

On the Development and Stability Analysis of Two Step Fractional Iterative Scheme with Applications

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Abstract. Fractional iterative techniques are pivotal for modeling complex dynamic systems with enhanced accuracy. Despite their significance, scarcity of two step fractional iterative methods presents critical research gap. To fill this void, we propose a novel class of conformable two step Newton type fractional algorithm based on Caputo derivative, designed to achieve higher computational efficiency and precision. Unlike existing single step fractional Newton type methods, our approach attains a convergence order of $(2\varphi + 1)$. This claim is thoroughly proved through theoretical analysis. The algorithm is computationally efficient requiring only two function evaluations and one fractional derivative calculation per iteration. Extensive numerical experiments conducted in Maple 2022, using test functions from various fields such as quarter-car suspension modeling, hanging object, spring-mass system analogue, bookshelf deflection analysis and acidity determination in chemical systems. Absolute error per iteration and elapsed time are key metrics that indicate its superiority over existing methods. Furthermore, a comprehensive fractal analysis using Möbius transformations for rational functions with fixed and critical point analysis deepens the understanding of dynamical properties and stability of the proposed algorithm. Parametric planes generated by MATLAB R2014a confirm enhanced stability. The results highlight algorithm's broad applicability in solving nonlinear equations across scientific and engineering domains, establishing it as a valuable contribution to numerical analysis.

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1. INTRODUCTION

Numerical analysis is one of the most important areas of applied mathematics and computational sciences. It presents a variety of numerical methods that can simplify solution of complex mathematical problems. One of the key issues in this domain is finding of roots of nonlinear equations $f(x) = 0$. Nonlinear equations emerged in different disciplines such as physics, engineering, chemistry, and biological modeling. Obtaining an analytical solution is very rare, thus requiring use of iterative methods with different generations ranging from single step to multistep techniques. Among them, Newton's method [13] is most important one as it guarantees a quadratic peak in its convergence:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (1.1)$$

Although it is a very efficient method, classical Newton method and similar methods still depend on integer-order derivatives, which might not be best option for modern systems characterized by memory effects, hereditary properties, and strong nonlinearities. Hence, one is led to consider fractional calculus, where differentiation is performed to noninteger orders, thereby enhancing the modeling capacity of iterative methods.

The concept of fractional differentiation emerged around 1695 between Leibniz and L'Hôpital, and was later developed by mathematicians such as Riemann, Liouville, and Euler. Among different types of fractional derivatives, ones defined by Caputo and Riemann–Liouville have turned out to be especially suitable for modeling long-term memory and anomalous diffusion processes, with applications spanning medicine, viscoelasticity, control theory, and economics [17]. Fractional differentiation has advantage of retaining properties of classical derivatives while providing additional flexibility for analyzing non-linear systems.

For a complete survey of fractional differential and fractional integral calculus explicitly, refer to texts [15,20] and references given in there. Consider a smooth function $f : \mathcal{R} \rightarrow \mathcal{R}$ where $f \in C^{+\infty}([a, x])$ and $-\infty < a < x < +\infty$. The Caputo fractional derivative of $f(x)$ of order φ is defined piecewise as:

$${}_a^c \mathcal{D}^\varphi f(x) = \begin{cases} \frac{1}{\tilde{\Gamma}^{[*]}(n-\varphi)} \int_a^x (x-t)^{n-\varphi-1} f^{(n)}(t) dt, & \varphi \notin \mathbb{N}, \\ f^{(n-1)}(x), & \varphi = n-1 \in \mathbb{N} \cup \{0\}, \end{cases} \quad (1.2)$$

where n is smallest integer that satisfies $n > \varphi$ (i.e., $n-1 < \varphi < n$), and $\tilde{\Gamma}^{[*]}(x)$ is Gamma function defined for $x > 0$. A key property of Caputo derivative, consistent with classical integer order calculus, is that derivative of a constant is zero: ${}_a^c \mathcal{D}^\varphi C = 0$ for any constant C .

Theorem 1.1 (Taylor series expansion by using Caputo derivative [18]). *Let ${}_a^c \mathcal{D}_a^{j\varphi} f(x) \in C([a, b])$ for $j = 1, 2, \dots, n+1$, with $\varphi \in (0, 1]$. Then, for any $x \in (a, b]$, function $f(x)$*

can be expanded as:

$$f(x) = \sum_{i=0}^n c \mathfrak{D}_a^{i\varphi} f(a) \frac{(x-a)^{i\varphi}}{\widetilde{\Gamma}^{[*]}(i\varphi+1)} + c \mathfrak{D}_a^{(n+1)\varphi} f(\xi) \frac{(x-a)^{(n+1)\varphi}}{\widetilde{\Gamma}^{[*]}((n+1)\varphi+1)}, \quad (1.3)$$

where $a \leq \xi \leq x$, and $c \mathfrak{D}_a^{n\varphi}$ represents n -fold composition of Caputo derivative operator.

An alternative formulation for fractional differentiation is provided by Riemann-Liouville derivative. Consider a function $f : \mathcal{R} \rightarrow \mathcal{R}$ that is Lebesgue integrable, i.e., $f \in L^1([a, x])$, with $-\infty < a < x < +\infty$. For a given order $\varphi \geq 0$, the definition of the Riemann-Liouville fractional derivative for $f(x)$ is provided below:

$$\mathfrak{D}_{a+}^{\varphi} f(x) = \begin{cases} \frac{1}{\widetilde{\Gamma}^{[*]}(1-\varphi)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\varphi}} dt, & 0 < \varphi < 1, \\ \frac{d}{dx} f(x), & \varphi = 1. \end{cases} \quad (1.4)$$

Unlike the Caputo fractional derivative, the Riemann-Liouville fractional derivative does not satisfy $\mathfrak{D}_{a+}^{\varphi} C = 0$, for a constant C . Consequently, the corresponding fractional Taylor expansion is formulated differently (see [26]). The following proposition provides the generalized Taylor expansion based on the Riemann-Liouville fractional derivative.

Proposition 1.2 (Riemann-Liouville fractional Taylor expansion [11]). *Let $f : \mathcal{R} \rightarrow \mathcal{R}$ be a continuous function possessing Riemann-Liouville fractional derivatives of order $k\varphi$ for every positive integer k , where $0 < \varphi \leq 1$. Then, for any h such that $x+h$ belongs to the domain of f , the following expansion holds:*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{k\varphi}}{\widetilde{\Gamma}^{[*]}(k\varphi+1)} \mathfrak{D}_{a+}^{k\varphi} f(x), \quad (1.5)$$

where $\mathfrak{D}_{a+}^{k\varphi}$ denotes the Riemann-Liouville fractional derivative of order $k\varphi$.

Using fractional derivatives in iterative methods for solving nonlinear equations is a promising and innovative research direction. Torres-Hernandez et al. [25] were first to propose Fractional Newton method, which can find complex roots using only real initial guesses. The Fractional Newton type iterative method, developed by Akgül et al. [4], was first to demonstrate feasibility and potential of this technique. Research in this area has not only been broadened but also deepened significantly. The effect of fractional derivatives on convergence order has been inquired into for one-point and multipoint schemes, which in turn have caused development of higher-order fractional iterative methods. This encompasses Chebyshev-type with convergence order 3φ [9], Traub-type fractional extensions [8], and Caputo-type family of methods with adjustable convergence order [5, 23]. Theoretical development in fractional calculus includes generalized Taylor formulas [18, 26], conformable derivatives [6] and their application to uncertain differential equations [2]. It is a comprehensive toolbox for future research in this area. Current path offers a remarkable opportunity to broaden scope of traditional iterative families in fractional domain [3, 21].

A thorough analysis of literature reveals a significant gap in development of multistep fractional iterative algorithms. Complex nature of fractional derivatives hinders the development of fractional-order multistep iterative schemes. As a consequence, both construction and convergence analysis become more difficult. The techniques that work for integer

order problems cannot be directly transferred to fractional domain problems because of their nontrivial nature. The gap needs scientific methods that will create efficient multistep fractional methods that achieve better convergence rates and higher computational efficiency.

The research tackles the existing gap by providing a complete framework for fractional iterative methods. We present a new two step Newton type fractional algorithm which uses Caputo fractional derivative. The proposed algorithm successfully generalizes to fractional domain, while maintaining computational efficiency. With only two function evaluations and one fractional derivative evaluation per iteration, proposed algorithm achieves local convergence order of $(2\wp + 1)$. Theoretical analysis is validated through extensive numerical experiments on various problems, including suspension model of quarter-car, hanging mass, spring-mass system, deflection of shelf and determination of chemical acidity. Furthermore, a thorough fractal analysis using Möbius transformations and fixed/critical point studies provides insight into dynamical properties and applicability of the proposed algorithm.

The rest of document is organized in the following manner. In Section 2, novel new two step fractional scheme is derived and its convergence properties are proved. In Section 3, we present results of numerical experiments that show method to be both effective and robust in comparison to existing alternatives. Section 4 conveys a fractal analysis through parametric planes. Section 5 concludes paper with overall remarks and future research directions.

2. METHODOLOGY OF FRACTIONAL ITERATIVE ALGORITHM

This section introduces a novel two step root finding algorithm for nonlinear equations that integrates principles of fractional calculus. The method utilizes Caputo derivative to generalize classical Newton method, potentially improving its fractional order convergence properties. The proposed technique, abbreviated as $M - FSI$, has convergence order at least $2\wp + 1$ and is constructed as follows.

$$\begin{cases} y_n = x_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \right)^{1/\wp}, \\ x_{n+1} = y_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \left(1 + \Upsilon^{[\circ]} \frac{f(y_n)}{f(x_n)} \right) \right)^{1/\wp}. \end{cases} \quad (2.6)$$

For iterative scheme presented in (2.6), we prove following theorem to establish its order of convergence.

Theorem 2.1. *Let $f : D \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a continuous function possessing fractional derivatives of order $k\wp$ for any positive integer k and any $\wp \in (0, 1]$, in an interval D containing a simple root $\tilde{x}$ of $f(x)$. Suppose $c\mathfrak{D}_{\tilde{x}}^{\wp} f(x)$ is continuous and non-zero at $\tilde{x}$. If an initial approximation x_0 is sufficiently close to $\tilde{x}$, then iterative method defined by (2.6) with $\Upsilon^{[\circ]} = 1$ has a local convergence order of at least $2\wp + 1$ for $0 < \wp \leq 1$. The*

asymptotic error constant is given by coefficient of $e_n^{2\wp+1}$ in error equation:

$$e_{n+1} = \left(\frac{B}{\wp A^{1-1/\wp} C_2^{\wp-1}} + \frac{\tilde{\Gamma}^{[*]}(\frac{1}{\wp} + 1)}{\tilde{\Gamma}^{[*]}(\frac{1}{\wp})} A \left(\frac{A \tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(\wp + 1))^2} - B \right) \right) e_n^{2\wp+1} \quad (2.7)$$

$$+ O(e_n^{\wp^2+2\wp+1}), \quad n = 0, 1, 2, \dots$$

where

$$C_j = \frac{\tilde{\Gamma}^{[*]}(1 + \wp)}{\tilde{\Gamma}^{[*]}(j\wp + 1)} \frac{c\mathcal{D}_{\tilde{x}}^{j\wp} f(\tilde{x})}{c\mathcal{D}_{\tilde{x}}^{\wp} f(\tilde{x})},$$

for $j \geq 2$.

$$A = \left(\frac{\tilde{\Gamma}^{[*]}(2\wp + 1) - (\tilde{\Gamma}^{[*]}(\wp + 1))^2}{\wp(\tilde{\Gamma}^{[*]}(\wp + 1))^2} \right)^{\wp} C_2^{\wp},$$

and

$$B = \wp \left(\frac{\tilde{\Gamma}^{[*]}(2\wp + 1) - (\tilde{\Gamma}^{[*]}(\wp + 1))^2}{\wp(\tilde{\Gamma}^{[*]}(\wp + 1))^2} \right)^{\wp-1} C_2^{\wp-1}$$

$$\times \left(\frac{1}{\wp} \left(\frac{\tilde{\Gamma}^{[*]}(3\wp + 1) - \tilde{\Gamma}^{[*]}(2\wp + 1)\tilde{\Gamma}^{[*]}(\wp + 1)}{\tilde{\Gamma}^{[*]}(2\wp + 1)} C_3 \right. \right.$$

$$\left. \left. + \tilde{\Gamma}^{[*]}(2\wp + 1) \frac{(\tilde{\Gamma}^{[*]}(\wp + 1))^2 - \tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(\wp + 1))^3} C_2^2 \right) \right)$$

$$+ \frac{1}{2\wp} \left(1 - \frac{1}{\wp} \right) \left(\frac{(\tilde{\Gamma}^{[*]}(\wp + 1))^2 - \tilde{\Gamma}^{[*]}(2\wp + 1)^2}{(\tilde{\Gamma}^{[*]}(\wp + 1))^4} C_2^2 \right).$$

The proof proceeds by analyzing the Taylor series expansions of $f(x)$ and its Caputo derivative around the root $\tilde{x}$, using the fractional Taylor formula. Let $e_n = x_n - \tilde{x}$ be the error at the n -th iteration.

Proof. The function $f(x_n)$ and its Caputo derivative $c\mathcal{D}_{\tilde{x}}^{\wp} f(x_n)$ are expanded around the root $\tilde{x}$ using fractional Taylor series:

$$f(x_n) = \frac{c\mathcal{D}_{\tilde{x}}^{\wp} f(\tilde{x})}{\tilde{\Gamma}^{[*]}(1 + \wp)} [e_n^{\wp} + C_2 e_n^{2\wp} + C_3 e_n^{3\wp} + C_4 e_n^{4\wp}] + O(e_n^{5\wp}), \quad (2.8)$$

where x_n represents the error term and $\wp$ denotes the order and

$$c\mathcal{D}_{\tilde{x}}^{\wp} f(x_n) = \frac{c\mathcal{D}_{\tilde{x}}^{\wp} f(\tilde{x})}{\tilde{\Gamma}^{[*]}(1 + \wp)} \left[\tilde{\Gamma}^{[*]}(1 + \wp) + \frac{\tilde{\Gamma}^{[*]}(1 + 2\wp)}{\tilde{\Gamma}^{[*]}(1 + \wp)} C_2 e_n^{\wp} + \frac{\tilde{\Gamma}^{[*]}(1 + 3\wp)}{\tilde{\Gamma}^{[*]}(1 + 2\wp)} C_3 e_n^{2\wp} \right.$$

$$\left. + \frac{\tilde{\Gamma}^{[*]}(1 + 4\wp)}{\tilde{\Gamma}^{[*]}(1 + 3\wp)} C_4 e_n^{3\wp} \right] + O(e_n^{4\wp}). \quad (2.9)$$

where $C_j = \frac{\tilde{\Gamma}^{[*]}(1+\wp)}{\tilde{\Gamma}^{[*]}(j\wp+1)} \frac{c\mathcal{D}_{\tilde{x}}^{j\wp} f(\tilde{x})}{c\mathcal{D}_{\tilde{x}}^{\wp} f(\tilde{x})}$, for $j \geq 2$. Here, $e_n = x_n - \tilde{x}$ is the error at the n -th iteration, and C_j are constants depending on the higher-order fractional derivatives of f at

$\tilde{x}$.

