigma^{-1})\geq\mathfrak{r}_\Phi(\varsigma)\\ \mathfrak{s}_\Phi(\varsigma^{-1})\geq\mathfrak{s}_\Phi(\varsigma)\\ \mathfrak{t}_\Phi(\varsigma^{-1})\leq\mathfrak{t}_\Phi(\varsigma)\end{cases}\text{and}\ \begin{cases}x_\Phi(\varsigma^{-1})\geq x_\Phi(\varsigma)\\ y_\Phi(\varsigma^{-1})\geq y_\Phi(\varsigma)\\ z_\Phi(\varsigma^{-1})\leq z_\Phi(\varsigma)\end{cases}.$$

This implies that

$$\begin{aligned}\varphi_\Phi(\varsigma\cdot\upsilon)&=\mathfrak{r}_\Phi(\varsigma\cdot\upsilon)e^{ix_\Phi(\varsigma\cdot\upsilon)}\\&\geq\min\{\mathfrak{r}_\Phi(\varsigma),\mathfrak{r}_\Phi(\upsilon)\}e^{i\min\{x_\Phi(\varsigma),x_\Phi(\upsilon)\}}\\&=\min\{\mathfrak{r}_\Phi(\varsigma)e^{ix_\Phi(\varsigma)},\mathfrak{r}_\Phi(\upsilon)e^{ix_\Phi(\upsilon)}\}\quad\text{(By homogeneity)}\\&=\min\{\varphi_\Phi(\varsigma),\varphi_\Phi(\upsilon)\}\end{aligned}$$

$$\implies\quad\varphi_\Phi(\varsigma\cdot\upsilon)\geq\min\{\varphi_\Phi(\varsigma),\varphi_\Phi(\upsilon)\}.$$

Similarly, $\psi_\Phi(\varsigma\cdot\upsilon)\geq\min\{\psi_\Phi(\varsigma),\psi_\Phi(\upsilon)\}$ and $\xi_\Phi(\varsigma\cdot\upsilon)\leq\max\{\xi_\Phi(\varsigma),\xi_\Phi(\upsilon)\}$. On the other hand,

$$\varphi_\Phi(\varsigma^{-1})=\mathfrak{r}_\Phi(\varsigma^{-1})e^{ix_\Phi(\varsigma^{-1})}\geq\mathfrak{r}_\Phi(\varsigma)e^{ix_\Phi(\varsigma)}=\varphi_\Phi(\varsigma).$$

Similarly, $\psi_\Phi(\varsigma^{-1})\geq\psi_\Phi(\varsigma)$ and $\xi_\Phi(\varsigma^{-1})\leq\xi_\Phi(\varsigma)$.

Hence, Φ is a CPFSG of Ξ. □

The next proposition discusses the relation of group identity element $\mathfrak{e}$ in a CPFSG.

Proposition 3.9. Consider $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ is CPFSG of the group $(\Xi, \cdot)$ having identity $\mathfrak{e}$. Then $\forall \varsigma \in \Xi$,

$$\varphi_\Phi(\mathfrak{e}) \geq \varphi_\Phi(\varsigma),\ \psi_\Phi(\mathfrak{e}) \geq \psi_\Phi(\varsigma),\ \text{and } \xi_\Phi(\mathfrak{e}) \leq \xi_\Phi(\varsigma).$$

Proof. Given that $\mathfrak{e}$ is the identity element for each $\varsigma \in \Xi$. This implies that $\mathfrak{e} = \varsigma \cdot \varsigma^{-1}$ for $\mathfrak{g}^{-1}$ representing inverse of ς in Ξ. Then,

$$\begin{cases} \varphi_\Phi(\mathfrak{e}) \ = \ \varphi_\Phi(\varsigma \cdot \varsigma^{-1}) \ \geq \ \min\{\varphi_\Phi(\varsigma), \varphi_\Phi(\varsigma^{-1})\} \ = \ \varphi_\Phi(\varsigma) \\ \psi_\Phi(\mathfrak{e}) \ = \ \psi_\Phi(\varsigma \cdot \varsigma^{-1}) \ \geq \ \min\{\psi_\Phi(\varsigma), \psi_\Phi(\varsigma^{-1})\} \ = \ \psi_\Phi(\varsigma)\ , \quad \forall \varsigma, \varsigma^{-1} \in \Xi. \\ \xi_\Phi(\mathfrak{e}) \ = \ \xi_\Phi(\varsigma \cdot \varsigma^{-1}) \ \leq \ \max\{\xi_\Phi(\varsigma), \xi_\Phi(\varsigma^{-1})\} \ = \ \xi_\Phi(\varsigma) \end{cases}$$

Since $\varphi_\Phi(\varsigma^{-1}) \geq \varphi_\Phi(\varsigma),\ \psi_\Phi(\varsigma^{-1}) \geq \psi_\Phi(\varsigma)$, and $\xi_\Phi(\varsigma^{-1}) \leq \xi_\Phi(\varsigma)$. □

The following proposes the equality of complex fuzzy memberships in case of inverse elements.

Proposition 3.10. Suppose that Φ is any CPFSG of a group $(\Xi, \cdot)$. Then $\forall \varsigma \in \Xi$,

$$\varphi_\Phi(\varsigma^{-1}) = \varphi_\Phi(\varsigma),\ \psi_\Phi(\varsigma^{-1}) = \psi_\Phi(\varsigma),\ \text{and } \xi_\Phi(\varsigma^{-1}) = \xi_\Phi(\varsigma).$$

Proof. Given that $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ is a CPFSG of Ξ. For all $\varsigma \in \Xi$,

$$\varphi_\Phi(\varsigma^{-1}) \geq \varphi_\Phi(\varsigma),\ \psi_\Phi(\varsigma^{-1}) \geq \psi_\Phi(\varsigma),\ \text{and } \xi_\Phi(\varsigma^{-1}) \leq \xi_\Phi(\varsigma). \tag{A}$$

Suppose that $\varsigma \cdot \varsigma^{-1} = \mathfrak{e} = \varsigma^{-1} \cdot \varsigma$ ($\mathfrak{e}$ symbolizing identity). Then, by substituting ς^{-1} in place of ς, the conditions in (A) yield the following:

$$\begin{cases} \varphi_\Phi((\varsigma^{-1})^{-1}) \geq \varphi_\Phi(\varsigma^{-1}) \implies \varphi_\Phi(\varsigma) \geq \varphi_\Phi(\varsigma^{-1}), \\ \psi_\Phi((\varsigma^{-1})^{-1}) \geq \psi_\Phi(\varsigma^{-1}) \implies \psi_\Phi(\varsigma) \geq \psi_\Phi(\varsigma^{-1}),\ \text{and} \\ \xi_\Phi((\varsigma^{-1})^{-1}) \leq \xi_\Phi(\varsigma^{-1}) \implies \xi_\Phi(\varsigma) \leq \xi_\Phi(\varsigma^{-1}). \end{cases} \tag{B}$$

Hence, it follows from (A) and (B) that $\forall \varsigma \in \Xi$,

$$\varphi_\Phi(\varsigma^{-1}) = \varphi_\Phi(\varsigma),\ \psi_\Phi(\varsigma^{-1}) = \psi_\Phi(\varsigma),\ \text{and } \xi_\Phi(\varsigma^{-1}) = \xi_\Phi(\varsigma).$$

□

The necessary and sufficient conditions for CPFSGs are provided in the upcoming theorem.

Theorem 3.11. Consider $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ is a homogeneous CPFS of a group Ξ. Then Φ is called a CPFSG of Ξ if and only if $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{cases} \varphi_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Phi(\upsilon)\} \\ \psi_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\psi_\Phi(\varsigma), \psi_\Phi(\upsilon)\} \\ \xi_\Phi(\varsigma \cdot \upsilon^{-1}) \leq \max\{\xi_\Phi(\varsigma), \xi_\Phi(\upsilon)\} \end{cases}$$

Proof. Suppose that Φ is CPFSG of the group Ξ. Then $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{cases} \varphi_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Phi(\upsilon^{-1})\} \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Phi(\upsilon)\} \\ \psi_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\psi_\Phi(\varsigma), \psi_\Phi(\upsilon^{-1})\} \geq \min\{\psi_\Phi(\varsigma), \psi_\Phi(\upsilon)\} \\ \xi_\Phi(\varsigma \cdot \upsilon^{-1}) \leq \max\{\xi_\Phi(\varsigma), \xi_\Phi(\upsilon^{-1})\} \leq \max\{\xi_\Phi(\varsigma), \xi_\Phi(\upsilon)\} \end{cases}$$

Conversely, let the given conditions satisfy. Then by homogeneity of CPFS Φ, it is obvious that $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\mathfrak{r}_\Phi(\varsigma), \mathfrak{r}_\Phi(\upsilon)\} \\ \mathfrak{s}_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\mathfrak{s}_\Phi(\varsigma), \mathfrak{s}_\Phi(\upsilon)\} \\ \mathfrak{t}_\Phi(\varsigma \cdot \upsilon^{-1}) \leq \max\{\mathfrak{t}_\Phi(\varsigma), \mathfrak{t}_\Phi(\upsilon)\} \end{cases} \quad \text{and} \quad \begin{cases} x_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{x_\Phi(\varsigma), x_\Phi(\upsilon)\} \\ y_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{y_\Phi(\varsigma), y_\Phi(\upsilon)\} \\ z_\Phi(\varsigma \cdot \upsilon^{-1}) \leq \max\{z_\Phi(\varsigma), z_\Phi(\upsilon)\} \end{cases}$$

Let $\hat{\Phi} = \{(\varsigma, \mathfrak{r}_\Phi(\varsigma), \mathfrak{s}_\Phi(\varsigma), \mathfrak{t}_\Phi(\varsigma)) \mid \varsigma \in \Xi\}$ be a PFS of Ξ and let $\breve{\Phi} = \{(\varsigma, x_\Phi(\varsigma), y_\Phi(\varsigma), z_\Phi(\varsigma)) \mid \varsigma \in \Xi\}$ represent a π-PFS of Ξ. Then by Propositions 2.11 and 3.4, $\hat{\Phi}$ and $\breve{\Phi}$ portray PFSG and π-PFSG of Ξ, respectively. This follows that, by Theorem 3.8, Φ is a CPFSG of Ξ. □

The forthcoming theorem elucidates that the intersection of CPFSGs again constitutes a CPFSG.

Theorem 3.12. Intersection of mutually homogeneous CPFSGs of a group Ξ is a CPFSG of Ξ.

Proof. Suppose that $\{\Phi_\mathfrak{i} \mid \mathfrak{i} \in \Delta\}$ represents mutually homogeneous CPFSGs of Ξ, such that $\forall \mathfrak{j}, \mathfrak{k} \in \Delta$,

$$\begin{cases} \mathfrak{r}_{\Phi_\mathfrak{j}}(\varsigma) \leq \mathfrak{r}_{\Phi_\mathfrak{k}}(\upsilon) \iff x_{\Phi_\mathfrak{j}}(\varsigma) \leq x_{\Phi_\mathfrak{k}}(\upsilon) \\ \mathfrak{s}_{\Phi_\mathfrak{j}}(\varsigma) \leq \mathfrak{s}_{\Phi_\mathfrak{k}}(\upsilon) \iff y_{\Phi_\mathfrak{j}}(\varsigma) \leq y_{\Phi_\mathfrak{k}}(\upsilon) \\ \mathfrak{t}_{\Phi_\mathfrak{j}}(\varsigma) \leq \mathfrak{t}_{\Phi_\mathfrak{k}}(\upsilon) \iff z_{\Phi_\mathfrak{j}}(\varsigma) \leq z_{\Phi_\mathfrak{k}}(\upsilon) \end{cases}$$

The intersection of these CPFSGs is given by $\bigcap_{\mathfrak{i}\in\Delta} \Phi_\mathfrak{i} = (\varphi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}, \psi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}, \xi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}})$, such that

$$\begin{cases} \varphi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}} = \min_{\mathfrak{i}\in\Delta}\{\varphi_{\Phi_\mathfrak{i}}\} = \min_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_\mathfrak{i}}\} e^{i\min_{\mathfrak{i}\in\Delta}\{x_{\Phi_\mathfrak{i}}\}} \\ \psi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}} = \min_{\mathfrak{i}\in\Delta}\{\psi_{\Phi_\mathfrak{i}}\} = \min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_\mathfrak{i}}\} e^{i\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_\mathfrak{i}}\}} \\ \xi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}} = \max_{\mathfrak{i}\in\Delta}\{\xi_{\Phi_\mathfrak{i}}\} = \max_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_\mathfrak{i}}\} e^{i\max_{\mathfrak{i}\in\Delta}\{z_{\Phi_\mathfrak{i}}\}} \end{cases}$$

