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Weighted Ostrowski–Type Inequalities for Generalized Convex Functions With Applications in Special Means

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Abstract. A new class of convex functions in mixed form is introduced, unifying and generalizing several existing classes of convex functions. By means of the weighted Montgomery identity, a well-known Ostrowski–type inequality is established for functions belonging to this class. Furthermore, various weighted Ostrowski–type inequalities are derived for functions whose derivatives, in absolute value and raised to certain powers, satisfy the proposed mixed convexity condition. The proofs are based on classical analytical tools, including Hölder's inequality (HI) and the power mean inequality (PMI). It is also shown that many previously known results can be obtained as special cases of the main theorems. In addition, several applications involving special means are presented.

AMS (MOS) Subject Classification Codes: 26A33; 26A51; 26D15; 26D99; 47A30.
Key Words: Ostrowski inequality; Montgomery identity; Convex functions; Special means.

1. Introduction

Inequalities are fundamental tools across nearly all branches of science. Although the subject is broad and extensive, our primary focus here is on Ostrowski–type inequalities. Alexander in 1938, Ostrowski introduced an important integral inequality for differentiable functions with bounded derivatives [14]. This result is widely recognized in the literature as the Ostrowski inequality.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable and $|f'(t)| \le M \ \forall t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le M(b-a)\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right], \tag{1. 1}$$

$\forall x \in (a, b)$.

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Weighted Ostrowski–type inequalities have garnered significant interest in recent years due to the advent of weight functions, which provide a more flexible representation of integral means and produce estimates that are sharper and more relevant. When the underlying issues involve variable coefficients, non-uniform distributions, or generalized integral operators, weighted representations are especially helpful. Thus, creating new weighted Ostrowski–type inequalities under more general convexity assumptions continues to be a significant and active field of study.

Using recently developed classes of convex functions, Hassan and Khan started a series of studies on generalized fractional Ostrowski–type inequalities. Under various convexity assumptions, their early works produced fractional and generalized fractional Ostrowski inequalities and showed that these methods produce more precise bounds than those obtained using traditional convexity [8]. The groundwork for future advancements in this field was established by these investigations. The authors then developed new variations of Ostrowski–type inequalities and expanded the theory by adding generalized fractional operators. These findings gave new estimates for larger classes of functions and greatly expanded the applicability of the classical inequality.

Additional generalized convexity frameworks were introduced, and corresponding fractional Ostrowski–type inequalities were derived, leading to further advancements. Specifically, under newly discovered convexity settings, a number of generalized fractional inequalities were derived, which enhanced the current theory and increased its applications. Fuzzy-valued functions and fuzzy Riemann integrals were used to develop fuzzy versions of Ostrowski–type inequalities around the same time period, extending the theory to fuzzy settings. The significance of integrating generalized convexity with uncertainty modeling was emphasized by these works. New Ostrowski–type inequalities and their fractional counterparts resulted from the introduction of increasingly complex convexity structures. By adding more functional parameters to the convexity framework, these research generated more flexible and general inequalities that extended a number of previously established findings [9].

In addition to generalized and fractional expansions that broadened the theory's scope, new inequalities were created under abstract convexity settings [10]. These studies show how Ostrowski–type inequalities continuously evolve toward more general integral operators and larger classes of functions. From generalized convexity-based Ostrowski inequalities to fractional, fuzzy, and mean-type convexity frameworks, the literature now in publication shows a methodical trend. Notwithstanding these noteworthy developments, there is still a great deal of room for creating new Ostrowski–type inequalities using recently published concepts of convexity, hybrid fractional operators, and unified methods that can cover a large number of current results as special cases.

Numerous integral inequalities for diverse classes of convex and non-convex functions have been established by numerous scholars. Integral inequalities for non-convex functions were established by Noor *et al.* [13], offering a more comprehensive framework than traditional convexity. Hussain *et al.* [11] expanded the use of fractional integral techniques by deriving Ostrowski-type fractional integral inequalities for r-times differentiable h-convex functions. Additionally, Hussain and Rafeeq [12] presented a number of related results with practical applications and constructed Hermite–Hadamard-type inequalities for Mittag–Leffler type convex functions. Additional research into novel classes of generalized convex functions and the integral inequalities that go along with them has been spurred by these contributions. The literature reveals a strong interplay between integral inequalities and convexity theory, motivating further under new assumptions research in extending these inequalities and broader mathematical settings. Here, we present the weighted version of Montgomery identity.

Theorem 1.2. [15] *Let* $f \in AC[a,b]$, *suppose that* $\omega : [a,b] \to [0,\infty)$, $\int_a^b \omega(t)dt = 1$, *and*

$$W(t) = \begin{cases} 0, & \text{if } t < a, \\ \displaystyle\int_a^t \omega(t)dt, & \text{if } t \in [a,b], \\ 0, & \text{if } t > b, \end{cases}$$

then

$$f(x) - \int_a^b \omega(t)f(t)dt = \int_a^b P_\omega(x,t)f'(t)dt$$

where P_ω *is the weighted Peano Kernel defined by:*

$$P_\omega(x,t) = \begin{cases} W(t), & \text{if } t \in [a,x], \\ W(t)-1, & \text{if } t \in (x,b]. \end{cases}$$

The primary objective of this work is to extend the Ostrowski inequality (1.1) to a broader class of functions, namely $\varphi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convex functions in the mixed sense, as introduced in Section 2. In addition, we derive several weighted Ostrowski–type inequalities for functions whose derivatives, in absolute value raised to certain powers, belong to this generalized convex class. Our approach relies on various analytical tools, including HI [18] and PMI [17]. Furthermore, we discuss particular instances of the obtained results and illustrate applications through midpoint-type inequalities associated with special means, which are presented in Section 3. The overall framework is described at the Figure 1. The final section concludes the paper with remarks and potential directions for future research.

2. Generalization of Weighted Ostrowski–Type inequalities via New convex function

From the existing literature, we recall and present several definitions related to different classes of convex functions, convexity is a fundamental and straightforward concept, yet it holds significant importance due to its wide-ranging applications in business and industry. Its influence extends into many aspects of everyday life. In fact, the concept of convexity plays a crucial role in solving numerous real-world problems.