The quotient $\frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp}f(x_n)}$ is computed by dividing the equation (2. 8) by equation (2. 9).

$$\begin{aligned} \frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp}f(x_n)} &= \frac{1}{\tilde{\Gamma}^{[*]}(1+\wp)} e_n^{\wp} + \frac{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^3} C_2^2 e_n^{2\wp} \\ &+ \left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp)\tilde{\Gamma}^{[*]}(1+\wp) - \tilde{\Gamma}^{[*]}(1+3\wp)}{\tilde{\Gamma}^{[*]}(1+2\wp)\tilde{\Gamma}^{[*]}(1+\wp)} C_3 - \frac{\tilde{\Gamma}^{[*]}(1+2\wp)}{\tilde{\Gamma}^{[*]}(1+\wp)} \right. \\ &\left. \left(\frac{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^3} \right) C_2^2 \right) e_n^{3\wp} + O(e_n^{4\wp}), \end{aligned} \quad (2. 10)$$

Subsequently, multiplying the resulting expression by $\tilde{\Gamma}^{[*]}(1+\wp)$ yields:

$$\begin{aligned} \tilde{\Gamma}^{[*]}(1+\wp) \frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp}f(x_n)} &= e_n^{\wp} + \frac{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^2} C_2^2 e_n^{2\wp} \\ &+ \left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp)\tilde{\Gamma}^{[*]}(1+\wp) - \tilde{\Gamma}^{[*]}(1+3\wp)}{\tilde{\Gamma}^{[*]}(1+2\wp)} C_3 \right. \\ &\left. - \tilde{\Gamma}^{[*]}(1+2\wp) \left(\frac{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{\left(\tilde{\Gamma}^{[*]}(1+\wp)\right)^3} \right) C_2^2 \right) e_n^{3\wp} \\ &+ O(e_n^{4\wp}). \end{aligned} \quad (2. 11)$$

After algebraic manipulation and raising to the power $1/\wp$:

$$\begin{aligned} \left(\tilde{\Gamma}^{[*]}(1+\wp) \frac{f(x_n)}{\mathcal{D}_{\tilde{x}}^{\wp} f(x_n)} \right)^{\frac{1}{\wp}} &= e_n + \frac{1}{\wp} e_n^{1-\wp} \left(\frac{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2 - \tilde{\Gamma}^{[*]}(2\wp+1)}{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2} C_2 e_n^{2\wp} \right. \\ &\quad + \left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp) \tilde{\Gamma}^{[*]}(1+\wp) - \tilde{\Gamma}^{[*]}(1+3\wp)}{\tilde{\Gamma}^{[*]}(1+2\wp)} C_3 \right. \\ &\quad \left. - \frac{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^3} \tilde{\Gamma}^{[*]}(1+2\wp) C_2^2 \right) e_n^{3\wp} \\ &\quad + \frac{\tilde{\Gamma}^{[*]}(1/\wp+1)}{2\tilde{\Gamma}^{[*]}(1/\wp-1)} e_n^{1-2\wp} \left(\frac{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2} \right)^2 \\ &\quad \times C_2^2 e_n^{4\wp} + O(e_n^{3\wp+1}). \end{aligned}$$

As $\tilde{\Gamma}^{[*]}(1/\wp+1) = \frac{1}{\wp} \tilde{\Gamma}^{[*]}(1+\wp) = \frac{1}{\wp} (\frac{1}{\wp}-1) \frac{1}{\wp} \tilde{\Gamma}^{[*]}(\frac{1}{\wp}-1)$, this implies that $\frac{\tilde{\Gamma}^{[*]}(\frac{1}{\wp}+1)}{2\tilde{\Gamma}^{[*]}(\frac{1}{\wp}-1)} = \frac{1}{2\wp} (\frac{1}{\wp}-1)$, and simplifying the expression yields:

$$\begin{aligned} \left(\tilde{\Gamma}^{[*]}(1+\wp) \frac{f(x_n)}{\mathcal{D}_{\tilde{x}}^{\wp} f(x_n)} \right)^{\frac{1}{\wp}} &= e_n + \left(\frac{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{\wp \left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2} \right) C_2 e_n^{\wp+1} \\ &\quad + \left(\frac{1}{\wp} \left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp) \tilde{\Gamma}^{[*]}(1+\wp)}{\tilde{\Gamma}^{[*]}(1+2\wp)} - \frac{\tilde{\Gamma}^{[*]}(1+3\wp)}{\tilde{\Gamma}^{[*]}(1+2\wp)} C_3 \right) \right. \\ &\quad \left. + \tilde{\Gamma}^{[*]}(1+2\wp) \frac{\tilde{\Gamma}^{[*]}(1+2\wp) - \left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2}{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^3} C_2^2 \frac{1}{2\wp} \right. \\ &\quad \times \left(\frac{1}{\wp} - 1 \right) + \left(\frac{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^2 - \tilde{\Gamma}^{[*]}(1+2\wp)^2}{\left(\tilde{\Gamma}^{[*]}(1+\wp) \right)^4} \right) C_2^2 \Bigg) \\ &\quad \times e_n^{2\wp+1} + O(e_n^{3\wp+1}). \end{aligned} \tag{2.12}$$

Let the error of the step $e_n^{(y)} = y_n - \tilde{x}$ is obtained as a series in powers of e_n :

$$\begin{aligned}
 e_n^{(y)} &= y_n - \tilde{x} = \left(\frac{(\tilde{\Gamma}^{[*]}(\wp + 1))^2 - \tilde{\Gamma}^{[*]}(1 + 2\wp)}{\wp (\tilde{\Gamma}^{[*]}(1 + \wp))^2} \right) C_2 e_n^{\wp+1} \\
 &\quad - \left(\frac{1}{\wp} \frac{\tilde{\Gamma}^{[*]}(1 + 2\wp) \tilde{\Gamma}^{[*]}(1 + \wp) - \tilde{\Gamma}^{[*]}(1 + 3\wp)}{\tilde{\Gamma}^{[*]}(1 + 2\wp)} C_3 \tilde{\Gamma}^{[*]}(1 + 2\wp) \right. \\
 &\quad \left. + \frac{\tilde{\Gamma}^{[*]}(1 + 2\wp) - (\tilde{\Gamma}^{[*]}(\wp + 1))^2}{(\tilde{\Gamma}^{[*]}(1 + \wp))^3} C_2^2 \right) e_n^{2\wp+1} + \frac{1}{2\wp} \left(\frac{1}{\wp} - 1 \right) \\
 &\quad \left(\frac{(\tilde{\Gamma}^{[*]}(1 + \wp))^2 - \tilde{\Gamma}^{[*]}(1 + 2\wp)^2}{(\tilde{\Gamma}^{[*]}(1 + \wp))^4} \right) C_2^2 e_n^{3\wp+1} + O(e_n^{3\wp+1}). \quad (2.13)
 \end{aligned}$$

The function value at the point, $f(y_n)$, is then expanded around $\tilde{x}$. Substituting the expression for $y_n - \tilde{x}$ into this expansion yields a series for $f(y_n)$ in terms of the original error e_n :

$$f(y_n) = \frac{c \mathfrak{D}_x^\wp f(\tilde{x})}{\tilde{\Gamma}^{[*]}(1 + \wp)} \left[(y_n - \tilde{x})^\wp + C_2 (y_n - \tilde{x})^{2\wp} \right] + O(e_n^{3\wp}). \quad (2.14)$$

By substituting equation (2.13) into equation (2.14), we get:

$$\begin{aligned}
 (y_n - \tilde{x})^\wp &= \left(\frac{\tilde{\Gamma}^{[*]}(1 + 2\wp) - (\tilde{\Gamma}^{[*]}(1 + \wp))^2}{\wp (\tilde{\Gamma}^{[*]}(1 + \wp))^2} C_2 e_n^{\wp+1} \right. \\
 &\quad + \left(\frac{1}{\wp} \left(\frac{\tilde{\Gamma}^{[*]}(1 + 3\wp) - \tilde{\Gamma}^{[*]}(2\wp + 1) \tilde{\Gamma}^{[*]}(1 + \wp)}{\tilde{\Gamma}^{[*]}(1 + 2\wp)} C_3 \right. \right. \\
 &\quad \left. \left. + \tilde{\Gamma}^{[*]}(1 + 2\wp) \frac{(\tilde{\Gamma}^{[*]}(1 + \wp))^2 - \tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(1 + \wp))^3} C_2^2 \right) \right) \\
 &\quad + \frac{1}{2\wp} \left(\frac{1}{\wp} - 1 \right) \left(\frac{(\tilde{\Gamma}^{[*]}(1 + \wp))^2 - \tilde{\Gamma}^{[*]}(1 + 2\wp)^2}{(\tilde{\Gamma}^{[*]}(1 + \wp))^4} \right) C_2^2 e_n^{2\wp+1} \\
 &\quad + O(e_n^{\wp^2+3\wp}).
 \end{aligned}$$

$$\begin{aligned}
(y_n - \tilde{x})^\varphi &= \left(\frac{\tilde{\Gamma}^{[*]}(2\varphi + 1) - (\tilde{\Gamma}^{[*]}(1 + \varphi))^2}{\varphi(\tilde{\Gamma}^{[*]}(1 + \varphi))^2} \right)^\varphi C_2^\varphi e_n^{\varphi^2 + \varphi} \\
&+ \varphi \left(\frac{\tilde{\Gamma}^{[*]}(1 + 2\varphi) - (\tilde{\Gamma}^{[*]}(1 + \varphi))^2}{\varphi(\tilde{\Gamma}^{[*]}(1 + \varphi))^2} \right)^{\varphi-1} C_2^{\varphi-1} e_n^{\varphi^2-1} \\
&\times \left(\frac{1}{\varphi} \left(\frac{\tilde{\Gamma}^{[*]}(1 + 3\varphi) - \tilde{\Gamma}^{[*]}(1 + 2\varphi)\tilde{\Gamma}^{[*]}(1 + \varphi)}{\tilde{\Gamma}^{[*]}(1 + 2\varphi)} C_3 \right. \right. \\
&\quad \left. \left. + \tilde{\Gamma}^{[*]}(1 + 2\varphi) \frac{(\tilde{\Gamma}^{[*]}(1 + \varphi))^2 - \tilde{\Gamma}^{[*]}(1 + 2\varphi)}{(\tilde{\Gamma}^{[*]}(1 + \varphi))^3} C_2^2 \right) \right) \\
&+ \frac{1}{2\varphi} \left(1 - \frac{1}{\varphi} \right) \left(\frac{(\tilde{\Gamma}^{[*]}(\varphi + 1))^2 - \tilde{\Gamma}^{[*]}(1 + 2\varphi)^2}{(\tilde{\Gamma}^{[*]}(1 + \varphi))^4} \right) C_2^2 e_n^{2\varphi+1} \\
&+ O\left(e_n^{\varphi^2+3\varphi}\right).
\end{aligned}$$

By simplifying:

$$\begin{aligned}
(y_n - \tilde{x})^\varphi &= \left(\frac{\tilde{\Gamma}^{[*]}(2\varphi + 1) - (\tilde{\Gamma}^{[*]}(1 + \varphi))^2}{\varphi(\tilde{\Gamma}^{[*]}(1 + \varphi))^2} \right)^\varphi C_2^\varphi e_n^{\varphi^2 + \varphi} \\
&+ \varphi \left(\frac{\tilde{\Gamma}^{[*]}(1 + 2\varphi) - (\tilde{\Gamma}^{[*]}(1 + \varphi))^2}{\varphi(\tilde{\Gamma}^{[*]}(1 + \varphi))^2} \right)^{\varphi-1} C_2^{\varphi-1} \\
&\times \left(\frac{1}{\varphi} \left(\frac{\tilde{\Gamma}^{[*]}(1 + 3\varphi) - \tilde{\Gamma}^{[*]}(1 + 2\varphi)\tilde{\Gamma}^{[*]}(1 + \varphi)}{\tilde{\Gamma}^{[*]}(1 + 2\varphi)} C_3 \right. \right. \\
&\quad \left. \left. + \tilde{\Gamma}^{[*]}(1 + 2\varphi) \frac{(\tilde{\Gamma}^{[*]}(1 + \varphi))^2 - \tilde{\Gamma}^{[*]}(1 + 2\varphi)}{(\tilde{\Gamma}^{[*]}(1 + \varphi))^3} C_2^2 \right) \right) \\
&+ \frac{1}{2\varphi} \left(1 - \frac{1}{\varphi} \right) \left(\frac{(\tilde{\Gamma}^{[*]}(1 + \varphi))^2 - \tilde{\Gamma}^{[*]}(1 + 2\varphi)^2}{(\tilde{\Gamma}^{[*]}(1 + \varphi))^4} \right) C_2^2 e_n^{\varphi^2+2\varphi} \\
&+ O\left(e_n^{\varphi^2+3\varphi}\right). \tag{2. 15}
\end{aligned}$$