Now, $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{aligned} \varphi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon^{-1}) &= \min_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon^{-1})\} e^{i\min_{\mathfrak{i}\in\Delta}\{x_{\Phi_\mathfrak{i}}(\varsigma\cdot\upsilon^{-1})\}} \\ &\geq \min_{\mathfrak{i}\in\Delta}\{\min\{\mathfrak{r}_{\Phi_\mathfrak{i}}(\varsigma), \mathfrak{r}_{\Phi_\mathfrak{i}}(\upsilon)\}\} e^{i\min_{\mathfrak{i}\in\Delta}\{\min\{x_{\Phi_\mathfrak{i}}(\varsigma), x_{\Phi_\mathfrak{i}}(\upsilon)\}\}} \\ &\qquad (\because \Phi_\mathfrak{i}\text{s are CPFSGs}) \\ &= \min\{\min_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_\mathfrak{i}}(\varsigma)\}, \min_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_\mathfrak{i}}(\upsilon)\}\} e^{i\min\{\min_{\mathfrak{i}\in\Delta}\{x_{\Phi_\mathfrak{i}}(\varsigma)\}, \min_{\mathfrak{i}\in\Delta}\{x_{\Phi_\mathfrak{i}}(\upsilon)\}\}} \\ &= \min\{\min_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_\mathfrak{i}}(\varsigma)\} e^{i\min_{\mathfrak{i}\in\Delta}\{x_{\Phi_\mathfrak{i}}(\varsigma)\}}, \min_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_\mathfrak{i}}(\upsilon)\} e^{i\min_{\mathfrak{i}\in\Delta}\{x_{\Phi_\mathfrak{i}}(\upsilon)\}}\} \\ &\qquad (\because \Phi_\mathfrak{i}\text{s are homogeneous}) \\ &= \min\{\varphi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\varsigma), \varphi_{\bigcap_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\upsilon)\} \end{aligned}$$

Similarly,

$$\begin{aligned}
\psi_{\underset{\mathfrak{i}\in\Delta}{\cap}\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1}) &= \min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1})\}e^{i\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1})\}}\\
&\geq \min_{\mathfrak{i}\in\Delta}\{\min\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\varsigma),\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}e^{i\min_{\mathfrak{i}\in\Delta}\{\min\{y_{\Phi_{\mathfrak{i}}}(\varsigma),y_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}}\\
&= \min\{\min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\varsigma)\},\min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}e^{i\min\{\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_{\mathfrak{i}}}(\varsigma)\},\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}}\\
&= \min\{\min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\varsigma)\}e^{i\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_{\mathfrak{i}}}(\varsigma)\}},\min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\upsilon)\}e^{i\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_{\mathfrak{i}}}(\upsilon)\}}\}\\
&= \min\{\psi_{\underset{\mathfrak{i}\in\Delta}{\cap}\Phi_{\mathfrak{i}}}(\varsigma),\psi_{\underset{\mathfrak{i}\in\Delta}{\cap}\Phi_{\mathfrak{i}}}(\upsilon)\}
\end{aligned}$$

and

$$\begin{aligned}
\xi_{\underset{\mathfrak{i}\in\Delta}{\cap}\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1}) &= \max_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1})\}e^{i\max_{\mathfrak{i}\in\Delta}\{z_{\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1})\}}\\
&\leq \max_{\mathfrak{i}\in\Delta}\{\max\{\mathfrak{t}_{\Phi_{\mathfrak{i}}}(\varsigma),\mathfrak{t}_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}e^{i\max_{\mathfrak{i}\in\Delta}\{\max\{z_{\Phi_{\mathfrak{i}}}(\varsigma),z_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}}\\
&= \max\{\max_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_{\mathfrak{i}}}(\varsigma)\},\max_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}e^{i\max\{\max_{\mathfrak{i}\in\Delta}\{z_{\Phi_{\mathfrak{i}}}(\varsigma)\},\max_{\mathfrak{i}\in\Delta}\{z_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}}\\
&= \max\{\max_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_{\mathfrak{i}}}(\varsigma)\}e^{i\max_{\mathfrak{i}\in\Delta}\{z_{\Phi_{\mathfrak{i}}}(\varsigma)\}},\max_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_{\mathfrak{i}}}(\upsilon)\}e^{i\max_{\mathfrak{i}\in\Delta}\{z_{\Phi_{\mathfrak{i}}}(\upsilon)\}}\}\\
&= \max\{\xi_{\underset{\mathfrak{i}\in\Delta}{\cap}\Phi_{\mathfrak{i}}}(\varsigma),\xi_{\underset{\mathfrak{i}\in\Delta}{\cap}\Phi_{\mathfrak{i}}}(\upsilon)\}
\end{aligned}$$

This proves that $\underset{\mathfrak{i}\in\Delta}{\cap}\Phi_{\mathfrak{i}}$ is a CPFSG of Ξ. □

A similar result is not possible for the union of CPFSGs. Proposition 3.13 discusses the specific conditions under which the union operation preserves CPSFG properties.

Proposition 3.13. Union of mutually homogeneous CPFSGs $\{\Phi_{\mathfrak{i}} \mid \mathfrak{i}\in\Delta\}$ of a group Ξ is again a CPFSG of Ξ, if $\exists\ \Phi^* \in \{\Phi_{\mathfrak{i}}\}$ such that $\varphi_{\Phi^*}(\varsigma) = \max_{\mathfrak{i}\in\Delta}\{\varphi_{\Phi_{\mathfrak{i}}}(\varsigma)\}$ and $\xi_{\Phi^*}(\varsigma) = \min_{\mathfrak{i}\in\Delta}\{\xi_{\Phi_{\mathfrak{i}}}(\varsigma)\},\ \forall\varsigma\in\Xi$.

Proof. Suppose that $\{\Phi_{\mathfrak{i}} \mid \mathfrak{i}\in\Delta\}$ represents mutually homogeneous CPFSGs of Ξ with some $\Phi^*\in\{\Phi_{\mathfrak{i}}\}$ such that $\varphi_{\Phi^*}(\varsigma) = \max_{\mathfrak{i}\in\Delta}\{\varphi_{\Phi_{\mathfrak{i}}}(\varsigma)\}$ and $\xi_{\Phi^*}(\varsigma) = \min_{\mathfrak{i}\in\Delta}\{\xi_{\Phi_{\mathfrak{i}}}(\varsigma)\},\ \forall\varsigma\in\Xi$. The union of these CPFSGs is given by $\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}} = (\varphi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}},\psi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}},\xi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}})$, such that

$$\begin{cases}
\varphi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}} = \max_{\mathfrak{i}\in\Delta}\{\varphi_{\Phi_{\mathfrak{i}}}\} = \max_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_{\mathfrak{i}}}\}e^{i\max_{\mathfrak{i}\in\Delta}\{x_{\Phi_{\mathfrak{i}}}\}} = \varphi_{\Phi^*}\\
\psi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}} = \min_{\mathfrak{i}\in\Delta}\{\psi_{\Phi_{\mathfrak{i}}}\} = \min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}\}e^{i\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_{\mathfrak{i}}}\}}\\
\xi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}} = \min_{\mathfrak{i}\in\Delta}\{\xi_{\Phi_{\mathfrak{i}}}\} = \min_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_{\mathfrak{i}}}\}e^{i\min_{\mathfrak{i}\in\Delta}\{z_{\Phi_{\mathfrak{i}}}\}} = \xi_{\Phi^*}
\end{cases}$$

Now, $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{aligned}\varphi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1}) &= \varphi_{\Phi^*}(\varsigma\cdot\upsilon^{-1})\\ &\geq \min\{\varphi_{\Phi^*}(\varsigma), \varphi_{\Phi^*}(\upsilon)\} \quad (\because \Phi^* \text{ is a CPFSG})\\ &= \min\{\varphi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\varsigma), \varphi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\upsilon)\}\end{aligned}$$

$$\begin{aligned}\psi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1}) &= \min_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1})\}e^{i\min_{\mathfrak{i}\in\Delta}\{y_{\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1})\}}\\ &\geq \min_{\mathfrak{i}\in\Delta}\{\min\{\mathfrak{s}_{\Phi_{\mathfrak{i}}}(\varsigma), \mathfrak{s}_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}e^{i\min_{\mathfrak{i}\in\Delta}\{\min\{y_{\Phi_{\mathfrak{i}}}(\varsigma), y_{\Phi_{\mathfrak{i}}}(\upsilon)\}\}}\\ &= \min\{\psi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\varsigma), \psi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\upsilon)\} \quad \text{(Similar to Theorem 3.12)}\end{aligned}$$

$$\begin{aligned}\xi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\varsigma\cdot\upsilon^{-1}) &= \xi_{\Phi^*}(\varsigma\cdot\upsilon^{-1})\\ &\leq \max\{\xi_{\Phi^*}(\varsigma), \xi_{\Phi^*}(\upsilon)\} \quad (\because \Phi^* \text{ is a CPFSG})\\ &= \max\{\xi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\varsigma), \xi_{\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}}(\upsilon)\}\end{aligned}$$

Hence, under the given conditions, $\underset{\mathfrak{i}\in\Delta}{\cup}\Phi_{\mathfrak{i}}$ forms a CPFSG of Ξ. □

4. Level Complex Picture Fuzzy Sets

This section discusses the level sets for CPFSs and the conditions relating these level CPFSs to CPFSGs. The complex picture fuzzy order of CPFSs is also discussed backing the CPFSG extension of Lagrange's theorem. The forthcoming definitions introduce different level CPFSs on a CPFS Φ.

Definition 4.1. Consider a complex fuzzy level $[\gamma e^{i\epsilon}, \delta e^{i\kappa}, \lambda e^{i\tau}]$ such that

$$0 \leq \gamma + \delta + \lambda \leq 1 \quad \text{and} \quad 0 \leq \epsilon + \kappa + \tau \leq 2\pi.$$

Then the $[\gamma e^{i\epsilon}, \delta e^{i\kappa}, \lambda e^{i\tau}]$-level CPFS (L-CPFS) of a CPFS Φ in Ξ is given by

$$\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda} = \left\{\varsigma \in \Xi \mid \begin{array}{l}\mathfrak{r}_\Phi(\varsigma) \geq \gamma,\ \mathfrak{s}_\Phi(\varsigma) \geq \delta,\ \mathfrak{t}_\Phi(\varsigma) \leq \lambda,\\ x_\Phi(\varsigma) \geq \epsilon,\ y_\Phi(\varsigma) \geq \kappa,\ z_\Phi(\varsigma) \leq \tau\end{array}\right\}$$

Let Ξ be a group having identity $\mathfrak{e}$, then $\mathfrak{r}_\Phi(\mathfrak{e}) \geq \gamma,\ \mathfrak{s}_\Phi(\mathfrak{e}) \geq \delta,\ \mathfrak{t}_\Phi(\mathfrak{e}) \leq \lambda, x_\Phi(\mathfrak{e}) \geq \epsilon,\ y_\Phi(\mathfrak{e}) \geq \kappa$, and $z_\Phi(\mathfrak{e}) \leq \tau$. The collection of all L-CPFSs of Φ is denoted by L_Φ.

Proposition 4.2. The crisp set $\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$ is called a $[\gamma e^{i\epsilon}, \delta e^{i\kappa}, \lambda e^{i\tau}]$-level CPFS of CPFS Φ if and only if

(1) The set $\hat{\Phi}_{\gamma,\delta,\lambda}$ is a $[\gamma, \delta, \lambda]$-level PFS of the PFS $\hat{\Phi}$.
(2) The set $\breve{\Phi}^{\epsilon,\kappa,\tau}$ is a $[\epsilon, \kappa, \tau]$-level π-PFS of the π-PFS $\hat{\Phi}$.

Proof. Clear from Definition 4.1. □

Definition 4.3. A level-CPFS of Φ is called a strict L-CPFS ($\overline{\text{L}}$-CPFS) if it is defined as

$$\overline{\Phi}^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda} = \left\{\varsigma \in \Xi \mid \begin{array}{l}\mathfrak{r}_\Phi(\varsigma) > \gamma,\ \mathfrak{s}_\Phi(\varsigma) > \delta,\ \mathfrak{t}_\Phi(\varsigma) < \lambda,\\ x_\Phi(\varsigma) > \epsilon,\ y_\Phi(\varsigma) > \kappa,\ z_\Phi(\varsigma) < \tau\end{array}\right\}$$

The following define the support and core of a CPFS in terms of its L-CPFSs.

Definition 4.4. The strict $[0, 0, e^{i2\pi}]$-level CPFS of a CPFS Φ in Ξ defines its support in Ξ. Mathematically,

$$Supp(\Phi) = \overline{\Phi}_{0,0,1}^{0,0,2\pi} = \left\{ \varsigma \in \Xi \mid \begin{array}{c} \mathfrak{r}_\Phi(\varsigma) > 0,\ \mathfrak{s}_\Phi(\varsigma) > 0,\ \mathfrak{t}_\Phi(\varsigma) < 1, \\ x_\Phi(\varsigma) > 0,\ y_\Phi(\varsigma) > 0,\ z_\Phi(\varsigma) < 2\pi \end{array} \right\}$$

Here, $Supp(\Phi)$ is the largest L-CPFS in L_Φ as it contains all the group elements fully or partially contained in Φ.