Definition 2.1. [2] *Let* $\varphi : J \subset \mathbb{R} \to \mathbb{R}$*, it is called convex if*

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.2. [2] *A* $\varphi : J \subset \mathbb{R} \to \mathbb{R}$ *is* $P-$*convex if* $\varphi(x) \ge 0$ *and*

$$\varphi(tx + (1-t)y) \le \varphi(x) + \varphi(y),$$

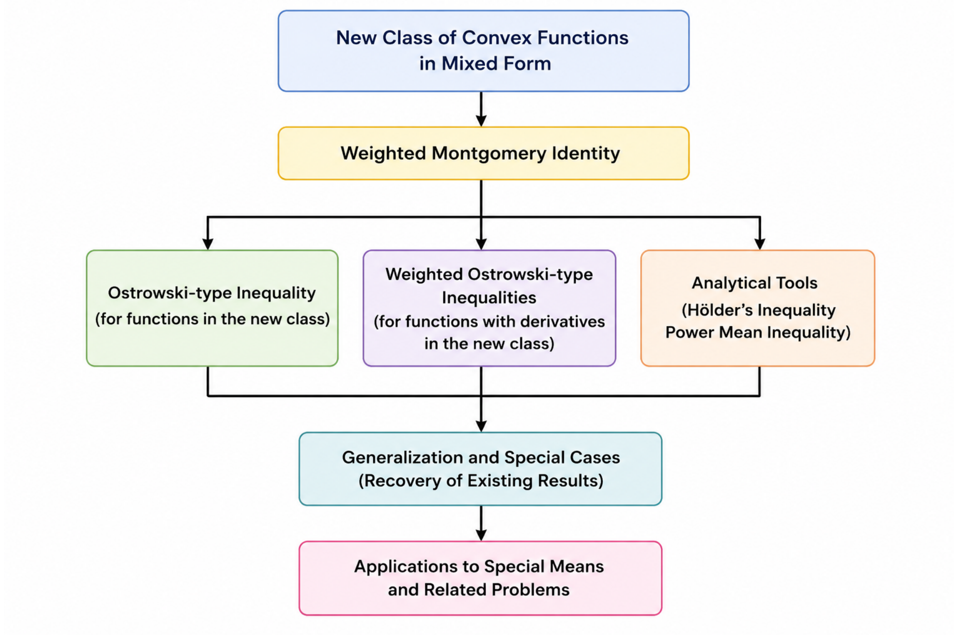

FIGURE 1. Overall Framework

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.3. [7] *A* $\varphi : J \subset \mathbb{R} \to \mathbb{R}$ *is called quasi-convex if*

$$\varphi\big(tx + (1-t)y\big) \leq \max\{\varphi(x), \varphi(y)\},$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.4. [3] *Let* $s \in (0,1]$. *A* $\varphi : J \subset [0,\infty) \to [0,\infty)$ *is* $s-$*convex in the first kind if*

$$\varphi\left(tx + (1-t)y\right) \leq t^s \varphi(x) + \left(1 - t^s\right)\varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Remark 2.5. *As* $s \to 0$, *this reduces to quasi-convexity.*

Definition 2.6. [3] *Let* $s \in (0,1]$. *A* $\varphi : J \subset [0,\infty) \to [0,\infty)$ *is called* $s-$*convex in the second kind if*

$$\varphi\left(tx + (1-t)y\right) \leq t^s \varphi(x) + (1-t)^s \varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Remark 2.7. *When* $s \to 0$, *this reduces to* $P-$*convexity.*

Definition 2.8. [9] *Let* $(s,r) \in (0,1]^2$. *A* $\varphi : J \subset [0,\infty) \to [0,\infty)$ *is* $(s,r)-$*convex of mixed form if*

$$\varphi\left(tx + (1-t)y\right) \leq t^{rs} \varphi(x) + \left(1 - t^r\right)^s \varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.9. [6] *Let* $(\alpha, \beta) \in (0,1]^2$. *A* $\varphi : J \subset [0,\infty) \to [0,\infty)$ *is called* $(\alpha,\beta)-$*convex of the first kind if*

$$\varphi(tx + (1-t)y) \leq t^{\alpha}\varphi(x) + \left(1 - t^{\beta}\right)\varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.10. [6] *Let* $(\alpha, \beta) \in (0,1]^2$. *A* $\varphi : J \subset [0,\infty) \to [0,\infty)$ *is* $(\alpha,\beta)-$*convex of the second kind if*

$$\varphi(tx + (1-t)y) \leq t^{\alpha}\varphi(x) + (1-t)^{\beta}\varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.11. [7] *A* $\varphi : J \subset \mathbb{R} \to \mathbb{R}$ *is Godunova–Levin convex if* $\varphi(x) \geq 0$ *and*

$$\varphi(tx + (1-t)y) \leq \frac{1}{t}\varphi(x) + \frac{1}{1-t}\varphi(y),$$

$\forall x, y \in J$ *and* $t \in (0,1)$.

Definition 2.12. [5] *A* $\varphi : J \subset \mathbb{R} \to [0,\infty)$ *is called Godunova–Levin* $s-$*convex, with* $s \in [0,1]$, *if*

$$\varphi(tx + (1-t)y) \leq \frac{1}{t^s}\varphi(x) + \frac{1}{(1-t)^s}\varphi(y),$$

$\forall x, y \in J$ *and* $t \in (0,1)$.

Definition 2.13. [16] *Let* $h : K \subseteq \mathbb{R} \to [0,\infty)$ *with* $h \neq 0$. *A* $\varphi : J \subseteq \mathbb{R} \to [0,\infty)$ *is* $h-$*convex if*

$$\varphi(tx + (1-t)y) \leq h(t)\varphi(x) + h(1-t)\varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.14. [2] *A* $\varphi : J \subset \mathbb{R} \to \mathbb{R}$ *is called* $MT-$*convex if* $\varphi(x) \geq 0$ *and*

$$\varphi(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\varphi(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\varphi(y),$$

$\forall x, y \in J$ *and* $t \in [0,1]$.

