Punjab University Journal of Mathematics (2025), 57(03), 309-339
https://doi.org/10.52280/pujm.2025.57(03)04

On Novel Extension of n-Polynomial Pre-invex Functions and Related Post-Quantum Integral Inequalities with Application

*Tariq Nawaz
Department of Mathematical Sciences,
UET, Taxila, Pakistan

Azeem Shahzad
Department of Mathematical Sciences,
UET, Taxila, Pakistan

*Corresponding Author

Email: tariqnawaz93@yahoo.com

Received:05 May, 2025 / Accepted: 25 August, 2025 / Published online: 23 September, 2025

Abstract. In this paper, we introduce a new class of n-polynomial pre-invex functions to investigate the refined bounds of Hermite-Hadamard type inequalities involving post-quantum integrals. We establish an ad-vanced and comprehensive multi-parameter identity and several post-quantum integrals to aid our main results. By applying this generic identity, we es-timate the unique and sharp bounds for the functions which are bounded and satisfy Lipschitz condition. In order to find the optimal bounds, we examine these results for the best choice of parameters involved and ex-press those graphically to validate our results. In the end, we calculate certain special means as the application of presented results.

AMS (MOS) Subject Classification Codes: 26A51; 26A33; 26D07; 26D10; 26D15
Key Words: Pre-invex function;Multi parameters;n-polynomial pre-invexity;Power mean inequality;Lipschitz condition.

1. Historical background

Infinitesimal calculus without the concept of limits is actually the quantum calculus, sometimes known as calculus without limit. In quantum calculus, there are two different forms of calculus: q-calculus and h-calculus. Both kinds seek to identify ”analogs” of mathematical things that yield the original object upon taking a given limit. The q-analog's limit as q goes to 1 is obtained in q-calculus and in h-calculus, the h-analog's limit is determined as h goes to 0. F.H. Jackson [11] developed the quantum calculus in the early 19th century, although Jacobi and Euler had already dealt with this kind of calculus as introductory concept. The geometrically increased requirement for mathematics that simulates quantum computing has recently sparked interest in this area and is emerged as a link between physics and mathematics. Numerous mathematical fields including number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences including quantum physics, theory of relativity, and mechanics, have found extensive uses for it. For further studies on quantum calculus, one can see ([3, 8, 9, 12, 14, 17, 29, 30, 31, 32, 33, 38]) and references therein. Recently, Tariboon and Ntouyas ([34],[35]) introduced the q-derivative and q-integrals over the finite interval and investigated several q-analogs of integral inequalities including Hermite-Hadamard inequality. In order to extend the idea of q-integrals, Chakarabarti and Jajonnathan [2] came up with the notion of (p, q)-derivatives and (p, q)-integrals by introducing a new parameter p to the quantum calculus to achieve what is known as post-quantum calculus and further investigated in [13]. Tunc and Gov [37] examined a new idea of (p, q)-derivatives over the finite interval and in ([4, 20, 27]), the authors reproduced fundamental theorem of calculus for $(p.q)$-integrals, (p, q)-Taylor's formula, (p, q)-beta function and (p, q)-gamma functions respectively to utilize these concepts in hyper geometric series and Lie algebra. To recover q-calculus from (p, q)-calculus, we utilize $p = 1$ and for classical calculus we further incorporate the condition $q \to 1^-$. In [7], Du et al. initiated the study of quantum integrals by making use of multi parameter identity to establish the novel inequalities for q-differentiable mappings and Raees et al. [26] further extended the idea for post-quantum integrals in the setting of n-polynomial convex functions.

Preinvex functions are one of the most important generalizations of convex function introduced by Wier and Mond [39] by considering the invex functions introduced by Hanson [10]. Several authors are of the view that there are numerous uses for these functions in mathematical programming and optimization theory. As compared to convexity, preinvexity is a more universal concept in the sense of generalization. The unique characterization of pre-invex function has led its usage in multiple directions. For a fruitful study of invex and pre-invex functions, we refer to ([5, 6, 18, 19, 21, 23, 25, 40]). In [36], Sarikaya et al. considered pre-invex and Log-pre-invex functions to estimate the bounds for left hand side of Hermite-Hadamard type inequalities for differentiable class of functions. In [15], Latif and Shoaib introduced the concept of m-pre-invex and (α, m)-pre-invex functions to investigate Hermite-Hadamard type inequalities for differentiable class of functions. As consequence of the presented results, the authors established these inequalities by taking $\eta(b, a) = b - a$ for both m-pre-invex and (α, m)-pre-invex functions to extend the several classical inequalities.

Motivated by post-quantum integrals and related results in ([1, 2, 3, 13, 20, 22, 24, 27, 34, 35, 36, 37]), specifically the results of [7, 26] and [39], as well as the pre-invex functions and corresponding integral inequalities in ([5, 6, 18, 19, 21, 23, 25, 40]) with the results of ([15, 36]) and [40]. Our goal in this paper is to present a generalized n-polynomial pre-invex function that gives rise to a number of new pre-invexities. Standard convex, invex, or pre-invex assumptions are too restrictive to capture the structural properties of real-world problems, which is why the class of n-polynomial pre-invex functions was introduced. This is done in order to increase the usefulness of generalized convexity in optimization theory. Non-convex and non-differentiable objective functions or constraints are present in many real-world optimization problems in applied sciences, engineering, and

economics, yet they nevertheless display some ”generalized convex-like” behavior. Specifically, polynomial-type functions which are frequently used to simulate biological, physical, and economic phenomena?frequently fall short of the rigorous requirements of convexity or even pre-invexity, yet they behave consistently inside a specific invex structure. With this expansion, we can apply well-known inequalities (such Hermite-Hadamard type) and optimality requirements to a broader class of functions that are not currently included in pre-invex or quasi-invex frameworks but are commonly seen in real-world models Since not every convex function is pre-invex, so the other motivation of this new class is the study of non-convex functions via polynomial functions. We develop a novel identity with multi parameters and establish a few post-quantum integrals to deal with our key result including applications and graphical representation.
The organization of the paper is as follows: in section 1, historical background and introduction is provided. Section 2 contains a brief literature survey on post-quantum integrals and pre-invex functions essential for this investigation. In section 3, new n-polynomial pre-invex functions and their possible outcomes along with their graphs to prove the validity of the presented results are given. Section 4 explains the treatment of main results to generate the applications involved in produced inequalities to explain the usability and usefulness therein. In last section, a comprehensive conclusion is given and furthermore, to seek the attention of researchers, few interesting open problems related to post-quantum integrals involving n-polynomial pre-invexty are suggested.

2. Preliminaries

We now recall some fundamental concepts from (p, q)-calculus which are essential for further investigation.

Definition 2.1. *([37]) Let $f : [\hat{e_1}, \hat{e_2}] \to \mathbb{R}$ be any continuous function along with $0 < q < p \leq 1$, then $(p, q)_{\hat{e_1}}$-derivative of f is characterized by the following quotient:*

$$ {}_{\hat{e_1}}\mathbf{D}_{p,q} f(\sigma) = \frac{f\left(p\sigma + (1-p)\hat{e_1}\right) - f\left(q\sigma + (1-q)\hat{e_1}\right)}{(p-q)(\sigma - \hat{e_1})}, \quad \sigma \neq \hat{e_1}. \tag{2. 1}$$

Remark 2.2. *It is pertinent to note that from above (p, q)-derivative, following deductions are evident:*

(1) *If $\hat{e_1} = 0$, then (2. 1) turns to classical sense as follows:*

$$ {}_{0}\mathbf{D}_{p,q} f(\sigma) = \mathbf{D}_{p,q} f(\sigma) = \frac{f(p\sigma) - f\left(q\sigma\right)}{p\sigma - q\sigma}, \quad \sigma \neq 0, \tag{2. 2}$$

(2) *If $p = 1$, then the $q_{\hat{e_1}}$-derivative in [34, 35] is obtained:*

$$ {}_{\hat{e_1}}\mathbf{D}_{q} f(\sigma) = \frac{f(\sigma) - f\left(q\sigma + (1-q)\hat{e_1}\right)}{(1-q)(\sigma - \hat{e_1})}, \quad \sigma \neq \hat{e_1}, \tag{2. 3}$$

(3) *For $p = 1$ and $\hat{e_1} = 0$, following q-derivative [14] is achieved:*

$$ \mathbf{D}_{q} f(\sigma) = \frac{f(q\sigma) - f\left(\sigma\right)}{q\sigma - \sigma}. \tag{2. 4}$$

Definition 2.3. *([37]) Let $f : [\hat{e_1}, \hat{e_2}] \to \mathbb{R}$ be any continuous function along with $0 < q < p \leq 1$. The definite $(p,q)_{\hat{e_1}}$-integral of the function f is characterized as:*

$$\int\limits_{\hat{e_1}}^{\hat{c}} f(\sigma)_{\hat{e_1}} d_{p,q}\sigma = (p-q)(\hat{c}-\hat{e_1}) \sum_{r=0}^{\infty} \frac{q^r}{p^{r+1}} f\left(\frac{q^r}{p^{r+1}}\hat{c} + \left(1 - \frac{q^r}{p^{r+1}}\right)\hat{e_1}\right), \hat{c} \in [\hat{e_1}, \hat{e_2}]. \tag{2. 5}$$

Definition 2.4. *([27]) Let f be an arbitrary function, $\hat{e_1}$ and $\hat{e_2}$ be two nonnegative numbers such that $\hat{e_1} < \hat{e_2}$, then we define*

$$\int\limits_{\hat{e_1}}^{\hat{e_2}} f(\sigma) d_{p,q}\sigma = \int\limits_{0}^{\hat{e_2}} f(\sigma) d_{p,q}\sigma - \int\limits_{0}^{\hat{e_1}} f(\sigma) d_{p,q}\sigma. \tag{2. 6}$$

Now, we depict few basic concepts from the pre-invex sets and its different generalization starting from definition.