Definition 4.5. The complex picture fuzzy core of a CPFS Φ in a group Ξ (with identity $\mathfrak{e}$) is represented by its $[\varphi_\Phi(\mathfrak{e}), \psi_\Phi(\mathfrak{e}), \xi_\Phi(\mathfrak{e})]$-level CPFS. That is,

$$Core(\Phi) = \Phi_{\mathfrak{r}_\Phi(\mathfrak{e}),\mathfrak{s}_\Phi(\mathfrak{e}),\mathfrak{t}_\Phi(\mathfrak{e})}^{x_\Phi(\mathfrak{e}),y_\Phi(\mathfrak{e}),z_\Phi(\mathfrak{e})} = \left\{ \varsigma \in \Xi \mid \begin{array}{c} \mathfrak{r}_\Phi(\varsigma) \geq \mathfrak{r}_\Phi(\mathfrak{e}),\ \mathfrak{s}_\Phi(\varsigma) \geq \mathfrak{s}_\Phi(\mathfrak{e}),\ \mathfrak{t}_\Phi(\varsigma) \leq \mathfrak{t}_\Phi(\mathfrak{e}), \\ x_\Phi(\varsigma) \geq x_\Phi(\mathfrak{e}),\ y_\Phi(\varsigma) \geq y_\Phi(\mathfrak{e}),\ z_\Phi(\varsigma) \leq z_\Phi(\mathfrak{e}) \end{array} \right\}$$

Here, $Core(\Phi)$ portrays the smallest $L - CPFS$ in L_Φ. If Ξ is not a group then $Core(\Phi)$ considers the group element with highest membership in Φ as $\mathfrak{e}$.

The upcoming theorems discuss the relation between L-CPFSs and CPFSG corresponding to the same CPFS.

Theorem 4.6. If Φ is CPFSG of a crisp group Ξ then each L-CPFS $\Phi_{\gamma,\delta,\lambda}^{\epsilon,\kappa,\tau} \in L_\Phi$ is a subgroup of Ξ.

Proof. Assume that Φ is a CPFSG of Ξ. Consider an arbitrary L-CPFS $\Phi_{\gamma,\delta,\lambda}^{\epsilon,\kappa,\tau} \in L_\Phi$. Then, essentially, $\mathfrak{e}$ belongs to the L-CPFS $\Phi_{\gamma,\delta,\lambda}^{\epsilon,\kappa,\tau}$. Let some elements $\varsigma, \upsilon \in \Phi_{\gamma,\delta,\lambda}^{\epsilon,\kappa,\tau}$, then

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma), \mathfrak{r}_\Phi(\upsilon) \geq \gamma \\ \mathfrak{s}_\Phi(\varsigma), \mathfrak{s}_\Phi(\upsilon) \geq \delta \\ \mathfrak{t}_\Phi(\varsigma), \mathfrak{t}_\Phi(\upsilon) \leq \lambda \end{cases} \quad \text{and} \quad \begin{cases} x_\Phi(\varsigma), x_\Phi(\upsilon) \geq \epsilon \\ y_\Phi(\varsigma), y_\Phi(\upsilon) \geq \kappa \\ z_\Phi(\varsigma), z_\Phi(\upsilon) \leq \tau \end{cases}$$

This implies that, for amplitude terms,

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\mathfrak{r}_\Phi(\varsigma), \mathfrak{r}_\Phi(\upsilon)\} \geq \min\{\gamma, \gamma\} = \gamma \\ \mathfrak{s}_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{\mathfrak{s}_\Phi(\varsigma), \mathfrak{s}_\Phi(\upsilon)\} \geq \min\{\delta, \delta\} = \delta \\ \mathfrak{t}_\Phi(\varsigma \cdot \upsilon^{-1}) \leq \max\{\mathfrak{t}_\Phi(\varsigma), \mathfrak{t}_\Phi(\upsilon)\} \leq \max\{\lambda, \lambda\} = \lambda \end{cases},$$

and for phase terms,

$$\begin{cases} x_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{x_\Phi(\varsigma), x_\Phi(\upsilon)\} \geq \min\{\epsilon, \epsilon\} = \epsilon \\ y_\Phi(\varsigma \cdot \upsilon^{-1}) \geq \min\{y_\Phi(\varsigma), y_\Phi(\upsilon)\} \geq \min\{\kappa, \kappa\} = \kappa \\ z_\Phi(\varsigma \cdot \upsilon^{-1}) \leq \max\{z_\Phi(\varsigma), z_\Phi(\upsilon)\} \leq \max\{\tau, \tau\} = \tau \end{cases}.$$

Hence, for any $\varsigma, \upsilon \in \Phi_{\gamma,\delta,\lambda}^{\epsilon,\kappa,\tau}$, $\varsigma \cdot \upsilon^{-1}$ is also in $\Phi_{\gamma,\delta,\lambda}^{\epsilon,\kappa,\tau}$, and therefore, it is a crisp subgroup of Ξ. Since, the L-CPFS $\Phi_{\gamma,\delta,\lambda}^{\epsilon,\kappa,\tau}$ was taken arbitrarily. Therefore, each L-CPFS in L_Φ is a subgroup of Ξ. □

Corollary 4.7. If Φ is a CPFSG of Ξ, then $Core(\Phi)$ and $Supp(\Phi)$ are crisp subgroups of Ξ.

Theorem 4.8. If Φ is a CPFSG of the group Ξ, then

(1) Each $[\gamma, \delta, \lambda]$-level PFS of $\hat{\Phi}$ forms a subgroup of Ξ.
(2) Each $[\epsilon, \kappa, \tau]$-level π-PFS of $\breve{\Phi}$ forms a subgroup of Ξ.

Proof. Directly follows from Proposition 4.2 and Theorem 4.6. □

5. Conjugacy in Complex Picture Fuzzy Subgroups

This section discusses conjugacy in terms of CPFSs and CPFSGs. The following introduces the notion of conjugate CPF elements in A CPFS Φ.

Definition 5.1. Any two elements $\varsigma, \upsilon \in \Xi$ are said to be conjugate complex picture fuzzy (CCPF) elements in CPFS Φ if $\exists\, \zeta \in \Xi$ such that

$$\varphi_\Phi(\varsigma) = \varphi_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta),\ \psi_\Phi(\varsigma) = \psi_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta),\ \text{and}\ \xi_\Phi(\varsigma) = \xi_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta).$$

Explicitly,

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma) = \mathfrak{r}_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta) \\ \mathfrak{s}_\Phi(\varsigma) = \mathfrak{s}_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta) \\ \mathfrak{t}_\Phi(\varsigma) = \mathfrak{t}_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta) \end{cases} \quad \text{and} \quad \begin{cases} x_\Phi(\varsigma) = x_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta) \\ y_\Phi(\varsigma) = y_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta) \\ z_\Phi(\varsigma) = z_\Phi(\zeta^{-1} \cdot \upsilon \cdot \zeta) \end{cases}$$

Symbolically, $\varsigma \overset{\zeta}{\sim} \upsilon$ in Φ.

Definition 5.2. Suppose Φ is a CPFSG of Ξ. Then CCPF class of any element $\varsigma \in \Xi$ corresponding to Φ is defined as

$$Cl^{\sim}_\Phi(\varsigma) = \{\upsilon \in \Xi \mid \varsigma \overset{\zeta}{\sim} \upsilon \text{ for some } \zeta \in \Xi\}.$$

Now, the condition for conjugate CPFSs in a group Ξ is presented.

Definition 5.3. Two CPFSs Φ and Ψ in a group Ξ are known as conjugate CPFSs (CCPFSs), if $\exists\, \zeta \in \Xi$ such that

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma) = \mathfrak{r}_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \\ \mathfrak{s}_\Phi(\varsigma) = \mathfrak{s}_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \\ \mathfrak{t}_\Phi(\varsigma) = \mathfrak{t}_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \end{cases} \quad \text{and} \quad \begin{cases} x_\Phi(\varsigma) = x_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \\ y_\Phi(\varsigma) = y_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \\ z_\Phi(\varsigma) = z_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \end{cases}, \ \forall \varsigma \in \Xi.$$

This relation is symbolized as $\Phi \overset{\zeta}{\sim} \Psi$ in Ξ, where Φ can be written as $\zeta \Psi \zeta^{-1}$.

Focusing on the conjugacy of CPFSs, forthcoming results discuss conjugate CPFSGs.

Theorem 5.4. A CPFS Φ is called a CPFSG of Ξ if and only if its CCPFS Ψ is a CPFSG of Ξ.

Proof. Suppose Φ is a CPFSG of Ξ and $\Phi \overset{\zeta}{\sim} \Psi$ in Ξ. Since $\varsigma, \upsilon, \zeta \in \Xi$, therefore the elements $\varsigma' = \zeta^{-1} \cdot \varsigma \cdot \zeta$ and $\varsigma' = \zeta^{-1} \cdot \varsigma \cdot \zeta$ also belong to Ξ. This implies that

$$\begin{cases} \varphi_\Phi(\varsigma) = \varphi_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) = \varphi_\Psi(\varsigma') \\ \psi_\Phi(\varsigma) = \psi_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) = \psi_\Psi(\varsigma') \\ \xi_\Phi(\varsigma) = \xi_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) = \xi_\Psi(\varsigma') \end{cases} \quad \text{and} \quad \begin{cases} \varphi_\Phi(\upsilon) = \varphi_\Psi(\zeta^{-1} \cdot \upsilon \cdot \zeta) = \varphi_\Psi(\upsilon') \\ \psi_\Phi(\upsilon) = \psi_\Psi(\zeta^{-1} \cdot \upsilon \cdot \zeta) = \psi_\Psi(\upsilon') \\ \xi_\Phi(\upsilon) = \xi_\Psi(\zeta^{-1} \cdot \upsilon \cdot \zeta) = \xi_\Psi(\upsilon') \end{cases}.$$

Now,

$$\begin{aligned}\varphi_\Psi(\varsigma'\cdot\upsilon') &= \varphi_\Psi((\zeta^{-1}\cdot\varsigma\cdot\zeta)\cdot(\zeta^{-1}\cdot\upsilon\cdot\zeta))\\ &= \varphi_\Psi(\zeta^{-1}\cdot(\varsigma\cdot\upsilon)\cdot\zeta)\\ &= \varphi_\Phi(\varsigma\cdot\upsilon)\\ &\geq \min\{\varphi_\Phi(\varsigma),\varphi_\Phi(\upsilon)\}\\ &= \min\{\varphi_\Psi(\varsigma'),\varphi_\Psi(\upsilon')\}\end{aligned}$$

$\implies \varphi_\Psi(\varsigma'\cdot\upsilon') \geq \min\{\varphi_\Psi(\varsigma'),\varphi_\Psi(\upsilon')\}$. Similarly, $\psi_\Psi(\varsigma'\cdot\upsilon') \geq \min\{\psi_\Psi(\varsigma'),\psi_\Psi(\upsilon')\}$ and $\xi_\Psi(\varsigma'\cdot\upsilon') \leq \max\{\xi_\Psi(\varsigma'),\xi_\Psi(\upsilon')\}$.

Consider $(\varsigma')^{-1} = (\zeta^{-1}\cdot\varsigma\cdot\zeta)^{-1} = \zeta^{-1}\cdot\varsigma^{-1}\cdot\zeta$ is the inverse of ς' in Ξ, then

$$\begin{cases}\varphi_\Psi((\varsigma')^{-1}) = \varphi_\Phi(\varsigma^{-1}) \geq \varphi_\Phi(\varsigma) = \varphi_\Psi(\varsigma')\\ \psi_\Psi((\varsigma')^{-1}) = \psi_\Phi(\varsigma^{-1}) \geq \psi_\Phi(\varsigma) = \psi_\Psi(\varsigma')\\ \xi_\Psi((\varsigma')^{-1}) = \xi_\Phi(\varsigma^{-1}) \leq \xi_\Phi(\varsigma) = \xi_\Psi(\varsigma')\end{cases}.$$

Moreover $\forall\varsigma\in\Xi$,

$$\varphi_\Phi(\mathfrak{e}) \geq \varphi_\Phi(\varsigma) \implies \varphi_\Psi(\zeta^{-1}\cdot\mathfrak{e}\cdot\zeta) = \varphi_\Psi(\mathfrak{e}) \geq \varphi_\Psi(\zeta^{-1}\cdot\varsigma\cdot\zeta).$$

Accordingly, $\psi_\Psi(\mathfrak{e}) \geq \psi_\Psi(\zeta^{-1}\cdot\varsigma\cdot\zeta)$ and $\xi_\Psi(\mathfrak{e}) \leq \xi_\Psi(\zeta^{-1}\cdot\varsigma\cdot\zeta)$. Hence, Ψ is a CPFSG of Ξ.

Converse of the theorem is similar to the preceding part. □

The following introduces the notion of CPF order of a CPFS Φ in Ξ.

Definition 5.5. The complex picture fuzzy order of a CPFS Φ is defined as the cardinality of its core in Ξ. Mathematically,

$$Ord(\Phi) = |Core(\Phi)|$$

All CPFSGs conjugate to each other are called conjugate CPFSGs (CCPFSGs). The upcoming theorem puts an order condition on CCPFSGs of a group Ξ.

Theorem 5.6. If Φ and Ψ are CCPFSGs in Ξ, then $Ord(\Phi) = Ord(\Psi)$.

