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Examining the Dynamic Behavior of $(3+1)$ Dimensional Yu Tuda Sasa Fukuyama Equation using Extended Fan sub-Equation Method

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Abstract. The solitary wave behavior of $(3+1)$ dimensional Yu Tuda Sasa Fukuyama equation is evaluated in this article. The equation describes the interaction between two incompatible layers moving with different velocities in fluid system and accounts its dispersive and non-linear effects. Using extended Fan sub-equation method, some kink type, bell type and periodic type traveling wave solutions are achieved that are considerable in engineering and physical sciences. At the end, graphical representations of these solutions are depicted in 3D, contour, and 2D figures by assigning numerical values to existed parameters. Extended fan sub-equation technique is used for the first time for this equation.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Yu Tuda Sasa Fukuyama equation; extended Fan sub-equation method; traveling wave solutions.

1. Introduction

Fluid dynamics provides a mathematical structure of the underlying system by evaluating its properties of flow measurement, depending upon time and space variables. Yu Tuda Sasa Fukuyama (YTSF) equation plays a vital role in fluid dynamics that describes the interaction between two incompatible layers of different densities of the fluid. The nonlinear and dispersive behaviors of these layers preserve a nonlinear partial differential equation (NLPDE).Evaluation of such NLPDEs has become an innovative part of the mathematics. Numerical solutions of such NLPDE's involve vast application in the field of physics, fluid dynamics, fiber optics and optical mechanics.Although, YTSF equation involves its application almost in each field of physical science. A large number of researchers have studied YTSF equation, evaluating its soliton solutions by using numerous mathematical technique. The direct bilinear method used in [6] and N-soliton solution of generalized $(3+1)$-dimensional YTSF equation are obtained. In [10] extended simplest equation method and modified Kudryashov method are used to find lump and breather wave solutions. (G'/G)-expansion method [1], improved (G'/G)-expansion method [2],

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homoclinic (heteroclinic) breather limit expansion method [3], algorithm–Riccati method [4], new modified simple equation method [5], Kadomtsev–Petviashvili hierarchy reduction method [7, 12, 26], $(m + 1/G')-$expansion and Adomian decomposition method [8], the multiplier and Noether's methods [9], extended hyperbolic function method [11], extended homoclinic test and method of separation of variables used in [13], He's semi-inverse and improved Hirota bilinear method [14, 22, 25], semi-inverse method [15], Lie symmetry analysis and power series expansion method [16], $(\frac{G'}{G}, \frac{1}{G})$ expansion method [18, 20], unified method [19, 24], generalized Kudryashov method [21], mapping method [23], (G'/G^2) method [27], long-wave limit method [28, 29], auxiliary equation method [30], Hirota bilinear method and extended homoclinic test [31], extended complex method and (G'/G) expansion method [32], $e^{\Phi}(\xi)$-expansion method [33], Picard's iterative scheme [34], generalized unified integral operator [35] and Sardar sub-equation method [36].
In this research paper, $(3+1)$ dimensional YTSF equation is under consideration. Using an appropriate traveling wave transformation, YTSF equation is transferred to a nonlinear ordinary differential equation (ODE). The balance between higher order derivative and nonlinear term is evaluated by homogeneous balancing principle. Analytical technique, extended Fan sub-equation method is used to find traveling wave solutions of proposed ODE. Setting an appropriate numerical values for existed parameters, 3D surface graphs, contour graphs and 2D line graphs are demonstrated.
The remaining research article is divided into different sections. Section 2 consists of governing model. Section 3 contains description of the proposed method. Section 4 includes application of the method. Section 5 defines graphical representation of the proposed solutions, while conclusion is discussed in Section 6.

2. Governing model

Rizvi et al. [17] defined YTSF equation, as

$$(a_0P_t + a_1P_{xxz} + a_2PP_z + a_3P_x\partial_x^{-1}P_z)_x + a_4P_{yy} = 0, \tag{2. 1}$$

where potential function $P = P(x, y, z, t), \partial_x^{-1} = \int dx$ and a_0, a_1, a_2, a_3 are constant coefficients of development term, dispersion term, nonlinear term and linear term respectively. Assigning $a_0 = -4, a_1 = 1, a_2 = 4, a_3 = 2$ and $a_4 = 3$, Eq.(2.1) is converted into $(3+1)$ dimensional YTSF equation, as

$$(-4P_t + P_{xxz} + 4PP_z + 2P_x\partial_x^{-1}P_z)_x + 3P_{yy} = 0. \tag{2. 2}$$

By substituting $P = R_x$, Eq.(2.2) yields to following form

$$-4R_{xt} + R_{xxxz} + 4R_xR_{xz} + 2R_{xx}R_z + 3R_{yy} = 0. \tag{2. 3}$$

Traveling wave transformation for Eq.(2.3) is

$$R(x, y, z, t) = Q(\zeta), \qquad \zeta = x + y + z + \epsilon t, \tag{2. 4}$$

where ϵ is speed of wave. Substituting Eq.(2.4) into Eq.(2.3), yields the following ODE

$$-4\epsilon Q^{''} + Q^{''''} + 6Q^{''}Q^{'} + 3Q^{''} = 0. \tag{2. 5}$$

Integrating Eq.(2.5)

$$\begin{aligned}\int\left(-4\epsilon Q''+Q''''+6Q''Q'+3Q''\right)\,d\zeta=0,\\ \int -4\epsilon Q''\,d\zeta+\int Q''''\,d\zeta+\int 6Q''Q'\,d\zeta+\int 3Q''\,d\zeta=0,\\ -4\epsilon Q'+Q'''+3(Q')^2+3Q'+C=0.\end{aligned} \tag{2. 6}$$

After simplification, we get

$$Q'''+3(Q')^2+(3-4\epsilon)Q'+C=0. \tag{2. 7}$$

Remaining postulate considers Eq.(2.7) where Q is potential function and C is constant of integration. Homogeneous balancing principle for highest order derivative term Q''' and nonlinear term $(Q')^2$ is $n+3=2(n+1)$ that extracts $n=1$.

3. Description of method

Extended Fan sub-equation method is proposed for the evaluation of exact solutions of Eq.(2.7). General algorithm of this method is given, as

$$Q(\zeta)=\sum_{k=0}^{n} d_k P^k(\zeta), \tag{3. 8}$$

where d_k $(k=0,1,...,n)$ are constants while $P(\zeta)$ satisfies generalized elliptic equation. Generalized elliptic equation is given, as

$$(P^{'}(\zeta))^2=\left(\frac{dP(\zeta)}{d\zeta}\right)^2=h_0+h_1P(\zeta)+h_2P^2(\zeta)+h_3P^3(\zeta)+h_4P^4(\zeta), \tag{3. 9}$$

where $h_i,(i=0,1,2,3,4)$ are constants. Setting $h_0=\alpha^2, h_1=2\alpha\beta, h_2=\beta^2+2\alpha\gamma, h_3=2\beta\gamma, h_4=\gamma^2$. Eq.(3.9) can be written, as

$$(P^{'}(\zeta))^2=(\alpha+\beta P(\zeta)+\gamma P^2(\zeta))^2, \tag{3. 10}$$

where α,β,γ are arbitrary constants. Inserting $h_0=h_1=0$ into Eq.(3.9), we obtain following auxiliary equation.

$$(P^{'}(\zeta))^2=\left(\frac{dP(\zeta)}{d\zeta}\right)^2=h_2P^2(\zeta)+h_3P^3(\zeta)+h_4P^4(\zeta), \tag{3. 11}$$

where $P(\zeta)$ satisfies following solutions.