Definition 2.15. [10] *A* $\varphi : J \subset [0,\infty) \to [0,\infty)$ *is* $(\alpha,\beta,\gamma,\delta)$*- convex of mixed form if*

$$\varphi(tx + (1-t)y) \leq t^{\alpha\gamma}\varphi(x) + \left(1 - t^{\beta}\right)^{\delta}\varphi(y),$$

$\forall x, y \in J$, $t \in [0,1]$ *and* $(\alpha,\beta,\gamma,\delta) \in (0,1]^4$.

Definition 2.16. [10] *Let* $\phi : (0,1) \to (0,\infty)$. *A* $\varphi : J \subset \mathbb{R} \to [0,\infty)$ *is called* $\phi-$*convex if*

$$\varphi(tx + (1-t)y) \leq t\phi(t)\varphi(x) + (1-t)\phi(1-t)\varphi(y),$$

$\forall x, y \in J$ *and* $t \in (0,1)$.

Here, we are introducing, for the very first time, the class of $\phi - (\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$ convex functions of a mixed kind.

Definition 2.17. *Let $(\alpha, \beta, \gamma, \delta, \sigma_1, \sigma_2) \in (0,1]^6$ and let $\phi : (0,1) \to (0,\infty)$. The $\varphi : J \subset [0,\infty) \to [0,\infty)$ is $\phi - (\alpha, \beta, \gamma, \delta, \sigma_1, \sigma_2)-$convex of mixed form if*

$$\varphi\left(tx + (1-t)y\right) \leq \frac{\sigma_1 \phi(t)}{t^{-\alpha\gamma}} \varphi\left[\frac{x}{\sigma_1}\right] + \frac{\sigma_2 \phi(1-t)}{\left(1-t^{\beta}\right)^{-\delta}} \varphi\left[\frac{y}{\sigma_2}\right], \tag{2. 2}$$

$\forall x, y \in J$ and $t \in [0,1]$. As shown in Table 1, several well-known classes of convexity are obtained as special cases.

TABLE 1. Special cases of the generalized convexity

No.	**Parameter Choices**	**Resulting Convexity Class**
1	$\phi(t) = 1$	$(\alpha, \beta, \gamma, \delta)-$convexity (mixed sense)
2	$\phi(t) = 1, \gamma = \delta = 1$	$(\alpha, \beta)-$convexity (first kind)
3	$\phi(t) = 1, \beta = \gamma = 1$	$(\alpha, \beta)-$convexity (second kind)
4	$\phi(t) = 1, \alpha = \delta = s, \beta = \gamma = r$	$(s, r)-$convexity (mixed sense)
5	$\alpha = \beta = s, \phi(t) = \gamma = \delta = 1$	$s-$convexity (first sense)
6	$\alpha = \beta = 0, \phi(t) = \gamma = \delta = 1$	Quasi-convexity
7	$\alpha = \delta = s, \phi(t) = \beta = \gamma = 1$	$s-$convexity (second sense)
8	$\alpha = \delta = 0, \phi(t) = \beta = \gamma = 1$	$P-$convexity
9	$\phi(t) = 1, \alpha = \beta = \gamma = \delta = 1$	Classical convexity
10	$\alpha = \beta = \gamma = \delta = 1$	$\phi-$convexity
11	$\alpha = \beta = \gamma = \delta = 1, h(t) = t\phi(t)$	$h-$convexity
12	$\alpha = \beta = \gamma = \delta = 1, \phi(t) = \frac{1}{t^{s+1}}$	Godunova–Levin $s-$convexity
13	$\alpha = \beta = \gamma = \delta = 1, \phi(t) = \frac{1}{t^2}$	Godunova–Levin convexity
14	$\alpha = \beta = \gamma = \delta = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$	$MT-$convexity

Example 2.18 (Constant Functions)**.** *Let $J \subset [0,\infty)$ be an interval. Assume that $\varphi(x) = c$ for some $c > 0$. If and only if the weight function $\phi : (0,1) \to (0,\infty)$ meets the structural requirement, the function φ is $\phi - (\alpha, \beta, \gamma, \delta)-$ convex of mixed form.*

$$\phi(t) + \left(1-t^{\beta}\right)^{\delta} \phi(1-t) \geq 1, \quad \forall t \in (0,1).$$

Specifically, if $\phi(t) \geq \frac{1}{2}$ for all $t \in (0,1)$ and $\alpha\gamma = \beta = \delta = 1$, condition holds trivially since $t + (1-t) = 1$.

Example 2.19 (Power Functions)**.** *Let $\varphi(x) = x^p$ on $J = [0,\infty)$ with $p \geq 1$. By classical convexity,*

$$\varphi\left(tx + (1-t)y\right) \leq t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in J$ and $t \in (0,1)$. For any $(\alpha, \beta, \gamma, \delta) \in (0,1]^4$, we have $t \leq t^{\alpha\gamma}$ and $1-t \leq 1-t^{\beta} \leq (1-t^{\beta})^{\delta}$. Since $\phi : (0,1) \to [1,\infty)$ satisfies $\phi \geq 1$,

$$\begin{aligned}\varphi\left(tx + (1-t)y\right) &\leq t^{\alpha\gamma}\varphi(x) + \left(1-t^{\beta}\right)^{\delta}\varphi(y)\\ &\leq t^{\alpha\gamma}\phi(t)\varphi(x) + \left(1-t^{\beta}\right)^{\delta}\phi(1-t)\varphi(y).\end{aligned}$$

Thus, $\varphi(x) = x^p$ is a $\phi - (\alpha, \beta, \gamma, \delta)$-convex function of mixed form on J.