Proof. Given that $\Phi \overset{\zeta}{\sim} \Psi$ for some $\zeta\in\Xi$. Then, by Definitions 4.5 and 5.3,

$$Core(\Phi) = \{\varsigma\in\Xi \mid \varphi_\Phi(\varsigma) \geq \varphi_\Phi(\mathfrak{e}),\ \psi_\Phi(\varsigma) \geq \psi_\Phi(\mathfrak{e}),\ \xi_\Phi(\varsigma) \leq \xi_\Phi(\mathfrak{e})\}$$

and

$$Core(\Psi) = \left\{\zeta^{-1}\cdot\varsigma\cdot\zeta\in\Xi \mid \begin{array}{c}\varphi_\Psi(\zeta^{-1}\cdot\varsigma\cdot\zeta) \geq \varphi_\Psi(\mathfrak{e}),\ \psi_\Psi(\zeta^{-1}\cdot\varsigma\cdot\zeta) \geq \psi_\Psi(\mathfrak{e}),\\ \xi_\Psi(\zeta^{-1}\cdot\varsigma\cdot\zeta) \leq \xi_\Psi(\mathfrak{e})\end{array}\right\}.$$

A function $f_\zeta : Core(\Phi) \to Core(\Psi)$ can be defined as $f_\zeta(\varsigma) = \zeta^{-1}\cdot\varsigma\cdot\zeta$. It can be observed that $\forall\varsigma,\upsilon\in\Xi$,

$$f_\zeta(\varsigma) = f_\zeta(\upsilon) \implies \zeta^{-1}\cdot\varsigma\cdot\zeta = \zeta^{-1}\cdot\upsilon\cdot\zeta \implies \varsigma = \upsilon$$

This implies that f is one-to-one. Moreover, by $\varphi_\Phi(\mathfrak{e}) = \varphi_\Psi(\mathfrak{e})$, for each $\zeta^{-1}\cdot\varsigma\cdot\zeta \in Core(\Psi)$, $\exists$ some $\varsigma\in Core(\Phi)$ such that $f_\zeta(\varsigma) = \zeta^{-1}\cdot\varsigma\cdot\zeta$. This implies that f is onto. Hence, f is a bijective function. Since domain and codomain of a bijection have the same cardinality, therefore, $|Core(\Phi)| = |Core(\Psi)|$. Hence, by using the notion of CPF order of CPFSGs (Definition 5.5), it follows that

$$Ord(\Phi) = Ord(\Psi).$$

□

Conjugacy among crisp subgroups is an equivalence relation. The same satisfies for CPFSGs as shown in the following theorem.

Theorem 5.7. CPF conjugacy between CPFSGs of a group is an equivalence relation.

Proof. Suppose Φ, Ψ, and Π represent CPFSGs of a group Ξ. It needs to be shown that CPF conjugacy among these CPFSGs is reflexive, transitive, and symmetric.

(1) For any $\varsigma \in \Xi$,

$$\begin{cases} \varphi_\Phi(\varsigma) = \varphi_\Phi(\mathfrak{e}^{-1} \cdot \varsigma \cdot \mathfrak{e}) \\ \psi_\Phi(\varsigma) = \psi_\Phi(\mathfrak{e}^{-1} \cdot \varsigma \cdot \mathfrak{e}) \\ \xi_\Phi(\varsigma) \ = \xi_\Phi(\mathfrak{e}^{-1} \cdot \varsigma \cdot \mathfrak{e}) \end{cases}, \ \mathfrak{e} \text{ is the identity in } \Xi.$$

That is, by trivial conjugation, $\Phi = \mathfrak{e}\Phi\mathfrak{e}^{-1}$ (or $\Phi \overset{\mathfrak{e}}{\sim} \Phi$). Hence, CCPF relation in CPFSGs is reflexive.

(2) Suppose $\Phi = \zeta\Psi\zeta^{-1}$ for some $\zeta \in \Xi$. Then $\forall \varsigma \in \Xi$,

$$\begin{aligned} &\varphi_\Phi(\varsigma) = \varphi_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \\ \implies &\varphi_\Phi(\zeta \cdot \varsigma \cdot \zeta^{-1}) = \varphi_\Psi(\zeta^{-1} \cdot \zeta \cdot \varsigma \cdot \zeta^{-1} \cdot \zeta) \\ \implies &\varphi_\Phi(\zeta \cdot \varsigma \cdot \zeta^{-1}) = \varphi_\Psi(\varsigma) \end{aligned}$$

Similarly, $\psi_\Phi(\zeta \cdot \varsigma \cdot \zeta^{-1}) = \psi_\Psi(\varsigma)$ and $\xi_\Phi(\zeta \cdot \varsigma \cdot \zeta^{-1}) = \xi_\Psi(\varsigma)$. Then for some $\upsilon = \zeta^{-1}$, $\Psi = \upsilon\Phi\upsilon^{-1}$. Hence, CPF conjugacy among CPFSGs is symmetric.

(3) Suppose $\Phi = \zeta\Psi\zeta^{-1}$ and $\Psi = \varpi\Pi\varpi^{-1}$ for some $\zeta, \varpi \in \Xi$. Then, $\forall \varsigma \in \Xi$,

$$\begin{cases} \varphi_\Phi(\varsigma) = \varphi_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \\ \psi_\Phi(\varsigma) = \psi_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \\ \xi_\Phi(\varsigma) \ = \xi_\Psi(\zeta^{-1} \cdot \varsigma \cdot \zeta) \end{cases} \quad \text{and} \quad \begin{cases} \varphi_\Psi(\varsigma) = \varphi_\Pi(\varpi^{-1} \cdot \varsigma \cdot \varpi) \\ \psi_\Psi(\varsigma) = \psi_\Pi(\varpi^{-1} \cdot \varsigma \cdot \varpi) \\ \xi_\Psi(\varsigma) \ = \xi_\Pi(\varpi^{-1} \cdot \varsigma \cdot \varpi) \end{cases}.$$

Let $\upsilon = \zeta^{-1} \cdot \varsigma \cdot \zeta$ be an element in Ξ. Then $\forall \varsigma \in \Xi$,

$$\varphi_\Phi(\varsigma) = \varphi_\Psi(\upsilon), \ \psi_\Phi(\varsigma) = \psi_\Psi(\upsilon), \text{ and } \xi_\Phi(\varsigma) = \xi_\Psi(\upsilon).$$

This implies that

$$\begin{cases} \varphi_\Phi(\varsigma) = \varphi_\Pi(\varpi^{-1} \cdot \upsilon \cdot \varpi) = \varphi_\Pi(\varpi^{-1} \cdot (\zeta^{-1} \cdot \varsigma \cdot \zeta) \cdot \varpi) = \varphi_\Pi((\zeta \cdot \varpi)^{-1} \cdot \varsigma \cdot (\zeta \cdot \varpi)) \\ \psi_\Phi(\varsigma) = \psi_\Pi(\varpi^{-1} \cdot \upsilon \cdot \varpi) = \psi_\Pi(\varpi^{-1} \cdot (\zeta^{-1} \cdot \varsigma \cdot \zeta) \cdot \varpi) = \psi_\Pi((\zeta \cdot \varpi)^{-1} \cdot \varsigma \cdot (\zeta \cdot \varpi)) \\ \xi_\Phi(\varsigma) \ = \xi_\Pi(\varpi^{-1} \cdot \upsilon \cdot \varpi) = \ \xi_\Pi(\varpi^{-1} \cdot (\zeta^{-1} \cdot \varsigma \cdot \zeta) \cdot \varpi) = \xi_\Pi((\zeta \cdot \varpi)^{-1} \cdot \varsigma \cdot (\zeta \cdot \varpi)) \end{cases}$$

Then for some $\alpha = \zeta \cdot \varpi$ in Ξ, $\Phi = \alpha\Pi\alpha^{-1}$. Hence, CPF conjugacy between CPFSGs is transitive. Consequently, CPF conjugacy among CCPFSGs is an equivalence relation. □

A normalizer measures how much a subgroup preserves conjugacy within a group. The following gives the notion of a normalizer for CPFSGs in a group.

Definition 5.8. Let Φ be the CPFSG of a group Ξ. Then, the normalizer of Φ in Ξ is defined as

$$\mathfrak{N}_{\Xi}(\Phi) = \{\varsigma \in \Xi \mid \Phi = \varsigma\Phi\varsigma^{-1}\}.$$

Next, the left and right CPF cosets are presented which will be helpful in understanding the work that follows.

Definition 5.9. Consider Φ is a CPFSG of Ξ. Then the left CPF coset of Φ by some $\varsigma \in \Xi$ is a CPFS $\varsigma\Phi = (\varphi_{\varsigma\Phi}, \psi_{\varsigma\Phi}, \xi_{\varsigma\Phi})$ in Ξ, such that $\forall \upsilon \in \Xi$,

$$\varphi_{\varsigma\Phi}(\upsilon) = \varphi_{\Phi}(\varsigma^{-1} \cdot \upsilon),\ \psi_{\varsigma\Phi}(\upsilon) = \psi_{\Phi}(\varsigma^{-1} \cdot \upsilon),\ \text{and } \xi_{\varsigma\Phi}(\upsilon) = \xi_{\Phi}(\varsigma^{-1} \cdot \upsilon).$$

Similarly, the right CPF coset is the CPFS $\Phi\varsigma = (\varphi_{\Phi\varsigma}, \psi_{\Phi\varsigma}, \xi_{\Phi\varsigma})$, such that $\forall \upsilon \in \Xi$,

$$\varphi_{\Phi\varsigma}(\upsilon) = \varphi_{\Phi}(\upsilon \cdot \varsigma^{-1}),\ \psi_{\Phi\varsigma}(\upsilon) = \psi_{\Phi}(\upsilon \cdot \varsigma^{-1}),\ \text{and } \xi_{\Phi\varsigma}(\upsilon) = \xi_{\Phi}(\upsilon \cdot \varsigma^{-1}).$$

Definition 5.10. The quantity of unique left (or right) CPF cosets of a CPFSG Φ in Ξ is given by its CPF index $\mathbb{I}_{\Xi}(\Phi)$ defined as

$$\mathbb{I}_{\Xi}(\Phi) = [\Xi : Core(\Phi)]$$

Here $[\Xi : Core(\Phi)]$ represents the index of crisp subgroup $Core(\Phi)$ in Ξ.

Remark 5.11. The index of a CPFSG's normalizer denoted by $[\Xi : \mathfrak{N}_{\Xi}(\Phi)]$ gives the number of distinct CCPFSGs of a CPFSG in Ξ.

The Lagrange's theorem of classical group theory relates the cardinality of a group to the cardinality and index of its subgroup. The coming theorem discusses Lagrange's theorem in terms of CPFSGs.

Theorem 5.12. (**CPF Lagrange's Theorem**) Consider a group Ξ (finite order). The CPF order of every CPFSG Φ of Ξ divides the cardinality of Ξ, with the CPF index of Φ in Ξ being the quotient. Mathematically,

$$\mathbb{I}_{\Xi}(\Phi) = \frac{|\Xi|}{Ord(\Phi)}.$$

Proof. Given that Φ is a CPFSG of Ξ. Then, $Core(\Phi)$ is the smallest L-CPFS of Φ. Also, $\mathbb{I}_{\Xi}(\Phi)$ gives the quantity of unique left cosets of Φ in Ξ. It is obvious by Theorem 4.6, that $Core(\Phi)$ is a crisp subgroup in Ξ. By Lagrange's theorem for crisp groups, the cardinality of every subgroup divides the group's cardinality. The quotient of this division equals the index of that subgroup. This implies that

$$\frac{|\Xi|}{|Core(\Phi)|} = [\Xi : Core(\Phi)]$$

Since $Ord(\Phi) = |Core(\Phi)|$ and $\mathbb{I}_{\Xi}(\Phi) = [\Xi : Core(\Phi)]$, therefore,

$$\mathbb{I}_{\Xi}(\Phi) = \frac{|\Xi|}{Ord(\Phi)}.$$

□

Corollary 5.13. If Φ is a CPFSG of Ξ, then $\forall \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda} \in L_{\Phi},\ |\Xi| = |\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}| \times [\Xi : \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}]$

6. Complex Picture Fuzzy Normal Subgroups

By utilizing the notions introduced in the preceding section, this section introduces and studies complex picture fuzzy normal subgroups. A complex picture fuzzy normal subgroup Φ in a group Ξ is defined below.

Definition 6.1. A CPFSG Φ of a group Ξ is called a complex picture fuzzy normal subgroup (CPFNSG) in Ξ if

$$\begin{cases} \varphi_\Phi(\varsigma) = \varphi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) \\ \psi_\Phi(\varsigma) = \psi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) \\ \xi_\Phi(\varsigma) \ = \xi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) \end{cases}, \ \forall \varsigma, \upsilon \in \Xi.$$

A CPFNSG Φ preserves CCPF relation throughout the group Ξ. Hence, this notion closely relates to the concepts in the previous section as presented below.

Proposition 6.2. Consider Φ is CPFSG of a group Ξ. The following assertions hold the same meaning.

(1) Φ is CPFNSG in Ξ
(2) $\Phi = \varsigma\Phi\varsigma^{-1}, \ \forall \varsigma \in \Xi$
(3) $\Phi\varsigma = \varsigma\Phi, \ \forall \varsigma \in \Xi$
(4) $\mathfrak{N}_\Xi(\Phi) = \Xi$

Proof.
(1) $\implies$ (2):
Obvious from Definitions 5.3 and 6.1.