Case I. Type 1: $\beta^2-4\alpha\gamma>0$ provided $(\alpha\gamma\neq 0)$ and $\alpha\neq 0$. $P(\zeta)$ assumes following solutions.

$$P_1^{I,1} = -\frac{1}{2\gamma}\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)\right),$$

$$P_2^{I,1} = -\frac{1}{2\gamma}\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)\right),$$

$$P_3^{I,1} = -\frac{1}{2\gamma}\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\left[\tanh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm i\,\mathrm{sech}\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)\right]\right),$$

$$P_4^{I,1} = -\frac{1}{2\gamma}\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\left[\coth\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm \mathrm{csch}\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)\right]\right),$$

$$P_5^{I,1} = -\frac{1}{4\gamma}\left(2\beta + \sqrt{\beta^2 - 4\alpha\gamma}\left[\tanh\left(\frac{\sqrt{\beta^2 - 4\alpha\gamma}\zeta}{4}\right) + \coth\left(\frac{\sqrt{\sigma^2 - 4\beta\gamma}\zeta}{4}\right)\right]\right),$$

$$P_6^{I,1} = \frac{1}{2\gamma}\left[-\beta + \frac{\sqrt{(A^2+B^2)(\beta^2 - 4\alpha\gamma)} - A\sqrt{\beta^2 - 4\alpha\gamma}\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{A\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) + B}\right],$$

$$P_7^{I,1} = \frac{1}{2\gamma}\left[-\beta - \frac{\sqrt{(B^2-A^2)(\beta^2 - 4\alpha\gamma)} + A\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{A\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) + B}\right], \tag{3. 12}$$

where A and B are constants and $B^2 - A^2 > 0$.

$$P_8^{I,1} = \frac{2\alpha\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)}{\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right) - \beta\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)},$$

$$P_9^{I,1} = \frac{-2\alpha\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)}{\beta\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right) - \sqrt{\beta^2 - 4\alpha\gamma}\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)},$$

$$P_{10}^{I,1} = \frac{2\alpha\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) - \beta\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm i\sqrt{\beta^2 - 4\alpha\gamma}},$$

$$P_{11}^{I,1} = \frac{2\alpha\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{-\beta\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) + \sqrt{\beta^2 - 4\alpha\gamma}\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm i\sqrt{\beta^2 - 4\alpha\gamma}}. \tag{3. 13}$$

$$P_{12}^{I,1} = \frac{4\alpha\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right)}{-2\beta\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right) + 2\sqrt{\beta^2-4\alpha\gamma}\cosh^2\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right) - \sqrt{\beta^2-4\alpha\gamma}}. \tag{3. 14}$$

Type 2: When $4\alpha\gamma-\beta^2>0$ provided $(\beta\gamma\neq 0)$. $Q(\zeta)$ assumes following solutions.

$$
\begin{aligned}
P_{13}^{I,2} &= \frac{1}{2\gamma}\left(-\beta+\sqrt{4\alpha\gamma-\beta^2}\tan\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)\right),\\
P_{14}^{I,2} &= -\frac{1}{2\gamma}\left(\beta+\sqrt{4\alpha\gamma-\beta^2}\cot\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)\right),\\
P_{15}^{I,2} &= \frac{1}{2\gamma}\left(-\beta+\sqrt{4\alpha\gamma-\beta^2}\left[\tan\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\pm\sec\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\right]\right),\\
P_{16}^{I,2} &= -\frac{1}{2\gamma}\left(\beta+\sqrt{4\alpha\gamma-\beta^2}\left[\cot\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\pm\csc\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\right]\right),\\
P_{17}^{I,2} &= \frac{1}{4\gamma}\left(-2\beta+\sqrt{4\alpha\gamma-\beta^2}\left[\tan\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)-\cot\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)\right]\right),\\
P_{18}^{I,2} &= \frac{1}{2\gamma}\left(-\beta+\frac{\pm\sqrt{(A^2-B^2)(4\alpha\gamma-\beta^2)}-A\sqrt{4\alpha\gamma-\beta^2}\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{A\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+B}\right),\\
P_{19}^{I,2} &= \frac{1}{2\gamma}\left(-\beta-\frac{\mp\sqrt{(A^2-B^2)(4\alpha\gamma-\beta^2)}-A\sqrt{4\alpha\gamma-\beta^2}\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{A\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+B}\right),
\end{aligned}
\tag{3. 15}
$$

where A and B are constants with $A^2-B^2>0$.

$$
\begin{aligned}
P_{20}^{I,2} &= -\frac{2\alpha\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)}{\sqrt{4\alpha\gamma-\beta^2}\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)+\beta\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)},\\
P_{21}^{I,2} &= \frac{2\alpha\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)}{-\beta\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)+\sqrt{4\alpha\gamma-\beta^2}\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)},\\
P_{22}^{I,2} &= -\frac{2\alpha\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{\sqrt{4\alpha\gamma-\beta^2}\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+P\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\pm\sqrt{4\alpha\gamma-\beta^2}},\\
P_{23}^{I,2} &= \frac{2\alpha\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{-\beta\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+\sqrt{4\alpha\gamma-\beta^2}\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\pm\sqrt{4\alpha\gamma-\beta^2}}.
\end{aligned}
\tag{3. 16}
$$

$$
P_{24}^{I,2} = \frac{4\alpha\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)}{-2\beta\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)+2\sqrt{4\alpha\gamma-\beta^2}\cos^2\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)-\sqrt{4\alpha\gamma-\beta^2}}.
\tag{3. 17}
$$

Case II. $\alpha\gamma < 0$ provided $\alpha\gamma \neq 0$. $P(\zeta)$ assumes following solution.