Proposition 2.20 (Non-negative Linear Combinations)**.** *Let $\varphi_1, \varphi_2 : J \to [0, \infty]$ be two mixed-form $\phi-(\alpha,\beta,\gamma,\delta)-$ convex functions. The function $\psi = c_1\varphi_1 + c_2\varphi_2$ is likewise a $\phi-(\alpha,\beta,\gamma,\delta)-$ convex function of mixed form for any non-negative constants $c_1, c_2 \geq 0$.*

Proposition 2.21 (Pointwise Supremum)**.** *Let $\{\varphi_i\}_{i\in I}$ be a family of mixed-form $\phi-(\alpha,\beta,\gamma,\delta)-$ convex functions defined on J. Define $\varphi(x) = \sup_{i\in I}\varphi_i(x)$. φ is a $\phi-(\alpha,\beta,\gamma,\delta)-$ convex function of mixed form if $\varphi(x) < \infty$ for every $x \in J$.*

Theorem 2.22. *Let $f \in AC[a,b]$, with $f' \in L_1[a,b]$, where $0 \leq a < b$, and $\omega \in C[a,b]$, such that $\int_a^b \omega(t)dt = 1$, $\sup\limits_{a\leq t\leq b} |f'(t)| = M$, and $|f'|$ is $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convex, then*

$$|\mathcal{W}(x)| \leq M\kappa_a^b(x) \max_{i\in\{1,2\}} \int_0^1 p_i(t)\left(\sigma_1 t^{\alpha\gamma}\phi(t) + \sigma_2(1-t^\beta)^\delta\phi(1-t)\right)dt, \tag{2. 3}$$

where $\kappa_a^b(x) = (x-a)^2 + (b-x)^2$, $p_1(t) = \int_0^t \omega(\varrho x + (1-\varrho)a)\,d\varrho$, and $p_2(t) = \int_0^t \omega(\varrho x + (1-\varrho)b)\,d\varrho$.

Proof. By applying Theorem 1.2 along with an appropriate change of variables, we obtain

$$\begin{aligned} |\mathcal{W}(x)| \quad \leq \quad & (x-a)^2 \int_0^1 p_1(t)\,|f'((1-t)a+tx)|\,dt \\ & +(b-x)^2 \int_0^1 p_2(t)\,|f'((1-t)b+tx)|\,dt. \end{aligned} \tag{2. 4}$$

Since $|f'|$ is $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convex and bounded by M, we estimate the first integral as

$$\begin{aligned} & \int_0^1 p_1(t)\,|f'(tx+(1-t)a)|\,dt \\ & \leq M\int_0^1 p_1(t)\left(\sigma_1 t^{\alpha\gamma}\phi(t) + \sigma_2(1-t^\beta)^\delta\phi(1-t)\right)dt, \end{aligned} \tag{2. 5}$$

and similarly, the second integral can be bounded by

$$\begin{aligned} & \int_0^1 p_2(t)\,|f'(tx+(1-t)b)|\,dt \\ & \leq M\int_0^1 p_2(t)\left(\sigma_1 t^{\alpha\gamma}\phi(t) + \sigma_2(1-t^\beta)^\delta\phi(1-t)\right)dt. \end{aligned} \tag{2. 6}$$

Finally, substituting inequalities (2.5) and (2.6) into (2.4), we obtain the desired result (2.3). □

Corollary 2.23. *If $\omega(t) = \frac{1}{b-a}, \sigma_1 = \sigma_2 = 1$ in Theorem 2.22, then it can be seen as follows.*

(1) *If $\alpha = \beta = \gamma = \delta = 1$ in (2.3), then*

$$|\mathcal{E}(x)| \leq M\left(\int_0^1 \left(t(1-t)\phi(1-t) + t^2\phi(t)\right)dt\right)\frac{\kappa_a^b(x)}{b-a}.$$

(2) *If* $\alpha=\beta=\gamma=\delta=1, l(t)=t$, *then if* $h=l\phi$, *in* (2.3), *then*

$$|\mathcal{E}(x)| \leq M\left(\int_0^1 (th(1-t)+th(t))\,dt\right)\,\frac{\kappa_a^b(x)}{b-a}.$$

(3) *If* $\alpha=\beta=\gamma=\delta=1, \phi(t)=t^{-(s+1)}$ *in* (2.3), *then*

$$|\mathcal{E}(x)| \leq M\left(\frac{1}{1-s}\right)\,\frac{\kappa_a^b(x)}{b-a}.$$

(4) *If* $\alpha=\beta=\gamma=\delta=1, \phi(t)=\frac{1}{2\sqrt{t(1-t)}}$ *in* (2.3), *then*

$$|\mathcal{E}(x)| \leq \frac{M\pi}{4}\,\frac{\kappa_a^b(x)}{b-a}.$$

(5) *If* $\phi(t)=\beta=\gamma=1$, $\alpha=\delta=s$ *where* $s\in[0,1]$, *then inequality* (2.1) *of Theorem* 2 *in* [1].

(6) *If* $\phi(t)=\alpha=\beta=\gamma=\delta=1$, *then* (2.3) *reduces inequality* (1.1).

(7) *If* $\phi(t)=1$, *in* (2.3), *then we have inequality* (2.1) *of Theorem* 2.1 *in* [8].

(8) *If* $\phi(t)=\gamma=\delta=1$, $\alpha\in[0,1]$ *and* $\beta\in(0,1]$, *in* (2.3), *then first inequality of Corollary* 2.2 *in* [8].

(9) *If* $\phi(t)=\beta=\gamma=1$, $\alpha\in[0,1]$ *and* $\delta\in[0,1]$, *in* (2.3), *then second inequality of Corollary* 2.2 *in* [8].

(10) *If* $\phi(t)=1, \alpha=\delta=s$, $\beta=\gamma=r$, *where* $s\in[0,1]$ *and* $r\in(0,1]$ *in* (2.3), *then third inequality of Corollary* 2.2 *in* [8].

(11) *If* $\alpha=\beta=s$ *and* $\phi(t)=\gamma=\delta=1$, *where* $s\in(0,1]$ *in* (2.3), *then fourth inequality of Corollary* 2.2 *in* [8].

(12) *If* $\alpha=\delta\to 0$ *and* $\phi(t)=\beta=\gamma=1$ *in* (2.3), *then fifth inequality of Corollary* 2.2 *in* [8].

Theorem 2.24. *Let* $f\in AC[a,b]$, *with* $f'\in L_1[a,b]$, *where* $0\leq a<b$, *and* $\omega\in C[a,b]$, $\int_a^b \omega(t)dt=1$, $\sup\limits_{a\leq t\leq b}|f'(t)|=M$, *and* $|f'|^q$ *is* $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$*convex for* $q>1$, *then*

$$|\mathcal{W}(x)| \leq M\kappa_a^b(x)\max_{i\in\{1,2\}}\frac{\left(\int_0^1 p_i(t)\left(\sigma_1 t^{\alpha\gamma}\phi(t)+\sigma_2(1-t^\beta)^\delta\phi(1-t)\right)dt\right)^{1/q}}{\left(\int_0^1 p_i(t)\,dt\right)^{1/q-1}}, \quad (2.\ 7)$$

where $\kappa_a^b(x)=(x-a)^2+(b-x)^2$, $p_1(t)=\int_0^t \omega(\varrho x+(1-\varrho)a)\,d\varrho$, *and* $p_2(t)=\int_0^t \omega(\varrho x+(1-\varrho)b)\,d\varrho$.