We can see that $\varphi^2 + 3\varphi \geq \varphi$ for all $\varphi \in [0, 1]$. It is also clear that if we choose the first term of the expansion $C_2 (y_n - \tilde{x})^{2\varphi}$, it will have the order $(\varphi + 1) 2\varphi = 2\varphi^2 + 2\varphi \geq 3\varphi$ for all $\varphi \in [0.5, 1]$. Therefore, substituting equation (2. 15) into equation (2. 14), we

get:

$$\begin{aligned}
 f(y_n) = & \frac{c\mathfrak{D}_{\tilde{x}}^{\wp} f(\tilde{x})}{\tilde{\Gamma}^{[*]}(1+\wp)} \left[\left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp) - (\tilde{\Gamma}^{[*]}(1+\wp))^2}{\wp(\tilde{\Gamma}^{[*]}(1+\wp))^2} \right)^{\wp} C_2^{\wp} e_n^{\wp^2+\wp} \right. \\
 & + \wp \left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp) - (\tilde{\Gamma}^{[*]}(1+\wp))^2}{\wp(\tilde{\Gamma}^{[*]}(1+\wp))^2} \right)^{\wp-1} C_2^{\wp-1} \\
 & \times \left(\frac{1}{\wp} \left(\frac{\tilde{\Gamma}^{[*]}(1+3\wp) - \tilde{\Gamma}^{[*]}(2\wp+1)\tilde{\Gamma}^{[*]}(1+\wp)}{\tilde{\Gamma}^{[*]}(1+2\wp)} C_3 \right. \right. \\
 & \quad \left. \left. + \tilde{\Gamma}^{[*]}(1+2\wp) \frac{(\tilde{\Gamma}^{[*]}(1+\wp))^2 - \tilde{\Gamma}^{[*]}(1+2\wp)}{(\tilde{\Gamma}^{[*]}(1+\wp))^3} C_2^2 \right) \right. \\
 & \left. + \frac{1}{2\wp} \left(1 - \frac{1}{\wp} \right) \left(\frac{(\tilde{\Gamma}^{[*]}(1+\wp))^2 - \tilde{\Gamma}^{[*]}(1+2\wp)^2}{(\tilde{\Gamma}^{[*]}(1+\wp))^4} C_2^2 \right) e_n^{\wp^2+2\wp} \right] \\
 & + O\left(e_n^{\wp^2+3\wp}\right).
 \end{aligned}$$

Let us call:

$$A = \left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp) - (\tilde{\Gamma}^{[*]}(1+\wp))^2}{\wp(\tilde{\Gamma}^{[*]}(1+\wp))^2} \right)^{\wp} C_2^{\wp},$$

and

$$\begin{aligned}
 B = & \wp \left(\frac{\tilde{\Gamma}^{[*]}(1+2\wp) - (\tilde{\Gamma}^{[*]}(1+\wp))^2}{\wp(\tilde{\Gamma}^{[*]}(1+\wp))^2} \right)^{\wp-1} C_2^{\wp-1} \\
 & \times \left(\frac{1}{\wp} \left(\frac{\tilde{\Gamma}^{[*]}(1+3\wp) - \tilde{\Gamma}^{[*]}(2\wp+1)\tilde{\Gamma}^{[*]}(1+\wp)}{\tilde{\Gamma}^{[*]}(1+2\wp)} C_3 \right. \right. \\
 & \quad \left. \left. + \tilde{\Gamma}^{[*]}(1+2\wp) \frac{(\tilde{\Gamma}^{[*]}(1+\wp))^2 - \tilde{\Gamma}^{[*]}(2\wp+1)}{(\tilde{\Gamma}^{[*]}(1+\wp))^3} C_2^2 \right) \right. \\
 & \left. + \frac{1}{2\wp} \left(1 - \frac{1}{\wp} \right) \left(\frac{(\tilde{\Gamma}^{[*]}(1+\wp))^2 - \tilde{\Gamma}^{[*]}(1+2\wp)^2}{(\tilde{\Gamma}^{[*]}(1+\wp))^4} C_2^2 \right) \right).
 \end{aligned}$$

then,

$$f(y_n) = \frac{c\mathfrak{D}_{\tilde{x}}^{\wp} f(\tilde{x})}{\tilde{\Gamma}^{[*]}(1+\wp)} \left[A e_n^{\wp^2+\wp} + B e_n^{\wp^2+2\wp} \right] + O\left(e_n^{\wp^2+3\wp}\right). \quad (2.16)$$

The quotient $\frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)}$ is computed again.

$$\frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} = \frac{A e_n^{\wp^2+\wp}}{\tilde{\Gamma}^{[*]}(1+\wp)} + \frac{1}{\tilde{\Gamma}^{[*]}(1+\wp)} \left(B - \frac{A \tilde{\Gamma}^{[*]}(1+2\wp)}{(\tilde{\Gamma}^{[*]}(1+\wp))^2} \right) e_n^{\wp^2+2\wp} + O\left(e_n^{\wp^2+3\wp}\right). \quad (2.17)$$

Now multiplying by $\tilde{\Gamma}^{[*]}(1 + \wp)$ with equation (2. 17).

$$\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} = Ae_n^{\wp^2 + \wp} + \left(B - \frac{A\tilde{\Gamma}^{[*]}(1 + 2\wp)}{(\tilde{\Gamma}^{[*]}(1 + \wp))^2} \right) e_n^{\wp^2 + 2\wp} + O(e_n^{\wp^2 + 3\wp})$$

This result is then used to make the step update, which includes the weight function $\left(1 + \frac{f(y_n)}{f(x_n)}\right)$.

$$\begin{aligned} 1 + \frac{f(y_n)}{f(x_n)} &= 1 + \Omega^{(8)} e_n^{\wp+1} + \Omega^{(9)} e_n^{2\wp+1} + \Omega^{(10)} e_n^{3\wp+1} + \Omega^{(11)} e_n^{4\wp+1} + \dots \\ \frac{\tilde{\Gamma}^{[*]}(\wp + 1) f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \left(1 + \frac{f(y_n)}{f(x_n)}\right) &= \left(Ae_n^{\wp^2 + \wp} + \left(B - \frac{A\tilde{\Gamma}^{[*]}(1 + 2\wp)}{(\tilde{\Gamma}^{[*]}(1 + \wp))^2} \right) e_n^{\wp^2 + 2\wp} \right) \\ &\quad \left(1 + \Omega^{(8)} e_n^{\wp+1} + \Omega^{(9)} e_n^{2\wp+1} + \Omega^{(10)} e_n^{3\wp+1} + \Omega^{(11)} e_n^{4\wp+1} + \dots \right) \\ &= Ae_n^{\wp^2 + \wp} + \left(B - \frac{A\tilde{\Gamma}^{[*]}(1 + 2\wp)}{(\tilde{\Gamma}^{[*]}(1 + \wp))^2} \right) e_n^{\wp^2 + 2\wp} + A\Omega^{(8)} e_n^{\wp^2 + 2\wp} + \Omega^{(8)} \left(B - \frac{A\tilde{\Gamma}^{[*]}(1 + 2\wp)}{(\tilde{\Gamma}^{[*]}(1 + \wp))^2} \right) \\ &\quad e_n^{\wp^2 + 3\wp} + \Omega^{(9)} Ae_n^{\wp^2 + 2\wp+1} + \Omega^{(9)} \left(B - \frac{A\tilde{\Gamma}^{[*]}(1 + 2\wp)}{(\tilde{\Gamma}^{[*]}(1 + \wp))^2} \right) e_n^{\wp^2 + 3\wp+1} + \dots \end{aligned}$$

Taking power $1/\wp$ on both sides:

$$\begin{aligned} \left(\tilde{\Gamma}^{[*]}(\wp + 1) \frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \left(1 + \frac{f(y_n)}{f(x_n)}\right) \right)^{1/\wp} &= \left(Ae_n^{\wp^2 + \wp} + Be_n^{\wp^2 + 2\wp} \right. \\ &\quad \left. + A\Omega^{(8)} e_n^{\wp^2 + 2\wp} + \Omega^{(8)} Be_n^{\wp^2 + 3\wp} + \dots \right)^{1/\wp} \\ &= A^{1/\wp} e_n^{\wp+1} + \frac{1}{\wp} e_n^{-\wp^2+1} \left(B - \frac{A\tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(\wp + 1))^2} \right) e_n^{\wp^2 + 2\wp} + O(e_n^{3\wp+1}) \\ &= A^{1/\wp} e_n^{\wp+1} + \frac{1}{\wp} \left(B - \frac{A\tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(\wp + 1))^2} \right) e_n^{2\wp+1} + O(e_n^{3\wp+1}). \end{aligned}$$

Therefore, substituting the series expansions for x_n, y_n and the various quotients calculated above into the second step of the proposed scheme (2. 6), i.e.,

$$x_{n+1} = y_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \left(1 + \frac{f(y_n)}{f(x_n)}\right) \right)^{1/\wp}, \quad (2. 19)$$

then the error for the next iteration $e_{n+1} = x_{n+1} - \tilde{x}$ is calculated. Let $x_{n+1} = e_{n+1} + \tilde{x}$ and $x_n = e_n + \tilde{x}$, we get

$$e_{n+1} + \tilde{x} = e_n + \tilde{x} + \left[-e_n + A^{1/\wp} e_n^{\wp+1} + \left(\frac{B}{\wp A^{1-1/\wp} C_2^{\wp-1}} \right) e_n^{2\wp+1} \right]$$

$$-\left[A^{1/\varphi}e_n^{\varphi+1} + \frac{1}{\varphi} \left(B - \frac{A\tilde{\Gamma}^{[*]}(2\varphi+1)}{(\tilde{\Gamma}^{[*]}(\varphi+1))^2}\right) e_n^{2\varphi+1}\right] + O(e_n^{3\varphi+1}).$$

The leading term in this error expression is shown to be proportional to $e_n^{2\varphi+1}$, conclusively proving that the method has a local convergence order of at least $2\varphi+1$.

$$e_{n+1} = \left(\frac{B}{\varphi A^{1-1/\varphi} C_2^{\varphi-1}} + \frac{1}{\varphi} \left(\frac{A\tilde{\Gamma}^{[*]}(2\varphi+1)}{(\tilde{\Gamma}^{[*]}(\varphi+1))^2} - B\right)\right) e_n^{2\varphi+1} + O(e_n^{3\varphi+1}). \quad (2. 20)$$

being

$$C_j = \frac{\tilde{\Gamma}^{[*]}(1+\varphi)}{\tilde{\Gamma}^{[*]}(j\varphi+1)} \frac{c\mathfrak{D}_{\tilde{x}}^{j\varphi} f(\tilde{x})}{c\mathfrak{D}_{\tilde{x}}^{\varphi} f(\tilde{x})},$$

for $j \geq 2$.

$$A = \left(\frac{\tilde{\Gamma}^{[*]}(1+2\varphi) - (\tilde{\Gamma}^{[*]}(1+\varphi))^2}{\varphi (\tilde{\Gamma}^{[*]}(1+\varphi))^2}\right)^{\varphi} C_2^{\varphi},$$

and

$$\begin{aligned} B = & \varphi \left(\frac{\tilde{\Gamma}^{[*]}(1+2\varphi) - (\tilde{\Gamma}^{[*]}(1+\varphi))^2}{\varphi (\tilde{\Gamma}^{[*]}(1+\varphi))^2}\right)^{\varphi-1} C_2^{\varphi-1} \\ & \times \left(\frac{1}{\varphi} \left(\frac{\tilde{\Gamma}^{[*]}(1+3\varphi) - \tilde{\Gamma}^{[*]}(1+2\varphi)\tilde{\Gamma}^{[*]}(1+\varphi)}{\tilde{\Gamma}^{[*]}(1+2\varphi)} C_3 \right. \right. \\ & \left. \left. + \tilde{\Gamma}^{[*]}(1+2\varphi) \frac{(\tilde{\Gamma}^{[*]}(1+\varphi))^2 - \tilde{\Gamma}^{[*]}(1+2\varphi)}{(\tilde{\Gamma}^{[*]}(1+\varphi))^3} C_2^2\right)\right) \\ & + \frac{1}{2\varphi} \left(1 - \frac{1}{\varphi}\right) \left(\frac{(\tilde{\Gamma}^{[*]}(1+\varphi))^2 - \tilde{\Gamma}^{[*]}(1+2\varphi)}{(\tilde{\Gamma}^{[*]}(1+\varphi))^4} C_2^2\right). \end{aligned}$$

With the error stated in equation (2. 20), we can derive and validate that our approach obtains a convergence order of $2\varphi+1$ combined with two functional and one fractional evaluations $f(x_n)$, $f(y_n)$, and $c\mathfrak{D}_{\tilde{x}}^{\varphi} f(x_n)$. This completes the proof. $\square$

Replacing the Caputo fractional derivative by the Riemann-Liouville fractional derivative defined in (1. 4), we obtain the following analogue of the proposed method, denoted by $M-FSI_{RL}$. For a function $f : \mathcal{R} \rightarrow \mathcal{R}$ that is Lebesgue integrable, i.e., $f \in L^1([a, x])$, with $-\infty < a < x < +\infty$, and for a given order $\varphi \geq 0$, the Riemann-Liouville derivative defined in (1. 4), we propose the following iterative scheme:

$$\begin{cases} y_n = x_n - \left(\tilde{\Gamma}^{[*]}(1+\varphi) \frac{f(x_n)}{\mathfrak{D}_{\tilde{x}}^{\varphi} f(x_n)}\right)^{1/\varphi}, \\ x_{n+1} = y_n - \left(\tilde{\Gamma}^{[*]}(1+\varphi) \frac{f(y_n)}{\mathfrak{D}_{\tilde{x}}^{\varphi} f(x_n)} \left(1 + \Upsilon^{[\circ]} \frac{f(y_n)}{f(x_n)}\right)\right)^{1/\varphi}. \end{cases} \quad (2. 21)$$

The following lemma provides the local Riemann-Liouville fractional Taylor expansions that are used in the convergence analysis of the proposed method.