Definition 2.5. *([39]) Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be any function and $Ķ \subseteq \mathbb{R}^n$, then $Ķ$ is an invex set if it satisfies the following condition:*

$$\hat{e_1} + \eta(\hat{e_2}, \hat{e_1}) \in Ķ \text{ for all } \hat{e_1}, \hat{e_2} \in Ķ,$$

for all $t \in [0,1]$.

It is worth noting that a convex set is pre-invex but converse is not true in general for example:

Example 2.6. *([40]) Let $Ķ = [-3,-2] \cup [-1,2]$ and*

$$\eta(\hat{e_1}, \hat{e_2}) = \begin{cases} \hat{e_1} - \hat{e_2} & \textit{if} \quad 2 \geq \hat{e_1} \geq -1, 2 \geq \hat{e_2} \geq -1; \\ \hat{e_1} - \hat{e_2} & \textit{if} \quad -3 \leq \hat{e_1} \leq -2, -3 \leq \hat{e_2} \leq -2; \\ -3 - \hat{e_2} & \textit{if} \quad -1 \leq \hat{e_1} \leq 2, -3 \leq \hat{e_2} \leq -2; \\ -1 - \hat{e_2} & \textit{if} \quad -3 \leq \hat{e_1} \leq -2, -1 \leq \hat{e_2} \leq 2. \end{cases}$$

Which is the example of invex set that is not convex.

Definition 2.7. *([39]) Let $Ķ \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : Ķ \times Ķ \to \mathbb{R}^n$. Then, the function $f : Ķ \to \mathbb{R}$ is pre-invex with respect to η, if for all $\hat{e_1}, \hat{e_2} \in Ķ$ and $\hat{t} \in [0,1]$*

$$f(\hat{e_1} + \hat{t}\eta(\hat{e_2}, \hat{e_1})) \leq (1-t)f(\hat{e_1}) + tf(\hat{e_2}). \tag{2. 7}$$

Pre-invex functions enjoy as the powerful extension of convex function since every convex function is pre-invex with respect to the mapping $\eta(\hat{e_2}, \hat{e_1}) = \hat{e_2} - \hat{e_1}$ but converse is not true. The most famous example in this context is $f(x) = -|x|$ which is pre-invex but not convex. In [21], Mohan and Neogy introduced a special condition $'\mathrm{C}'$ under which a differentiable invex function becomes a pre-invex function.

Condition C: The function $\eta : Ķ \times Ķ \to \mathbb{R}^n$ claim to satisfy condition $'\mathrm{C}'$ if following identities hold:

$$\eta(\hat{e_2}, \hat{e_2} + \hat{t}\eta(\hat{e_1}, \hat{e_2})) = -\hat{t}\eta(\hat{e_1}, \hat{e_2}),$$
$$\eta(\hat{e_1}, \hat{e_2} + \hat{t}\eta(\hat{e_1}, \hat{e_2})) = (1-\hat{t})\eta(\hat{e_1}, \hat{e_2}).$$

Definition 2.8. *([15]) Let f be any function on an invex set $Ķ$, then f is claimed to be (α, m)-pre-invex function with respect to η if*

$$f(\hat{e_1} + \eta(\hat{e_1}, \hat{e_2})) \leq (1 - \hat{t}^{\alpha})f(\hat{e_1}) + m\hat{t}^{\alpha} f\big(\frac{\hat{e_2}}{m}\big), \tag{2. 8}$$

holds for all $\hat{e_1}, \hat{e_2} \in Ķ, \hat{t} \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

Remark 2.9. *It is significant to note that, from (2. 8), one can achieve m-preinvex function [15] by setting $\alpha = 1$ and the classical pre-invexity by substituting $\alpha = 1 = m$.*

Definition 2.10. *([16]) Assume that $f : Ķ \to \mathbb{R}$ is any function on an invex set $Ķ$ w.r.t the mapping η. Further, let h be any non negative real valued function, then for $n \in \mathbb{N}$, f is called (n, h)-polynomial convex function if it satisfies following inequality:*

$$f\left(\hat{t}\hat{e_1} + (1 - \hat{t})\hat{e_2}\right) \leq \frac{1}{n}\sum_{k=1}^{n}(1 - (h(\hat{t}))^k)f(\hat{e_1}) + \frac{1}{n}\sum_{k=1}^{n}\left(h(\hat{t})\right)^k f(\hat{e_2}), \tag{2. 9}$$

for all $\hat{e_1}, \hat{e_2} \in Ķ$ and $\hat{t}, \in [0, 1]$.

Remark 2.11. *If we take $n = 1$ and $h(\hat{t}) = 1 - \hat{t}$ in (2. 9), classic convex function is reproduced and for $h(\hat{t}) = 1 - \hat{t}$, modified n-polynomial convex function is achieved.*

In [33], the authors established following lemma based on quantum integrals and estimated corresponding estimates.

Lemma 2.12. *Assume that $f : Ķ \to \mathbb{R}$ is q differentiable function on the interior of $Ķ$ such that $0 < q \leq 1$. Further, if ${}_{\hat{e_1}}D_q f$ is integrable on $Ķ$, then we have the following identity:*

$$\frac{qf(\hat{e_1}) + f(\hat{e_2})}{1 + q} - \frac{1}{\hat{e_2} - \hat{e_2}}\int_{\hat{e_1}}^{\hat{e_2}} f(\hat{t})\,{}_{\hat{e_1}}d_q\hat{t}$$
$$= \frac{q(\hat{e_2} - \hat{e_1})}{1 + q}\int_{0}^{1}((1 + q)\hat{t} - 1)\,{}_{\hat{e_1}}D\left((1 - \hat{t})\hat{e_1} + \hat{t}\hat{e_2}\right)\,{}_0d_q\hat{t}. \tag{2. 10}$$

Theorem 2.13. *Assume that $f : Ķ \to \mathbb{R}$ is q differentiable function on the interior of $Ķ$ such that $0 < q \leq 1$. Further, if $|{}_{\hat{e_1}}D_q f|^{r_2}$ is integrable on $Ķ$ with $r_2 \geq 1$, then we have the following identity:*

$$\left|\frac{qf(\hat{e_1}) + f(\hat{e_2})}{1 + q} - \frac{1}{\hat{e_2} - \hat{e_2}}\int_{\hat{e_1}}^{\hat{e_2}} f(\hat{t})\,{}_{\hat{e_1}}d_q\hat{t}\right| \leq \frac{q^2(q^2 + q + 2)(\hat{e_2} - \hat{e_1})}{(1 + q)^4}$$
$$\times\left[\frac{(q^2 + 4q + 1)\,|{}_{\hat{e_1}}D_q f(\hat{e_2})|^{r_2} + (2q^3 + 3q^2 + 1)\,|{}_{\hat{e_1}}D_q f(\hat{e_1})|^{r_2}}{(q^2 + q + 1)(q^3 + q + 2)}\right]^{\frac{1}{r_2}}. \tag{2. 11}$$

For further investigation on quantum integrals, Tunç et al [38] presented the lemma and its main result based on quantum estimates.

Lemma 2.14. *Assume that $f : Ķ \to \mathbb{R}$ is q differentiable function on the interior of Ķ such that $0 < q \leq 1$. Further, if ${}_{\hat{e_1}}D_q f$ is integrable on Ķ, then we have the following identity:*

$$\frac{1}{6}\left[f(\hat{e_1}) + f\left(\frac{\hat{e_1}+\hat{e_2}}{2}\right) + f(\hat{e_2})\right] - \frac{1}{\hat{e_2}-\hat{e_2}}\int_{\hat{e_1}}^{\hat{e_2}} f(\hat{t})\, {}_{\hat{e_1}}d_q\hat{t}$$
$$= (\hat{e_2}-\hat{e_1})\left\{\int_0^{\frac{1}{2}} \left(q\hat{t}-\frac{1}{6}\right) {}_{\hat{e_1}}D\left((1-\hat{t})\hat{e_1}+\hat{t}\hat{e_2}\right)\, {}_0d_q\hat{t}\right.$$
$$\left. + \int_{\frac{1}{2}}^{1} \left(q\hat{t}-\frac{5}{6}\right) {}_{\hat{e_1}}D\left((1-\hat{t})\hat{e_1}+\hat{t}\hat{e_2}\right)\, {}_0d_q\hat{t}\right\}. \quad (2.\ 12)$$

Theorem 2.15. *Assume that $f : Ķ \to \mathbb{R}$ is q differentiable function on the interior of Ķ such that $0 < q \leq 1$. Further, if ${}_{\hat{e_1}}D_q f$ is integrable on Ķ, then we have the following identity:*

$$\left|\frac{1}{6}\left[f(\hat{e_1}) + f\left(\frac{\hat{e_1}+\hat{e_2}}{2}\right) + f(\hat{e_2})\right] - \frac{1}{\hat{e_2}-\hat{e_2}}\int_{\hat{e_1}}^{\hat{e_2}} f(\hat{t})\, {}_{\hat{e_1}}d_q\hat{t}\right|$$
$$\leq \frac{(\hat{e_2}-\hat{e_1})}{12}\left[\frac{6q^3+4q^2+4q+1}{3(q^3+2q^2+2q+1)}\,|{}_{\hat{e_1}}D_q f(\hat{e_1})| + \frac{2q^2+2q+1}{(q^3+2q^2+2q+1)}\,|{}_{\hat{e_1}}D_q f(\hat{e_2})|\right]. \quad (2.\ 13)$$

3. Main Results

We start this section by introducing a new concept namely generalized n-polynomial pre-invex functions.