Proof. By Theorem 1.2, change of variables, and PMI, we have

$$\begin{aligned}|\mathcal{W}(x)| \leq{}& (x-a)^2\left(\int_0^1 p_1(t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 p_1(t)\left|f'((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}\\ &+(b-x)^2\left(\int_0^1 p_2(t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 p_2(t)\left|f'((1-t)b+tx)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convex and $|f'(x)|\leq M$, we get (2.7). $\square$

Corollary 2.25. *If $\omega(t) = \frac{1}{b-a}, \sigma_1 = \sigma_2 = 1$ in Theorem 2.24, then it can be seen as follows.*

(1) *If $\alpha = \beta = \gamma = \delta = 1$, in (2.7), then*

$$|\mathcal{E}(x)| \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\int_0^1 \left(t(1-t)\phi(1-t) + t^2\phi(t) \right) dt \right)^{\frac{1}{q}} \frac{\kappa_a^b(x)}{b-a}.$$

(2) *If $\alpha = \beta = \gamma = \delta = 1, l(t) = t$, then if $h = l\phi$, in (2.7), then*

$$|\mathcal{E}(x)| \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\int_0^1 (th(1-t) + th(t))\, dt \right)^{\frac{1}{q}} \frac{\kappa_a^b(x)}{b-a}.$$

(3) *If $\alpha = \beta = \gamma = \delta = 1, \phi(t) = t^{-(s+1)}$ in (2.7), then*

$$|\mathcal{E}(x)| \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\frac{1}{1-s} \right)^{\frac{1}{q}} \frac{\kappa_a^b(x)}{b-a}.$$

(4) *If $\alpha = \beta = \gamma = \delta = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (2.7), then*

$$|\mathcal{E}(x)| \leq \frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} \frac{\kappa_a^b(x)}{b-a}.$$

(5) *If $\phi(t) = \beta = \gamma = 1$, $\alpha = \delta = s$ where $s \in [0,1]$, then inequality (2.3) of Theorem 4 in [1].*

(6) *If $\phi(t) = \alpha = \beta = \gamma = \delta = q = 1$, then (2.7) reduces inequality (1.1).*

(7) *If $\phi(t) = 1$, in (2.7), then inequality of Theorem 2.3 in [8].*

(8) *If $\phi(t) = \gamma = \delta = 1$, $\alpha \in [0,1]$ and $\beta \in (0,1]$, in (2.7), then first inequality of Corollary 2.4 in [8].*

(9) *If $\phi(t) = \beta = \gamma = 1$, $\alpha \in [0,1]$ and $\delta \in [0,1]$, in (2.7), then second inequality of Corollary 2.4 in [8].*

(10) *If $\phi(t) = 1, \alpha = \delta = s$, $\beta = \gamma = r$, where $s \in [0,1]$ and $r \in (0,1]$ in (2.7), then third inequality of Corollary 2.4 in [8].*

(11) *If $\alpha = \beta = s$ and $\phi(t) = \gamma = \delta = 1$, where $s \in (0,1]$ in (2.7), then fourth inequality of Corollary 2.4 in [8].*

(12) *If $\alpha = \delta \to 0$ and $\phi(t) = \beta = \gamma = 1$ in (2.7), then fifth inequality of Corollary 2.4 in [8].*

(13) *Theorem 2.22, for $q = 1$.*

Theorem 2.26. *Let $f \in AC[a,b]$ with $f' \in L_1[a,b]$ ($0 \leq a < b$), and $\omega \in C[a,b]$ such that $\int_a^b \omega(t)\, dt = 1$. Let $M = \sup_{t\in[a,b]} |f'(t)|$, $p^{-1} + q^{-1} = 1$ ($q > 1$), and suppose $|f'|^q$ is $\phi - (\alpha, \beta, \gamma, \delta, \sigma_1, \sigma_2)$-convex. Then*

$$|\mathcal{W}(x)| \leq M\, \kappa_a^b(x)\, \max\{\|p_1\|_p, \|p_2\|_p\}\, (I_\phi)^{1/q}, \tag{2. 8}$$

where $\kappa_a^b(x) = (x-a)^2 + (b-x)^2$, $I_\phi = \int_0^1 \left(\sigma_1 t^{\alpha\gamma}\phi(t) + \sigma_2(1-t^\beta)^\delta\phi(1-t)\right) dt$, $p_1(t) = \int_0^t \omega\big(\varrho x + (1-\varrho)a\big) d\varrho$, and $p_2(t) = \int_0^t \omega\big(\varrho x + (1-\varrho)b\big) d\varrho$.

Proof. By Theorem 1.2, changing variables, and (HI), we have

$$|\mathcal{W}(x)| \leq (x-a)^2 \left(\int_0^1 [p_1(t)]^p \, dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx+(1-t)a)|^q \, dt\right)^{\frac{1}{q}} + (b-x)^2 \left(\int_0^1 [p_2(t)]^p \, dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx+(1-t)b)|^q \, dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convex and $|f'(x)| \leq M$, we get (2.8). □

Corollary 2.27. *If $\omega(t) = \frac{1}{b-a}, \sigma_1 = \sigma_2 = 1$ in Theorem 2.26, then it can be seen as follows.*

(1) *If $\alpha=\beta=\gamma=\delta=1$, in (2.8), then*

$$|\mathcal{E}(x)| \leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\int_0^1 ((1-t)\phi(1-t)+t\phi(t)) \, dt\right)^{\frac{1}{q}} \frac{\kappa_a^b(x)}{b-a}.$$

(2) *If $\alpha=\beta=\gamma=\delta=1, l(t)=t$, then if $h=l\phi$, in (2.8), then*

$$|\mathcal{E}(x)| \leq \frac{M}{(1+p)^{\frac{1}{p}}} \left(\int_0^1 (h(1-t)+h(t)) \, dt\right)^{\frac{1}{q}} \frac{\kappa_a^b(x)}{b-a}.$$