Lemma 2.2. *Let $f : D \subseteq \mathcal{R} \rightarrow \mathcal{R}$ possess a simple root $\tilde{x}$ and assume that the hypotheses of Proposition 1.2 hold for $0 < \wp \leq 1$. Then, for $x_n = \tilde{x} + e_n$ sufficiently close to $\tilde{x}$, the function $f(x_n)$ and its Riemann-Liouville fractional derivative $\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)$ admit the local expansions:*

$$f(x_n) = \frac{\mathfrak{D}_{\tilde{x}}^{\wp} f(\tilde{x})}{\tilde{\Gamma}^{[\star]}(1 + \wp)} [e_n^{\wp} + C_2 e_n^{2\wp} + C_3 e_n^{3\wp} + C_4 e_n^{4\wp}] + O(e_n^{5\wp}), \quad (2.22)$$

and

$$\begin{aligned} \mathfrak{D}_{\tilde{x}}^{\wp} f(x_n) = \frac{\mathfrak{D}_{\tilde{x}}^{\wp} f(\tilde{x})}{\tilde{\Gamma}^{[\star]}(1 + \wp)} & \left[\tilde{\Gamma}^{[\star]}(1 + \wp) + \frac{\tilde{\Gamma}^{[\star]}(1 + 2\wp)}{\tilde{\Gamma}^{[\star]}(1 + \wp)} C_2 e_n^{\wp} + \frac{\tilde{\Gamma}^{[\star]}(1 + 3\wp)}{\tilde{\Gamma}^{[\star]}(1 + 2\wp)} C_3 e_n^{2\wp} \right. \\ & \left. + \frac{\tilde{\Gamma}^{[\star]}(1 + 4\wp)}{\tilde{\Gamma}^{[\star]}(1 + 3\wp)} C_4 e_n^{3\wp} \right] + O(e_n^{4\wp}). \end{aligned} \quad (2.23)$$

where

$$C_j = \frac{\tilde{\Gamma}^{[\star]}(1 + \wp)}{\tilde{\Gamma}^{[\star]}(j\wp + 1)} \frac{\mathfrak{D}_{\tilde{x}}^{j\wp} f(\tilde{x})}{\mathfrak{D}_{\tilde{x}}^{\wp} f(\tilde{x})}, \quad j \geq 2.$$

Proof. Let $x = \tilde{x}$ and $h = e_n = x_n - \tilde{x}$, and applying Proposition 1.2 gives the Riemann-Liouville fractional Taylor series expansion as:

$$f(x_n) = \sum_{k=0}^{\infty} \frac{e_n^{k\wp}}{\tilde{\Gamma}^{[\star]}(k\wp + 1)} \mathfrak{D}_{\tilde{x}}^{k\wp} f(\tilde{x}).$$

Since $\tilde{x}$ is a simple root of f , we have $f(\tilde{x}) = 0$, and therefore the first nonzero term corresponds to $k = 1$. Factoring $\mathfrak{D}_{\tilde{x}}^{\wp} f(\tilde{x}) / \tilde{\Gamma}^{[\star]}(1 + \wp)$ yields expansion (2.22), where C_j is defined in Lemma 2.2. Applying the Riemann-Liouville fractional derivative of order $\wp$ term-by-term to the expansion (2.22), together with the identity

$$\mathfrak{D}_{\tilde{x}}^{\wp} ((x - \tilde{x})^{k\wp}) = \frac{\tilde{\Gamma}^{[\star]}(k\wp + 1)}{\tilde{\Gamma}^{[\star]}((k - 1)\wp + 1)} (x - \tilde{x})^{(k-1)\wp}, \quad k \geq 1,$$

yields expansion (2.23). $\square$

Theorem 2.3. *Let $f : D \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a continuous function possessing Riemann-Liouville fractional derivatives of order $k\wp$ for any positive integer k and any $\wp \in (0, 1]$, in an interval D containing a simple root $\tilde{x}$ of $f(x)$. Suppose $\mathfrak{D}_{\tilde{x}}^{\wp} f(x)$ is continuous and non-zero at $\tilde{x}$. If an initial approximation x_0 is sufficiently close to $\tilde{x}$, then the iterative method defined by (2.21) with $\Upsilon^{[0]} = 1$ has a local convergence order of at least $2\wp + 1$ for $0 < \wp \leq 1$. The asymptotic error constant is given by the coefficient of $e_n^{2\wp+1}$ in the error equation:*

$$e_{n+1} = \left(\frac{B}{\wp A^{1-1/\wp} C_2^{\wp-1}} + \frac{\tilde{\Gamma}^{[*]}(\frac{1}{\wp} + 1)}{\tilde{\Gamma}^{[*]}(\frac{1}{\wp})} A \left(\frac{A \tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(\wp + 1))^2} - B \right) \right) e_n^{2\wp+1} \quad (2.24)$$

$$+ O(e_n^{2\wp+1}), \quad n = 0, 1, 2, \dots$$

where

$$C_j = \frac{\tilde{\Gamma}^{[*]}(1 + \wp)}{\tilde{\Gamma}^{[*]}(j\wp + 1)} \frac{\mathfrak{D}_{\tilde{x}}^{j\wp} f(\tilde{x})}{\mathfrak{D}_{\tilde{x}}^{\wp} f(\tilde{x})},$$

for $j \geq 2$,

$$A = \left(\frac{\tilde{\Gamma}^{[*]}(2\wp + 1) - (\tilde{\Gamma}^{[*]}(\wp + 1))^2}{\wp (\tilde{\Gamma}^{[*]}(\wp + 1))^2} \right)^{\wp} C_2^{\wp},$$

and

$$B = \wp \left(\frac{\tilde{\Gamma}^{[*]}(2\wp + 1) - (\tilde{\Gamma}^{[*]}(\wp + 1))^2}{\wp (\tilde{\Gamma}^{[*]}(\wp + 1))^2} \right)^{\wp-1} C_2^{\wp-1} \left(\frac{1}{\wp} \left(\frac{\tilde{\Gamma}^{[*]}(3\wp + 1) - \tilde{\Gamma}^{[*]}(2\wp + 1)\tilde{\Gamma}^{[*]}(\wp + 1)}{\tilde{\Gamma}^{[*]}(2\wp + 1)} C_3 \right. \right.$$

$$\left. \left. + \tilde{\Gamma}^{[*]}(2\wp + 1) \frac{(\tilde{\Gamma}^{[*]}(\wp + 1))^2 - \tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(\wp + 1))^3} C_2^2 \right) \right)$$

$$+ \frac{1}{2\wp} \left(1 - \frac{1}{\wp} \right) \left(\frac{(\tilde{\Gamma}^{[*]}(\wp + 1))^2 - \tilde{\Gamma}^{[*]}(2\wp + 1)}{(\tilde{\Gamma}^{[*]}(\wp + 1))^4} C_2^2 \right).$$

Proof. By Lemma 2.2, the local asymptotic expansions of $f(x_n)$ and $\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)$ are given by (2.22) and (2.23). Substituting the expansions (2.22)-(2.23) into the first step of (2.21), expanding the fractional power by the generalized binomial theorem, and collecting like powers of e_n yields the error equation for the first step. Repeating the same procedure for the second step produces the error equation (2.24). Therefore, the proposed method possesses local convergence order at least $2\wp + 1$. $\square$

3. NUMERICAL RESULTS

This section validates efficacy, robustness and convergence behavior of proposed two-step fractional iterative scheme (2.6). Numerical experiments are carried out for different values of fractional order $\wp$. Practical applicability is demonstrated through several real-world problems, including quarter-car suspension model, hanging object, spring-mass system analogue, bookshelf deflection analysis and acidity determination in chemical systems.

Performance of proposed scheme is compared with existing fractional iterative methods reported in literature. First, we consider method proposed by Shams *et al.* [22], denoted by

M -SC, which possesses convergence order at least $(2\wp + 1)$ and is defined by

$$\begin{cases} y_n = x_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \right)^{1/\wp}, \\ x_{n+1} = x_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \left(\frac{f(y_n)}{f(x_n)} + 2 \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \Upsilon^{[\circ]} \left(\frac{f(y_n)}{f(x_n)} \right)^4 \right) \right)^{1/\wp}. \end{cases} \quad (3.25)$$

Next, two special cases from fractional two-step family introduced by Shams *et al.* [23] are employed for comparison.

First scheme, denoted by M -BB1, is given by

$$\begin{cases} y_n = x_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \right)^{1/\wp}, \\ x_{n+1} = y_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \left(1 + \frac{\frac{f(y_n)}{f(x_n)}}{1 - \Upsilon^{[\circ]} \left(\frac{f(y_n)}{f(x_n)} \right)^2} \right) \right)^{1/\wp}. \end{cases} \quad (3.26)$$

Second scheme, denoted by M -BB2, is defined as

$$\begin{cases} y_n = x_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(x_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \right)^{1/\wp}, \\ x_{n+1} = y_n - \left(\tilde{\Gamma}^{[*]}(1 + \wp) \frac{f(y_n)}{c\mathfrak{D}_{\tilde{x}}^{\wp} f(x_n)} \left(1 + 2 \frac{f(y_n)}{f(x_n)} - \frac{\frac{f(y_n)}{f(x_n)}}{1 - \Upsilon^{[\circ]} \left(\frac{f(y_n)}{f(x_n)} \right)^2} \right) \right)^{1/\wp}, \end{cases} \quad (3.27)$$

where $\Upsilon^{[\circ]} \in \mathbb{R}$.

Comparison is based on number of iterations required to satisfy tolerance 10^{-5} with maximum limit of 500 iterations, absolute difference between two consecutive approximations, $|x_{n+1} - x_n|$, elapsed computational time, and computational order of convergence (COC). Results for integer-order derivative ($\wp = 1$) are also reported. Computational order of convergence is computed using formula proposed by Jay [10]:

$$\text{COC} = \frac{\ln |x_{n+1} - \tilde{x}/x_n - \tilde{x}|}{\ln |x_n - \tilde{x}/x_{n-1} - \tilde{x}|}, \quad (3.28)$$

where $\tilde{x}$ denotes exact root.

Fractional orders $\wp \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$ are selected to investigate influence of fractional differentiation on convergence characteristics. These values are treated as numerical fractional parameters, following standard practice in development of fractional iterative methods, allowing systematic assessment without requiring physical interpretation for individual test problems.

Parameter $\Upsilon^{[\circ]} = 1$ is adopted in both proposed and comparison methods throughout all experiments. Computations are performed using Maple 2022 with at least 1000 significant digits to minimize round-off errors and ensure numerical reliability. For better readability, values reported in Tables 1-5 are rounded to suitable decimal precision.

Each real-world problem includes corresponding nonlinear function, fractional derivative, exact root $\tilde{x}$, and initial approximation x_0 . Complete numerical comparisons are presented in corresponding tables.

Example 3.1. Quarter-car suspension model [7]:

Responsible for controlling motion of the chassis relative to the wheels, the damper (or shock absorber) is a fundamental component of car's suspension system, affecting both sprung and unsprung masses [12, 19]. It has very complex and nonlinear behavior, which makes it one of the most advanced parts of the whole system. As evidenced by the asymmetric and nonlinear hysteresis loop which defines the shock absorber's damping force, this nonlinear behavior is significant [16]. A quarter-car model featuring two degrees of freedom is utilized to simulate vehicle dynamics and investigate these phenomena, thereby enabling a direct performance assessment of linear versus nonlinear damping methods.

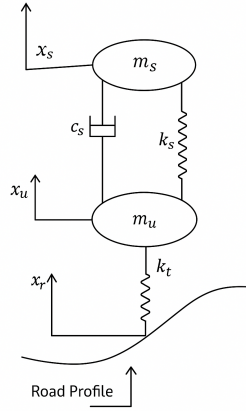


FIGURE 1. Quarter-car suspension model.

Common simplifications, including purely linear or piecewise-linear models, are frequently inadequate for accurately representing the true behavior of a damper. The system's dynamics are described by the following equations of motion (3. 29):

$$\begin{cases} \ddot{v}_u m_u &= (-v_u + v_p)k_p - (v_u - v_g)k_\delta + F, \\ \ddot{v}_p m_p &= -(-v_u + v_p)k_p - F. \end{cases} \quad (3. 29)$$

where k_p denotes the suspension spring stiffness, k_δ represents the tire stiffness, m_p and m_u are the sprung and unsprung masses, and v_p , v_u their respective displacements. Following [7], the damping force F in (3. 29) is modeled by the quartic polynomial:

$$f_1(x) = -77.14x^4 + 23.14x^3 + 342.7x^2 + 956.7x + 124.5, \quad (3. 30)$$

For a function $f_1(x)$, the Caputo fractional derivative of order $\wp$ is defined by following expression:

$$\begin{aligned} c\mathcal{D}_x^\wp f_1(x) = & -77.14 \frac{\tilde{\Gamma}^{[*]}(5)}{\tilde{\Gamma}^{[*]}(5-\wp)} x^{4-\wp} + 23.14 \frac{\tilde{\Gamma}^{[*]}(4)}{\tilde{\Gamma}^{[*]}(4-\wp)} x^{3-\wp} + 342.7 \frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(3-\wp)} x^{2-\wp} \\ & + 956.7 \frac{\tilde{\Gamma}^{[*]}(2)}{\tilde{\Gamma}^{[*]}(2-\wp)} x^{1-\wp} + 124.5 \frac{1}{\tilde{\Gamma}^{[*]}(1-\wp)} x^{-\wp}. \end{aligned} \quad (3.31)$$

derived from experimental measurements of mass acceleration, velocity, and displacement (see Figure 1). The exact roots of (3.30) are $\tilde{x}_1 = 3.090556803$, $\tilde{x}_2 = -1.326919946 + 1.434668028i$, $\tilde{x}_3 = -0.1367428388$, and $\tilde{x}_4 = -1.326919946 - 1.434668028i$. We take $\tilde{x}_1 = 3.090556803$ with the initial approximation $x_0 = 3.5$ for this test problem.

For function $f_1(x)$, all methods exhibit improved convergence as fractional order $\wp$ approaches 1. The proposed $M-FSI_{RL}$ and $M-FSI_C$ methods generally require fewer iterations and lower computational time than existing schemes while maintaining comparable or smaller absolute errors. For lower fractional orders ($0.50 \leq \wp \leq 0.80$), $M-FSI_{RL}$ demonstrates fastest convergence, whereas $M-FSI_C$ achieves competitive accuracy with reduced execution time. At $\wp = 1.00$, both proposed methods converge in only three iterations with computational order of convergence (COC) equal to 2.90944, matching highest convergence order among all methods. Overall, results presented in Table 1 confirm superior efficiency and competitive accuracy of proposed fractional iterative schemes.