Definition 3.1. *Assume that $f : Ķ \to \mathbb{R}$ be any function on an invex set Ķ w.r.t the mapping η. Further, let h be any non negative real valued function, then for $n \in \mathbb{N}$, f is called generalized n-polynomial pre-invex function(may be denoted by $P_n^{(\alpha m h)-\delta}-$prinvex) if it satisfies the following inequality:*

$$f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})) \leq \frac{1}{n}\sum_{k=1}^{n}(1-(h(\hat{t}))^{\alpha k})f(\hat{e_1}) + \frac{m}{n}\sum_{k=1}^{n}(h(\hat{t}))^{\alpha k} f\left(\frac{\hat{e_2}}{m}\right) + \delta, \quad (3.\ 14)$$

where $(\alpha, m) \in [0,1]^2$ and δ is any non negative real number. If the inequality (3. 14) is reversed, the function f is claimed to be generalized n-polynomial pre-concave.

Remark 3.2. *It is worth mentioning that the generalized 1-polynomial pre-invex function f reduces to classical pre-invexity when accompanied with the condition $h(\hat{t}) = \hat{t}$ and $\delta = 0, \alpha = 1 = m$, while further reduced to convex function when $\eta(\hat{e_2},\hat{e_1}) = \hat{e_2}-\hat{e_1}$*

Remark 3.3. *Every non negative m-pre-invex function is generalized 1-polynomial preinvex function for $h(\hat{t}) \geq \hat{t}$.*

For instance, assume that f is non-negative m-pre-invex, then

$$\begin{aligned} f\left(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})\right) &\leq (1-\hat{t})f(\hat{e_1})+\frac{1}{m}\hat{t}f\big(\frac{\hat{e_2}}{m}\big) \\ &\leq (1-(h(\hat{t}))^{\alpha})f(\hat{e_1})+(h(\hat{t}))^{\alpha}\frac{1}{m}f\big(\frac{\hat{e_2}}{m}\big) \\ &\leq (1-(h(\hat{t}))^{\alpha})f(\hat{e_1})+(h(\hat{t}))^{\alpha}\frac{1}{m}f\big(\frac{\hat{e_2}}{m}\big)+\delta. \end{aligned}$$

This shows every non-negative m-pre-invex function is generalized 1-polynomial pre-invex.

Some Consequences: It is paramount to note that the above class of pre-invex functions contains several new pre-invexities and some of those are mandatory to be mentioned here as follows:

(1) Set $h(\hat{t})=\hat{t}^{s}$ to obtain

$$f\left(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})\right)\leq \frac{1}{n}\sum_{k=1}^{n}(1-\hat{t}^{\alpha sk})f(\hat{e_1})+\frac{m}{n}\sum_{k=1}^{n}\hat{t}^{\alpha sk}f\big(\frac{\hat{e_2}}{m}\big)+\delta, \tag{3. 15}$$

we name it as $P_n^{(\alpha sm)-\delta}-$prinvex function.

(2) Taking $\alpha=1$ in (3. 15), we achieve

$$f\left(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})\right)\leq \frac{1}{n}\sum_{k=1}^{n}(1-\hat{t}^{sk})f(\hat{e_1})+\frac{m}{n}\sum_{k=1}^{n}\hat{t}^{sk}f\big(\frac{\hat{e_2}}{m}\big)+\delta, \tag{3. 16}$$

we call this $P_n^{(sm)-\delta}-$prinvex function.

(3) For $m=1$ in (3. 16), we attain

$$f\left(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})\right)\leq \frac{1}{n}\sum_{k=1}^{n}(1-\hat{t}^{sk})f(\hat{e_1})+\frac{1}{n}\sum_{k=1}^{n}\hat{t}^{sk}f\left(\hat{e_2}\right)+\delta, \tag{3. 17}$$

we claim this as $P_n^{s-\delta}-$prinvex function.

(4) For the choice of $s=1$ in (3. 15), we acquire

$$f\left(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})\right)\leq \frac{1}{n}\sum_{k=1}^{n}(1-\hat{t}^{\alpha k})f(\hat{e_1})+\frac{m}{n}\sum_{k=1}^{n}\hat{t}^{\alpha k}f\big(\frac{\hat{e_2}}{m}\big)+\delta, \tag{3. 18}$$

we name it as$P_n^{(\alpha m)-\delta}-$prinvex function.

(5) Replacing $m=1$ in above inequality we come across with the following:

$$f\left(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})\right)\leq \frac{1}{n}\sum_{k=1}^{n}(1-\hat{t}^{\alpha k})f(\hat{e_1})+\frac{1}{n}\sum_{k=1}^{n}\hat{t}^{\alpha k}f\left(\hat{e_2}\right)+\delta, \tag{3. 19}$$

we name it as $P_n^{\alpha-\delta}-$prinvex function.

Now, we prove a generic identity involving post-quantum integrals for (p,q)-differentiable functions.

Lemma 3.4. *Assume that $f : Ķ \to \mathbb{R}$ is (p,q) differentiable function on the interior of Ķ such that $0 < q < p \leq 1$. Further, if ${}_{\hat{e_1}}D_{p,q}f$ is integrable on Ķ, then for the parameters $\hat{A}, \hat{B} \in \mathbb{R}$ we have the following identity:*

$$\mathcal{S}_1 = \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}}(pq\hat{t}-\hat{A})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t} + \int_{\frac{1}{2}}^{1}(pq\hat{t}-\hat{B})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\Bigg], \tag{3. 20}$$

where

$$\mathcal{S}_1 = \hat{A}f(\hat{e_1}) + (\hat{B}-\hat{A})f\left(\frac{2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})}{2}\right) + (p-\hat{B})f(\hat{e_2}) - \frac{1}{\eta(\hat{e_2},\hat{e_1})}\int_{\hat{e_1}}^{\hat{e_1}+p\eta(\hat{e_2},\hat{e_1})} f(\hat{t})\ {}_{\hat{e_1}}d_{p,q}\hat{t}.$$

Proof. By making use of the definition of (p,q) derivative and (p,q)-integral

$$\int_0^1(pq\hat{t}-\hat{B})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}$$

$$= \int_0^1(pq\hat{t}-\hat{B})\left[\frac{f(\hat{e_1}+p\hat{t}\eta(\hat{e_2},\hat{e_1}))-f(\hat{e_1}+q\hat{t}\eta(\hat{e_2},\hat{e_1}))}{(p-q)\hat{t}\eta(\hat{e_2},\hat{e_1})}\right]\ {}_0d_{p,q}\hat{t}$$

$$= \frac{pq}{\eta(\hat{e_2},\hat{e_1})}\left[\sum_{r=o}^{\infty}\frac{q^r}{p^{r+1}}f(\hat{e_1}+p\frac{q^r}{p^{r+1}}\eta(\hat{e_2},\hat{e_1})) - \sum_{r=o}^{\infty}\frac{q^r}{p^{r+1}}f(\hat{e_1}+\frac{q^r}{q^{r+1}}q\eta(\hat{e_2},\hat{e_1})\right]$$

$$-\frac{\hat{B}}{\eta(\hat{e_2},\hat{e_1})}\left[\frac{\sum_{r=o}^{\infty}\frac{q^r}{p^{r+1}}f(\hat{e_1}+p\frac{q^r}{p^{r+1}}\eta(\hat{e_2},\hat{e_1})) - \sum_{r=o}^{\infty}\frac{q^r}{p^{r+1}}f(\hat{e_1}+\frac{q^r}{q^{r+1}}q\eta(\hat{e_2},\hat{e_1}))}{\frac{q^r}{p^{r+1}}}\right]$$

$$= -p(p-q)\left[\frac{\sum_{0}^{\infty}\frac{q^r}{p^{r+1}}f\left(\hat{e_1}+\frac{q^r}{p^r}\eta(\hat{e_2},\hat{e_1})\right)}{\eta(\hat{e_2},\hat{e_1})}\right] + \frac{(p-\hat{B})f\left(\hat{e_1}+\frac{\eta(\hat{e_2},\hat{e_1})}{2}\right)}{\eta(\hat{e_2},\hat{e_1})} + \frac{\hat{B}f(\hat{e_1})}{\eta(\hat{e_2},\hat{e_1})}.$$

Finally, we have

$$\int_0^1(pq\hat{t}-\hat{B})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}$$

$$= \frac{-1}{\eta^2(\hat{e_2},\hat{e_1})}\int_{\hat{e_1}}^{\hat{e_1}+p\eta(\hat{e_2},\hat{e_1})} f(\hat{t})\ {}_{\hat{e_1}}d_{p,q} + \frac{(p-\hat{B})f\left(\hat{e_1}+\frac{\eta(\hat{e_2},\hat{e_1})}{2}\right)}{\eta(\hat{e_2},\hat{e_1})} + \frac{\hat{B}f(\hat{e_1})}{\eta(\hat{e_2},\hat{e_1})}. \tag{3. 21}$$

Similarly

$$\int_0^{\frac{1}{2}} {}_{\hat{e_1}}D_{p,q} f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t}$$
$$= \frac{\sum_{r=0}^{\infty} f(\hat{e_1}+\frac{q^r}{2p^r}\eta(\hat{e_2},\hat{e_1})) - \sum_{r=0}^{\infty} f(\hat{e_1}+\frac{q^{r+1}}{2p^{r+1}}\eta(\hat{e_2},\hat{e_1}))}{\eta(\hat{e_2},\hat{e_1})}.$$

So, we have

$$\int_0^{\frac{1}{2}} {}_{\hat{e_1}}D_{p,q} f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t} = \frac{f(\hat{e_1}+\eta(\hat{e_2},\hat{e_1}) - f(\hat{e_1})}{\eta(\hat{e_2},\hat{e_1})}. \tag{3. 22}$$

Now consider

$$\int_0^{\frac{1}{2}} (pq\hat{t}-\hat{A})\, {}_{\hat{e_1}}D_{p,q} f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t}$$
$$+ \int_{\frac{1}{2}}^{1} {}_{\hat{e_1}}D_{p,q}(pq\hat{t}-\hat{B}) f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t}$$
$$= \int_0^{\frac{1}{2}} (\hat{B}-\hat{A})\, {}_{\hat{e_1}}D_{p,q} f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t}$$
$$+ \int_0^{1} (pq\hat{t}-\hat{B})\, {}_{\hat{e_1}}D_{p,q} f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t}.$$

Substituting the values from (3. 21) and (3. 22) in above equation to obtain the required identity (3. 20). □

Corollary 3.5. *For $\eta(\hat{e_2},\hat{e_1}) = \hat{e_2} - \hat{e_1}$ and $\delta = 0$, we have the following identity:*

$$\mathcal{S} = \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (pq\hat{t}-\hat{A})\, {}_{\hat{e_1}}D_{p,q} f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t}$$
$$+ \int_{\frac{1}{2}}^{1} (pq\hat{t}-\hat{B})\, {}_{\hat{e_1}}D_{p,q} f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\, {}_0d_{p,q}\hat{t}\Bigg],$$

which is Lemma 2.1 *in* [**?**].