(2) $\implies$ (3):
Consider $\Phi = \varsigma\Phi\varsigma^{-1}$ for all $\varsigma \in \Xi$. Then,

$$\begin{cases} \varphi_\Phi(\varsigma) = \varphi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) \implies \varphi_\Phi(\varsigma \cdot \upsilon^{-1}) = \varphi_\Phi(\upsilon^{-1} \cdot \varsigma) \implies \varphi_{\Phi\upsilon}(\varsigma) = \varphi_{\upsilon\Phi}(\varsigma) \\ \psi_\Phi(\varsigma) = \psi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) \implies \psi_\Phi(\varsigma \cdot \upsilon^{-1}) = \psi_\Phi(\upsilon^{-1} \cdot \varsigma) \implies \psi_{\Phi\upsilon}(\varsigma) = \psi_{\upsilon\Phi}(\varsigma) \\ \xi_\Phi(\varsigma) \ = \xi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) \implies \xi_\Phi(\varsigma \cdot \upsilon^{-1}) = \xi_\Phi(\upsilon^{-1} \cdot \varsigma) \implies \xi_{\Phi\upsilon}(\varsigma) = \ \xi_{\upsilon\Phi}(\varsigma) \end{cases}$$

for all $\varsigma, \upsilon \in \Xi$. Hence, $\Phi\varsigma = \varsigma\Phi, \ \forall \varsigma \in \Xi$.

(2) $\implies$ (4):
By Definition 5.8, $\mathfrak{N}_\Xi(\Phi) = \{\varsigma \in \Xi \mid \Phi = \varsigma\Phi\varsigma^{-1}\}$. If $\Phi = \varsigma\Phi\varsigma^{-1}, \ \forall \varsigma \in \Xi$, then clearly, $\mathfrak{N}_\Xi(\Phi) = \Xi$.

(4) $\implies$ (1):
Clear. □

The following gives a more convenient representation for CPFNSGs in terms of the group elements.

Theorem 6.3. A CPFSG Φ is called a CPFNSG of the group Ξ if and only if

$$\begin{cases} \varphi_\Phi(\varsigma \cdot \upsilon) = \varphi_\Phi(\upsilon \cdot \varsigma) \\ \psi_\Phi(\varsigma \cdot \upsilon) = \psi_\Phi(\upsilon \cdot \varsigma) \\ \xi_\Phi(\varsigma \cdot \upsilon) \ = \xi_\Phi(\upsilon \cdot \varsigma) \end{cases}, \ \forall \varsigma, \upsilon \in \Xi.$$

Proof. Straightforward. □

The next result discusses the intersection of CPFNSGs of a group.

Theorem 6.4. Intersection of mutually homogeneous CPFNSGs of a group Ξ is a CPFNSG of Ξ.

Proof. Consider $\{\Phi_\mathfrak{i} \mid \mathfrak{i} \in \Delta\}$ shows the collection of mutually homogeneous CPFNSGs of a group Ξ. Then their intersection is given by $\bigcap\limits_{\mathfrak{i}\in\Delta} \Phi_\mathfrak{i} = (\varphi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}, \psi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}, \xi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}})$, such that

$$\begin{cases} \varphi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}} = \min\limits_{\mathfrak{i}\in\Delta}\{\varphi_{\Phi_\mathfrak{i}}\} = \min\limits_{\mathfrak{i}\in\Delta}\{\mathfrak{r}_{\Phi_\mathfrak{i}}\} e^{i\min\limits_{\mathfrak{i}\in\Delta}\{x_{\Phi_\mathfrak{i}}\}} \\ \psi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}} = \min\limits_{\mathfrak{i}\in\Delta}\{\psi_{\Phi_\mathfrak{i}}\} = \min\limits_{\mathfrak{i}\in\Delta}\{\mathfrak{s}_{\Phi_\mathfrak{i}}\} e^{i\min\limits_{\mathfrak{i}\in\Delta}\{y_{\Phi_\mathfrak{i}}\}} \\ \xi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}} \ = \max\limits_{\mathfrak{i}\in\Delta}\{\xi_{\Phi_\mathfrak{i}}\} = \max\limits_{\mathfrak{i}\in\Delta}\{\mathfrak{t}_{\Phi_\mathfrak{i}}\} e^{i\max\limits_{\mathfrak{i}\in\Delta}\{z_{\Phi_\mathfrak{i}}\}} \end{cases}$$

By Theorem 3.12, $\bigcap\limits_{\mathfrak{i}\in\Delta} \Phi_\mathfrak{i}$ is a CPFSG of Ξ. Moreover, by Theorem 6.3,

$$\begin{cases} \varphi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon) = \min\limits_{\mathfrak{i}\in\Delta}\{\varphi_{\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon)\} = \min\limits_{\mathfrak{i}\in\Delta}\{\varphi_{\Phi_\mathfrak{i}}(\upsilon \cdot \varsigma)\} = \varphi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\upsilon \cdot \varsigma) \\ \psi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon) = \min\limits_{\mathfrak{i}\in\Delta}\{\psi_{\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon)\} = \min\limits_{\mathfrak{i}\in\Delta}\{\psi_{\Phi_\mathfrak{i}}(\upsilon \cdot \varsigma)\} = \psi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\upsilon \cdot \varsigma) \\ \xi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon) \ = \max\limits_{\mathfrak{i}\in\Delta}\{\xi_{\Phi_\mathfrak{i}}(\varsigma \cdot \upsilon)\} = \max\limits_{\mathfrak{i}\in\Delta}\{\xi_{\Phi_\mathfrak{i}}(\upsilon \cdot \varsigma)\} = \xi_{\bigcap\limits_{\mathfrak{i}\in\Delta}\Phi_\mathfrak{i}}(\upsilon \cdot \varsigma) \end{cases}, \ \forall \varsigma, \upsilon \in \Xi.$$

Hence, CPF normality holds for $\bigcap\limits_{\mathfrak{i}\in\Delta} \Phi_\mathfrak{i}$. □

The CPF normality of a CPFNSG is tied to the normality of its corresponding PFS and π-PFS as stated below.

Theorem 6.5. Consider $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ is a CPFSG of group Ξ. Then Φ is called a CPFSNG of Ξ if and only if:

(1) The PFS $\hat{\Phi} = \{(\varsigma, \mathfrak{r}_\Phi(\varsigma), \mathfrak{s}_\Phi(\varsigma), \mathfrak{t}_\Phi(\varsigma) \mid \varsigma \in \Xi)\}$ is a PFSNG of Ξ.
(2) The π-PFS $\breve{\Phi} = \{(\varsigma, x_\Phi(\varsigma), y_\Phi(\varsigma), z_\Phi(\varsigma)) \mid \varsigma \in \Xi\}$ is a π-PFSNG of Ξ.

Proof. Straightforward. □

The ensuing theorem translates CPF normality of a CPFNSG to the crisp group normality of its L-CPFSs.

Theorem 6.6. If Φ is CPFNSG of a crisp group Ξ then each L-CPFS $\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda} \in L_\Phi$ forms a normal subgroup of Ξ.

Proof. Consider Φ is a CPFNSG of Ξ. By Theorem 4.6, each L-CPFS $\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda} \in L_\Phi$ is a subgroup of Ξ. Now, it needs to be proved that each L-CPFS of Φ is normal in Ξ. Consider $\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$ is an arbitrary L-CPFS of Φ, then $\forall \varsigma \in \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$,

$$\begin{cases} \mathfrak{r}_\Phi(\varsigma) \geq \gamma \\ \mathfrak{s}_\Phi(\varsigma) \geq \delta \\ \mathfrak{t}_\Phi(\varsigma) \leq \lambda \end{cases} \quad \text{and} \quad \begin{cases} x_\Phi(\varsigma) \geq \epsilon \\ y_\Phi(\varsigma) \geq \kappa \\ z_\Phi(\varsigma) \leq \tau \end{cases}$$

By CPF normality of $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ in Ξ,

$$\varphi_\Phi(\varsigma) = \varphi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon), \ \psi_\Phi(\varsigma) = \psi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon), \text{ and } \xi_\Phi(\varsigma) = \xi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon),$$

$\forall \varsigma, \upsilon \in \Xi$. By subsethood of $\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$ in Ξ, the above condition also holds for any $\varsigma \in \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$. Hence, by homogeneity of Φ, for any element $\varsigma \in \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$.

$$\begin{cases} \mathfrak{r}_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = \mathfrak{r}_\Phi(\varsigma) \geq \gamma \\ \mathfrak{s}_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = \mathfrak{s}_\Phi(\varsigma) \geq \delta \\ \mathfrak{t}_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = \mathfrak{t}_\Phi(\varsigma) \leq \lambda \end{cases} \quad \text{and} \quad \begin{cases} x_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = x_\Phi(\varsigma) \geq \epsilon \\ y_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = y_\Phi(\varsigma) \geq \kappa \\ z_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = z_\Phi(\varsigma) \leq \tau \end{cases}, \forall \upsilon \in \Xi.$$

This implies that an element $\upsilon^{-1} \cdot \varsigma \cdot \upsilon$ exists in $\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$ for each $\varsigma \in \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$. Consequently, the L-CPFS is closed under conjugation and hence,

$$\upsilon^{-1} \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda} \upsilon = \Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}, \forall \upsilon \in \Xi.$$

Finally, the arbitrary consideration of $\Phi^{\epsilon,\kappa,\tau}_{\gamma,\delta,\lambda}$ proves that each L-CPFS in L_Φ is a normal subgroup of Ξ. □

The homogeneity of CPFSGs translates the properties of CPFSG to their corresponding PFSGs and π-PFSGs. Considering this, the next theorem relates the CPF-normality of CPFNSGs to the normality of their level PFSs and level π-PFSs in the crisp group.

Theorem 6.7. If Φ is a CPFNSG of the group Ξ, then

(1) Each $[\gamma, \delta, \lambda]$-level PFS of $\hat{\Phi}$ is a normal subgroup of Ξ.
(2) Each $[\epsilon, \kappa, \tau]$-level π-PFS of $\breve{\Phi}$ is a normal subgroup of Ξ.

Proof. Directly follows from Proposition 4.2 and Theorem 6.6. □

Definition 6.8 gives the notion of a relative CPFNSG as a CPFSG holding normality within another CPFSG containing it.

Definition 6.8. Consider two CPFSGs Φ and Ψ of a group Ξ, where $\Phi \subseteq \Psi$. Then Φ is called a CPFNSG relative to Ψ in Ξ if

$$\begin{cases} \varphi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Psi(\upsilon)\} \\ \psi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) \geq \min\{\psi_\Phi(\varsigma), \psi_\Psi(\upsilon)\} \\ \xi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) \leq \max\{\xi_\Phi(\varsigma), \xi_\Psi(\upsilon)\} \end{cases}, \ \forall \varsigma, \upsilon \in \Xi.$$

The above notion of relative CPFNSG is related to that of a CPFNSG in the following theorem.

Theorem 6.9. Consider two CPFSGs Φ and Ψ of a group Ξ with $\Phi \subseteq \Psi$. If Φ is a CPFNSG of Ξ, then Φ is a CPFNSG relative to Ψ in Ξ.

Proof. Suppose that $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ is a CPFNSG of Ξ, and $\Psi = (\varphi_\Psi, \psi_\Psi, \xi_\Psi)$ is another CPFSG of Ξ containing Φ. Then for all $\varsigma, \upsilon \in \Xi$,

$$\begin{cases} \varphi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) = \varphi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = \varphi_\Phi(\varsigma) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Psi(\upsilon)\} \\ \psi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) = \psi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = \psi_\Phi(\varsigma) \geq \min\{\psi_\Phi(\varsigma), \psi_\Psi(\upsilon)\} \\ \xi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) = \xi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) = \xi_\Phi(\varsigma) \leq \max\{\xi_\Phi(\varsigma), \xi_\Psi(\upsilon)\} \end{cases}$$

Hence, by Definition 6.8, Φ is a CPFNSG relative to Ψ in Ξ. □

Proposition 6.10. If Φ is a CPFNSG relative to Ψ in Ξ, then

$$\begin{cases} \varphi_{\upsilon\Phi}(\varsigma) \geq \min\{\varphi_{\Phi\upsilon}(\varsigma), \varphi_\Psi(\upsilon^{-1})\} \\ \psi_{\upsilon\Phi}(\varsigma) \geq \min\{\psi_{\Phi\upsilon}(\varsigma), \psi_\Psi(\upsilon^{-1})\} \\ \xi_{\upsilon\Phi}(\varsigma) \leq \max\{\xi_{\Phi\upsilon}(\varsigma), \xi_\Psi(\upsilon^{-1})\} \end{cases}, \ \forall \varsigma, \upsilon \in \Xi.$$

Proof. Let Φ be the CPFNSG relative to Ψ in Ξ, then $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{aligned} & \varphi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Psi(\upsilon)\} \\ \Longrightarrow \quad & \varphi_\Phi(\upsilon^{-1} \cdot \varsigma \cdot \upsilon) \geq \min\{\varphi_\Phi(\varsigma), \varphi_\Psi(\upsilon^{-1})\} \\ \Longrightarrow \quad & \varphi_\Phi(\upsilon^{-1} \cdot (\varsigma \cdot \upsilon^{-1}) \cdot \upsilon) \geq \min\{\varphi_\Phi(\varsigma \cdot \upsilon^{-1}), \varphi_\Psi(\upsilon^{-1})\} \\ \Longrightarrow \quad & \varphi_{\upsilon\Phi}(\varsigma) \geq \min\{\varphi_{\Phi\upsilon}(\varsigma), \varphi_\Psi(\upsilon^{-1})\} \end{aligned}$$

Similarly,

$$\psi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) \geq \min\{\varphi_\Phi(\varsigma), \psi_\Psi(\upsilon)\} \quad \Longrightarrow \quad \psi_{\upsilon\Phi}(\varsigma) \geq \min\{\psi_{\Phi\upsilon}(\varsigma), \psi_\Psi(\upsilon^{-1})\},$$

and

$$\xi_\Phi(\upsilon \cdot \varsigma \cdot \upsilon^{-1}) \leq \max\{\xi_\Phi(\varsigma), \xi_\Psi(\upsilon)\} \quad \Longrightarrow \quad \xi_{\upsilon\Phi}(\varsigma) \leq \max\{\xi_{\Phi\upsilon}(\varsigma), \xi_\Psi(\upsilon^{-1})\},$$

for all $\varsigma, \upsilon \in \Xi$. □

7. HOMOMORPHISM

This section discusses the properties of CPFSGs under homomorphism. The definition below recalls the notion of homomorphism between groups.