Example 3.2. Hanging object:

Based on a physical system where a constant force lifts a chain vertically from the ground against gravity, we arrive at the following nonlinear initial value problem:

$$\begin{cases} \frac{d^2 v}{du^2} = \left(\frac{dv}{du}\right)^2 - 13u - 1, \\ v(0) = 0, \quad v'(0) = \frac{e^2 - 1}{e^2 + 1}. \end{cases} \quad (3.32)$$

Following the methodology in [24], Equation (3.32) is approximated using the polynomial:

$$f_2(x) = -0.02590111180x^4 - 0.1066166681x^3 - 0.2099871708x^2 + 0.7615941560x, \quad (3.33)$$

The fractional derivative of $f_2(x)$ according to the Caputo interpretation, with order $\wp$, is given as:

$$\begin{aligned} c\mathcal{D}_x^\wp f_2(x) = & -0.02590111180 \frac{\tilde{\Gamma}^{[*]}(5)}{\tilde{\Gamma}^{[*]}(5-\wp)} x^{4-\wp} - 0.1066166681 \frac{\tilde{\Gamma}^{[*]}(4)}{\tilde{\Gamma}^{[*]}(4-\wp)} x^{3-\wp} \\ & - 0.2099871708 \frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(3-\wp)} x^{2-\wp} + 0.7615941560 \frac{\tilde{\Gamma}^{[*]}(2)}{\tilde{\Gamma}^{[*]}(2-\wp)} x^{1-\wp}. \end{aligned} \quad (3.34)$$

whose exact roots are $\tilde{x}_1 = 0$, $\tilde{x}_2 = 1.660954916$, $\tilde{x}_3 = -2.8886 + 3.0592i$, and $\tilde{x}_4 = -2.8886 - 3.0592i$. We take $\tilde{x}_2 = 1.660954916$ with the initial approximation $x_0 = 1.7$ for this test problem. Table 2 summarizes performance of all iterative schemes

$f_1(x) = -77.14x^4 + 23.14x^3 + 342.7x^2 + 956.7x + 124.5$					
Fract. order($\wp$)	Method	Iterations(n)	$(x_{n+1} - x_n)$	CPU time	COC
0.50	$M - SC$	500	8.14550×10^{-5}	3.250	—
	$M - BB1$	500	3.18908×10^{-1}	2.235	—
	$M - BB2$	500	3.08976×10^{-1}	2.209	—
	$M - FSI_{RL}$	426	9.97500×10^{-6}	2.010	—
	$M - FSI_C$	500	2.5486×10^{-5}	1.846	—
0.60	$M - SC$	417	9.94500×10^{-6}	1.093	—
	$M - BB1$	500	3.27659×10^{-3}	1.593	—
	$M - BB2$	500	3.14851×10^{-3}	1.172	—
	$M - FSI_{RL}$	184	9.90200×10^{-6}	1.32	—
	$M - FSI_C$	292	9.92500×10^{-6}	1.019	—
0.70	$M - SC$	138	9.94500×10^{-6}	0.422	—
	$M - BB1$	500	1.76340×10^{-5}	1.579	—
	$M - BB2$	500	1.66980×10^{-5}	1.110	—
	$M - FSI_{RL}$	81	9.78300×10^{-6}	0.184	—
	$M - FSI_C$	112	9.87500×10^{-6}	0.422	—
0.80	$M - SC$	50	9.32600×10^{-6}	0.203	—
	$M - BB1$	67	9.57800×10^{-6}	0.266	—
	$M - BB2$	50	9.62778×10^{-6}	0.187	—
	$M - FSI_{RL}$	36	9.28200×10^{-6}	0.133	—
	$M - FSI_C$	44	9.90300×10^{-6}	0.140	—
0.90	$M - SC$	18	8.94600×10^{-6}	0.157	—
	$M - BB1$	18	8.77700×10^{-6}	0.078	—
	$M - BB2$	17	9.50200×10^{-6}	0.172	—
	$M - FSI_{RL}$	15	9.66000×10^{-6}	0.106	—
	$M - FSI_C$	17	6.89700×10^{-6}	0.094	—
1.00	$M - SC$	4	1.33000×10^{-7}	0.031	2.02966
	$M - BB1$	3	5.56300×10^{-6}	0.032	2.89976
	$M - BB2$	3	5.99100×10^{-6}	0.031	2.90902
	$M - FSI_{RL}$	3	5.77400×10^{-6}	0.015	2.90944
	$M - FSI_C$	3	5.77400×10^{-6}	0.015	2.90944

TABLE 1. Performance evaluation of the proposed $M - FSI_{RL}$ and $M - FSI_C$ methods against $M - SC$, $M - BB1$, and $M - BB2$ for solving $f_1(x)$ with different fractional orders $\wp$.

for solving $f_2(x)$ under different fractional orders $\wp$. Proposed $M - FSI_{RL}$ and $M - FSI_C$ methods exhibit stable convergence across all tested values of $\wp$, with noticeable reductions in iteration count and execution time, particularly for $0.50 \leq \wp \leq 0.80$. As $\wp$ increases, convergence becomes faster for all convergent methods. At $\wp = 1.00$, both proposed schemes attain convergence in only three iterations with absolute error of 1.0×10^{-9} and COC of 2.59598, while both $M - BB1$ and $M - BB2$ fail to converge. These results

$f_2(x) = -0.02590111180x^4 - 0.1066166681x^3 - 0.2099871708x^2 + 0.7615941560x$					
Fract. order($\wp$)	Method	Iterations(n)	$(x_{n+1} - x_n)$	CPU time	COC
0.50	$M - SC$	500	8.11800×10^{-5}	4.093	—
	$M - BB1$	500	3.18879×10^{-1}	3.344	—
	$M - BB2$	500	3.09344×10^{-1}	2.422	—
	$M - FSI_{RL}$	345	9.99500×10^{-6}	1.828	—
	$M - FSI_C$	500	2.26370×10^{-5}	2.375	—
0.60	$M - SC$	416	9.95000×10^{-6}	1.421	—
	$M - BB1$	500	3.20301×10^{-3}	1.031	—
	$M - BB2$	500	3.08045×10^{-3}	1.406	—
	$M - FSI_{RL}$	154	9.90400×10^{-6}	0.891	—
	$M - FSI_C$	280	9.92700×10^{-6}	1.011	—
0.70	$M - SC$	138	9.78100×10^{-6}	0.703	—
	$M - BB1$	500	1.78010×10^{-5}	2.844	—
	$M - BB2$	500	1.68791×10^{-5}	1.094	—
	$M - FSI_{RL}$	69	9.90000×10^{-5}	0.375	—
	$M - FSI_C$	106	9.89000×10^{-6}	0.344	—
0.80	$M - SC$	49	9.69400×10^{-6}	0.203	—
	$M - BB1$	63	9.88900×10^{-6}	0.219	—
	$M - BB2$	46	9.90666×10^{-6}	0.250	—
	$M - FSI_{RL}$	31	9.28000×10^{-6}	0.132	—
	$M - FSI_C$	41	9.58000×10^{-6}	0.141	—
0.90	$M - SC$	18	7.77200×10^{-6}	0.125	—
	$M - BB1$	16	8.66500×10^{-6}	0.063	—
	$M - BB2$	15	9.36800×10^{-6}	0.078	—
	$M - FSI_{RL}$	13	8.18700×10^{-6}	0.056	—
	$M - FSI_C$	15	6.79200×10^{-6}	0.062	—
1.00	$M - SC$	4	5.90000×10^{-8}	0.032	2.02683
	$M - BB1$	500	<i>Div</i>	2.265	<i>Div</i>
	$M - BB2$	500	<i>Div</i>	2.172	<i>Div</i>
	$M - FSI_{RL}$	3	1.00000×10^{-9}	0.001	2.59598
	$M - FSI_C$	3	1.00000×10^{-9}	0.001	2.59598

TABLE 2. Performance evaluation of the proposed $M - FSI_{RL}$ and $M - FSI_C$ methods against $M - SC$, $M - BB1$, and $M - BB2$ for solving $f_2(x)$ with different fractional orders $\wp$. Here, *Div* denotes divergence (method failed to converge).

demonstrate robust convergence behavior and improved computational efficiency of proposed fractional methods over existing approaches.

Example 3.3. Spring-mass system analogue:

This analysis considers a vertical spring, fixed at one end and with a mass m attached to the other. The spring's elongation depends directly on the value of the attached mass. The

system's behavior is fundamentally described by Hooke's law, which posits that the spring exerts a restoring force F proportional to its elongation s and opposite in direction. The spring constant k is mentioned in this relationship, which is defined as $F = -ks$. If we consider the case of an undamped spring-mass system, the corresponding motion will be described by the differential equation $F(x)$ as pointed out in [24].

$$\begin{cases} \frac{d^2v}{du^2} + v^3 = 0, \\ v(0) = 1, \quad v'(0) = 1. \end{cases} \quad (3.35)$$

Following the formulated approach, equation (3.35) is approximated using the polynomial:

$$f_3(x) = -0.5x^3 - 0.5x^2 + x + 1. \quad (3.36)$$

Let the Caputo fractional derivative of order $\wp$ acting on $f_3(x)$ be defined as follows:

$$\begin{aligned} {}^c\mathcal{D}_x^\wp f_3(x) = & -0.5 \frac{\tilde{\Gamma}^{[*]}(4)}{\tilde{\Gamma}^{[*]}(4-\wp)} x^{3-\wp} - 0.5 \frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(3-\wp)} x^{2-\wp} + \frac{\tilde{\Gamma}^{[*]}(2)}{\tilde{\Gamma}^{[*]}(2-\wp)} x^{1-\wp} \\ & + \frac{1}{\tilde{\Gamma}^{[*]}(1-\wp)} x^{-\wp}. \end{aligned} \quad (3.37)$$

The exact roots of (3.36) are $\tilde{x}_1 = -1.0$, $\tilde{x}_2 = -1.414213562$, $\tilde{x}_3 = 1.414213562$. We take $\tilde{x}_1 = -1.0$ with the initial approximation $x_0 = -1.5$ for this test problem.

Results in Table 3 indicate that convergence accelerates as fractional order $\wp$ increases from 0.50 to 1.00. Proposed M -FSI_{RL} method consistently attains smaller absolute errors than competing schemes over most fractional orders, whereas M -FSI_C achieves comparable accuracy with lower execution time. For $\wp \geq 0.80$, both proposed methods converge in fewer iterations than or equal to existing approaches, demonstrating improved computational efficiency. At $\wp = 1.00$, all methods converge within four iterations; however, M -FSI_{RL} and M -FSI_C produce smallest absolute error, 2.23738×10^{-8} , with COC of 2.17245, confirming reliable and efficient convergence for solving $f_3(x)$.

Example 3.4. Bookshelf deflection analysis:

With a total length of 29 inches and a height that can be adjusted between 8.5 and 11 inches, the bookshelf is constructed from a wooden board. This board has a width of 12 inches, a thickness of $\frac{3}{8}$ inches, and a Young's modulus of $E = 3.66$ Msi. The vertical deflection $z(x)$ at any point x along the shelf's length (in inches) is modeled by the following equation:

$$z(x) = \xi_1 x^3 + \xi_2 x^5 + \xi_3 x^4 + \xi_4 x, \quad (3.38)$$

where the coefficients are given by $\xi_1 = 0.42493 \times 10^{-4}$, $\xi_2 = -0.13533 \times 10^{-8}$, $\xi_3 = -0.66722 \times 10^{-6}$, and $\xi_4 = -0.018507$. The location of the maximum deflection is found where the condition $dz/dx = 0$ is satisfied.

$$\frac{dz}{dx} = (0.12748 \times 10^{-3})x^2 - (0.67665 \times 10^{-8})x^4 - (0.26689 \times 10^{-5})x^3 - 0.018507 = 0. \quad (3.39)$$

$f_3(x) = -0.5x^3 - 0.5x^2 + x + 1$					
Fract. order($\wp$)	Method	Iterations(n)	$(x_{n+1} - x_n)$	CPU time	COC
0.50	$M - SC$	500	7.85110×10^{-5}	4.890	—
	$M - BB1$	500	3.18975×10^{-1}	3.250	—
	$M - BB2$	500	3.08617×10^{-1}	2.094	—
	$M - FSI_{RL}$	500	1.33850×10^{-6}	1.465	—
	$M - FSI_C$	500	2.67451×10^{-5}	1.922	—
0.60	$M - SC$	420	9.99910×10^{-6}	1.266	—
	$M - BB1$	500	3.29770×10^{-3}	1.162	—
	$M - BB2$	500	3.16778×10^{-3}	1.156	—
	$M - FSI_{RL}$	332	1.57843×10^{-7}	1.301	—
	$M - FSI_C$	295	9.99260×10^{-6}	1.122	—
0.70	$M - SC$	140	9.83990×10^{-6}	0.609	—
	$M - BB1$	500	1.79705×10^{-5}	0.953	—
	$M - BB2$	500	1.70148×10^{-5}	1.219	—
	$M - FSI_{RL}$	117	2.56421×10^{-6}	0.562	—
	$M - FSI_C$	114	9.7454×10^{-6}	0.453	—
0.80	$M - SC$	50	9.98890×10^{-6}	0.281	—
	$M - BB1$	67	9.96890×10^{-6}	0.375	—
	$M - BB2$	50	9.79990×10^{-6}	0.172	—
	$M - FSI_{RL}$	46	5.65327×10^{-6}	0.202	—
	$M - FSI_C$	45	9.71960×10^{-6}	0.118	—
0.90	$M - SC$	19	7.58200×10^{-6}	0.078	—
	$M - BB1$	19	7.49480×10^{-6}	0.125	—
	$M - BB2$	18	7.98760×10^{-6}	0.078	—
	$M - FSI_{RL}$	17	1.53827×10^{-6}	0.056	—
	$M - FSI_C$	17	8.42000×10^{-6}	0.031	—
1.00	$M - SC$	4	5.40000×10^{-8}	0.015	2.04284
	$M - BB1$	4	2.29296×10^{-4}	0.172	2.17229
	$M - BB2$	4	2.33312×10^{-4}	0.016	2.17238
	$M - FSI_{RL}$	4	2.23738×10^{-8}	0.011	2.17245
	$M - FSI_C$	4	2.23738×10^{-8}	0.011	2.17245

TABLE 3. Performance evaluation of the proposed $M - FSI_{RL}$ and $M - FSI_C$ methods against $M - SC$, $M - BB1$, and $M - BB2$ for solving $f_3(x)$ with different fractional orders $\wp$.