$$\begin{aligned}
P_1^{II} &= -\frac{1}{2\gamma}\left[\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\tanh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)\right],\\
P_2^{II} &= -\frac{1}{2\gamma}\left[\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\coth\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)\right],\\
P_3^{II} &= -\frac{1}{2\gamma}\left(\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\left[\tanh\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm i\,\mathrm{sech}\left(\sqrt{-6\alpha\gamma}\zeta\right)\right]\right),\\
P_4^{II} &= -\frac{1}{2\gamma}\left(\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\left[\coth\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm i\,\mathrm{csch}\left(\sqrt{-6\alpha\gamma}\zeta\right)\right]\right),\\
P_5^{II} &= -\frac{1}{4\gamma}\left(\pm 2\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\left[\tanh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right) + \coth\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)\right]\right),\\
P_6^{II} &= \frac{1}{2\gamma}\left[\pm\sqrt{-2\alpha\gamma} + \frac{\sqrt{(P^2+Q^2)(-6\alpha\gamma)} - A\sqrt{-6\alpha\gamma}\cos\left(\sqrt{-6\alpha\gamma}\zeta\right)}{A\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right) + B}\right],\\
P_7^{II} &= \frac{1}{2\gamma}\left[\pm\sqrt{-2\alpha\gamma} - \frac{\sqrt{(Q^2-P^2)(-6\alpha\gamma)} + A\sqrt{-6\alpha\gamma}\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right)}{A\cos\left(\sqrt{-6\alpha\gamma}\zeta\right) + B}\right],\\
P_8^{II} &= \frac{2\alpha\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)}{\sqrt{-6\alpha\gamma}\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right) \pm \sqrt{-2\alpha\gamma}\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)},\\
P_9^{II} &= \frac{2\alpha\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)}{\pm\sqrt{-2\alpha\gamma}\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right) - \sqrt{-6\alpha\gamma}\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)},\\
P_{10}^{II} &= \frac{2\alpha\cosh\left(\sqrt{-6\alpha\gamma}\zeta\right)}{\sqrt{-6\alpha\gamma}\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm \sqrt{-2\alpha\gamma}\cosh\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm i\sqrt{-6\alpha\gamma}},\\
P_{11}^{II} &= \frac{2\alpha\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right)}{\pm\sqrt{-2\alpha\gamma}\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right) + \sqrt{-6\alpha\gamma}\cosh\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm \sqrt{-6\alpha\gamma}}.
\end{aligned} \tag{3. 18}$$

$$P_{12}^{II} = \frac{4\alpha\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)}{\pm 2\sqrt{-2\alpha\gamma}\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right) + 2\sqrt{-6\alpha\gamma}\cosh^2\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right) - \sqrt{-6\alpha\gamma}}. \tag{3. 19}$$

Considering the relations $h_0 = \alpha^2, h_1 = 2\alpha\beta, h_2 = \beta^2 + 2\alpha\gamma, h_3 = 2\beta\gamma, h_4 = \gamma^2$. $P(\zeta)$ satisfies following solutions for Eq.(9).

Case III. Type 1. Setting $h_2 = 1, h_3 = \frac{-2c}{a}, h_4 = \frac{c^2-b^2}{a^2}$,

$$P_1^{III} = \frac{\frac{a}{\cosh(\zeta)}}{b + \frac{c}{\cosh(\zeta)}}. \tag{3. 20}$$

Type 2. Setting $h_2 = 1, h_3 = \frac{-2c}{a}, h_4 = \frac{c^2+b^2}{a^2}$,

$$P_2^{III} = \frac{\frac{a}{\sinh(\zeta)}}{b + \frac{c}{\sinh(\zeta)}}. \tag{3. 21}$$

Type 3. Setting $h_2 = 4, h_3 = -\frac{4(2b+d)}{a}, h_4 = \frac{c^2+4b^2+4bd}{a^2}$,

$$P_3^{III} = \frac{\frac{a}{\cosh^2(\zeta)}}{b + \frac{c}{\cosh(\zeta)}}. \tag{3. 22}$$

Type 4. Setting $h_2 = 4, h_3 = -\frac{4(d-2b)}{a}, h_4 = \frac{c^2+4b^2-4bd}{a^2}$,

$$P_4^{III} = \frac{\frac{a}{\sinh^2(\zeta)}}{\frac{b}{\sinh^2(\zeta)} + c\coth(\zeta) + d}. \tag{3. 23}$$

Type 5. Setting $h_2 = a_1^2, h_3 = 2ab, h_4 = b_1^2$,

$$\begin{aligned} P_5^{III} &= -\frac{ac}{b(c+\cosh(a\zeta)-\sinh(a\zeta))}, \\ P_6^{III} &= \frac{a(\cosh(a\zeta)+\sinh(a\zeta))}{b(c+\cosh(a\zeta)+\sinh(a\zeta))}. \end{aligned} \tag{3. 24}$$

Type 6. Setting $h_2 = -1, h_3 = \frac{2c}{a}, h_4 = -\frac{c^2-b^2}{a^2}$,

$$\begin{aligned} P_7^{III} &= \frac{a\sec(\zeta)}{b+c\sec(\zeta)}, \\ P_8^{III} &= \frac{a\csc(\zeta)}{b+c\csc(\zeta)}. \end{aligned} \tag{3. 25}$$

Type 7. Setting $h_2 = -4, h_3 = -\frac{4(2b+d)}{a}, h_4 = -\frac{c^2+4b^2+4bd}{a^2}$,

$$\begin{aligned} P_9^{III} &= \frac{a\sec^2(\zeta)}{b\sec^2(\zeta)+c\cot(\zeta)+d}, \\ P_{10}^{III} &= \frac{a\csc^2(\zeta)}{b\csc^2(\zeta)+c\cot(\zeta)+d}. \end{aligned} \tag{3. 26}$$

where a, b, c, d are arbitrary constants.

4. Application of method

This section comprises the implementation of proposed method. Eq.(2.8) got following form for Eq.(2.7).

$$Q(\zeta) = d_0 + d_1 P(\zeta). \tag{4. 27}$$

Replacing Eq.(2.7) by Eq.(4.27) along with Eq.(3.10). Following system of algebraic equations is achieved by equating higher power of $P(\zeta)$.

$$\begin{aligned} 6\gamma^3 d_1 + 3\gamma^2 d_1^2 &= 0, \\ 12\beta\gamma^2 d_1 + 6\beta\gamma d_1^2 &= 0, \\ 8\alpha\gamma^2 d_1 + 6\alpha\gamma d_1^2 + 7\beta^2\gamma d_1 + 3\beta^2 d_1^2 - 4\epsilon\gamma d_1 + 3\gamma d_1 &= 0, \\ 8\alpha\beta\gamma d_1 + 6\alpha\beta d_1^2 + \beta^3 d_1 - 4\beta\epsilon d_1 + 3\beta d_1 &= 0, \\ 2\alpha^2\gamma d_1 + 3\alpha^2 d_1^2 + \alpha\beta^2 d_1 - 4\alpha\epsilon d_1 + 3\alpha d_1 + C &= 0. \end{aligned} \tag{4. 28}$$

Solving this system simultaneously yields a following solution set of values of existed constants.

$$\left(C = 0, \quad \epsilon = -\alpha\gamma + \frac{\beta^2}{4} + \frac{3}{4}, \quad d_0 = d_0, \quad d_1 = -2\gamma \right).$$

Substituting d_0, d_1 along with Eqs.(3.12-3.26) into Eq.(4.27), following wave solutions are achieved. For $\beta^2 - 4\alpha\gamma > 0$.