Remark 3.6. *It is worth mentioning that the following special cases can be re obtained from the identity proved in* (3. 20)*, for example:*

(1) *Taking $\eta(\hat{e_2},\hat{e_1}) = \hat{e_2} - \hat{e_1}$ and $p = 1$ gives Lemma* 2.1 *in* [7].

(2) *For* $\eta(\hat{e_2},\hat{e_1}) = \hat{e_2} - \hat{e_1}$ *along with* $\hat{A} = \frac{1}{6}$, $\hat{B} = \frac{5}{6}$ *and* $p = 1$, *we can recapture Lemma* 3 *in* [38].

(3) *If we choose* $\eta(\hat{e_2},\hat{e_1}) = \hat{e_2} - \hat{e_1}$ *along with* $\hat{A} = \frac{1}{4}$ *and* $\hat{B} = \frac{3}{4}$, *we attain the following identity:*

$$\frac{1}{4}\left[f(\hat{A}) + 2f\left(\frac{\hat{e_1}+\hat{e_2}}{2}\right) + 4\big(p - \frac{3}{4}\big)f(\hat{e_2})\right] - \frac{1}{p(\hat{e_2}-\hat{e_1})}\int_{\hat{e_1}}^{(1-p)\hat{e_1}+p\hat{e_2}} f(\hat{t})\ {}_{\hat{e_1}}d_{p,q}\hat{t}$$

$$= \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}} \big(pq\hat{t} - \frac{1}{4}\big)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}$$

$$+ \int_{\frac{1}{2}}^{1} \big(pq\hat{t} - \frac{3}{4}\big)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\Bigg],$$

which is the Remark 2.1(2) *in* [**?**].

(4) *For* $p = 1$ *along with* $\eta(\hat{e_2},\hat{e_1}) = \hat{e_2} - \hat{e_1}$, $\hat{A} = \frac{1}{4}$ *and* $\hat{B} = \frac{3}{4}$, *Remark*$2.1(ii)$ *in* [7] *is obtained.*

In order to utilize in our upcoming results, following (p,q)-integral have been caculated.

(1) $\mathcal{I}_1^k(\hat{t},h;\alpha) := \int_0^{\frac{1}{2}} (1 - (h(\hat{t}))^{\alpha k})\ {}_0d_{p,q}\hat{t}$.

(2) $\mathcal{I}_2^k(\hat{t},h,\alpha;\hat{B}) := \int_0^1 |pq\hat{t} - \hat{B}|(1 - (h(\hat{t}))^{\alpha k})\ {}_0d_{p,q}\hat{t}$

(3) $\mathcal{I}_3^k(\hat{t},h;\alpha) := \int_0^{\frac{1}{2}} (h(\hat{t}))^{\alpha k}\ {}_0d_{p,q}\hat{t}$.

(4) $\mathcal{I}_4^k(\hat{t},h,\alpha;\hat{B}) := \int_0^1 |pq\hat{t} - \hat{B}|(h(\hat{t}))^{\alpha k}\ {}_0d_{p,q}\hat{t}$.

(5) $\mathcal{I}_5^k(q,p;\hat{B}) := \int_0^1 |pq\hat{t} - \hat{B}|_0d_{p,q}\hat{t}$

$$= \begin{cases} \frac{2\hat{B}^2(p+q-1)+pq[pq-\hat{B}(p+q)]}{pq(p+q)}, & \text{if} \quad 0 \leq \frac{\hat{B}}{pq} \leq \frac{1}{2}, \\ \frac{8\hat{B}^2(p+q-1)+5p^2q^2-6\hat{B}pq(p+q)}{4pq(p+q)}, & \text{if} \quad \frac{1}{2} < \frac{\hat{B}}{pq} \leq 1, \\ \frac{B(p+q)-pq}{(p+q)}, & \text{if} \quad \frac{B}{pq} > 1. \end{cases}$$

(6) $\mathcal{I}_6^k(q,p,\alpha,s;\hat{B}) := \int\limits_0^1 |pq\hat{t}-\hat{B}|(1-\hat{t}^{\alpha sk})_0 d_{p,q}\hat{t}$

$$= \begin{cases} \frac{2\hat{B}^2(p+q-1)+pq\left[pq-\hat{B}(p+q)\right]}{pq(p+q)} \\ -\frac{2^{\alpha sk+3}\hat{B}^{\alpha sk+2}(p-q)\left[\left(p^{\alpha sk+2}-p^{\alpha sk+1}\right)+q^{\alpha sk+1}\left(p-2\hat{B}\right)\right]}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})} \\ -\frac{(pq)^{\alpha sk+1}(p-q)\left[q^{\alpha sk+2}\left\{2\hat{B}\left(1+2^{\alpha sk+1}\right)-p\left(1-2^{\alpha sk+2}\right)\right\}\right]}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})}, & \text{if} \quad 0 \le \frac{\hat{B}}{pq} \le \frac{1}{2}, \\ \frac{8\hat{B}^2(p+q-1)+5p^2q^2-6\hat{B}pq(p+q)}{4pq(p+q)} \\ -\frac{2p^{\alpha sk+1}\hat{B}^{\alpha sk+2}(p-q)(p-1)+2q^{\alpha sk+1}\hat{B}^{\alpha sk+2}(p-q)(1-q)}{p^{\alpha sk+1}q^{\alpha sk+1}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})} \\ -\frac{p^{\alpha sk+2}(p-q)(q-\hat{B})+q^{\alpha sk+2}(p-q)(\hat{B}-p)}{(p^{\alpha sk+2}-q^{\alpha sk+2})(p^{\alpha sk+1}-q^{\alpha sk+1})} \\ -\frac{p^{\alpha sk+2}(p-q)(q-2\hat{B})+q^{\alpha sk+2}(p-q)(2\hat{B}-p)}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})}, & \text{if} \quad \frac{1}{2} < \frac{\hat{B}}{pq} \le 1, \\ \frac{2\hat{B}(p+q)-3pq}{4(p+q)} - \frac{p^{\alpha sk+2}(p-q)(\hat{B}-q)+pq^{\alpha sk+2}(p-q)(p-\hat{B})}{(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})} \\ -\frac{p^{\alpha sk+2}(p-q)(q-2\hat{B})+q^{\alpha sk+2}(p-q)(2\hat{B}-p)}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})}, & \text{if} \quad \frac{\hat{B}}{pq} > 1. \end{cases}$$

(7) $\mathcal{I}_7^k(q,p,\alpha,s;\hat{B}) := \int\limits_0^1 |pq\hat{t}-\hat{B}|\hat{t}^{\alpha sk}{}_0 d_{p,q}\hat{t}$

$$= \begin{cases} \frac{2^{\alpha sk+3}\hat{B}^{\alpha sk+2}(p-q)\left[\left(p^{\alpha sk+2}-p^{\alpha sk+1}\right)+q^{\alpha sk+1}\left(p-2\hat{B}\right)\right]}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})} \\ +\frac{(pq)^{\alpha sk+1}(p-q)\left[q^{\alpha sk+2}\left\{2\hat{B}\left(1+2^{\alpha sk+1}\right)-p\left(1-2^{\alpha sk+2}\right)\right\}\right]}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})}, & \text{if} \quad 0 \le \frac{\hat{B}}{pq} \le \frac{1}{2}, \\ \frac{2p^{\alpha sk+1}\hat{B}^{\alpha sk+2}(p-q)(p-1)+2q^{\alpha sk+1}\hat{B}^{\alpha sk+2}(p-q)(1-q)}{p^{\alpha sk+1}q^{\alpha sk+1}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})} \\ +\frac{p^{\alpha sk+2}(p-q)(q-\hat{B})+q^{\alpha sk+2}(p-q)(\hat{B}-p)}{(p^{\alpha sk+2}-q^{\alpha sk+2})(p^{\alpha sk+1}-q^{\alpha sk+1})} \\ +\frac{p^{\alpha sk+2}(p-q)(q-2\hat{B})+q^{\alpha sk+2}(p-q)(2\hat{B}-p)}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})}, & \text{if} \quad \frac{1}{2} < \frac{\hat{B}}{pq} \le 1, \\ \frac{p^{\alpha sk+2}(p-q)(\hat{B}-q)+pq^{\alpha sk+2}(p-q)(p-\hat{B})}{(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})} \\ +\frac{p^{\alpha sk+2}(p-q)(q-2\hat{B})+q^{\alpha sk+2}(p-q)(2\hat{B}-p)}{2^{\alpha sk+2}(p^{\alpha sk+1}-q^{\alpha sk+1})(p^{\alpha sk+2}-q^{\alpha sk+2})}, & \text{if} \quad \frac{\hat{B}}{pq} > 1. \end{cases}$$