This critical condition corresponds to the polynomial:

$$f_4(x) = (0.12748 \times 10^{-3})x^2 - (0.67665 \times 10^{-8})x^4 - (0.26689 \times 10^{-5})x^3 - 0.018507, \quad (3.40)$$

In the context of fractional calculus, the Caputo-type derivative of $f_4(x)$ (of order $\wp$) takes the form:

$$\begin{aligned} c\mathcal{D}_{\tilde{x}}^{\wp} f_4(x) = & 0.12748 \times 10^{-3} \frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(3-\wp)} x^{2-\wp} - 0.67665 \times 10^{-8} \frac{\tilde{\Gamma}^{[*]}(5)}{\tilde{\Gamma}^{[*]}(5-\wp)} x^{4-\wp} \\ & - 0.26689 \times 10^{-5} \frac{\tilde{\Gamma}^{[*]}(4)}{\tilde{\Gamma}^{[*]}(4-\wp)} x^{3-\wp} - 0.018507 \frac{1}{\tilde{\Gamma}^{[*]}(1-\wp)} x^{-\wp}. \end{aligned} \quad (3.41)$$

whose roots identify potential locations of maximum deflection, to be verified by the second derivative test. We take $\tilde{x}_1 = 14.57245222$ with the initial approximation $x_0 = 14.9$ for this test problem. Table 4 compares convergence characteristics of all methods for solving $f_4(x)$ over different fractional orders $\wp$. Decreasing iteration count with increasing $\wp$ is observed for every convergent scheme, indicating enhanced convergence at higher fractional orders. Proposed $M-FSI_C$ method consistently delivers lower execution time while maintaining accuracy comparable to or better than existing methods. Although $M-FSI_{RL}$ requires slightly more iterations for some fractional orders, it achieves competitive error levels throughout computation. At $\wp = 1.00$, all methods converge within two iterations; thus COC is not evaluated due to an insufficient number of iterative steps required by empirical formula. Overall, proposed fractional schemes provide efficient and reliable performance across all tested values of $\wp$.

Example 3.5. Acidity determination in chemical systems [1]:

In a solution where magnesium hydroxide ($Mg(OH)_2$) is dissolved in hydrochloric acid (HCl), the resulting acidity depends directly on the concentration of hydronium ions, $[H_3O^+]$. This relationship is defined by the equilibrium expression:

$$\frac{3.64 \times 10^{-11}}{[H_3O^+]} = [H_3O^+] + 3.6 \times 10^{-4}, \quad (3.42)$$

where the terms on the right-hand side account for the competing chemical influences in the system. By introducing the dimensionless variable $x = 10^4[H_3O^+]$, we obtain the equivalent cubic polynomial:

$$f_5(x) = x^3 - 3.6x^2 - 36.4, \quad (3.43)$$

The Caputo fractional derivative of order $\wp$ for a given function $f_5(x)$ is defined by:

$$c\mathcal{D}_{\tilde{x}}^{\wp} f_5(x) = \frac{\tilde{\Gamma}^{[*]}(4)}{\tilde{\Gamma}^{[*]}(4-\wp)} x^{3-\wp} - 3.6 \frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(3-\wp)} x^{2-\wp} - 36.4 \frac{1}{\tilde{\Gamma}^{[*]}(1-\wp)} x^{-\wp}. \quad (3.44)$$

whose complete set of roots characterizes the system's behavior are $\tilde{x}_1 = -0.7177636689 + 2.591033089i$, $\tilde{x}_2 = -0.7177636689 - 2.591033089i$, $\tilde{x}_3 = 5.035527338$. We take $\tilde{x}_3 = 5.035527338$ with the initial approximation $x_0 = 5.5$ for this test problem. Performance results for $f_5(x)$ are presented in Table 5. All methods benefit from increasing fractional order $\wp$, reflected by substantial reductions in iteration count and absolute error. Among fractional schemes, $M-FSI_C$ frequently attains fewer iterations and lower execution time, while $M-FSI_{RL}$ provides comparable accuracy with stable convergence. For $\wp = 1.00$, both proposed methods converge in only three iterations, yielding the smallest

$f_4(x) = (0.12748 \times 10^{-3})x^2 - (0.67665 \times 10^{-8})x^4 - (0.26689 \times 10^{-5})x^3 - 0.018507$					
Fract. order($\wp$)	Method	Iterations(n)	$(x_{n+1} - x_n)$	CPU time	COC
0.50	$M - SC$	500	7.85100×10^{-5}	2.672	—
	$M - BB1$	500	3.17784×10^{-1}	2.063	—
	$M - BB2$	500	3.07873×10^{-1}	2.500	—
	$M - FSI_{RL}$	500	2.50490×10^{-5}	2.360	—
	$M - FSI_C$	500	2.45700×10^{-5}	1.969	—
0.60	$M - SC$	413	9.98000×10^{-6}	1.578	—
	$M - BB1$	500	3.23673×10^{-3}	2.438	—
	$M - BB2$	500	3.11059×10^{-3}	1.453	—
	$M - FSI_{RL}$	340	2.97200×10^{-6}	1.405	—
	$M - FSI_C$	290	9.97000×10^{-6}	1.031	—
0.70	$M - SC$	137	9.78000×10^{-6}	0.516	—
	$M - BB1$	500	1.41900×10^{-5}	2.313	—
	$M - BB2$	500	1.33071×10^{-5}	2.313	—
	$M - FSI_{RL}$	130	9.96000×10^{-6}	1.438	—
	$M - FSI_C$	110	9.9500×10^{-6}	0.328	—
0.80	$M - SC$	49	9.21000×10^{-6}	0.203	—
	$M - BB1$	66	9.46000×10^{-6}	0.203	—
	$M - BB2$	48	9.98185×10^{-6}	0.234	—
	$M - FSI_{RL}$	47	9.84000×10^{-6}	0.230	—
	$M - FSI_C$	43	9.76000×10^{-6}	0.187	—
0.90	$M - SC$	17	9.08000×10^{-6}	0.047	—
	$M - BB1$	17	8.91000×10^{-6}	0.063	—
	$M - BB2$	16	9.65000×10^{-6}	0.093	—
	$M - FSI_{RL}$	16	9.92000×10^{-6}	0.109	—
	$M - FSI_C$	16	7.01000×10^{-6}	0.063	—
1.00	$M - SC$	2	9.57000×10^{-6}	0.016	*
	$M - BB1$	2	5.00000×10^{-8}	0.017	*
	$M - BB2$	2	2.00000×10^{-8}	0.008	*
	$M - FSI_{RL}$	2	1.00000×10^{-8}	0.001	*
	$M - FSI_C$	2	1.00000×10^{-8}	0.001	*

TABLE 4. Performance evaluation of the proposed $M - FSI_{RL}$ and $M - FSI_C$ methods against $M - SC$, $M - BB1$, and $M - BB2$ for solving $f_4(x)$ with different fractional orders $\wp$. * denotes convergence in 2 iterations; COC requires ≥ 3 iterations.

absolute error of 2.8×10^{-8} and highest COC of 2.95649. These observations demonstrate that proposed methods offer efficient and accurate solutions, particularly at higher fractional orders.

$f_5(x) = x^3 - 3.6x^2 - 36.4$					
Fract. order($\wp$)	Method	Iterations(n)	$(x_{n+1} - x_n)$	CPU time	COC
0.50	$M - SC$	500	7.98720×10^{-5}	3.219	—
	$M - BB1$	500	3.18909×10^{-1}	3.328	—
	$M - BB2$	500	3.08890×10^{-1}	2.844	—
	$M - FSI_{RL}$	500	2.75180×10^{-5}	2.763	—
	$M - FSI_C$	500	2.49900×10^{-5}	1.734	—
0.60	$M - SC$	415	9.96400×10^{-6}	1.578	—
	$M - BB1$	500	3.24392×10^{-3}	1.312	—
	$M - BB2$	500	3.11700×10^{-3}	1.375	—
	$M - FSI_{RL}$	305	9.94800×10^{-6}	1.278	—
	$M - FSI_C$	290	9.95600×10^{-6}	1.609	—
0.70	$M - SC$	138	9.77600×10^{-6}	0.562	—
	$M - BB1$	500	1.97480×10^{-5}	1.938	—
	$M - BB2$	500	1.87744×10^{-5}	1.609	—
	$M - FSI_{RL}$	117	9.83800×10^{-6}	0.532	—
	$M - FSI_C$	111	9.94500×10^{-6}	0.172	—
0.80	$M - SC$	49	9.76700×10^{-6}	0.234	—
	$M - BB1$	66	9.28300×10^{-6}	0.281	—
	$M - BB2$	49	9.87744×10^{-6}	0.250	—
	$M - FSI_{RL}$	46	9.45400×10^{-6}	0.232	—
	$M - FSI_C$	44	9.48100×10^{-6}	0.156	—
0.90	$M - SC$	18	7.94200×10^{-6}	0.156	—
	$M - BB1$	18	7.83300×10^{-6}	0.210	—
	$M - BB2$	16	8.89200×10^{-6}	0.203	—
	$M - FSI_{RL}$	18	7.25700×10^{-6}	0.296	—
	$M - FSI_C$	16	8.36800×10^{-6}	0.094	—
1.00	$M - SC$	4	2.10000×10^{-8}	0.056	2.03299
	$M - BB1$	3	2.90000×10^{-8}	0.031	2.94464
	$M - BB2$	3	2.90000×10^{-8}	0.031	2.94223
	$M - FSI_{RL}$	3	2.80000×10^{-8}	0.011	2.95649
	$M - FSI_C$	3	2.80000×10^{-8}	0.011	2.95649

TABLE 5. Performance evaluation of the proposed $M - FSI_{RL}$ and $M - FSI_C$ methods against $M - SC$, $M - BB1$, and $M - BB2$ for solving $f_5(x)$ with different fractional orders $\wp$.

4. COMPLEX FRACTAL STUDY OF ITERATIVE METHOD

Although local convergence analysis Theorem 2.1 guarantees convergence for initial guesses sufficiently close to root, it does not reveal global behavior of proposed method (2. 6) or how it performs for arbitrary initial points in complex plane. Fractal dynamics provide a powerful framework for this global analysis by visualizing basins of attraction,

sets of initial guesses that converge to each root. The geometry of algorithm's basins of attraction and fractal nature of their boundaries, known as Julia set, provide profound insight into its stability and reliability. Möbius transformations and parameter plane visualizations receive special attention from us as they allow us to find areas of stable convergence and avoid unstable parameter selection.

For this analysis, we begin with a rational function $\mathcal{R} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ defined in Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, where function $\mathcal{R}(x) = P(x)/Q(x)$ with P and Q as coprime polynomials. The *orbit* of a point $x_0 \in \hat{\mathbb{C}}$ is the sequence:

$$\text{orb}(x_0) = \{x_0, \mathcal{R}(x_0), \mathcal{R}^2(x_0), \dots, \mathcal{R}^n(x_0), \dots\}.$$

A point x_0 is called a *fixed point* if $\mathcal{R}(x_0) = x_0$, a *critical point* if $\mathcal{R}'(x_0) = 0$, and a *periodic point* of minimal period $p > 1$ if $\mathcal{R}^p(x_0) = x_0$ and $\mathcal{R}^k(x_0) \neq x_0$ for $0 < k < p$. Fixed points are further classified by their *multiplier* $|\mathcal{R}'(x_0)|$:

- attracting if $|\mathcal{R}'(x_0)| < 1$;
- super-attracting if $|\mathcal{R}'(x_0)| = 0$;
- repelling if $|\mathcal{R}'(x_0)| > 1$;
- neutral (or parabolic) if $|\mathcal{R}'(x_0)| = 1$.

The stability of an iterative method is related to dynamic stability of corresponding rational operator $\mathcal{R}_f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. The set of points converging to an attracting fixed point α defines its *basin of attraction*:

$$\mathcal{A}(\alpha) = \left\{ x_0 \in \hat{\mathbb{C}} \mid \mathcal{R}_f^n(x_0) \rightarrow \alpha \text{ as } n \rightarrow \infty \right\}.$$

The Julia set $\mathcal{J}(\mathcal{R})$ is fractal boundary that divides different basins of attraction, while Fatou set $\mathcal{F}(\mathcal{R}) = \hat{\mathbb{C}} \setminus \mathcal{J}(\mathcal{R})$ consists of open domains where dynamics are stable. In order to see these patterns, we take samples from complex plane in a rectangular grid pattern. Each starting point is subjected to a maximum of 25 iterations, and convergence is defined when $|f(x_n)| \leq 10^{-3}$. The output image shows basins of attraction with different colors and at their borders, complex fractal patterns of Julia set are revealed.

The scaling theorem [14] is an important tool used to analyze dynamics of this technique through coordinate transformations. In particular, for an affine map $T(x) = \alpha x + \beta$ (a specific type of Möbius map) and transformed function $g(x) = f(T(x))$, following conjugacy holds:

$$T \circ \mathcal{R}_g \circ T^{-1}(x) = \mathcal{R}_f(x),$$

This relationship shows that $\mathcal{R}_g$ and $\mathcal{R}_f$ are conjugated with T . The iterative method (2.6) applied to quadratic polynomials $f(x) = (x - a)(x - b)$ gives us a view of conjugacy that makes analysis very easy, since it shows that method's core dynamical structure. Julia set is universal under Möbius transformation. The stability analysis confirms that suitable parameter choices can enhance convergence properties and avoid undesirable dynamic behavior, as revealed by corresponding parameter planes.

The dynamical analysis in this section is developed under the assumption $\wp \approx 1$. Therefore, the resulting stability analysis and fractal structures are intended to provide qualitative insight near the integer-order case and do not constitute a complete characterization for all $\wp \in (0, 1]$.