$$\begin{aligned} Q_1(x,y,z,t) &= d_0 + \left(\beta + \sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\zeta\right)\right), \\ Q_2(x,y,z,t) &= d_0 + \left(\beta + \sqrt{\beta^2-4\alpha\gamma}\coth\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\zeta\right)\right). \end{aligned} \tag{4. 29}$$

$$Q_3(x,y,z,t) = d_0 + \left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\left[\tanh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm i\,\mathrm{sech}\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)\right]\right),$$
$$Q_4(x,y,z,t) = d_0 + \left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\left[\coth\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm \mathrm{csch}\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)\right]\right). \tag{4. 30}$$

$$Q_5(x,y,z,t) = d_0 + \frac{1}{2}\left(2\beta + \sqrt{\beta^2 - 4\alpha\gamma}\left[\tanh\left(\frac{\sqrt{\beta^2 - 4\alpha\gamma}\zeta}{4}\right) + \coth\left(\frac{\sqrt{\sigma^2 - 4\beta\gamma}\zeta}{4}\right)\right]\right),$$
$$Q_6(x,y,z,t) = d_0 - \left[-\beta + \frac{\sqrt{(A^2+B^2)(\beta^2 - 4\alpha\gamma)} - A\sqrt{\beta^2 - 4\alpha\gamma}\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{A\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) + B}\right],$$
$$Q_7(x,y,z,t) = d_0 - \left[-\beta - \frac{\sqrt{(B^2-A^2)(\beta^2 - 4\alpha\gamma)} + A\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{A\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) + B}\right]. \tag{4. 31}$$

$$Q_8(x,y,z,t) = d_0 - 2\gamma\left[\frac{2\alpha\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)}{\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right) - \beta\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)}\right],$$
$$Q_9(x,y,z,t) = d_0 - 2\gamma\left[\frac{-2\alpha\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)}{\beta\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right) - \sqrt{\beta^2 - 4\alpha\gamma}\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{2}\right)}\right]. \tag{4. 32}$$

$$Q_{10}(x,y,z,t) = d_0 - 2\gamma\left[\frac{2\alpha\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) - \beta\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm i\sqrt{\beta^2 - 4\alpha\gamma}}\right],$$
$$Q_{11}(x,y,z,t) = d_0 - 2\gamma\left[\frac{2\alpha\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right)}{-\beta\sinh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) + \sqrt{\beta^2 - 4\alpha\gamma}\cosh\left(\sqrt{\beta^2 - 4\alpha\gamma}\zeta\right) \pm i\sqrt{\beta^2 - 4\alpha\gamma}}\right]. \tag{4. 33}$$

$$Q_{12}(x,y,z,t) = d_0 - 2\gamma\left[\frac{4\alpha\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right)}{-2\beta\cosh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right) + 2\sqrt{\beta^2-4\alpha\gamma}\cosh^2\left(\frac{\sqrt{\beta^2-4\alpha\gamma}\zeta}{4}\right) - \sqrt{\beta^2-4\alpha\gamma}}\right]. \tag{4. 34}$$

For $4\alpha\gamma - \beta^2 < 0$.

$$Q_{13}(x,y,z,t) = d_0 - \left(-\beta + \sqrt{4\alpha\gamma - \beta^2}\tan\left(\frac{\sqrt{4\alpha\gamma - \beta^2}\zeta}{2}\right)\right),$$
$$Q_{14}(x,y,z,t) = d_0 + \left(\beta + \sqrt{4\alpha\gamma - \beta^2}\cot\left(\frac{\sqrt{4\alpha\gamma - \beta^2}\zeta}{2}\right)\right),$$
$$Q_{15}(x,y,z,t) = d_0 - \left(-\beta + \sqrt{4\alpha\gamma - \beta^2}\left[\tan\left(\sqrt{4\alpha\gamma - \beta^2}\zeta\right) \pm \sec\left(\sqrt{4\alpha\gamma - \beta^2}\zeta\right)\right]\right),$$
$$Q_{16}(x,y,z,t) = d_0 + \left(\beta + \sqrt{4\alpha\gamma - \beta^2}\left[\cot\left(\sqrt{4\alpha\gamma - \beta^2}\zeta\right) \pm \csc\left(\sqrt{4\alpha\gamma - \beta^2}\zeta\right)\right]\right). \tag{4. 35}$$

$$
\begin{aligned}
Q_{17}(x,y,z,t) &= d_0 - \frac{1}{2}\left(-2\beta + \sqrt{4\alpha\gamma-\beta^2}\left[\tan\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right) - \cot\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)\right]\right),\\
Q_{18}(x,y,z,t) &= d_0 - \left(-\beta + \frac{\pm\sqrt{(A^2-B^2)(4\alpha\gamma-\beta^2)} - A\sqrt{4\alpha\gamma-\beta^2}\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{A\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+B}\right),\\
Q_{19}(x,y,z,t) &= d_0 - \left(-\beta - \frac{\mp\sqrt{(A^2-B^2)(4\alpha\gamma-\beta^2)} - A\sqrt{4\alpha\gamma-\beta^2}\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{A\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+B}\right),\\
Q_{20}(x,y,z,t) &= d_0 + 2\gamma\left[\frac{2\alpha\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)}{\sqrt{4\alpha\gamma-\beta^2}\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)+\beta\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)}\right],\\
Q_{21}(x,y,z,t) &= d_0 - 2\gamma\left[\frac{2\alpha\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)}{-\beta\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)+\sqrt{4\alpha\gamma-\beta^2}\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{2}\right)}\right],\\
Q_{22}(x,y,z,t) &= d_0 + 2\gamma\left[\frac{2\alpha\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{\sqrt{4\alpha\gamma-\beta^2}\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+P\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\pm\sqrt{4\alpha\gamma-\beta^2}}\right],\\
Q_{23}(x,y,z,t) &= d_0 - 2\gamma\left[\frac{2\alpha\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)}{-\beta\sin\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)+\sqrt{4\alpha\gamma-\beta^2}\cos\left(\sqrt{4\alpha\gamma-\beta^2}\zeta\right)\pm\sqrt{4\alpha\gamma-\beta^2}}\right],\\
Q_{24}(x,y,z,t) &= d_0 2\gamma\left[\frac{4\alpha\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)}{-2\beta\cos\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)\sin\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)+2\sqrt{4\alpha\gamma-\beta^2}\cos^2\left(\frac{\sqrt{4\alpha\gamma-\beta^2}\zeta}{4}\right)-\sqrt{4\alpha\gamma-\beta^2}}\right].
\end{aligned}
\tag{4. 36}
$$

For $\alpha\gamma < 0$.