(8) $\mathcal{I}_8^k(q,p,s;\hat{B}) = \int\limits_0^1 |pq\hat{t}-\hat{B}|(1-\hat{t}^{sk})_0 d_{p,q}\hat{t}$

$$= \begin{cases} \frac{2\hat{B}^2(p+q-1)+pq\left[pq-\hat{B}(p+q)\right]}{pq(p+q)} \\ -\frac{2^{sk+3}\hat{B}^{sk+2}(p-q)\left[\left(p^{sk+2}-p^{sk+1}\right)+q^{sk+1}\left(p-2\hat{B}\right)\right]}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})} \\ -\frac{(pq)^{sk+1}(p-q)\left[q^{sk+2}\left\{2\hat{B}\left(1+2^{sk+1}\right)-p\left(1-2^{sk+2}\right)\right\}\right]}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})}, & \text{if} \quad 0 \le \frac{\hat{B}}{pq} \le \frac{1}{2}, \\ \frac{8\hat{B}^2(p+q-1)+5p^2q^2-6\hat{B}pq(p+q)}{4pq(p+q)} \\ -\frac{2p^{sk+1}\hat{B}^{sk+2}(p-q)(p-1)+2q^{sk+1}\hat{B}^{sk+2}(p-q)(1-q)}{p^{sk+1}q^{sk+1}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})} \\ -\frac{p^{sk+2}(p-q)(q-\hat{B})+q^{sk+2}(p-q)(\hat{B}-p)}{(p^{sk+2}-q^{sk+2})(p^{sk+1}-q^{sk+1})} \\ -\frac{p^{sk+2}(p-q)(q-2\hat{B})+q^{sk+2}(p-q)(2\hat{B}-p)}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})}, & \text{if} \quad \frac{1}{2} < \frac{\hat{B}}{pq} \le 1, \\ \frac{2\hat{B}(p+q)-3pq}{4(p+q)} - \frac{p^{sk+2}(p-q)(\hat{B}-q)+pq^{sk+2}(p-q)(p-\hat{B})}{(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})} \\ -\frac{p^{sk+2}(p-q)(q-2\hat{B})+q^{sk+2}(p-q)(2\hat{B}-p)}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})}, & \text{if} \quad \frac{\hat{B}}{pq} > 1. \end{cases}$$

(9) $\mathcal{I}_9^k(q,p,s,;\hat{B}) := \int\limits_0^1 |pq\hat{t}-\hat{B}|\hat{t}^{sk}{}_0d_{p,q}\hat{t}$

$$
= \begin{cases}
\frac{2^{sk+3}\hat{B}^{sk+2}(p-q)\left[\left(p^{sk+2}-p^{sk+1}\right)+q^{sk+1}\left(p-2\hat{B}\right)\right]}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})} \\
+\frac{(pq)^{sk+1}(p-q)\left[q^{sk+2}\left\{2\hat{B}\left(1+2^{sk+1}\right)-p\left(1-2^{sk+2}\right)\right\}\right]}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})}, & \text{if} \quad 0 \leq \frac{\hat{B}}{pq} \leq \frac{1}{2}, \\
\frac{2p^{sk+1}\hat{B}^{sk+2}(p-q)(p-1)+2q^{sk+1}\hat{B}^{sk+2}(p-q)(1-q)}{p^{sk+1}q^{sk+1}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})} \\
+\frac{p^{sk+2}(p-q)(q-\hat{B})+q^{sk+2}(p-q)(\hat{B}-p)}{(p^{sk+2}-q^{sk+2})(p^{sk+1}-q^{sk+1})} \\
+\frac{p^{sk+2}(p-q)(q-2\hat{B})+q^{sk+2}(p-q)(2\hat{B}-p)}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})}, & \text{if} \quad \frac{1}{2} < \frac{\hat{B}}{pq} \leq 1, \\
\frac{p^{sk+2}(p-q)(\hat{B}-q)+pq^{sk+2}(p-q)(p-\hat{B})}{(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})} \\
+\frac{p^{sk+2}(p-q)(q-2\hat{B})+q^{sk+2}(p-q)(2\hat{B}-p)}{2^{sk+2}(p^{sk+1}-q^{sk+1})(p^{sk+2}-q^{sk+2})}, & \text{if} \quad \frac{\hat{B}}{pq} > 1.
\end{cases}
$$

(10) $\mathcal{I}_{10}^k(q,p,\alpha;\hat{B}) = \int\limits_0^1 |pq\hat{t}-\hat{B}|(1-\hat{t}^{\alpha k})_0d_{p,q}\hat{t}$

$$
= \begin{cases}
\frac{2\hat{B}^2(p+q-1)+pq\left[pq-\hat{B}(p+q)\right]}{pq(p+q)} \\
-\frac{2^{\alpha k+3}\hat{B}^{\alpha k+2}(p-q)\left[\left(p^{\alpha k+2}-p^{\alpha k+1}\right)+q^{\alpha k+1}\left(p-2\hat{B}\right)\right]}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})} \\
-\frac{(pq)^{\alpha k+1}(p-q)\left[q^{\alpha k+2}\left\{2\hat{B}\left(1+2^{\alpha k+1}\right)-p\left(1-2^{\alpha k+2}\right)\right\}\right]}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})}, & \text{if} \quad 0 \leq \frac{\hat{B}}{pq} \leq \frac{1}{2}, \\
\frac{8\hat{B}^2(p+q-1)+5p^2q^2-6\hat{B}pq(p+q)}{4pq(p+q)} \\
-\frac{2p^{\alpha k+1}\hat{B}^{\alpha k+2}(p-q)(p-1)+2q^{\alpha k+1}\hat{B}^{\alpha k+2}(p-q)(1-q)}{p^{\alpha k+1}q^{\alpha k+1}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})} \\
-\frac{p^{\alpha k+2}(p-q)(q-\hat{B})+q^{\alpha k+2}(p-q)(\hat{B}-p)}{(p^{\alpha k+2}-q^{\alpha k+2})(p^{\alpha k+1}-q^{\alpha k+1})} \\
-\frac{p^{\alpha k+2}(p-q)(q-2\hat{B})+q^{\alpha k+2}(p-q)(2\hat{B}-p)}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})}, & \text{if} \quad \frac{1}{2} < \frac{\hat{B}}{pq} \leq 1, \\
\frac{2\hat{B}(p+q)-3pq}{4(p+q)} - \frac{p^{\alpha k+2}(p-q)(\hat{B}-q)+pq^{\alpha k+2}(p-q)(p-\hat{B})}{(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})} \\
-\frac{p^{\alpha k+2}(p-q)(q-2\hat{B})+q^{\alpha k+2}(p-q)(2\hat{B}-p)}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})}, & \text{if} \quad \frac{\hat{B}}{pq} > 1.
\end{cases}
$$

(11) $\mathcal{I}_{11}^k(q,p,\alpha;\hat{B}) := \int\limits_0^1 |pq\hat{t}-\hat{B}|\hat{t}^{\alpha k}{}_0d_{p,q}\hat{t}$

$$
= \begin{cases}
\frac{2^{\alpha k+3}\hat{B}^{\alpha k+2}(p-q)\left[\left(p^{\alpha k+2}-p^{\alpha k+1}\right)+q^{\alpha k+1}\left(p-2\hat{B}\right)\right]}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})} \\
+\frac{(pq)^{\alpha k+1}(p-q)\left[q^{\alpha k+2}\left\{2\hat{B}\left(1+2^{\alpha k+1}\right)-p\left(1-2^{\alpha k+2}\right)\right\}\right]}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})}, & \text{if} \quad 0 \leq \frac{\hat{B}}{pq} \leq \frac{1}{2}, \\
\frac{2p^{\alpha k+1}\hat{B}^{\alpha k+2}(p-q)(p-1)+2q^{\alpha k+1}\hat{B}^{\alpha k+2}(p-q)(1-q)}{p^{\alpha k+1}q^{\alpha k+1}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})} \\
+\frac{p^{\alpha k+2}(p-q)(q-\hat{B})+q^{\alpha k+2}(p-q)(\hat{B}-p)}{(p^{\alpha k+2}-q^{\alpha k+2})(p^{\alpha k+1}-q^{\alpha k+1})} \\
+\frac{p^{\alpha k+2}(p-q)(q-2\hat{B})+q^{\alpha k+2}(p-q)(2\hat{B}-p)}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})}, & \text{if} \quad \frac{1}{2} < \frac{\hat{B}}{pq} \leq 1, \\
\frac{p^{\alpha k+2}(p-q)(\hat{B}-q)+pq^{\alpha k+2}(p-q)(p-\hat{B})}{(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})} \\
+\frac{p^{\alpha k+2}(p-q)(q-2\hat{B})+q^{\alpha k+2}(p-q)(2\hat{B}-p)}{2^{\alpha k+2}(p^{\alpha k+1}-q^{\alpha k+1})(p^{\alpha k+2}-q^{\alpha k+2})}, & \text{if} \quad \frac{\hat{B}}{pq} > 1.
\end{cases}
$$

(12) $\mathcal{I}_{12}^k(\hat{t},q,p,h;\alpha) := \int\limits_0^{\frac{1}{2}} (\frac{1}{2}-\hat{t})[1-\left(h(\hat{t})\right)^{\alpha k}]\,{}_0d_{p,q}\hat{t}.$

(13) $\mathcal{I}_{13}^{k}(\hat{t},q,p,h;\alpha) := \int\limits_{0}^{\frac{1}{2}} (\frac{1}{2}-\hat{t})\big[\big(h(\hat{t})\big)^{\alpha k}\big]\ {}_{0}d_{p,q}\hat{t}.$