Definition 7.1. Consider two groups $(\Xi, \cdot)$ and $(\Theta, \circ)$. A function $\Upsilon : \Xi \to \Theta$ preserving the group operations is called a homomorphism if

$$\Upsilon(\varsigma \cdot \upsilon) = \Upsilon(\varsigma) \circ \Upsilon(\upsilon), \ \forall \varsigma, \upsilon \in \Xi.$$

Moreover, a one-one and onto homomorphism between groups is called a group isomorphism.

Now, motivated by the studies of homomorphism on fuzzy subgroups and extensions, the subsequent definitions elucidate the notions of image and inverse-image of CPFSGs under homomorphisms.

Definition 7.2. Consider a homomorphism $\Upsilon : \Xi \to \Theta$. Let $\Phi = (\varphi_\Phi, \psi_\Phi, \xi_\Phi)$ be the CPFSG of Ξ. Then, the image of Φ under Υ given by $\Upsilon(\Phi) = (\varphi_{\Upsilon(\Phi)}, \psi_{\Upsilon(\Phi)}, \xi_{\Upsilon(\Phi)})$ is a CPFS of Θ, such that

$$\begin{cases} \varphi_{\Upsilon(\Phi)}(\mathfrak{p}) = \max\limits_{\varsigma \in \Upsilon^{-1}(\mathfrak{p})} \{\varphi_\Phi(\varsigma)\} \\ \psi_{\Upsilon(\Phi)}(\mathfrak{p}) = \min\limits_{\varsigma \in \Upsilon^{-1}(\mathfrak{p})} \{\psi_\Phi(\varsigma)\} \\ \xi_{\Upsilon(\Phi)}(\mathfrak{p}) \ = \min\limits_{\varsigma \in \Upsilon^{-1}(\mathfrak{p})} \{\xi_\Phi(\varsigma)\} \end{cases}, \ \forall \mathfrak{p} \in \Theta.$$

If nothing maps from Ξ to $\mathfrak{p}$, then $\varphi_{\Upsilon(\Phi)}(\mathfrak{p}) = 0$, $\psi_{\Upsilon(\Phi)}(\mathfrak{p}) = 0$, and $\xi_{\Upsilon(\Phi)}(\mathfrak{p}) = e^{2\pi i}$.

Definition 7.3. Consider a homomorphism $\Upsilon : \Xi \to \Theta$. Let $\Psi = (\varphi_\Psi, \psi_\Psi, \xi_\Psi)$ be the CPFSG of Θ. Then, the pre-image of Ψ given by $\Upsilon^{-1}(\Psi) = (\varphi_{\Upsilon^{-1}(\Psi)}, \psi_{\Upsilon^{-1}(\Psi)}, \xi_{\Upsilon^{-1}(\Psi)})$ is a CPFS of Ξ, such that

$$\begin{cases} \varphi_{\Upsilon^{-1}(\Psi)}(\varsigma) = \varphi_\Psi(\Upsilon(\varsigma)) \\ \psi_{\Upsilon^{-1}(\Psi)}(\varsigma) = \psi_\Psi(\Upsilon(\varsigma)) \\ \xi_{\Upsilon^{-1}(\Psi)}(\varsigma) \ = \xi_\Psi(\Upsilon(\varsigma)) \end{cases}, \ \forall \varsigma \in \Xi.$$

The ensuing theorems discuss the conditions under which image and preimage of CPFSGs return CPFSGs.

Theorem 7.4. Consider two groups $(\Xi, \cdot)$ and $(\Theta, \circ)$ with an isomorphism $\Upsilon : \Xi \to \Theta$. Suppose Φ is a CPFSG of Ξ, then its image $\Upsilon(\Phi)$ is a CPFSG of Θ.

Proof. Given that $\Upsilon : \Xi \to \Theta$ is a group isomorphism. This implies that $\Upsilon(\Xi) = \Theta$. In other words, for each $\mathfrak{p} \in \Theta, \exists\, \varsigma \in \Xi : \Upsilon(\varsigma) = \upsilon$. Let Φ be a CPFSG of Ξ, then by Definition 7.2,

$$\begin{cases} \varphi_{\Upsilon(\Phi)}(\mathfrak{p}) = \varphi_{\Upsilon(\Phi)}(\Upsilon(\varsigma)) = \max\limits_{\varsigma \in \Upsilon^{-1}(\mathfrak{p})} \{\varphi_\Phi(\varsigma)\} = \varphi_\Phi(\varsigma) \\ \psi_{\Upsilon(\Phi)}(\mathfrak{p}) = \psi_{\Upsilon(\Phi)}(\Upsilon(\varsigma)) = \min\limits_{\varsigma \in \Upsilon^{-1}(\mathfrak{p})} \{\psi_\Phi(\varsigma)\} = \psi_\Phi(\varsigma) \\ \xi_{\Upsilon(\Phi)}(\mathfrak{p}) \ = \xi_{\Upsilon(\Phi)}(\Upsilon(\varsigma)) = \min\limits_{\varsigma \in \Upsilon^{-1}(\mathfrak{p})} \{\xi_\Phi(\varsigma)\} = \xi_\Phi(\varsigma) \end{cases}, \ \forall \mathfrak{p} \in \Theta.$$

Consider $\mathfrak{p}, \mathfrak{q} \in \Theta$ arbitrarily, such that $\Upsilon^{-1}(\mathfrak{p}) = \varsigma$ and $\Upsilon^{-1}(\mathfrak{q}) = \upsilon$ for some $\varsigma, \upsilon \in \Xi$. Then,

$$\begin{aligned} \varphi_{\Upsilon(\Phi)}(\mathfrak{p} \circ \mathfrak{q}) &= \varphi_{\Upsilon(\Phi)}(\Upsilon(\varsigma) \circ \Upsilon(\upsilon)) \\ &= \varphi_{\Upsilon(\Phi)}(\Upsilon(\varsigma \cdot \upsilon)) \qquad \text{(By Definition 7.1)} \\ &= \varphi_\Phi(\varsigma \cdot \upsilon) \qquad (\because \varphi_{\Upsilon(\Phi)}(\Upsilon(\zeta)) = \varphi_\Phi(\zeta)) \\ &\geq \min\{\varphi_\Phi(\varsigma), \varphi_\Phi(\upsilon)\} \ (\because \Phi \text{ is a CPFSG}) \\ &= \min\{\varphi_{\Upsilon(\Phi)}(\mathfrak{p}), \varphi_{\Upsilon(\Phi)}(\mathfrak{q})\} \end{aligned}$$

$$\implies \varphi_{\Upsilon(\Phi)}(\mathfrak{p} \circ \mathfrak{q}) \geq \min\{\varphi_{\Upsilon(\Phi)}(\mathfrak{p}), \varphi_{\Upsilon(\Phi)}(\mathfrak{q})\}, \ \forall \mathfrak{p}, \mathfrak{q} \in \Theta.$$

Similarly, $\forall \mathfrak{p}, \mathfrak{q} \in \Theta$,

$$\psi_{\Upsilon(\Phi)}(\mathfrak{p} \circ \mathfrak{q}) \geq \min\{\psi_{\Upsilon(\Phi)}(\mathfrak{p}), \psi_{\Upsilon(\Phi)}(\mathfrak{q})\} \quad \text{and} \quad \xi_{\Upsilon(\Phi)}(\mathfrak{p} \circ \mathfrak{q}) \leq \max\{\xi_{\Upsilon(\Phi)}(\mathfrak{p}), \xi_{\Upsilon(\Phi)}(\mathfrak{q})\}.$$

Since the group Θ is isomorphic to Ξ by Υ. This implies that for each $\mathfrak{p} \in \Theta$, there exists an inverse $\mathfrak{p}^{-1} = \Upsilon(\varsigma^{-1})$. Then,

$$\begin{cases} \varphi_{\Upsilon(\Phi)}(\mathfrak{p}^{-1}) = \varphi_{\Phi}(\varsigma^{-1}) \geq \varphi_{\Phi}(\varsigma) = \varphi_{\Upsilon(\Phi)}(\mathfrak{p}) \\ \psi_{\Upsilon(\Phi)}(\mathfrak{p}^{-1}) = \psi_{\Phi}(\varsigma^{-1}) \geq \psi_{\Phi}(\varsigma) = \psi_{\Upsilon(\Phi)}(\mathfrak{p}) \ , \ \forall \mathfrak{p} \in \Theta. \\ \xi_{\Upsilon(\Phi)}(\mathfrak{p}^{-1}) \ = \xi_{\Phi}(\varsigma^{-1}) \leq \ \xi_{\Phi}(\varsigma) = \xi_{\Upsilon(\Phi)}(\mathfrak{p}) \end{cases}$$

Hence, for a given CPFSG Φ of Ξ, its isomorphic image $\Upsilon(\Phi)$ is a CPFSG of Θ. □

Theorem 7.5. Consider two groups $(\Xi, \cdot)$ and $(\Theta, \circ)$ with a homomorphism $\Upsilon : \Xi \to \Theta$. Let Ψ be a CPFSG of Θ, then its inverse image $\Upsilon^{-1}(\Psi)$ in Ξ is a CPFSG of Ξ.

Proof. Consider that Υ is a homomorphism from $(\Xi, \cdot)$ to $(\Theta, \circ)$. Then for each $\varsigma \in \Xi$, $\exists$ a unique image $\Upsilon(\varsigma) \in \Theta$. Let $\Psi = (\varphi_\Psi, \psi_\Psi, \xi_\Psi)$ be a CPFSG of Θ having its inverse image $\Upsilon^{-1}(\mathfrak{B}) = (\varphi_{\Upsilon^{-1}(\Psi)}, \psi_{\Upsilon^{-1}(\Psi)}, \xi_{\Upsilon^{-1}(\Psi)})$ in Ξ, given by Definition 7.3. Then $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{aligned} \varphi_{\Upsilon^{-1}(\Psi)}(\varsigma \cdot \upsilon^{-1}) &= \varphi_\Psi(\Upsilon(\varsigma \cdot \upsilon^{-1})) \\ &= \varphi_\Psi(\Upsilon(\varsigma) \circ \Upsilon(\upsilon^{-1})) \qquad \text{(By Definition 7.1)} \\ &\geq \min\{\varphi_\Psi(\Upsilon(\varsigma)), \varphi_\Psi(\Upsilon(\upsilon))\} \ (\because \Psi \text{ is a CPFSG}) \\ &= \min\{\varphi_{\Upsilon^{-1}(\Psi)}(\varsigma), \varphi_{\Upsilon^{-1}(\Psi)}(\upsilon)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} \psi_{\Upsilon^{-1}(\Psi)}(\varsigma \cdot \upsilon^{-1}) &= \psi_\Psi(\Upsilon(\varsigma \cdot \upsilon^{-1})) \\ &= \psi_\Psi(\Upsilon(\varsigma) \circ \Upsilon(\upsilon^{-1})) \\ &\geq \min\{\psi_\Psi(\Upsilon(\varsigma)), \psi_\Psi(\Upsilon(\upsilon))\} \\ &= \min\{\psi_{\Upsilon^{-1}(\Psi)}(\varsigma), \psi_{\Upsilon^{-1}(\Psi)}(\upsilon)\}. \end{aligned}$$

$$\begin{aligned} \xi_{\Upsilon^{-1}(\Psi)}(\varsigma \cdot \upsilon^{-1}) &= \xi_\Psi(\Upsilon(\varsigma \cdot \upsilon^{-1})) \\ &= \xi_\Psi(\Upsilon(\varsigma) \circ \Upsilon(\upsilon^{-1})) \\ &\leq \max\{\xi_\Psi(\Upsilon(\varsigma)), \xi_\Psi(\Upsilon(\upsilon))\} \\ &= \max\{\xi_{\Upsilon^{-1}(\Psi)}(\varsigma), \xi_{\Upsilon^{-1}(\Psi)}(\upsilon)\}. \end{aligned}$$

Hence, $\Upsilon^{-1}(\Psi)$ is a CPFSG of Ξ. □

Theorem 7.6. Consider a isomorphism $\Upsilon : \Xi \to \Theta$. Suppose Φ and Ψ are CPFNSGs of Ξ and Θ, respectively. Then

(1) The image $\Upsilon(\Phi)$ is a CPFNSG of Θ.
(2) The inverse image $\Upsilon^{-1}(\Psi)$ is a CPFNSG of Ξ.