$$
\begin{aligned}
Q_{25}(x,y,z,t) &= d_0 + \left[\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\tanh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)\right],\\
Q_{26}(x,y,z,t) &= d_0 + \left[\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\coth\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)\right],\\
Q_{27}(x,y,z,t) &= d_0 + \left(\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\left[\tanh\left(\sqrt{-6\alpha\gamma}\zeta\right)\pm i\,\mathrm{sech}\left(\sqrt{-6\alpha\gamma}\zeta\right)\right]\right),\\
Q_{28}(x,y,z,t) &= d_0 + \left(\pm\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\left[\coth\left(\sqrt{-6\alpha\gamma}\zeta\right)\pm i\,\mathrm{csch}\left(\sqrt{-6\alpha\gamma}\zeta\right)\right]\right),\\
Q_{29}(x,y,z,t) &= d_0 + \frac{1}{2}\left(\pm2\sqrt{-2\alpha\gamma} + \sqrt{-6\alpha\gamma}\left[\tanh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)+\coth\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)\right]\right),\\
Q_{30}(x,y,z,t) &= d_0 - \left[\pm\sqrt{-2\alpha\gamma} + \frac{\sqrt{(P^2+Q^2)(-6\alpha\gamma)} - A\sqrt{-6\alpha\gamma}\cos\left(\sqrt{-6\alpha\gamma}\zeta\right)}{A\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right)+B}\right],\\
Q_{31}(x,y,z,t) &= d_0 - \left[\pm\sqrt{-2\alpha\gamma} - \frac{\sqrt{(Q^2-P^2)(-6\alpha\gamma)} + A\sqrt{-6\alpha\gamma}\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right)}{A\cos\left(\sqrt{-6\alpha\gamma}\zeta\right)+B}\right],\\
Q_{32}(x,y,z,t) &= d_0 - 2\gamma\left[\frac{2\alpha\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)}{\sqrt{-6\alpha\gamma}\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)\pm\sqrt{-2\alpha\gamma}\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)}\right].
\end{aligned}
\tag{4. 37}
$$

$$Q_{33}(x,y,z,t) = d_0 - 2\gamma \left[\frac{2\alpha \sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)}{\pm\sqrt{-2\alpha\gamma}\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right) - \sqrt{-6\alpha\gamma}\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{2}\right)} \right],$$

$$Q_{34}(x,y,z,t) = d_0 - 2\gamma \left[\frac{2\alpha \cosh\left(\sqrt{-6\alpha\gamma}\zeta\right)}{\sqrt{-6\alpha\gamma}\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm \sqrt{-2\alpha\gamma}\cosh\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm i\sqrt{-6\alpha\gamma}} \right],$$

$$Q_{35}(x,y,z,t) = d_0 - 2\gamma \left[\frac{2\alpha \sinh\left(\sqrt{-6\alpha\gamma}\zeta\right)}{\pm\sqrt{-2\alpha\gamma}\sinh\left(\sqrt{-6\alpha\gamma}\zeta\right) + \sqrt{-6\alpha\gamma}\cosh\left(\sqrt{-6\alpha\gamma}\zeta\right) \pm \sqrt{-6\alpha\gamma}} \right]. \tag{4. 38}$$

$$Q_{36}(x,y,z,t) = d_0 - 2\gamma \left[\frac{4\alpha\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)}{\pm 2\sqrt{-2\alpha\gamma}\cosh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right)\sinh\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right) + 2\sqrt{-6\alpha\gamma}\cosh^2\left(\frac{\sqrt{-6\alpha\gamma}\zeta}{4}\right) - \sqrt{-6\alpha\gamma}} \right]. \tag{4. 39}$$

For $h_2 = 1, h_3 = \frac{-2c}{a}, h_4 = \frac{c^2-b^2}{a^2}$,

$$Q_{37}(x,y,z,t) = d_0 - 2\gamma \left[\frac{\frac{a}{\cosh(\zeta)}}{b + \frac{c}{\cosh(\zeta)}} \right]. \tag{4. 40}$$

For $h_2 = 1, h_3 = \frac{-2c}{a}, h_4 = \frac{c^2+b^2}{a^2}$,

$$Q_{38}(x,y,z,t) = d_0 - 2\gamma \left[\frac{\frac{a}{\sinh(\zeta)}}{b + \frac{c}{\sinh(\zeta)}} \right]. \tag{4. 41}$$

For $h_2 = 4, h_3 = -\frac{4(2b+d)}{a}, h_4 = \frac{c^2+4b^2+4bd}{a^2}$,

$$Q_{39}(x,y,z,t) = d_0 - 2\gamma \left[\frac{\frac{a}{\cosh^2(\zeta)}}{b + \frac{c}{\cosh(\zeta)}} \right]. \tag{4. 42}$$

For $h_2 = 4, h_3 = -\frac{4(d-2b)}{a}, h_4 = \frac{c^2+4b^2-4bd}{a^2}$,

$$Q_{40}(x,y,z,t) = d_0 - 2\gamma \left[\frac{\frac{a}{\sinh^2(\zeta)}}{\frac{b}{\sinh^2(\zeta)} + c\coth(\zeta) + d} \right]. \tag{4. 43}$$

For $h_2 = a_1^2, h_3 = 2ab, h_4 = b_1^2$,

$$\begin{aligned} Q_{41}(x,y,z,t) &= d_0 + 2\gamma \left[\frac{ac}{b(c + \cosh(a\zeta) - \sinh(a\zeta))} \right], \\ Q_{42}(x,y,z,t) &= d_0 - 2\gamma \left[\frac{a(\cosh(a\zeta) + \sinh(a\zeta))}{b(c + \cosh(a\zeta) + \sinh(a\zeta))} \right]. \end{aligned} \tag{4. 44}$$

For $h_2 = -1, h_3 = \frac{2c}{a}, h_4 = -\frac{c^2-b^2}{a^2}$,

$$\begin{aligned} Q_{43}(x,y,z,t) &= d_0 - 2\gamma \left[\frac{a\sec(\zeta)}{b + c\sec(\zeta)} \right], \\ Q_{44}(x,y,z,t) &= d_0 - 2\gamma \left[\frac{a\csc(\zeta)}{b + c\csc(\zeta)} \right]. \end{aligned} \tag{4. 45}$$

For $h_2 = -4, h_3 = -\frac{4(2b+d)}{a}, h_4 = -\frac{c^2+4b^2+4bd}{a^2}$,

$$\begin{aligned} Q_{45}(x,y,z,t) &= d_0 - 2\gamma \left[\frac{a\sec^2(\zeta)}{b\sec^2(\zeta) + c\cot(\zeta) + d} \right], \\ Q_{46}(x,y,z,t) &= d_0 - 2\gamma \left[\frac{a\csc^2(\zeta)}{b\csc^2(\zeta) + c\cot(\zeta) + d} \right], \end{aligned} \tag{4. 46}$$

where a, b, c, d are arbitrary constants.

5. GRAPHICAL REPRESENTATION

This section comprises the graphical description of the above proposed solutions.