(14) $\mathcal{I}_{14}^{k}(\hat{t},q,p,h;\alpha) := \int\limits_{0}^{\frac{1}{2}} \hat{t}[1-(h(\hat{t}))^{\alpha k}]\ {}_{0}d_{p,q}\hat{t}.$

(15) $\mathcal{I}_{15}^{k}(\hat{t},q,p,h;\alpha) := \int\limits_{0}^{\frac{1}{2}} \hat{t}[(h(\hat{t}))^{\alpha k}]\ {}_{0}d_{p,q}\hat{t}.$

(16) $\mathcal{I}_{16}^{k}(\hat{t},q,p;\hat{B}) = \int\limits_{0}^{1} |pq\hat{t}-\hat{B}|(1-\hat{t})_{0}d_{p,q}\hat{t}$

$$= \begin{cases} \frac{16\hat{B}^2(p+q-1)+5pq\left[pq-2\hat{B}(p+q)\right]}{4pq(p+q)} \\ -\frac{2\hat{B}^2(p-q)^2(p+q)[(p-1)+(1-q)]}{p^2q^2(p^2-q^2)(p^3-q^3)} \\ -\frac{(pq+8\hat{B}^2)(p^2-q^2)-2\hat{B}(p^3-q^3)+17\hat{B}p^2q^5}{8(p+q)(p^3-q^3)}, & \text{if} \quad 0\le \frac{\hat{B}}{pq}\le \frac{1}{2}, \\ \frac{8\hat{B}^2(p+q-1)+5p^2q^2-6\hat{B}pq(p+q)}{4pq(p+q)} \\ -\frac{2p^2\hat{B}^3(p-q)(p-1)+2q^2\hat{B}^3(p-q)(1-q)}{p^2q^2(p^2-q^2)(p^3-q^3)} \\ -\frac{p^3(p-q)(q-\hat{B})+q^3(p-q)(\hat{B}-p)}{(p^3-q^3)(p^2-q^2)} \\ -\frac{p^3(p-q)(q-2\hat{B})+q^3(p-q)(2\hat{B}-p)}{8(p^2-q^2)(p^3-q^3)}, & \text{if} \quad \frac{1}{2}<\frac{\hat{B}}{pq}\le 1, \\ \frac{2\hat{B}(p+q)-3pq}{4(p+q)} - \frac{p^3(p-q)(\hat{B}-q)+pq3(p-q)(p-\hat{B})}{(p^2-q^2)(p^3-q^3)} \\ -\frac{p^3(p-q)(q-2\hat{B})+q^3(p-q)(2\hat{B}-p)}{8(p^2-q^2)(p^3-q^3)}, & \text{if} \quad \frac{\hat{B}}{pq}>1. \end{cases}$$

(17) $\mathcal{I}_{17}^{k}(\hat{t},q,p,h\alpha;\hat{B}) := \int\limits_{0}^{1} (1-\hat{t})|pq\hat{t}-\hat{B}|(1-(h(\hat{t}))^{\alpha k})\ {}_{0}d_{p,q}\hat{t}.$

(18) $\mathcal{I}_{18}^{k}(\hat{t},q,p,h\alpha;\hat{B}) := \int\limits_{0}^{1} (1-\hat{t})|pq\hat{t}-\hat{B}|(h(\hat{t}))^{\alpha k}\ {}_{0}d_{p,q}\hat{t}.$

(19) $\mathcal{I}_{19}^{k}(q,p;\hat{B}) := \int\limits_{0}^{1} \hat{t}|pq\hat{t}-\hat{B}|_{0}d_{p,q}\hat{t}$

$$= \begin{cases} \frac{2\hat{B}^3\left[p^2(p-1)+q^2(1-q)\right]+p^2q^2\left[p^3(q-\hat{B})+q^3(\hat{B}-p))\right]}{(p+q)(p^3-q^3)}, & \text{if} \quad 0\le \frac{\hat{B}}{pq}\le \frac{1}{2}, \\ \frac{2p^2\hat{B}^3(p-q)(p-1)+2q^2\hat{B}^3(p-q)(1-q)}{p^2q^2(p^2-q^2)(p^3-q^3)} \\ +\frac{p^3(p-q)(q-\hat{B})+q^3(p-q)(\hat{B}-p)}{(p^3-q^3)(p^2-q^2)} \\ +\frac{p^3(p-q)(q-2\hat{B})+q^3(p-q)(2\hat{B}-p)}{8(p^2-q^2)(p^3-q^3)}, & \text{if} \quad \frac{1}{2}<\frac{\hat{B}}{pq}\le 1, \\ \frac{p^3(\hat{B}-q)+pq^3(p-\hat{B})}{(p+q)(p^3-q^3)}, & \text{if} \quad \frac{\hat{B}}{pq}>1. \end{cases}$$

(20) $\mathcal{I}_{20}^{k}(\hat{t},q,p,\alpha;\hat{B}) := \int\limits_{0}^{1} \hat{t}|pq\hat{t}-\hat{B}|(1-(h(\hat{t}))^{\alpha k})\ {}_{0}d_{p,q}\hat{t}.$

(21) $\mathcal{I}_{21}^{k}(\hat{t},q,p,\alpha;\hat{B}) := \int\limits_{0}^{1} \hat{t}|pq\hat{t}-\hat{B}|(h(\hat{t}))^{\alpha k}\ {}_{0}d_{p,q}\hat{t}.$

(22) $\mathcal{I}_{22}^{k}(\hat{t},q,p;\hat{B}) := \int\limits_{0}^{1} |pq\hat{t}-\hat{B}||\hat{t}-\frac{(\hat{e_2}-\hat{e_1})}{\eta(\hat{e_2},\hat{e_1})}|\ {}_{0}d_{p,q}\hat{t}.$

Theorem 3.7. *Assume that $f : Ķ \to \mathbb{R}$ be (p,q)-differentiable function on interior of Ķ such that $0 < q < p \leq 1$ and η is pre-invex function on the invex set Ķ. If $|\ _{\hat{e_1}}D_{p,q}f|$ is integrable and $P_n^{(\alpha mh)-\delta}$−pre-invex function, then for the parameters $\hat{A}$, $\hat{B} \in \mathbb{R}$, following inequality is satisfied:*

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1})\Bigg[\frac{1}{n}\sum_{k=1}^{n}\Bigg\{|\hat{B}-\hat{A}|\mathcal{I}_1^k\left(\hat{t},h;\alpha\right)+\mathcal{I}_2^k(\hat{t},h,\alpha;\hat{B})\Bigg\}\ |_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})| + \frac{m}{n}\sum_{k=1}^{n}\Bigg\{|\hat{B}-\hat{A}|\mathcal{I}_3^k\left(\hat{t},h;\alpha\right)+\mathcal{I}_4^k(\hat{t},h,\alpha;\hat{B})\Bigg\}\big|\ _{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big| + \Bigg\{\frac{|\hat{B}-\hat{A}|}{2}+\mathcal{I}_5^k(q,p;\hat{B})\Bigg\}\delta\Bigg]. \tag{3. 23}$$

Proof. Making use of Lemma 3.4 along with modulus property and generalized n-polynomial pre-invexity, we have:

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}}|\hat{B}-\hat{A}|\,\big|\ _{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|\ _0d_{p,q}\hat{t} + \int_0^{1}|pq\hat{t}-\hat{B}|\,\big|\ _{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|\ _0d_{p,q}\hat{t}\Bigg]$$

$$\leq \eta(\hat{e_2},\hat{e_1})\Bigg[|\ _{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|\,|\hat{B}-\hat{A}|\frac{1}{n}\sum_{k=1}^{n}\int_0^{\frac{1}{2}}(1-\big(h(\hat{t})\big)^{\alpha k})\ _0d_{p,q}\hat{t} + \big|\ _{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big||\hat{B}-\hat{A}|\frac{m}{n}\sum_{k=1}^{n}\int_0^{\frac{1}{2}}(h(\hat{t}))^{\alpha k}\ _0d_{p,q}\hat{t} + |\hat{B}-\hat{A}|\int_0^{\frac{1}{2}}\delta\ _0d_{p,q}\hat{t} + |\ _{\hat{e_1}}D_{p,q}f(\hat{e_1})|\,\frac{1}{n}\sum_{k=1}^{n}\int_0^{1}|pq\hat{t}-\hat{B}|(1-(h(\hat{t}))^{\alpha k})\ _0d_{p,q}\hat{t} + \big|\ _{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|\frac{m}{n}\sum_{k=1}^{n}\int_0^{1}|pq\hat{t}-\hat{B}|((h(\hat{t}))^{\alpha k})\ _0d_{p,q}\hat{t} + \int_0^{1}\delta|pq\hat{t}-\hat{B}|\ _0d_{p,q}\hat{t}\Bigg].$$

Finally, we are left with

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1})\Bigg[\frac{1}{n}\sum_{k=1}^{n}\Bigg\{|\hat{B}-\hat{A}|\int_0^{\frac{1}{2}}(1-\big(h(\hat{t})\big)^{\alpha k})\ {}_0d_{p,q}\hat{t}$$

$$+\int_0^1 |pq\hat{t}-\hat{B}|(1-\big(h(\hat{t})\big)^{\alpha k})\ {}_0d_{p,q}\hat{t}\Bigg\}\ \big|_{\hat{e_1}}D_{p,q}f\left(\hat{e_1}\right)\big|$$

$$+\frac{m}{n}\sum_{k=1}^{n}\Bigg\{|\hat{B}-\hat{A}|\int_0^{\frac{1}{2}}((\big(h(\hat{t})\big)^{\alpha k})\ {}_0d_{p,q}\hat{t}$$

$$+\int_0^1 |pq\hat{t}-\hat{B}|((h(\hat{t}))^{\alpha k})\ {}_0d_{p,q}\hat{t}\Bigg\}\big|_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|$$

$$+\Bigg\{\frac{|\hat{B}-\hat{A}|}{2}+\int_0^1|pq\hat{t}-\hat{B}|\ {}_0d_{p,q}\hat{t}\Bigg\}\delta\Bigg]. \tag{3. 24}$$