The iterative method (2. 6) satisfies the scaling theorem, which allows for dynamic analysis of iterative function defined by polynomial $f(x) = (x - a)(x - b)$, where $a, b \in \mathcal{R}$. The fractional Caputo-type derivative of order $\wp \in (0, 1]$ is defined as:

$${}^c\mathcal{D}^\wp f(x) = \frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(3 - \wp)} x^{2-\wp} - \frac{\tilde{\Gamma}^{[*]}(2)}{\tilde{\Gamma}^{[*]}(2 - \wp)} x^{1-\wp}(a + b) + \frac{1}{\tilde{\Gamma}^{[*]}(1 - \wp)} x^{-\wp} ab \quad (4. 45)$$

The Möbius transformation and its inverse are:

$$M(u) = \frac{u - a}{u - b}, \quad M^{-1}(u) = \frac{ub - a}{u - 1} \quad (4. 46)$$

The method $M - FSI$ exhibits a universal Julia set if there exists a rational map that is conjugated by a Möbius transformation. For the rational map $\Theta_{M-FSI}(x, \Upsilon^{[o]})$ derived from the fractional two-step scheme (2. 6) applied to $f(x) = (x - a)(x - b)$, the Möbius transformation (4. 46) conjugates the iteration to:

$$\Theta_{M-FSI}(x, \Upsilon^{[o]}) = \frac{\xi_9 \left(\frac{\left(\xi_1 \left(\frac{\xi_2}{\xi_4} \right)^{\frac{2}{\wp}} + \xi_5 \left(\frac{\xi_2}{\xi_4} \right)^{\frac{3}{\wp}} + \Upsilon^{[o]} \xi_{12}^4 \left(\frac{\xi_2}{\xi_4} \right)^{\frac{4}{\wp}} + \xi_{13} \right) \xi_{10} \xi_3}{((\xi_7) \tilde{\Gamma}^{[*]}(2 - \wp) + \tilde{\Gamma}^{[*]}(3 - \wp) \tilde{\Gamma}^{[*]}(2) \tilde{\Gamma}^{[*]}(1 - \wp) \xi_{13}) \xi_{11}^2 x} \right)^{\frac{1}{\wp}} + \xi_8 + x \xi_{11}}{\xi_9 \left(\frac{\left(\xi_1 \left(\frac{\xi_2}{\xi_4} \right)^{\frac{2}{\wp}} + \xi_5 \left(\frac{\xi_2}{\xi_4} \right)^{\frac{3}{\wp}} + \Upsilon^{[o]} \xi_{12}^4 \left(\frac{\xi_2}{\xi_4} \right)^{\frac{4}{\wp}} + \xi_{13} \right) \xi_{10} \xi_3}{((\xi_7) \tilde{\Gamma}^{[*]}(2 - \wp) + \tilde{\Gamma}^{[*]}(3 - \wp) \tilde{\Gamma}^{[*]}(2) \tilde{\Gamma}^{[*]}(1 - \wp) \xi_{13}) \xi_{11}^2 x} \right)^{\frac{1}{\wp}} + \xi_8 + a - b} \quad (4. 47)$$

$$\begin{aligned} \xi_1 &= (\Upsilon^{[o]} x^2 + (4\Upsilon^{[o]} + 1)x + \Upsilon^{[o]}) (-1 + x)^2 (-b + a)^2 \\ \xi_2 &= \tilde{\Gamma}^{[*]}(\wp + 1) x (-b + a)^2 \tilde{\Gamma}^{[*]}(3 - \wp) \tilde{\Gamma}^{[*]}(2 - \wp) \xi_{10} \\ \xi_3 &= \tilde{\Gamma}^{[*]}(2 - \wp) \tilde{\Gamma}^{[*]}(3 - \wp) \\ \xi_4 &= \left(\tilde{\Gamma}^{[*]}(2) (-1 + x) (a + b) \tilde{\Gamma}^{[*]}(3 - \wp) + \tilde{\Gamma}^{[*]}(3) \tilde{\Gamma}^{[*]}(2 - \wp) (-bx + a) \right) \\ &\quad (-bx + a) \tilde{\Gamma}^{[*]}(1 - \wp) + ab(\xi_3) (-1 + x)^2 \\ \xi_5 &= 2\Upsilon^{[o]} (x + 1) (-1 + x)^3 (-b + a) \\ \xi_6 &= ((2\Upsilon^{[o]} + 1) x^2 - 2\Upsilon^{[o]} - 1) \left(\frac{\xi_2}{\xi_4} \right)^{\frac{1}{\wp}} + x (\Upsilon^{[o]} + 1) (-b + a) \\ \xi_7 &= (-1 + x)^2 ab \tilde{\Gamma}^{[*]}(3 - \wp) + \tilde{\Gamma}^{[*]}(3) \tilde{\Gamma}^{[*]}(1 - \wp) (-bx + a)^2 \\ \xi_8 &= (-1 + x) \left(\frac{\xi_2}{\xi_4} \right)^{\frac{1}{\wp}} \\ \xi_9 &= (-1 + x) \tilde{\Gamma}^{[*]}(\wp + 1) \\ \xi_{10} &= \tilde{\Gamma}^{[*]}(1 - \wp) \left(\frac{bx - a}{x - 1} \right)^{\wp} \\ \xi_{11} &= (-b + a) \\ \xi_{12} &= (-1 + x) \\ \xi_{13} &= \xi_{12} (a + b) (-bx + a) \\ \xi_{14} &= (\xi_6) \xi_{11}^3 x \end{aligned}$$

Taking the limit $\wp \rightarrow 1$ in the fractional derivative (4. 45):

$$\begin{aligned} \lim_{\wp \rightarrow 1} {}^c \mathfrak{D}^\wp f(x) &= \lim_{\wp \rightarrow 1} \left[\frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(3-\wp)} x^{2-\wp} - \frac{\tilde{\Gamma}^{[*]}(2)}{\tilde{\Gamma}^{[*]}(2-\wp)} x^{1-\wp} (a+b) + \frac{ab x^{-\wp}}{\tilde{\Gamma}^{[*]}(1-\wp)} \right] \\ &= \frac{\tilde{\Gamma}^{[*]}(3)}{\tilde{\Gamma}^{[*]}(2)} x^1 - \frac{\tilde{\Gamma}^{[*]}(2)}{\tilde{\Gamma}^{[*]}(1)} x^0 (a+b) + ab \lim_{\wp \rightarrow 1} \frac{x^{-\wp}}{\tilde{\Gamma}^{[*]}(1-\wp)} \\ &= 2x - (a+b) + ab \cdot 0 \\ &= 2x - a - b = f'(x) \end{aligned} \quad (4. 48)$$

where we used $\lim_{\wp \rightarrow 1} 1/\tilde{\Gamma}^{[*]}(1-\wp) = 0$ since $\tilde{\Gamma}^{[*]}$ has a pole at 0. The following theorem represents conjugate via Möbius transformation for $\wp \approx 1$.

Theorem 4.1. For a rational map $\Theta_{M-FSI}(x)$ derived from (2. 6) and applied to $f(x)$ when $\wp \approx 1$, $\Theta_{M-FSI}(x)$ is conjugate via a Möbius transformation:

$$\Theta_{M-FSI}(x, \Upsilon^{[\circ]}) = - \frac{(x^3 + 4x^2 - \Upsilon^{[\circ]} + 5x + 2) x^3}{-1 + (\Upsilon^{[\circ]} - 2) x^3 - 5x^2 - 4x} \quad (4. 49)$$

Proof. Let $f(x) = (x-a)(x-b)$, where $a, b \in \mathcal{R}$. The Möbius transformation is given by

$$M(u) = \frac{u-a}{u-b}, \quad (4. 50)$$

with inverse

$$[M(u)]^{-1} = \frac{ub-a}{u-1}, \quad (4. 51)$$

which is considered as a map from $\mathbb{C} \cup \infty$. Then we have

$$\Theta_{M-FSI}(x, \Upsilon^{[\circ]}) = - \frac{(x^3 + 4x^2 - \Upsilon^{[\circ]} + 5x + 2) x^3}{-1 + (\Upsilon^{[\circ]} - 2) x^3 - 5x^2 - 4x}. \quad (4. 52)$$

□

Note that parameters a and b have been omitted in the expression $\Theta_{M-FSI}(x, \Upsilon^{[\circ]})$. The union of stability function corresponding to all strange fixed points of rational map (4. 49) is illustrated in Figure 2. To analyze stability of critical points of rational function

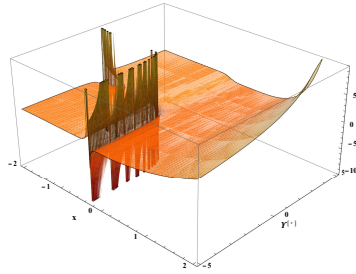


FIGURE 2. Union formed by the stability function corresponding to all strange fixed points of the rational map.

(4. 49) for fractional iterative method $M-FSI$ in (2. 6), we calculate $\Theta'_{M-FSI}(x, \Upsilon^{[\circ]})$ as:

$$\Theta'_{M-FSI}(x, \Upsilon^{[\circ]}) = - \frac{3x^2 (x+1)^4 \left((\Upsilon^{[\circ]} - 2) x^2 + \left(-\frac{4\Upsilon^{[\circ]}}{3} - 4 \right) x + \Upsilon^{[\circ]} - 2 \right)}{(-1 + (\Upsilon^{[\circ]} - 2) x^3 - 5x^2 - 4x)^2}. \quad (4. 53)$$

The graph of function given in (4. 53) is shown in Figure 3. In the study of rational

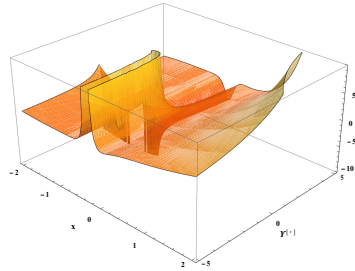


FIGURE 3. Derivative of the union formed by the stability function of the rational map.

functions obtained from fractional iterative method (2. 6) to solve nonlinear equations, fixed points are values that remain unchanged when iteration function is applied.

Theorem 4.2. Using equation $|\Theta_{M-FSI}(x, \Upsilon^{[\circ]})| = x$ when $\wp \approx 1$, we arrive at the following assertions for fixed points of $\Theta_{M-FSI}(\tilde{\rho})$.

- The super-attracting fixed points $\tilde{\rho}_0 = 0$ and $\tilde{\rho}_\infty = \infty$ correspond to the actual roots of the polynomial being solved.
- $\tilde{\rho}_1 = 1$ is the strange fixed point. The other extraneous fixed points depend on the solution of the below given 4th degree polynomial.

$$P(d) = d^4 + 5d^3 + (\Upsilon^{[\circ]} + 8) d^2 + 5d + 1. \quad (4. 54)$$

To analyze stability of extraneous fixed points, we exploit symmetry of polynomial (4. 54). Specifically, we reduce it to a second-degree polynomial under following conditions:

- $d = 0$ is not a root.
- If $d = \omega$ is a root, then $d = \frac{1}{\omega}$ is also a root.

First, we rewrite (4. 54) in terms of symmetric expressions:

$$d^4 + 5d^3 + (\Upsilon^{[\circ]} + 8) d^2 + 5d + 1 = 0,$$

Let $x = d + \frac{1}{d}$. Then, we have:

$$d^2 + \frac{1}{d^2} = x^2 - 2,$$

Dividing by d^2 (assuming $d \neq 0$):

$$d^2 + 5d + (\Upsilon^{[0]} + 8) + \frac{5}{d} + \frac{1}{d^2} = 0.$$

$$\left(d^2 + \frac{1}{d^2}\right) + 5\left(d + \frac{1}{d}\right) + (\Upsilon^{[0]} + 8) = 0,$$

Rewriting in terms of x :

$$(x^2 - 2) + 5x + (\Upsilon^{[0]} + 8) = 0,$$

$$x^2 + 5x + (\Upsilon^{[0]} + 6) = 0.$$

Now, we calculate roots of equation of 2^{nd} degree as:

$$x_{1,2}(\Upsilon^{[0]}) = \frac{-5 \pm \sqrt{25 - 4(\Upsilon^{[0]} + 6)}}{2} = \frac{-5 \pm \sqrt{1 - 4\Upsilon^{[0]}}}{2}.$$

Now, recall that $x = d + \frac{1}{d}$, so:

$$d^2 - xd + 1 = 0.$$

For each x_i , we get two roots:

$$d = \frac{x_i(\Upsilon^{[0]}) \pm \sqrt{(x_i(\Upsilon^{[0]}))^2 - 4}}{2}.$$

Thus, four roots that are extraneous fixed points are:

$$Ex_1(\Upsilon^{[0]}) = \frac{x_1(\Upsilon^{[0]}) + \sqrt{(x_1(\Upsilon^{[0]}))^2 - 4}}{2},$$

$$Ex_2(\Upsilon^{[0]}) = \frac{(x_1(\Upsilon^{[0]})) - \sqrt{(x_1(\Upsilon^{[0]}))^2 - 4}}{2},$$

$$Ex_3(\Upsilon^{[0]}) = \frac{(x_2(\Upsilon^{[0]})) + \sqrt{(x_2(\Upsilon^{[0]}))^2 - 4}}{2},$$

$$Ex_4(\Upsilon^{[0]}) = \frac{(x_2(\Upsilon^{[0]})) - \sqrt{(x_2(\Upsilon^{[0]}))^2 - 4}}{2},$$

where:

$$x_{1,2}(\Upsilon^{[0]}) = \frac{-5 \pm \sqrt{1 - 4\Upsilon^{[0]}}}{2}.$$

These extraneous fixed points, also called strange fixed points that do not correspond to roots of rational function and vary depending on the value of parameter $\Upsilon^{[0]}$.

To investigate stability of strange fixed points and identify critical points, we employ differential operator associated with iterative method (2.6). When $x = 1$, operator $\Theta'_{M-FSI}(x, \Upsilon^{[0]})$ given in (4.53) results in

$$\left| \Theta'_{M-FSI}(1, \Upsilon^{[0]}) \right| = \left| \frac{32}{\Upsilon^{[0]} - 12} \right|. \quad (4.55)$$

Analyzing equation (4.55), as $\Upsilon^{[0]} \rightarrow \pm\infty$, we observe horizontal asymptotes $\Upsilon^{[0]} = 12$, where $\Theta'_{M-FSI}(1, \Upsilon^{[0]}) \rightarrow 0$. In following result, we discuss stability of strange fixed point at $x = 1$. Figure 4 shows zones of stability of strange fixed points in $\Theta'_{M-FSI}(1, \Upsilon^{[0]})$.