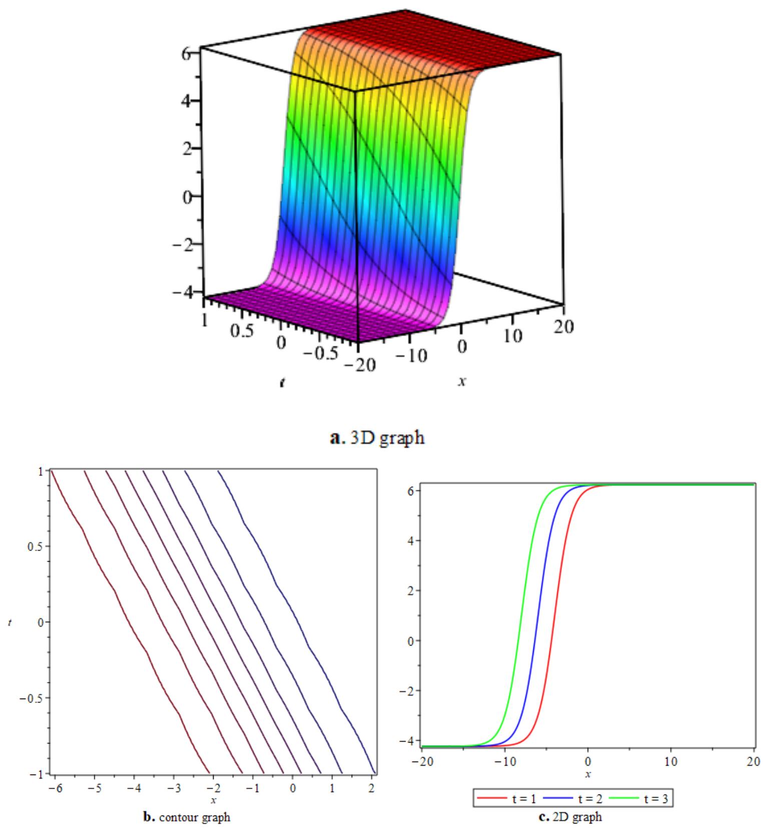


Fig.1. Kink soliton represented by $Q_1(x, y, z, t)$ for $d_0 = 1, \alpha = 1, \beta = 3, \gamma = 1, z = 1, y = 1$.

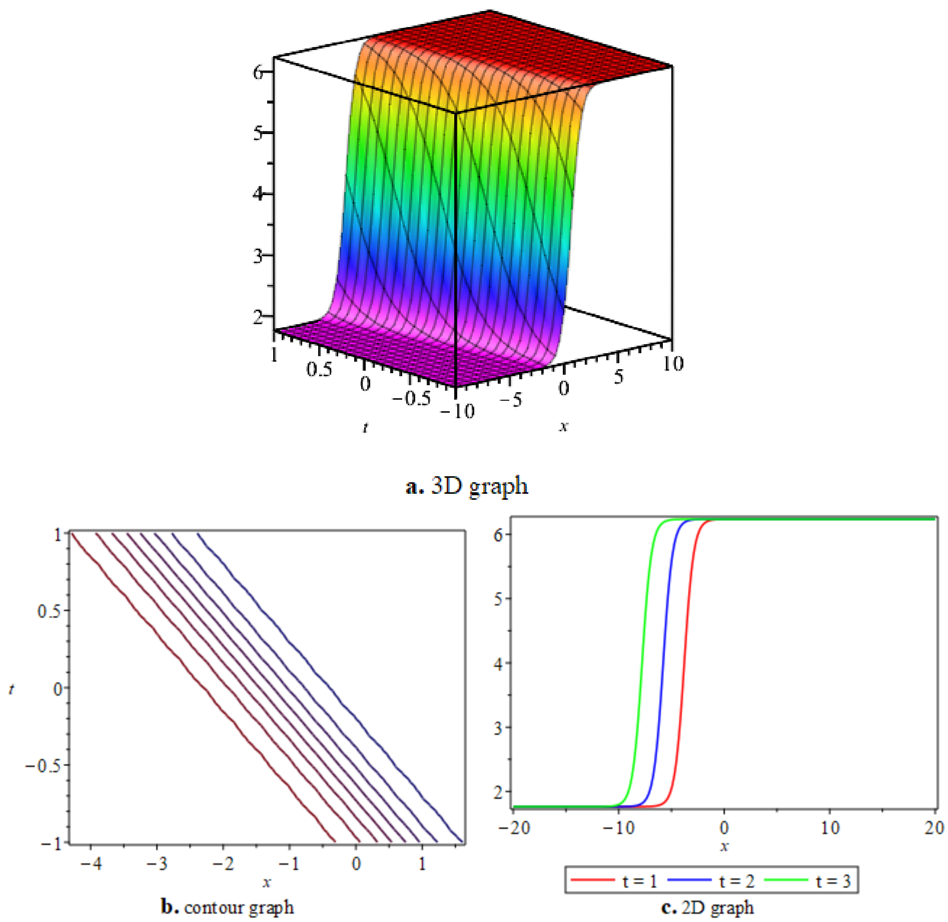
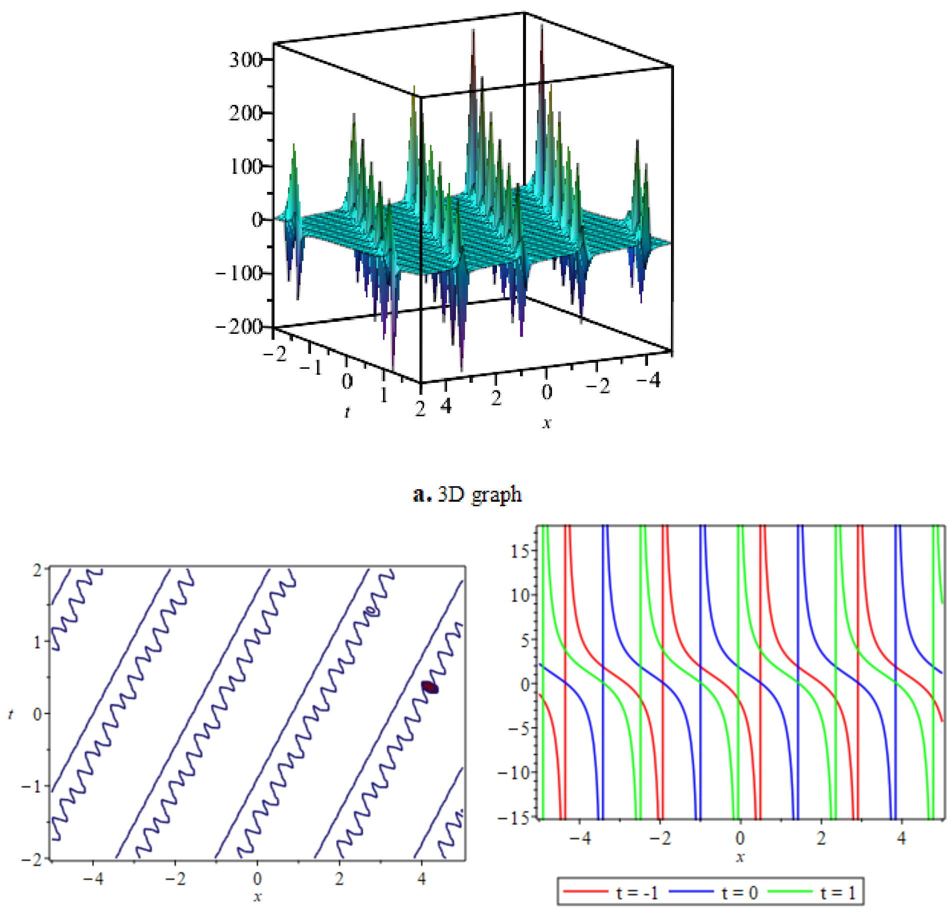