Substituting the values of (p,q)-integrals in (3. 24) to obtain the desired inequality (3. 23). □

Corollary 3.8. *For the choice of $h(\hat{t}) = \hat{t}^s$ in (3. 24), we come across the bounds of $P_n^{(\alpha sm)-\delta}$−prinvex function as follows:*

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1})$$
$$\times\Bigg[\frac{1}{n}\sum_{k=1}^{n}\Bigg\{\frac{2^{\alpha sk}(p^{\alpha sk+1}-q^{\alpha sk+1})-(p-q)}{2^{\alpha sk+1}(p^{\alpha sk+1}-q^{\alpha sk+1})}+\mathcal{I}_6^k(q,p,s,\alpha;\hat{B})\Bigg\}\big|\ {}_{\hat{e_1}}D_{p,q}f\left(\hat{e_1}\right)\big|$$
$$+\frac{m}{n}\sum_{k=1}^{n}\Bigg\{\frac{p-q}{(p^{\alpha sk+1}-q^{\alpha sk+1})}+\mathcal{I}_7^k(q,p,\alpha,s;\hat{B})\Bigg\}\big|\ {}_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|$$
$$+\Bigg\{\frac{|\hat{B}-\hat{A}|}{2}+\mathcal{I}_5^k(q,p;\hat{B})\Bigg\}\delta\Bigg]. \tag{3. 25}$$

Corollary 3.9. *To attain the bounds of $P_n^{(sm)-\delta}$−prinvex function, set $\alpha = 1$ in (3. 25)*

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1})$$
$$\times\Bigg[\frac{1}{n}\sum_{k=1}^{n}\Bigg\{\frac{2^{sk}(p^{sk+1}-q^{sk+1})-(p-q)}{2^{sk+1}(p^{sk+1}-q^{sk+1})}+\mathcal{I}_8^k(q,p,s;\hat{B})\Bigg\}\big|\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}\big|$$
$$+\frac{m}{n}\sum_{k=1}^{n}\Bigg\{\frac{p-q}{(p^{sk+1}-q^{sk+1})}+\mathcal{I}_9^k(q,p,s;\hat{B})\Bigg\}\big|\ {}_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|$$
$$+\Bigg\{\frac{|\hat{B}-\hat{A}|}{2}+\mathcal{I}_5^k(q,p;\hat{B})\Bigg\}\delta\Bigg]. \tag{3. 26}$$

Corollary 3.10. *To estimate the bounds of $P_n^{s-\delta}$−prinvex function, set $m=1$ in (3. 26)*

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1}) \times \left[\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{2^{sk}(p^{sk+1}-q^{sk+1})-(p-q)}{2^{sk+1}(p^{sk+1}-q^{sk+1})}+\mathcal{I}_8^k(q,p,s;\hat{B})\right\}|\,{}_{\hat{e_1}}D_{p,q}f(\hat{e_1})| + \frac{1}{n}\sum_{k=1}^{n}\left\{\frac{p-q}{(p^{sk+1}-q^{sk+1})}+\mathcal{I}_9^k(q,p,s;\hat{B})\right\}|\,{}_{\hat{e_1}}D_{p,q}f(\hat{e_2})| + \left\{\frac{|\hat{B}-\hat{A}|}{2}+\mathcal{I}_5^k(q,p;\hat{B})\right\}\delta\right]. \quad (3.\ 27)$$

Corollary 3.11. *By taking $s=1$ in (3. 25), the inequality admits the bounds of $P_n^{(\alpha m)-\delta}$−prinvex function as follows:*

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1}) \times \left[\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{2^{\alpha k}(p^{\alpha k+1}-q^{\alpha k+1})-(p-q)}{2^{\alpha k+1}(p^{\alpha k+1}-q^{\alpha k+1})}+\mathcal{I}_{10}^k(q,p,s,\alpha;\hat{B})\right\}|\,{}_{\hat{e_1}}D_{p,q}f(\hat{e_1})| + \frac{m}{n}\sum_{k=1}^{n}\left\{\frac{p-q}{(p^{\alpha k+1}-q^{\alpha k+1})}+\mathcal{I}_{11}^k(q,p,\alpha,s;\hat{B})\right\}|\,{}_{\hat{e_1}}D_{p,q}f(\frac{\hat{e_2}}{m})| + \left\{\frac{|\hat{B}-\hat{A}|}{2}+\mathcal{I}_5^k(q,p;\hat{B})\right\}\delta\right]. \quad (3.\ 28)$$

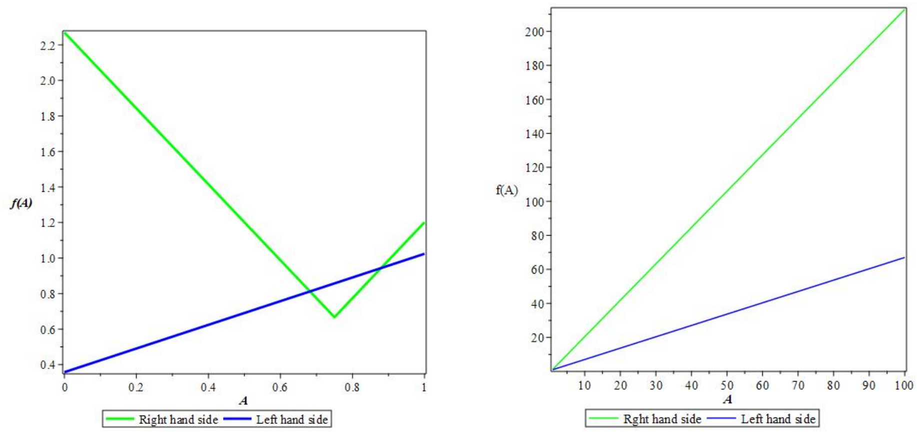
Corollary 3.12. *For the choice of $m=1$ in (3. 28), the estimates of $P_n^{\alpha-\delta}$−prinvex function are obtained as:*

$$|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1}) \times \left[\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{2^{\alpha k}(p^{\alpha k+1}-q^{\alpha k+1})-(p-q)}{2^{\alpha k+1}(p^{\alpha k+1}-q^{\alpha k+1})}+\mathcal{I}_{10}^k(q,p,\alpha,s;\hat{B})\right\}|\,{}_{\hat{e_1}}D_{p,q}f\ (\hat{e_1})| + \frac{1}{n}\sum_{k=1}^{n}\left\{\frac{p-q}{(p^{\alpha k+1}-q^{\alpha k+1})}+\mathcal{I}_{11}^k(q,p,\alpha,s;\hat{B})\right\}|\,{}_{\hat{e_1}}D_{p,q}f(\hat{e_2})| + \left\{\frac{|\hat{B}-\hat{A}|}{2}+\mathcal{I}_5^k(q,p;\hat{B})\right\}\delta\right]. \quad (3.\ 29)$$

Remark 3.13. *A Theorem 3.7 produces Theorem 2.2 of* [**?**] *by using h as identity function, $\eta(\hat{e_2},\hat{e_1})=\hat{e_2}-\hat{e_1}$ and $\delta=0$.*

From now onwards, to check the validity of all the results. we will use the following function and set of parameters:
$f(\sigma)=\frac{\sigma^2}{p+q}$, $\hat{e_1}=0, \hat{e_2}=2, p=1, q=\frac{1}{2}$. The graphical representation of above theorem is as follows:

Remark 3.14. *From the Figure 1, it is evident that whenever the function f is generalized 2-polynomial preinvex, the inequality refuses to hold for $\hat{A}\in[0.7,0.9]$ and permits to hold for all positive real numbers other than previously mentioned interval.*



(A) For $\hat{A} \in [0, 1]$ (B) For $\hat{A} \in [0, 100]$

FIGURE 1. Graphical illustration of inequality (3. 23).

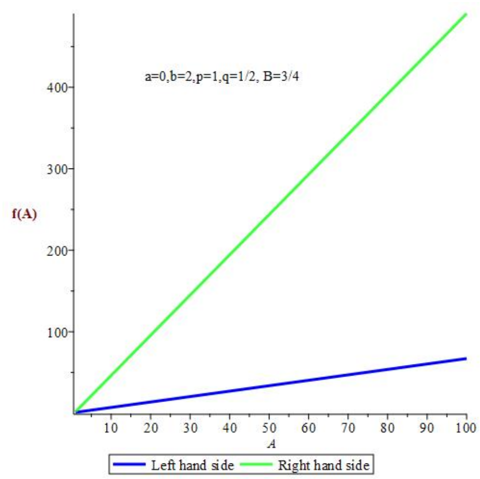


FIGURE 2. Inequality (3. 23) for $\hat{A} \in [0, 100]$.

Remark 3.15. *From Figure* 2*, it can be seen that the inequality* (3. 23) *is valid for generalized* 3*-polynomial pre-invex functions under the mentioned values of parameters.*

Conclusion: Following analysis can be made from above graphs:

(1) When $\hat{A} \in [0.7, 0.9]$, the inequality (3. 23) is not true for generalized 2-polynomial pre-invex functions under the values of parameters specified.
(2) Inequality holds for $[1, \infty)$ for generalized 2-polynomial pre-invex functions.
(3) Finally, whenever $n \geq 3$ then Theorem 3.7 is true for generalized n-polynomial pre-invex functions.