(3) *If $\alpha=\beta=\gamma=\delta=1, \phi(t)=t^{-(s+1)}$ where $s \in [0,1)$ in (2.8), then*

$$|\mathcal{E}(x)| \leq \frac{M}{(1+p)^{\frac{1}{p}}} \left(\frac{2}{1-s}\right)^{\frac{1}{q}} \frac{\kappa_a^b(x)}{b-a}.$$

(4) *If $\alpha=\beta=\gamma=\delta=1, \phi(t)=\frac{1}{2\sqrt{t(1-t)}}$ in (2.8), then*

$$|\mathcal{E}(x)| \leq \frac{M\left(\frac{\pi}{2}\right)^{\frac{1}{q}}}{(1+p)^{\frac{1}{p}}} \frac{\kappa_a^b(x)}{b-a}.$$

(5) *If $\phi(t)=\beta=\gamma=1$, $\alpha=\delta=s$ where $s \in [0,1]$, then inequality (2.2) of Theorem 3 in [1].*

(6) *If $\phi(t)=1$, in (2.8), then inequality (2.8) of Theorem 2.5 in [8].*

(7) *If $\phi(t)=\gamma=\delta=1$, $\alpha \in [0,1]$ and $\beta \in (0,1]$, in (2.8), then first inequality of Corollary 2.8 in [8].*

(8) *If $\phi(t)=\beta=\gamma=1$, $\alpha \in [0,1]$ and $\delta \in [0,1]$, in (2.8), then second inequality of Corollary 2.8 in [8].*

(9) *If $\phi(t)=1, \alpha=\delta=s, \beta=\gamma=r$, where $s \in [0,1]$ and $r \in (0,1]$ in (2.8), then third inequality of Corollary 2.8 in [8].*

(10) *If $\alpha=\beta=s$ and $\phi(t)=\gamma=\delta=1$, where $s \in (0,1]$ in (2.8), then fourth inequality of Corollary 2.8 in [8].*

(11) *If $\alpha=\delta \to 0$ and $\phi(t)=\beta=\gamma=1$ in (2.8), then fifth inequality of Corollary 2.8 in [8].*

(12) *If $\phi(t)=\alpha=\beta=\gamma=\delta=1$ in (2.8), then first inequality of Corollary 2.8 in [8].*

Theorem 2.28. *Let Theorem 1.2 be satisfied. If* $\tau\colon [a,b] \to \mathbb{R}$ *is* $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$*convex, then for all* $x \in [a,b]$*:*

$$\begin{aligned} \tau\left(\mathcal{W}(x)\right) &\leq \frac{\sigma_1\phi(\Omega_a(x))}{\Omega_a(x)^{-\alpha\gamma+1}}\int_a^x \tau\left(\frac{\Omega_a(t)f'(t)}{\sigma_1}\right)dt \\ &\quad + \frac{\sigma_2\phi(\Omega_b(x))}{\Omega_b(x)\left(1-\Omega_a(x)^{\beta}\right)^{-\delta}}\int_x^b \tau\left(\frac{\Omega_b(t)f'(t)}{\sigma_2}\right)dt, \end{aligned} \tag{2. 9}$$

where $\Omega_a(x) = \int_a^x \omega(t)\,dt$ *and* $\Omega_b(x) = \int_x^b \omega(t)\,dt$.

Proof. By Theorem 1.2, we obtain

$$\begin{aligned} \mathcal{W}(x) &= \left(\int_a^x \omega(t)dt\right)\left[\frac{1}{\int_a^x \omega(t)dt}\int_a^x W(t)f'(t)dt\right] \\ &\quad + \left(\int_x^b \omega(t)dt\right)\left[\frac{1}{\int_x^b \omega(t)dt}\int_x^b \left(W(t)-1\right)f'(t)dt\right], \end{aligned}$$

using the $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convexity of $\tau : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned} \tau\left(\mathcal{W}(x)\right) &\leq \frac{\sigma_1\phi\left(\int_a^x \omega(t)dt\right)}{\left(\int_a^x \omega(t)dt\right)^{-\alpha\gamma}}\tau\left[\frac{1}{\int_a^x \omega(t)dt}\int_a^x \frac{W(t)f'(t)}{\sigma_1}dt\right] \\ &\quad + \frac{\sigma_2\phi\left(\int_x^b \omega(t)dt\right)}{\left(1-\left(\int_a^x \omega(t)dt\right)^{\beta}\right)^{-\delta}}\tau\left[\frac{1}{\int_x^b \omega(t)dt}\int_x^b \frac{\left(W(t)-1\right)f'(t)}{\sigma_2}dt\right], \end{aligned}$$

$\forall x \in [a,b]$, which is an inequality of interest in itself as well. If we use Jensen's integral inequality with weights, we get (2.9). □

Corollary 2.29. *If* $\omega(t) = \frac{1}{b-a}, \sigma_1 = \sigma_2 = 1$ *in Theorem* 2.28, *then it can be seen as follows.*

(1) *If* $\alpha = \beta = \gamma = \delta = 1$, *in* (2.9), *then*

$$\begin{aligned} \tau\left(\mathcal{E}(x)\right) &\leq \frac{1}{b-a}\left[\phi\left(\frac{x-a}{b-a}\right)\int_a^x \tau\left[(t-a)\,f'(t)\right]dt\right. \\ &\quad \left.+\phi\left(\frac{b-x}{b-a}\right)\int_x^b \tau\left[(t-b)\,f'(t)\right]dt\right]. \end{aligned}$$

(2) *If* $\alpha = \beta = \gamma = \delta = 1, l(t) = t$, *and* $h = l\phi$ *in* (2.9), *then*

$$\begin{aligned} \tau\left(\mathcal{E}(x)\right) &\leq h\left(\frac{x-a}{b-a}\right)\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right] \\ &\quad +h\left(\frac{b-x}{b-a}\right)\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right]. \end{aligned}$$