Fig.2. Kink soliton represented by $Q_6(x, y, z, t)$ for
$d_0 = 1, \alpha = 1, \beta = 3, \gamma = 1, z = 1, y = 1, A = 1, B = 2.$

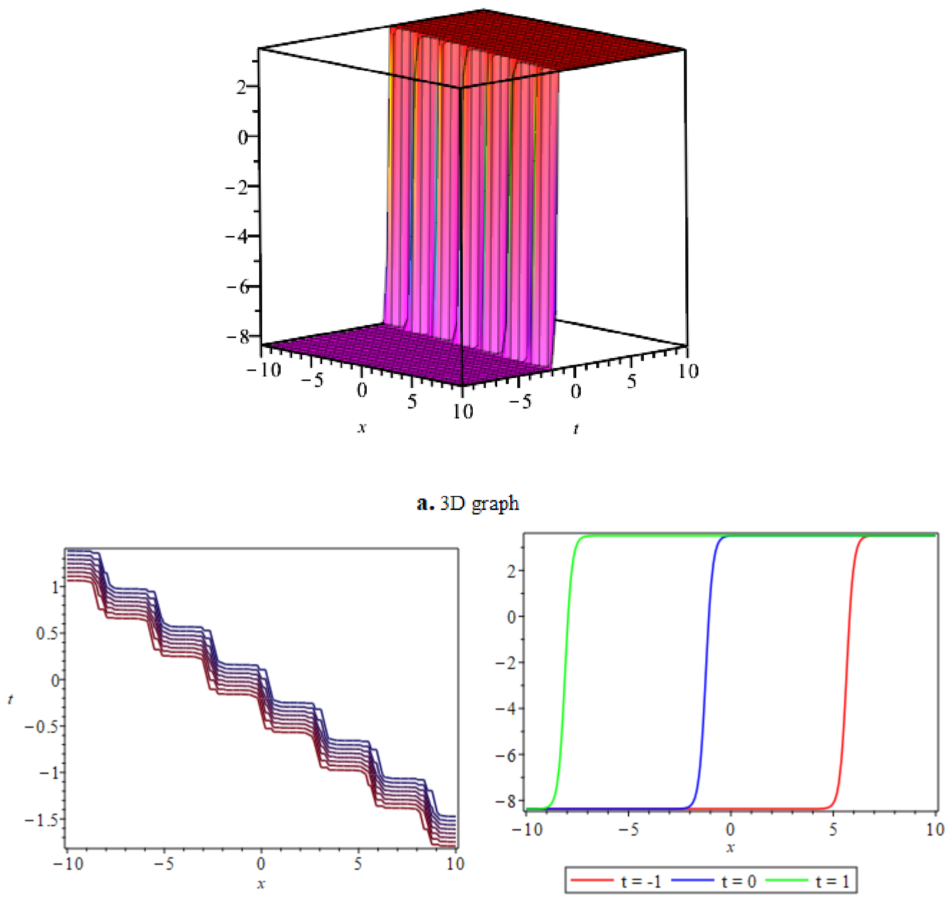


a. 3D graph

b. contour graph

c. 2D graph

Fig.3. Singular soliton represented by $Q_{17}(x, y, z, t)$ for $d_0 = 1, \alpha = 1.25, \beta = 0.1, \gamma = 1.35, z = 0.7, y = 0.3.$

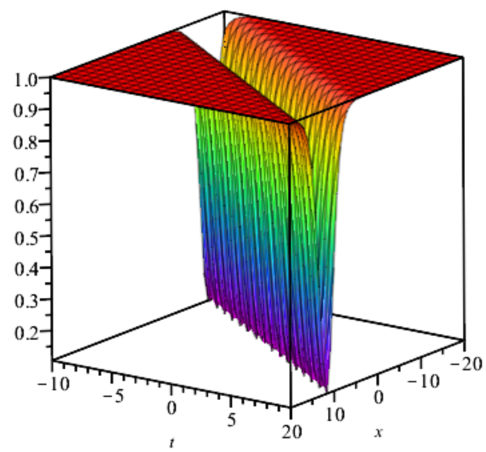


a. 3D graph

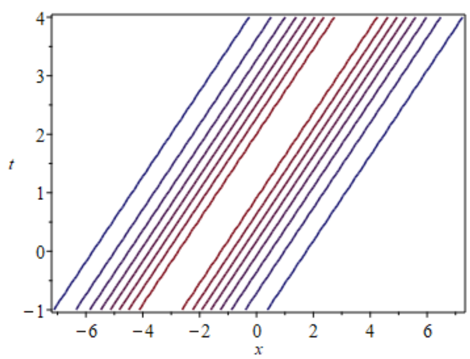
b. contour graph

c. 2D graph

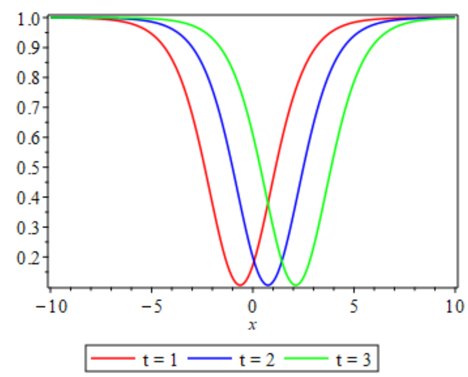
Fig.4. Kink soliton represented by $Q_{35}(x, y, z, t)$ for
$d_0 = 1, \alpha = 2.5, \beta = 1, \gamma = -2.35, z = 0.5, y = 0.5, Q = 3, P = 2.$



a. 3D graph



b. contour graph



c. 2D graph

Fig.5. Dark soliton represented by $Q_{37}(x, y, z, t)$ for $d_0 = 1, a = 1, b = 2, c = 3, y = 1, z = 3, h_0 = 1.$