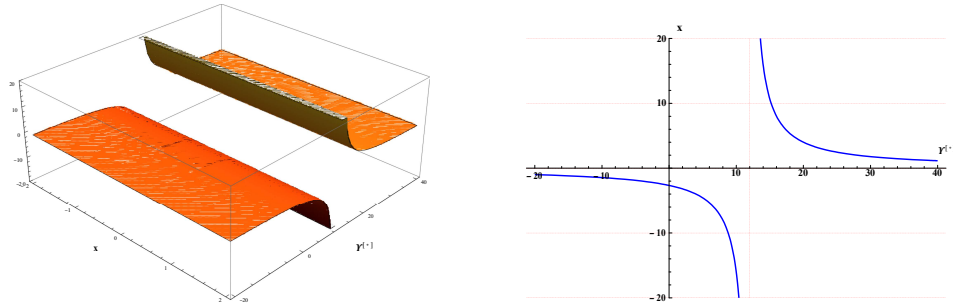


FIGURE 4. Three-dimensional view and corresponding two-dimensional projection of stability regions associated with strange fixed points of $\Theta'_{M-FSI}(1, \Upsilon^{[0]})$.

Theorem 4.3. For dynamical system governed by parameter $\Upsilon^{[0]} \in \hat{\mathbb{C}}$, fixed point $x = 1$ exhibits distinct stability properties determined by its position relative to circle $|\Upsilon^{[0]} - 12| = 32$ in complex plane. Specifically:

- (1) $x = 1$ is repulsive when $\left| \frac{32}{\Upsilon^{[0]} - 12} \right| > 1$, equivalent to $|\Upsilon^{[0]} - 12| < 32$.
- (2) $x = 1$ is attractive when $\left| \frac{32}{\Upsilon^{[0]} - 12} \right| < 1$, equivalent to $|\Upsilon^{[0]} - 12| > 32$.
- (3) $x = 1$ is neutral when $\left| \frac{32}{\Upsilon^{[0]} - 12} \right| = 1$, equivalent to $|\Upsilon^{[0]} - 12| = 32$.
- (4) $x = 1$ is never superattracting since $\left| \frac{32}{\Upsilon^{[0]} - 12} \right| \neq 0$, as $\frac{32}{\Upsilon^{[0]} - 12}$ cannot be zero.

Proof. Starting from stability condition:

$$\left| \frac{32}{\Upsilon^{[0]} - 12} \right| \leq 1 \iff |\Upsilon^{[0]} - 12| \geq 32. \quad (4.56)$$

Consider a complex number $\Upsilon^{[0]}$ expressed as $\Upsilon^{[0]} = \alpha + i\beta$ where $\alpha, \beta \in \mathcal{R}$. Then we compute

$$|\Upsilon^{[0]} - 12|^2 = \left(\sqrt{(\alpha - 12)^2 + \beta^2} \right)^2 = (\alpha - 12)^2 + \beta^2.$$

Substituting into (4.56) gives:

$$1024 \leq (\alpha - 12)^2 + \beta^2.$$

This inequality describes all points in complex plane outside or on circle centered at $(12, 0)$ with radius 32. For fixed point $x = 1$ in dynamical system has:

Repulsive point: When $\left| \frac{32}{\Upsilon^{[0]} - 12} \right| > 1$ ($|\Upsilon^{[0]} - 12| < 32$), trajectories near $x = 1$ are repelled. This occurs inside circle of radius 32 centered at $\Upsilon^{[0]} = 12$.

Attractive point: When $\left| \frac{32}{\Upsilon^{[0]} - 12} \right| < 1$ ($|\Upsilon^{[0]} - 12| > 32$), trajectories are attracted to $x = 1$. This occurs outside circle.

Parabolic point: When $\left| \frac{32}{\Upsilon^{[0]} - 12} \right| = 1$ ($|\Upsilon^{[0]} - 12| = 32$), point is neutrally stable (parabolic). This occurs exactly on circle boundary.

Superattractive point: The condition $\frac{d}{dz} \left(\frac{32}{\Upsilon^{[0]} - 12} \right) = 0$ has no solution, so no superattractive point exists. $\square$

4.4. Analysis of critical points. We analyze critical points of equation (4.49) satisfy $|\Theta'_{M-FSI}(x, \Upsilon^{[0]})| = 0$, where $\Upsilon^{[0]} \neq 0$. For $\Theta'_{M-FSI}(\tilde{\sigma})$ when $\wp \approx 1$, critical points are $\tilde{\sigma} = 0$, $\tilde{\sigma} = \infty$ and $\tilde{\sigma} = -1$, and

$$Cr_1(\Upsilon^{[0]}) = \frac{2\Upsilon^{[0]} + 6 + \sqrt{-5(\Upsilon^{[0]})^2 + 60\Upsilon^{[0]}}}{3(\Upsilon^{[0]} - 2)},$$

$$Cr_2(\Upsilon^{[0]}) = -\frac{-2\Upsilon^{[0]} - 6 + \sqrt{-5(\Upsilon^{[0]})^2 + 60\Upsilon^{[0]}}}{3(\Upsilon^{[0]} - 2)}.$$

where $\Upsilon^{[0]} \neq 2$.

Lemma 4.5. Analyzing equation $|\Theta'_{M-FSI}(x, \Upsilon^{[0]})| = 0$, one obtains following results.

- (a) If $\Upsilon^{[0]} = 0$, there is one free critical point $Cr_{1,2}(0) = -1$.
- (b) If $\Upsilon^{[0]} = 1$, then there are three free critical points:

$$\tilde{\sigma} = -1, \quad Cr_1(1) = -\frac{8 + \sqrt{55}}{3}, \quad Cr_2(1) = -\frac{8 - \sqrt{55}}{3}.$$

- (c) In case of $\Upsilon^{[0]} = 12$, free critical point is $Cr_{1,2}(12) = 1$.
- (d) In any other case of $\Upsilon^{[0]}$, free critical points are $\tilde{\sigma} = -1$, $Cr_1(\Upsilon^{[0]})$, and $Cr_2(\Upsilon^{[0]})$.

4.6. Parametric planes. Parametric planes serve as a crucial visual aid for analyzing stability and convergence of iterative root finding methods. By illustrating basins of attraction for each root and highlighting regions of divergence or chaos. The system identifies essential dynamic characteristics which include critical points and fixed points. The graphical feedback mechanism helps to enhance algorithm performance through improved speed and dependable operation.

We employ these dynamical planes to assess the stability of a single root finding method which operates at fractional orders within the range of $\Upsilon^{[0]} \in (0, 1]$. The illustrations use different colors to show various behaviors: orange indicates convergence to a root that maps $\Theta(x)$ to 0 while black shows the path to infinity. The black points show that the process did not reach a conclusion while the white stars show the locations of fixed points. The analysis creates parametric planes by testing initial values within the range of $[-6, 6] \times [-6, 6]$ using MATLAB R2014a.

The study examines specific examples within stability region for parameter values $\Upsilon^{[0]} \in \{1, -1, 0.5, -0.5, \text{ and } 0\}$ and $\Upsilon^{[0]} \in \{-2\sqrt{-1}, -2\sqrt{-1}, 4 + \sqrt{15}, \sqrt{-55}, \frac{16 + 5\sqrt{7}}{9}, \text{ and } \frac{1}{4} \left(-\frac{4}{3} + \frac{8\sqrt{-2}}{3} \right)\}$, with their corresponding dynamic planes and attraction basins illustrated in Figures (5a-5e) and (6a-6d). The dynamics yield only two basins of attraction: one for root in $x = 0$ (shown in orange) and another for $x = \infty$ (shown in black). The corresponding basin plots display large, contiguous regions with smooth, well-defined boundaries. This organized structure points to a highly stable method for these parameters. Vast majority of starting points in plotted area will converge quickly to one of the two roots, with little dependence on exact initial guess. In contrast, the method shows clear instability

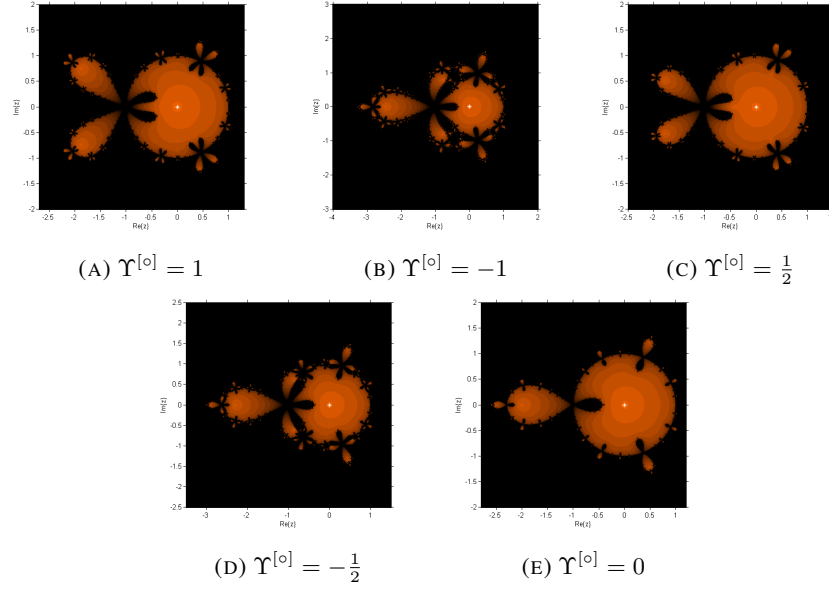


FIGURE 5. Parametric planes exhibiting stable behavior for $\Upsilon^{[0]} = 1, -1, 0.5, -0.5$, and 0

for values of $\Upsilon^{[0]} \in \{-10, \text{ and } -20\}$. In Figures (7a-7b), color coding distinguishes dynamic regions, and white asterisks ('*') mark attracting points. The basin structure exposed by these images is extremely fragmented and chaotic. The large black regions signify divergence of many initial points, whereas complex fractal borders signify a very high sensitivity to starting position. Such a visual proof strongly classifies $\Upsilon^{[0]} = -10$ as an improper parameter selection for practical use. The contrast between stable and unstable behaviors is vividly seen in Figures 5, 6, and 7.

5. CONCLUSION

This study successfully addresses a significant gap in existing research by constructing a multistep fractional iterative algorithm. We introduce a two step Newton type fractional iterative method based on Caputo fractional derivative, which is a considerable extension of ideas within framework of fractional calculus. The resulting algorithm not only achieves local convergence order of at least $2\varphi + 1$, but also remains computationally efficient. It requires only two function evaluations and one fractional derivative per iteration. Our theoretical convergence analysis is robustly validated through extensive numerical experimentation on a diverse range of real-world problems, spanning from engineering models, such as quarter-car suspension system, to scientific applications, including determination of chemical equilibrium. The consistent performance of algorithm, as evidenced by low absolute error and reduced elapsed time, highlights its stability and practical benefits compared to current approaches. This demonstrated effectiveness is further solidified by in-depth fractal studies of critical and fixed points, which verify algorithm's reliable stability and

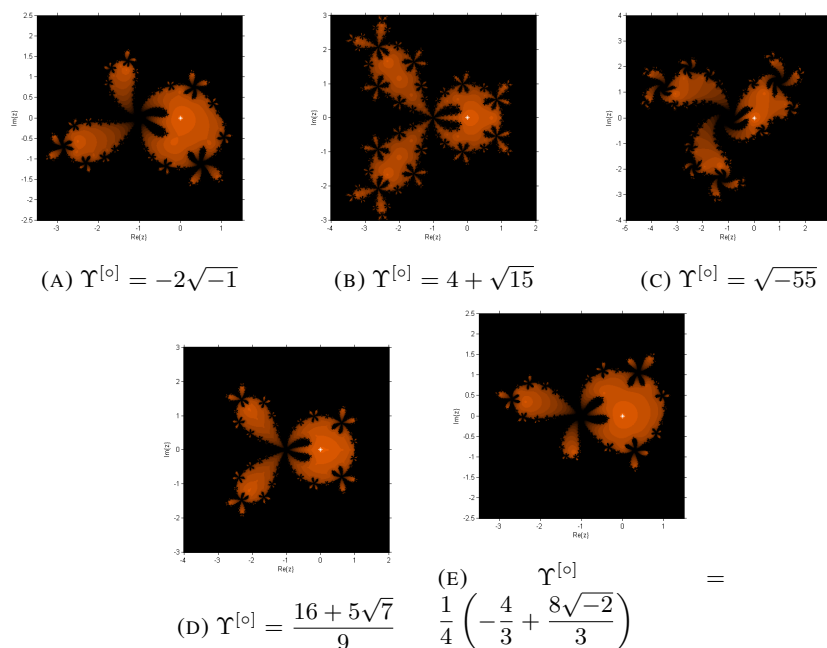


FIGURE 6. Parametric planes exhibiting stable behavior for $\Upsilon^{[0]} = -2\sqrt{-1}$, $4 + \sqrt{15}$, $\sqrt{-55}$, $\frac{16 + 5\sqrt{7}}{9}$, and $\frac{1}{4} \left(-\frac{4}{3} + \frac{8\sqrt{-2}}{3} \right)$

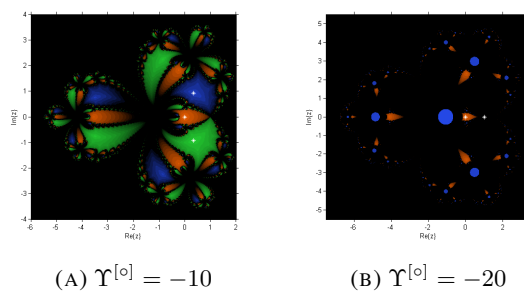


FIGURE 7. Parametric planes exhibiting unstable behavior for $\Upsilon^{[0]} = -10$, and -20

strong convergence behavior. Therefore, this work makes a substantial contribution to numerical analysis, providing a powerful and stable tool for tackling nonlinear equations and laying a foundational framework for future advances in increasing fractional order of fractional iterative methods. The proposed methods are developed for solving scalar nonlinear equations using Caputo and Riemann-Liouville fractional derivatives. Future research will investigate their extension to systems of nonlinear equations, other fractional derivative

definitions (e.g., Atangana-Baleanu and Caputo-Fabrizio), and the construction of optimal higher-order fractional iterative schemes with improved convergence properties.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Faiza Akram: Conceptualization, Data Curation, Formal Analysis, Investigation, Methodology, Software, Writing-Original Draft, Visualization. Saima Akram: Conceptualization, Supervision, Formal Analysis, Software, Validation, Writing-Review & Editing. Muhammad Ibrahim: Formal Analysis, Software, Supervision, Validation, Writing-Review & Editing. All authors have read and agreed to the published version of the manuscript.

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