(3) *If* $\alpha = \beta = \gamma = \delta = 1, \phi(t) = \frac{1}{t^{s+1}}$ *with* $s \in [0,1]$ *in* (2.9), *then*

$$\begin{aligned}\tau\left(\mathcal{E}(x)\right) \quad \leq \quad & (b-a)^s \left[\frac{1}{(x-a)^{s+1}} \int_a^x \tau[(t-a)f'(t)]dt \right. \\ & \left. + \frac{1}{(b-x)^{s+1}} \int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

(4) *If* $\alpha = \beta = \gamma = \delta = 1, \phi(t) = \frac{1}{t^2}$ *in* (2.9), *then*

$$\begin{aligned}\tau\left(\mathcal{E}(x)\right) \quad \leq \quad & (b-a) \left[\frac{1}{(x-a)^2} \int_a^x \tau[(t-a)f'(t)]dt \right. \\ & \left. + \frac{1}{(b-x)^2} \int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

(5) *If* $\alpha = \beta = \gamma = \delta = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ *in* (2.9), *then*

$$\tau\left(\mathcal{E}(x)\right) \leq \frac{\left[\int_a^x \tau[(t-a)f'(t)]dt + \int_x^b \tau[(t-b)f'(t)]dt\right]}{2\sqrt{(x-a)(b-x)}}.$$

(6) *If* $\phi(t) = 1$, *in* (2.9), *then inequality of Remark* 2.8 *in* [8].

(7) *If* $\phi(t) = \gamma = \delta = 1$, *and* $\alpha, \beta \in (0,1]$ *in* (2.9), *then inequality of Remark* 2.10 *in* [8].

(8) *If* $\phi(t) = \beta = \gamma = 1$, *and* $\alpha, \delta \in (0,1]$ *in* (2.9), *then inequality of Remark* 2.11 *in* [8].

(9) *If* $\phi(t) = 1, \alpha = \delta = s$, *and* $\beta = \gamma = r$, *where* $r, s \in (0,1]$ *in* (2.9), *then inequality of Remark* 2.12 *in* [8].

(10) *If* $\alpha = \beta = s$ *and* $\phi(t) = \gamma = \delta = 1$, *where* $s \in (0,1]$ *in* (2.9), *then inequality of Remark* 2.13 *in* [8].

(11) *If* $\alpha = \beta \to 0$ *and* $\phi(t) = \gamma = \delta = 1$ *in* (2.9), *then inequality of Remark* 2.14 *in* [8].

(12) *If* $\alpha = \delta = s$, *and* $\phi(t) = \beta = \gamma = 1$, *where* $s \in [0,1]$ *in* (2.9), *then inequality of Remark* 2.15 *in* [8].

(13) *If* $\alpha = \delta \to 0$ *and* $\phi(t) = \beta = \gamma = 1$ *in* (2.9), *then inequality of Remark* 2.16 *in* [8].

(14) *If* $\phi(t) = \alpha = \beta = \gamma = \delta = 1$ *in* (2.9), *then inequality* (2.1) *of Theorem* 7 *in* [4].

3. Midpoint Ostrowski–Type Inequalities with applications

In this section, we provide some compelling applications of our main theoretical results to the theory of special means for positive real numbers. Let $[a,b] \subseteq (0,+\infty)$, the special means are given below:

(1) **Classical Means:** For $a, b > 0$, the Arithmetic (A), Geometric (G), and Harmonic (H) means are given by:

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}, \quad H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}.$$

(2) **Logarithmic Mean (L):**

$$L(a,b)=\begin{cases} a & \text{if } a=b,\\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b.\end{cases}$$

(3) **Identric Mean (I):**

$$I(a,b)=\begin{cases} a & \text{if } a=b,\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b.\end{cases}$$

(4) **Generalized p-Logarithmic Mean (L_p):** For $p\in\mathbb{R}\setminus\{0,-1\}$:

$$L_p(a,b)=\begin{cases} a & \text{if } a=b,\\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\neq b.\end{cases}$$

If we replace $\omega(t)=\frac{1}{b-a}$, f by $-f$ and $x=\frac{a+b}{2}$ in Theorem 2.28, then Ostrowski midpoint inequality for $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convex functions:

$$\begin{aligned}
&\tau\left(\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\right)\\
&\leq\frac{\phi\left(\frac{1}{2}\right)}{b-a}\left[\frac{\sigma_1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{(a-t)f'(t)}{\sigma_1}\right]dt\right.\\
&\left.+\frac{\sigma_2\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau\left[\frac{(b-t)f'(t)}{\sigma_2}\right]dt\right]. \qquad (3.\ 10)
\end{aligned}$$

Remark 3.1. *Let τ is a $\phi-(\alpha,\beta,\gamma,\delta,\sigma_1,\sigma_2)-$convex on $[a,b]\subseteq\mathbb{R}$.*

(1) *If $f(t)=\frac{1}{t}$ in (3.10), then*

$$\begin{aligned}
&\tau\left[\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right]\\
&\leq\frac{\phi\left(\frac{1}{2}\right)}{b-a}\left[\frac{\sigma_1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{t-a}{\sigma_1t^2}\right]dt+\frac{\sigma_2(2^\beta-1)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau\left[\frac{t-b}{\sigma_2t^2}\right]dt\right].
\end{aligned}$$

(2) *If $f(t)=-\ln t$ in (3.10), then*

$$\begin{aligned}
&\tau\left[\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right]\\
&\leq\frac{\phi\left(\frac{1}{2}\right)}{b-a}\left[\frac{\sigma_1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{t-a}{\sigma_1t}\right]dt+\frac{\sigma_2(2^\beta-1)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau\left[\frac{t-b}{\sigma_2t}\right]dt\right].
\end{aligned}$$

(3) *If $f(t)=t^p, p\neq0,-1$ in (3.10), then*

$$\begin{aligned}
&\tau\left[L_p^p(a,b)-A^p(a,b)\right]\\
&\leq\frac{\phi\left(\frac{1}{2}\right)}{b-a}\left[\frac{\sigma_1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{p(a-t)}{\sigma_1t^{1-p}}\right]dt+\frac{\sigma_2(2^\beta-1)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau\left[\frac{p(b-t)}{\sigma_2t^{1-p}}\right]dt\right].
\end{aligned}$$

Remark 3.2. *If $\omega(t) = \frac{1}{b-a}$, in Theorem 2.24, we have the following:*