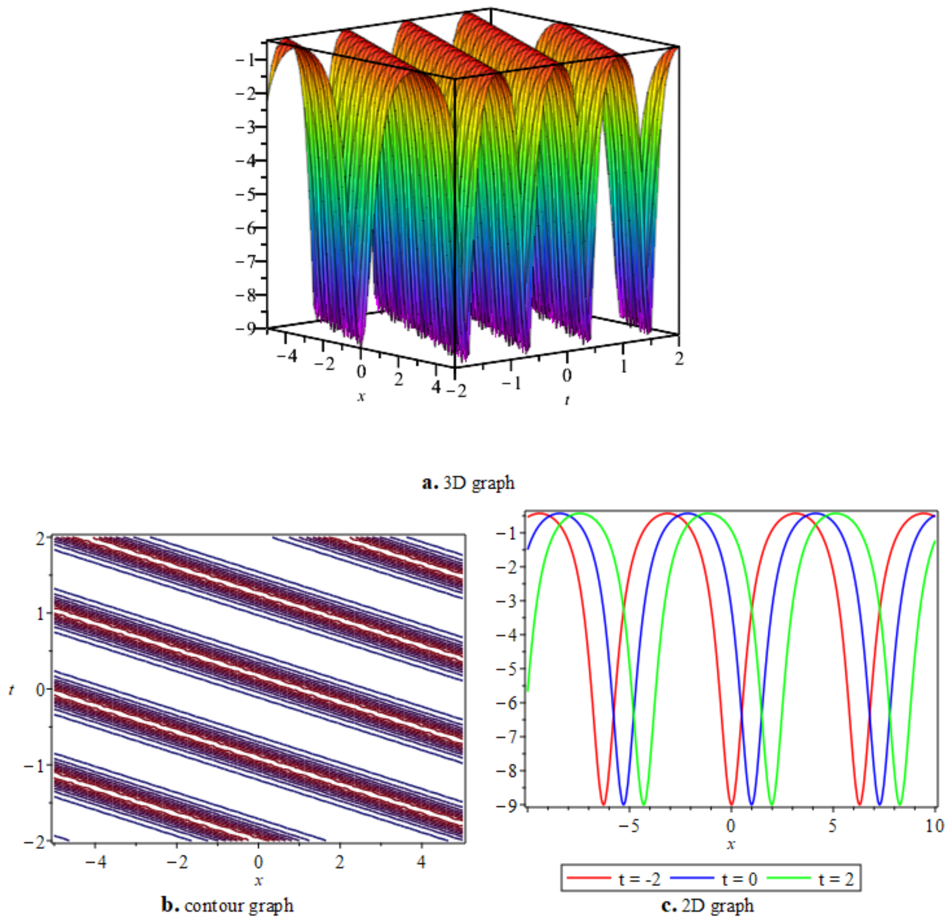
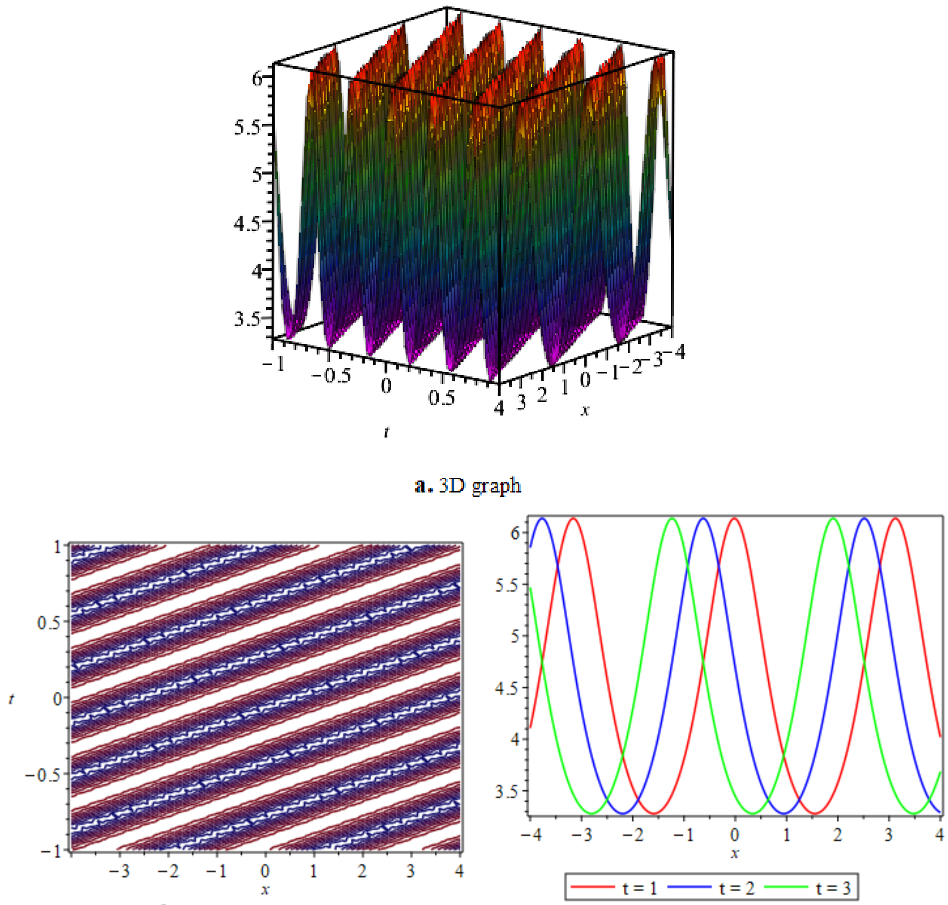


Fig.6. Periodic wave soliton represented by $Q_{44}(x, y, z, t)$ for $d_0 = 1, a = -2, b = 3, c = 4, y = 5, z = 5, h_0 = 4.$



a. 3D graph

b. contour graph

c. 2D graph

Fig.7. Periodic wave soliton represented by $Q_{46}(x, y, z, t)$ for $d_0 = 1, a = 1, b = 3, c = 2, d = 1, y = 1, z = 1, h_0 = 2$.

6. Conclusion

In this research paper, the solitary wave behavior of $(3+1)$ dimensional YTSF equation has been determined. YTSF equation has vast application in fluid mechanics, hydro dynamics and plasma physics where interaction between two layers of the fluid is evaluated by using appropriate technique. At first, traveling wave solutions of $(3+1)$ dimensional YTSF are obtained using extended Fan sub-equation method. This method satisfies general elliptic equation and auxiliary ordinary equation for its exact solutions. These solutions get bell type, periodic and hyperbolic type depictions. Special kink type solutions are also obtained by assigning suitable numerical values to existed parameters. Furthermore, This method can be used to find exact wave solutions of complex problems formed in mathematical physics, biology and chemistry and vice versa.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Faisal Hayat Shaheen: participated in the conceptualization, data curation, investigation, methodology, software implementation, validation, visualization and writing of the manuscript. Masoomah Sadaf: participated in the formal analysis, investigation, supervision, review and editing of the manuscript. Both authors read and approved the the final manuscript.

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Data Availability: All the data used during this study is accessible within the manuscript.

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