Proof. Given that $\Upsilon : \Xi \to \Theta$ is an isomorphism. Then for every $\mathfrak{p}, \mathfrak{q} \in \Theta$, $\exists$ unique $\varsigma, \upsilon \in \Xi$, such that, $\Upsilon^{-1}(\mathfrak{p}) = \varsigma$ and $\Upsilon^{-1}(\mathfrak{q}) = \upsilon$.

(1) Let Φ be a CPFNSG of Ξ. Then $\forall \mathfrak{p}, \mathfrak{q} \in \Theta$,

$$\begin{cases} \varphi_{\Upsilon(\Phi)}(\mathfrak{p} \circ \mathfrak{q}) = \varphi_{\Upsilon(\Phi)}(\Upsilon(\varsigma \cdot \upsilon)) = \varphi_{\Upsilon(\Phi)}(\Upsilon(\upsilon \cdot \varsigma)) = \varphi_{\Upsilon(\Phi)}(\mathfrak{q} \circ \mathfrak{p}) \\ \psi_{\Upsilon(\Phi)}(\mathfrak{p} \circ \mathfrak{q}) = \psi_{\Upsilon(\Phi)}(\Upsilon(\varsigma \cdot \upsilon)) = \psi_{\Upsilon(\Phi)}(\Upsilon(\upsilon \cdot \varsigma)) = \psi_{\Upsilon(\Phi)}(\mathfrak{q} \circ \mathfrak{p}) \\ \xi_{\Upsilon(\Phi)}(\mathfrak{p} \circ \mathfrak{q}) = \xi_{\Upsilon(\Phi)}(\Upsilon(\varsigma \cdot \upsilon)) = \xi_{\Upsilon(\Phi)}(\Upsilon(\upsilon \cdot \varsigma)) = \xi_{\Upsilon(\Phi)}(\mathfrak{q} \circ \mathfrak{p}) \end{cases}$$

$\Longrightarrow \Upsilon(\Phi)$ is a CPFNSG of Θ.

(2) Let Ψ be a CPFNSG of Θ. Then $\forall \varsigma, \upsilon \in \Xi$,

$$\begin{cases} \varphi_{\Upsilon^{-1}(\Psi)}(\varsigma \cdot \upsilon) = \varphi_{\Psi}(\mathfrak{p} \circ \mathfrak{q}) = \varphi_{\Psi}(\mathfrak{q} \circ \mathfrak{p}) = \varphi_{\Upsilon^{-1}(\Psi)}(\upsilon \cdot \varsigma) \\ \psi_{\Upsilon^{-1}(\Psi)}(\varsigma \cdot \upsilon) = \psi_{\Psi}(\mathfrak{p} \circ \mathfrak{q}) = \psi_{\Psi}(\mathfrak{q} \circ \mathfrak{p}) = \psi_{\Upsilon^{-1}(\Psi)}(\upsilon \cdot \varsigma) \\ \xi_{\Upsilon^{-1}(\Psi)}(\varsigma \cdot \upsilon) = \xi_{\Psi}(\mathfrak{p} \circ \mathfrak{q}) = \xi_{\Psi}(\mathfrak{q} \circ \mathfrak{p}) = \xi_{\Upsilon^{-1}(\Psi)}(\upsilon \cdot \varsigma) \end{cases}$$

$\Longrightarrow \Upsilon^{-1}(\Psi)$ is a CPFNSG of Ξ.

□

8. Application in Modeling of a Parallel Robotic Grippers

A parallel robotic gripper is a type of gripper that uses two jaws which move symmetrically towards an object while keeping parallel. A parallel robotic gripper is a type of a robotic gripper which comprises two jaws that move symmetrically towards an object and keep parallel. Ideally the motion of the gripper is deterministic. In actual robotic environments, however, uncertainty in the sensors' readings, vibration, slip, force imbalance, alignment error, uncertainty in the surface, and small time-dependent disturbances will affect grasping performance. The 3-D rigid-body pose of a robotic gripper is most naturally represented by the *Special Euclidean Group* or $SE(3)$. This group includes all rigid transformations (rotations) of 3D space together with translations. In robotics, rigid-body motions are composed by matrix multiplication and $SE(3)$ is often represented by homogeneous transformation matrices. Importantly, this multiplication is in general non-commutative, that is, "rotate then translate" is not the same as "translate then rotate".

An element of $SE(3)$ is written as $T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$, where $R \in SO(3), \quad t \in \mathbb{R}^3$. In this case, R is a rotation matrix that defines the orientation of the gripper, and t is a translation vector that defines the position of the gripper in space. The subgroup $SO(3)$ corresponds to pure rotations, e.g. aligning the gripper fingers with the object surface, whereas $\mathbb{R}^3$ represents translations along the x, y, and z axes.

In general, deformation is not in $SE(3)$. The group $SE(3)$ is the group of *rigid-body motion*. In all cases of deformation, compliance, vibration, uncertainties etc., should be considered as an additional deformation or uncertainty variable, not as a direct part of $SE(3)$. Let's assume a two-finger parallel gripper, then the poses of the two fingers can be stored as the product group at first. $SE(3) \times SE(3)$. This implies that the left finger and the right finger each have their own pose as a rigid-body: $T_L = (R_L, t_L), \quad T_R = (R_R, t_R)$. But a genuine parallel-jaw gripper will not enable both fingers to move freely. Fingers must be parallel and have a mechanically controlled separation distance. Thus the feasible configuration space is a limited subset of $SE(3) \times SE(3)$. We can impose the following for a symmetric parallel gripper: $R_L = R_R = R$, and $t_R - t_L = dRe_y$, where d is the distance of the jaw opening and e_y is the unit vector in the direction of the jaw opening.

Therefore, the two jaws have the same orientation but there is a controlled offset. Assuming that the opening distance d is constant, the motion of the whole gripper can be modeled as a constrained subgroup embedded in $SE(3) \times SE(3)$ which is essentially isomorphic to $SE(3)$. When opening and closing, the configuration becomes more accurately expressed as $SE(3) \times [d_{\min}, d_{\max}]$, in this equation, the scalar variable whose value is the depth of the jaw opening, is d. For detail study see [22, 23, 32]

Complex Picture Fuzzy Set Representation. In the classical model, the gripper pose is considered exact and deterministic. Robotic grasping is, however, very seldom precise. Each time the robot has some proof of success, some doubt, and some proof of failure. The fuzzy sets are useful here since the positive membership, neutral membership, and negative membership degrees are possible, as well as a refusal or hesitation degree. In this case study, a CPFS over the feasible gripper motion space is defined. A CPFS is an extension of this concept, by using complex values to express the degrees of membership, neutrality of membership and non-membership, the amplitude representing the intensity and the phase representing the periodic or dynamic behavior. Consider a set $\Xi \subseteq SE(3) \times SE(3)$ to be the constrained motion space of the parallel gripper. A CPFS Φ on Ξ is defined by:

$$\Phi(T) = \left(\varphi_\Phi(T), \psi_\Phi(T), \xi_\Phi(T)\right),$$

where $\varphi_\Phi(T) = \mathfrak{r}_\Phi(T)e^{ix_\Phi(T)}$, $\psi_\Phi(T) = \mathfrak{s}_\Phi(T)e^{iy_\Phi(T)}$, $\xi_\Phi(T) = \mathfrak{t}_\Phi(T)e^{iz_\Phi(T)}$, $\varphi_\Psi(T) = \mathfrak{r}_\Psi(T)e^{ix_\Psi(T)}$, $\psi_\Psi(T) = \mathfrak{s}_\Psi(T)e^{iy_\Psi(T)}$, and $\xi_\Psi(T) = \mathfrak{t}_\Psi(T)e^{iz_\Psi(T)}$, $\forall T \in \Xi$.

$\varphi_\Phi(T)$ is the amount of successful grasping,

$\psi_\Phi(T)$ represents the degree of uncertainty or hesitation, $\xi_\Phi(T)$ represents the degree of failed grasping. The amplitude condition is $0 \leq \mathfrak{r}_\Phi(T) + \mathfrak{s}_\Phi(T) + \mathfrak{t}_\Phi(T) \leq 1$. The degree of refusal is represented by

$$\pi(T) = 1 - \{\mathfrak{r}_\Phi(T) + \mathfrak{s}_\Phi(T) + \mathfrak{t}_\Phi(T)\}.$$

The phase terms

$$x_\Phi(T), y_\Phi(T), z_\Phi(T)$$

Capture dynamic aspects of grasping, including vibration, force synchronization, sensor delay, oscillatory instability and time-dependent ambiguity.

Sensor-Based Membership Modeling. In a parallel robotic gripper, the fuzzy evaluation can be evaluated with several sensor signals, like: F_L, F_R are the left and right contact forces, respectively, s is the slip measurement, e_a is the alignment error, e_p is the positioning error. Level of vibrations/oscillations: v. The success membership can be modeled to be high if the contact forces are balanced, the slip is low, and the alignment error is small:

$$\mathfrak{r}_\Phi(T) = \exp\left[-x_1|F_L - F_R| - x_2 s - x_3 e_a - x_4 e_p\right].$$

Failure degree can rise as slip, force imbalance and misalignment rises:

$$\mathfrak{t}_\Phi(T) = 1 - \exp\left[-z_1 s - z_2|F_L - F_R| - z_3 e_a\right].$$

Neutral membership may be used for ambiguous cases including weak contact, unclear tactile response, and visual estimation:

$$\mathfrak{s}_\Phi(T) = y_1\sigma_{\text{vision}} + y_2\sigma_{\text{force}} + y_3\sigma_{\text{tactile}},$$

The uncertainty in the vision, force, and tactile sensors are denoted as $\sigma_{\text{vision}}, \sigma_{\text{force}}, \sigma_{\text{tactile}}$, respectively.

The phase term can be a

$$\theta(T) = \omega t + \phi_s + \phi_f,$$

Here, the time-dependent oscillation is given by ωt, sensor phase delay is given by ϕ_s, and force synchronization phase delay is given by ϕ_f. This enables the model to not only predict the success rate for grasping but also if the gripper is dynamically stable.

This is an overview of the topic of group action and pose-invariant grasp evaluation. One of the most significant benefits of $SE(3)$ is that the fuzzy grasp evaluation can be investigated in the context of rigid transformations. If the pose of the gripper is changed by another rigid transformation $S \in SE(3)$, then

$$T \mapsto ST.$$

The transformation of the Complex Picture Fuzzy Set is given by

$$\Xi_S(T) = \Xi(S^{-1}T).$$

This provides pose-invariant grasp analysis. Put simply, the robot can assess the quality of the grip, regardless of the perception of the object or gripper in a different coordinate frame. Useful in robotic vision, object manipulation, pose estimation and grasp planning.

For sensor fusion, we use complex picture fuzzy aggregation in sensor fusion.

In real-world robotic systems, grasping decisions are made using more than one sensor. The vision system can be used to estimate the pose of the object; the force sensor can measure the contact pressure; and tactile sensors can detect the slip or surface contact. Let $\Xi_v, \quad \Xi_f, \quad \Xi_t$ denote CPF evaluations based on vision, force, and tactile sensors, respectively. A weighted aggregation can be represented as

$$\Xi_{agg}(T) = w_v \Xi_v(T) + w_f \Xi_f(T) + w_t \Xi_t(T),$$

where

$$w_v + w_f + w_t = 1.$$

During contact, for instance, if tactile information is more reliable then w_t may be raised. If the object hasn't been touched, its vision weight w_v can be larger. This adaptive aggregation allows the model to be more realistic for real-time grasping by the robot.

Stability Index with Phase Penalty. A grasp may have high force contact but still be unstable due to vibration or sensor desynchronization. Therefore, we define a stability index:

$$SI(T) = \mathfrak{r}_\Phi(T) - \mathfrak{t}_\Phi(T) - \lambda \mathfrak{s}_\Phi(T) - \rho P_\theta(T),$$

where

$$P_\theta(T) = 1 - \cos(\Delta\theta).$$

Here, $\Delta\theta$ represents the phase difference between the two gripper fingers or between sensor modalities. When the phases are synchronized, When $\Delta\theta \approx 0$, then $P_\theta(T) \approx 0$, and the stability index becomes high. When the phases are desynchronized, the penalty increases, indicating possible vibration, slip, or unstable grasping.