(1) *For $f(t) = t^n$ and x as the midpoint, the inequality reduces to*

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{M(b-a)}{2^{2-1/q}} \left(\int_0^1 \left(\sigma_1 t^{\alpha\gamma+1} \phi(t) + \sigma_2 t (1-t^\beta)^\delta \phi(1-t) \right) dt \right)^{1/q}.$$

(2) *For $f(t) = -\ln t$, we have*

$$\left| \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right| \leq \frac{M(b-a)}{2^{2-1/q}} \left(\int_0^1 \left(\sigma_1 t^{\alpha\gamma+1} \phi(t) + \sigma_2 t (1-t^\beta)^\delta \phi(1-t) \right) dt \right)^{1/q}.$$

Remark 3.3. *If $\omega(t) = \frac{1}{b-a}$, in Theorem 2.26, we have the following:*

(1) *For $f(t) = t^n$ and $p^{-1} + q^{-1} = 1$, the inequality becomes*

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{M(b-a)}{2(p+1)^{1/p}} \left(\int_0^1 \left(\sigma_1 t^{\alpha\gamma} \phi(t) + \sigma_2 (1-t^\beta)^\delta \phi(1-t) \right) dt \right)^{1/q}.$$

(2) *For $f(t) = -\ln t$ and $p^{-1} + q^{-1} = 1$, we have*

$$\left| \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right| \leq \frac{M(b-a)}{2(p+1)^{1/p}} \left(\int_0^1 \left(\sigma_1 t^{\alpha\gamma} \phi(t) + \sigma_2 (1-t^\beta)^\delta \phi(1-t) \right) dt \right)^{1/q}.$$

4. Conclusion

The Ostrowski inequality is one of the most widely studied inequality, with numerous generalizations and variations appearing in the literature. In this work, we introduce the concept of $\phi - (\alpha, \beta, \gamma, \delta, \sigma_1, \sigma_2)-$convex functions in the mixed sense, a class that encompasses many important sub-classes of convex functions. In Section 2, we present our first main result: a generalization of the Ostrowski inequality using the weighted Montgomery identity for this new convex function in the mixed sense. Additionally, we employ various techniques, including (HI) and the (PMI), to further extend the Ostrowski inequality [14]. Finally, we present applications of these results to various special means including the geometric, arithmetic, harmonic, identric, logarithmic, and $p-$logarithmic means via midpoint type inequalities.

Credit authorship contribution's statement

Conceptualization, methodology, validation, formal analysis, investigation, writing original draft preparation, writing, review and editing, visualization, and funding acquisition were performed by Ali Hassan. The author has read and agreed to the published version of the manuscript.

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REFERENCES

[1] M. Alomari, M. Darus, S. S. Dragomir and P. Cerone. *Ostrowski–type inequalities for functions whose derivatives are s-convex in the second sense*. Applied Mathematics Letters, **23** No.1, (2010) 1071–1076.
[2] E. F. Beckenbach. *Convex functions*, Bulletin of the American Mathematical Society, **54** No.1, (1948) 439–460.
[3] W. W. Breckner. *Stetigkeitsaussagen Fur Eine Klasse Verallgemeinerter Konvexer Funktionen in Topologischen Linearen Raumen*. Publications de l'Institut Mathématique, **23** No.3, (1978) 13–20.
[4] S. S. Dragomir. *A functional generalization of Ostrowski inequality via Montgomery identity*. Acta Mathematica Universitatis Comenianae, **84** No.1, (2015) 63–78.
[5] S. S. Dragomir. *Integral inequalities of Jensen type for varrho-convex functions*. Proceedings of RGMIA Research Report Collection, **17** No.18, (2014) 1–12.
[6] A. Ekinci. *Klasik Eşitsizlikler Yoluyla Konveks Fonksiyonlar için Integral Eşitsizlikler*. Ph.D. Thesis, Atatürk University, (2014) Thesis ID: 361162.
[7] E. K. Godunova and V. I. Levin. *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*. Numerical Mathematics and Mathematical Physics, **166** No.1, (1985) 138–142.
[8] A. Hassan and A. R. Khan. *Ostrowski–type inequalities for* $(A, B, G, D)-$*convex function*. International Journal of Nonlinear Analysis and Applications, **40** No.3, (2025) 441–487.
[9] A. Hassan and A. R. Khan. *Ostrowski–type inequalities via* ψ-$(\alpha, \beta, \gamma, \delta)-$*convex functions*. Studia Universitatis Babes Bolyai Mathematica, **69** No.2, (2024) 247–265.
[10] A. Hassan, A. R. Khan, N. Irshad and S. Khatoon. *Fractional Ostrowski–type inequalities via* $(\alpha, \beta, \gamma, \delta)-$*convex function*. Sahand Communications in Mathematical Analysis, **20** No.4, (2023) 1–20.
[11] S. Hussain, F. Azhar, M. A. Latif, and J. Khalid. *Ostrowski–type fractional integral inequalities for r-times differentiable h-convex functions*. Punjab Univ. J. Math., **53** No.7, (2021), 459–474.
[12] S. Hussain and S. Rafeeq. *Hermite-Hadamard type inequality and some related results for Mittag-Leffler type convex functions with applications*. Punjab Univ. J. Math., **55** No.4, (2023) 159–183.
[13] M. A. Noor, K. I. Noor, and S. Iftikhar. *Nonconvex functions and integral inequalities*. Punjab Univ. J. Math., **47** No.2, (2015) 19–27.
[14] A. M. Ostrowski. *Uber die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*. Commentarii Mathematici Helvetici, **10**, (1938) 226–227.
[15] J. E. Pecarić. *On the Cebysev inequality*. Scientific Bulletin of the Politehnica University of Timisoara, **25** No.1, (1980) 5–9.
[16] S. Varosanec. *On h-convexity*. Journal of Mathematical Analysis and Applications, **326** No.1, (2007) 303–311.
[17] Z. G. Xiao and A. H. Zhang. *Mixed power mean inequalities*. Research Communications in Inequalities, **8** No.1, (2002) 15–17.
[18] X. Yang. *A note on Holder inequality*. Applied Mathematics and Computation, **134** No.1, (2003) 319–322.