Optimization Problem. The grasp planning problem can now be written as an optimization problem: $\max_{T,d} SI(T)$ subject to $T \in SE(3)$, $d_{\min} \leq d \leq d_{\max}$, $R_L = R_R$, $t_R - t_L = dRe_y$, $F_{\min} \leq F_L, F_R \leq F_{\max}$, $s \leq s_{\max}$. This formulation selects the gripper pose and jaw opening that maximize grasp success while minimizing failure, uncertainty, slip, and phase instability.

Limitations and Impact of the Study. The proposed framework is mainly theoretical and depends on the proper selection of fuzzy values, phase information, and sensor data. Its practical performance may vary depending on the accuracy of sensors, the complexity of the object, and the real-time control system used in the robotic gripper. This work can improve uncertainty-aware robotic grasp planning by giving a more detailed evaluation of grasp quality. It may help robots make better decisions in precision manipulation tasks where vibration, slip, force imbalance, and sensor ambiguity are present.

Discussion. In classical models, the gripper pose is usually described precisely and deterministically while the proposed model uses the group structure of $SE(3)$ for the pose description, so that a mathematically consistent treatment of rigid-body motions and transformations can be done. Typical methods either don't consider or simply fail to solve uncertainty issues due to sensor noise, contact variation and environmental disturbances. The CPF model, on the other hand, takes into account uncertainty by using complex-valued positive, neutral and negative membership functions.

Furthermore, typical notions of grasp success and failure are binary or probabilistic based, inadequate to describe hesitation, uncertainty, and incomplete sensor data. The proposed CPF framework extends this representation, by modelling success, uncertainty, failure, and refusal states at the same time, in a unified algebraic structure. The treatment of dynamics and synchronization is also an important difference. The phase part of the complex picture fuzzy representation can effectively reveal the behavior of synchronization and dynamic instability, which is usually neglected by classical gripper models, while the oscillatory and phase-dependent effects, such as vibration and asynchronous finger motion, are not included in the classical gripper models.

In the conventional systems, sensor fusion generally involves that visual, force, and tactile information are processed separately, creating disjointed decision making. The proposed CPF model combines multisensory information into one complex picture fuzzy model, which helps to enhance the robustness and consistency of grasp analysis. Likewise, classical stability analysis is predominantly a force closure and contact mechanics analysis, while the proposed approach takes into account uncertainty, slip behavior and phase penalties, providing a more comprehensive stability analysis.

Thirdly, classical models of robots are generally frame-dependent and easily affected by coordinate transformations, while the proposed CPF group-theoretic models are pose invariant due to CPF group actions on $SE(3)$. Thus, in the uncertain, dynamic and multi-dimensional environments, the CPF model can offer a more generalized, flexible and realistic solution for robotic grasping.

9. Conclusion

Complex picture fuzzy sets have been widely used in the previous few years, however, they have not been applied to group theory. This work extended the complex fuzzy subgroups and picture fuzzy subgroups to complex picture fuzzy subgroups. The important results in the forms of theorems and propositions were presented and proved. Separate sections intended for level CPFSGs, CPF conjugacy, CPFNSGs, and homomorphism discussed all the included notions, their properties and interconnections in detail. Meanwhile the definition of important concepts like CPF cosets, conjugacy classes, CPF normalizer, CPF order, CPF index, CPF Core and CPF Support helped in achieving and explaining the results including the CPFSG interpretation of Lagrange's theorem. An application of the proposed group structure is presented in context of quality assessment. Considering the applicability of this work, the presented results can be extended to complex spherical fuzzy subgroups or complex T-spherical fuzzy subgroups due to their structural similarities with the presented CPFSGs. Moreover, the future works may discuss the applications of CPFSGs on Sylow theorems, characterizations, norms, etc. Due to their multi-dimensional structure, the same may be applied to clustering problems and pattern analysis where amplitude and phase terms in three different states may depict the scenario effectively.

10. Future Directions

CPFSG theory can also be extended to other geometric structures, like complex picture fuzzy topological groups, metric spaces in future studies. Under complex picture fuzzy environments, the research of continuity, compactness, connectedness, convergence and symmetry will create more connections between fuzzy algebra, topology, Lie groups and differential geometry. In particular, the use of phase-sensitive uncertainty in the topological and geometric group structures could find new fields for research in modern algebraic topology and geometric analysis.

CPFSGs could provide novel mathematical frameworks for modelling the oscillatory dynamics, vibration analysis, synchronization effects and phase-sensitive uncertainty that occur in signal processing and communication systems. The phase component of CPFSGs has a special role in representing periodic phenomena and time varying phenomena, which often occur in real-world engineering systems.

One of the other areas of research that has great potential is the development of quantum-inspired computational models and quantum information systems using CPFSGs. CPFSGs can be a very useful algebraic structure to model multidimensional probabilistic reasoning, entanglement-related ambiguity, and quantum transitions with uncertainties that are inherently complex-valued, probabilistic, and phase-dependent.

Future work could also focus on the optimization algorithms, computational methods, and software systems to be used in the CPFSG-based systems, to facilitate their practical implementation in the field of quantum data, intelligent control, autonomous navigation, health care analytics, cyber security, and smart decision-support systems. The advances in these developments suggest that CPFSGs can be an important link between the abstract fuzzy algebra, computational intelligence, and high-level engineering applications.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

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REFERENCES

[1] E. A. Abuhijleh, M. Massa'deh, A. Sheimat, A. Alkouri, *Complex fuzzy groups based on Rosenfeld's approach*, WSEAS Transactions on Mathematics **20**, No. 1 (2021): 368–377.

[2] M. Akram, A. Bashir and H. Garg, *Decision-making model under complex picture fuzzy Hamacher aggregation operators*, Computational and Applied Mathematics, **39**, No. 3 (2020): 1–38.

[3] G. Ali, M. Z. U. Abidin, Q. Xin, F. M. Tawfiq, *An Innovative Hybrid Multi-Criteria Decision-Making Approach under Picture Fuzzy Information*, Symmetry **14**, No. 11 (2022): 2434.

[4] A. Ali, M. H. Mateen, Q. Xin, T. Alsuraiheed, G. Alhamzi, *(ϵ, δ)-complex anti fuzzy subgroups and their applications,* AIMS Mathematics **9**, No. 5 (2024): 11580–11595.

[5] H. M. H. Alolaiyan, H. A. Alshehri, M. H. Mateen, D. Pamucar, M. Gulzar, *A novel algebraic structure of (α, β)-complex fuzzy subgroups*, Entropy **23**, No. 8 (2021): 992.

[6] M. O. Alsarahead, A. A. Ahmad, *Complex Fuzzy Subgroups*, Applied Mathematical Sciences, **11**, No. 41 (2017): 2011–2021.

[7] M. O. Alsarahead, A. Al-Husban, *Complex multi–fuzzy subgroups*, Journal of Discrete Mathematical Sciences and Cryptography **25**, No. 8 (2022): 2707–2716.

[8] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20**, No. 1 (1986): 87–96.

[9] M. Bani-Doumi, J. Serrano-Guerrero, F. Chiclana, F. P. Romero, J. A. Olivas, *A picture fuzzy set multi-criteria decision-making approach to customize hospital recommendations based on patient feedback*, Applied Soft Computing **153** (2024): 111331.

[10] R. Biswas, *Intuitionistic fuzzy subgroups*, Mathematical Forum **10** (1989): 37–46.

[11] B. C. Cuong, V. Kreinovich, *Picture fuzzy sets*, Journal of Computer Science and Cybernetics **30**, No. 4 (2014): 409–420.

[12] P. S. Das, *Fuzzy groups and level subgroups*, Journal of Mathematical Analysis and Applications **84**, No. 1 (1981): 264–269.

[13] H. Dhumras, V. Shukla, R. K. Bajaj, M. Driss, W. Boulila, *On Similarity Measures of Complex Picture Fuzzy Sets with Applications in the Field of Pattern Recognition*, IEEE Access (2024).

[14] S. Dogra, M. Pal, Picture fuzzy subgroup, Kragujevac Journal of Mathematics **47**, No. 6 (2023): 911–933.

[15] L. Hayat, A. Naseem, M. Riaz, *Intuitionistic Fuzzy Credibility Implicative Ideals in BCK-algebra for Efficiency Evaluation, Supply Chain and Risk Analysis*, Punjab Univ. J. Math. **57**, No. 10 (2025): 1066–1103.

[16] S. He, Y. Wang, *Evaluating new energy vehicles by picture fuzzy sets based on sentiment analysis from online reviews*, Artificial Intelligence Review **56**, No. 3 (2023): 2171–2192.

[17] A. Imtiaz, H. Alolaiyan, U. Shuaib, A. Razaq, J. B. Liu, *Applications of conjunctive complex fuzzy subgroups to Sylow theory*, AIMS Mathematics **9**, No. 1 (2024): 38–54.

[18] N. Jan, J. Gwak, D. Pamucar, *Mathematical analysis of generative adversarial networks based on complex picture fuzzy soft information*, Applied Soft Computing **137** (2023): 110088.

[19] C. Kahraman, *Proportional picture fuzzy sets and their AHP extension: Application to waste disposal site selection, Expert Systems with Applications* **238** (2024): 122354.

[20] H. Khan, I. Ahmad, A. Khan, *On Fuzzy Bipolar Soft Ordered Semigroups*, Punjab Univ. J. Math. **53**, No. 4 (2021).

[21] Z. Khan, F. Hussain, T. Rahim, R. Jan, S. Boulaaras, *Distance measure and its application to decision making, medical diagnosis, and pattern recognition problems under complex picture fuzzy sets*, The European Physical Journal Plus **139**, No. 3 (2024): 243.

[22] K. M. Lynch, F. C. Park, *Modern Robotics*, Cambridge University Press (2017).

[23] R. M. Murray, Z. Li, S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, CRC Press (2017).

[24] S. Petchimuthu, B. Palpandi, T. Senapati, *Exploring pharmacological therapies through complex q-rung picture fuzzy Aczel–Alsina prioritized ordered operators in adverse drug reaction analysis*, Engineering Applications of Artificial Intelligence **133** (2024): 107996.

[25] M. Qiyas, M. Naeem, N. Khan, S. Khan, F. Khan, *Confidence levels bipolar complex fuzzy aggregation operators and their application in decision making problem*, IEEE Access (2024).

[26] D. Ramot, R. Milo, M. Friedman, A. Kandel, *Complex fuzzy sets*, IEEE Transactions on Fuzzy Systems **10**, No. 2 (2002): 171–186.

[27] R. Rasuli, *Norms over intuitionistic fuzzy subgroups on direct product of groups*, Communications in Combinatorics, Cryptography, and Computer Science **1** (2023): 39–54.

[28] R. Rasuli, *T-norms over complex fuzzy subgroups*, Mathematical Analysis and Its Contemporary Applications **5**, No. 1 (2023): 33–49.

[29] A. Rosenfeld, *Fuzzy groups*, Journal of Mathematical Analysis and Applications **35**, No. 3 (1971): 512–517.

[30] M. Şahin, V. Ulucay, S. A. Edalatpanah, F. A. A. Elsebaee, H. A. E. W. Khalifa, (α, γ)*-Anti-Multi-Fuzzy Subgroups and Some of Its Properties*, Computers, Materials & Continua **74**, No. 2 (2023).

[31] M. Şahin, N. Olgun, A. Kargın, V. Uluçay, *Isomorphism theorems for soft G-modules*, Afrika Matematika **29**, No. 7 (2018): 1237–1244.

[32] J. Seo, N. P. S. Prakash, A. Rose, J. Choi, R. Horowitz, *Geometric impedance control on SE(3) for robotic manipulators*, IFAC-PapersOnLine **56**, No. 2 (2023): 276–283.

[33] S. Shoaib, W. Mahmood, Q. Xin, F. Tchier, F. M. Tawfiq, *Certain operations on complex picture fuzzy graphs*, IEEE Access **10** (2022): 114284–114296.

[34] M. N. K. Tanoli, M. Gulistan, F. Amin, M. M. Al-Shamiri, *Innovative discussion of decision-making model based on complex cubic picture fuzzy information and geometric aggregation operators with applicatio*ns, Complex & Intelligent Systems **10**, No. 2 (2024): 1801–1843.

[35] R. Verma, B. Rohtagi, *Novel similarity measures between picture fuzzy sets and their applications to pattern recognition and medical diagnosis*, Granular Computing **7**, No. 4 (2022): 761–777.

[36] S. Vijayabalaji, S. Kalaiselvan, A. O. Atagün, H. Kamacı, *A study on intuitionistic fuzzy soft expert groups*, International Journal of Open Problems in Computer Science & Mathematics **17**, No. 2 (2024).

[37] W. Wu, *Normal fuzzy subgroups*, Fuzzy Math **1**, No. 1 (1981): 21–30.

[38] X. Yang, T. Mahmood, U. ur Rehman, *Bipolar complex fuzzy subgroups*, Mathematics **10**, No. 16 (2022): 2882.

[39] R. Yasar, A. Hatip, *The Structure of the Binary Two Fold Algebra Based on Intuitionistic Fuzzy Groups*, Neutrosophic Sets and Systems **72**, No. 1 (2024): 5.

[40] L. A. Zadeh, *Fuzzy sets*, Information and Control **8**, No. 3 (1965): 338–353.