Theorem 3.16. *Assume that $f : \c{K} \to \mathbb{R}$ be (p,q)-differentiable function on the interior of $\c{K}$. If $|_{\hat{e_1}}D_{p,q}f|^{r_2}$ is $P_n^{(\alpha m h)-\delta}-$pre-invex function with $r_2 \geq 1$, then for the parameters $\hat{A}$, $\hat{B} \in \mathbb{R}$, following inequality is satisfied:*

$$\begin{aligned}
|\mathcal{S}_1| \leq \eta(\hat{e_2},\hat{e_1})\Bigg[2|\hat{B}-\hat{A}|\Bigg\{\Big(\frac{p}{4(p+q)}\Big)^{1-\frac{1}{r_2}}\Bigg(\big|\,_{\hat{e_1}}D_{p,q}f(\hat{e_1})\big|^{r_2}\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{12}^{k}(\hat{t},q,p,h;\alpha) \\
+\frac{m}{n}\big|\,_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\sum_{k=1}^{n}\mathcal{I}_{13}^{k}(\hat{t},q,p,h;\alpha)+\frac{p\delta}{4(p+q)}\Bigg)^{\frac{1}{r_2}} \\
+\left(\frac{1}{4(p+q)}\right)^{1-\frac{1}{r_2}}\Bigg(|_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|^{r_2}\,\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{14}^{k}\left(\hat{t},q,p,h;\alpha\right) \\
+\big|\,_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\frac{m}{n}\sum_{k=1}^{n}\mathcal{I}_{15}^{k}(\hat{t},q,p,h;\alpha)+\frac{\delta}{4(p+q)}\Bigg)^{\frac{1}{r_2}} \\
+\big(\mathcal{I}_{16}^{k}(\hat{t},q,p;\hat{B})\big)^{1-\frac{1}{r_2}}\Bigg(|_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|^{r_2}\,\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{17}^{k}(\hat{t},q,p,h,\alpha;\hat{B}) \\
+\big|\,_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\frac{m}{n}\sum_{k=1}^{n}\mathcal{I}_{18}^{k}(\hat{t},q,p,h,\alpha;\hat{B})+\delta\mathcal{I}_{16}^{k}(\hat{t},q,p;\hat{B})\Bigg)^{\frac{1}{r_2}} \\
+\big(\mathcal{I}_{19}^{k}(q,p;\hat{B})\big)^{1-\frac{1}{r_2}}\Bigg(|_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|^{r_2}\,\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{20}^{k}(\hat{t},q,p,\alpha;\hat{B}) \\
+\big|\,_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\frac{m}{n}\sum_{k=1}^{n}\mathcal{I}_{21}^{k}(\hat{t},q,p,\alpha;\hat{B})+\delta\mathcal{I}_{19}^{k}(q,p;\hat{B})\Bigg)^{\frac{1}{r_2}}\Bigg\}\Bigg]. \quad (3.\,30)
\end{aligned}$$

Proof. By utilizing Lemma 3. 20 along with modulus property, we have

$$\begin{aligned}
|\mathcal{S}| \leq \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}}|\hat{B}-\hat{A}|\;\big|_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|\;{}_0d_{p,q}\hat{t} \\
+\int_0^{1}|pq\hat{t}-\hat{B}|\;\big|_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|\;{}_0d_{p,q}\hat{t}\Bigg]. \quad (3.\,31)
\end{aligned}$$

By taking into account the improved power mean inequality, we have

$$
\begin{aligned}
|\mathcal{S}| \leq \eta(\hat{e_2},\hat{e_1})\Big[2|\hat{B}-\hat{A}|\Big\{&\Big(\int_0^{\frac{1}{2}}\big(\frac{1}{2}-\hat{t}\big)\ {}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\\
&\times\Big(\int_0^{\frac{1}{2}}\big(\frac{1}{2}-\hat{t}\big)\ \big|{}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|^{r_2}\ {}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}\\
&+\Big(\int_0^{\frac{1}{2}}\hat{t}\ {}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\Big(\int_0^{\frac{1}{2}}\hat{t}\ \big|{}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|^{r_2}\ {}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}\Big\}\\
&+\Big\{\Big(\int_0^{1}\big(1-\hat{t}\big)\,|pq\hat{t}-\hat{B}|\ {}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\\
&\times\Big(\int_0^{1}\big(1-\hat{t}\big)\,|pq\hat{t}-\hat{B}|\ \big|{}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|^{r_2}\ {}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}\\
&+\Big(\int_0^{1}\hat{t}|pq\hat{t}-\hat{B}|\ {}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\\
&\times\Big(\int_0^{1}\hat{t}|pq\hat{t}-\hat{B}|\ \big|{}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|^{r_2}\ {}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}\Big\}\Big]. \qquad (3.\ 32)
\end{aligned}
$$

Since $|{}_{\hat{e_1}}D_{p,q}f|^{r_2}$ is $P_n^{(\alpha m h)-\delta}-$pre-invex, so

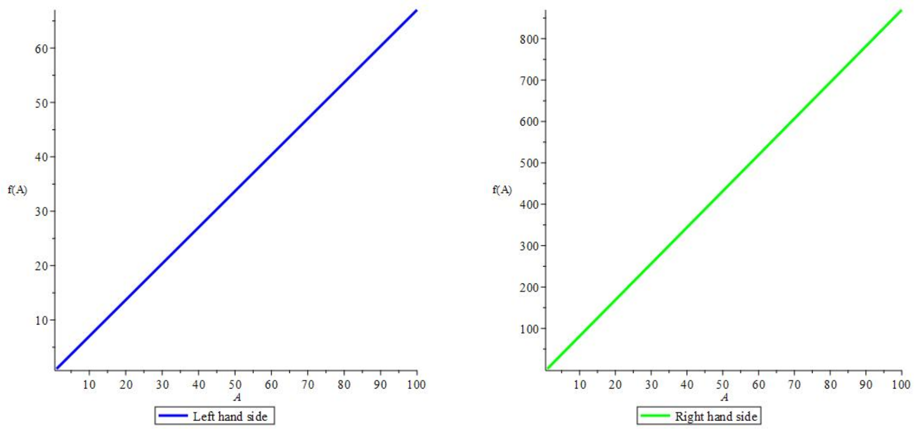
$$
\begin{aligned}
\big|{}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\big|^{r_2} \leq\ & \frac{1}{n}\sum_{k=1}^{n}(1-\big(h(\hat{t})\big)^{\alpha k})\,|{}_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|^{r_2}\\
&+\ \frac{m}{n}\sum_{k=1}^{n}(h(\hat{t}))^{\alpha k})\Big|\ {}_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\Big|^{r_2}+\delta. \qquad (3.\ 33)
\end{aligned}
$$

By making use of (3. 33) in (3. 32) , we have

$$
\begin{aligned}
|\mathcal{S}_1| \le \eta(\hat{e_2},\hat{e_1})\Big[2|\hat{B}-\hat{A}|\Big\{\Big(\int_0^{\frac{1}{2}}\big(\frac{1}{2}-\hat{t}\big)\,{}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\\
\times\Big(\,|_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|^{r_2}\,\frac{1}{n}\sum_{k=1}^{n}\int_0^{\frac{1}{2}}\big(\frac{1}{2}-\hat{t}\big)(1-\big(h(\hat{t})\big)^{\alpha k})\,{}_0d_{p,q}\hat{t}\\
+\big|\,{}_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\frac{m}{n}\sum_{k=1}^{n}\int_0^{\frac{1}{2}}\big(\frac{1}{2}-\hat{t}\big)(h(\hat{t}))^{\alpha k}\,{}_0d_{p,q}\hat{t}\\
+\delta\int_0^{\frac{1}{2}}\big(\frac{1}{2}-\hat{t}\big)\,{}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}+\Big(\int_0^{\frac{1}{2}}\hat{t}\,{}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\\
\times\Big(\,|_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|^{r_2}\,\frac{1}{n}\sum_{k=1}^{n}\int_0^{\frac{1}{2}}\hat{t}(1-(h(\hat{t}))^{\alpha k})\,{}_0d_{p,q}\hat{t}\\
+\big|\,{}_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\frac{m}{n}\sum_{k=1}^{n}\int_0^{\frac{1}{2}}\hat{t}(h(\hat{t}))^{\alpha k}\,{}_0d_{p,q}\hat{t}\\
+\delta\int_0^{\frac{1}{2}}\hat{t}\,{}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}\Big\}+\Big(\int_0^{1}(1-\hat{t})|pq\hat{t}-\hat{B}|\,{}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\\
\times\Big(\,|_{\hat{e_1}}D_{p,q}f\,(\hat{e_1})|^{r_2}\,\frac{1}{n}\sum_{k=1}^{n}\int_0^{1}(1-\hat{t})|pq\hat{t}-\hat{B}|(1-\big(h(\hat{t})\big)^{\alpha k})\,{}_0d_{p,q}\hat{t}\\
+\big|\,{}_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\frac{m}{n}\sum_{k=1}^{n}\int_0^{1}(1-\hat{t})|pq\hat{t}-\hat{B}|(h(\hat{t}))^{\alpha k}\,{}_0d_{p,q}\hat{t}\\
+\delta\int_0^{1}(1-\hat{t})|pq\hat{t}-\hat{B}|\,{}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}+\int_0^{1}\hat{t}|pq\hat{t}-\hat{B}|\,{}_0d_{p,q}\hat{t}\Big)^{1-\frac{1}{r_2}}\\
\times\Big(\,|_{\hat{e_1}}D_{p,q}f(\hat{e_1})|^{r_2}\,\frac{1}{n}\sum_{k=1}^{n}\int_0^{1}\hat{t}|pq\hat{t}-\hat{B}|(1-(h(\hat{t}))^{\alpha k})\,{}_0d_{p,q}\hat{t}\\
+\big|\,{}_{\hat{e_1}}D_{p,q}f\big(\frac{\hat{e_2}}{m}\big)\big|^{r_2}\frac{m}{n}\sum_{k=1}^{n}\int_0^{1}\hat{t}|pq\hat{t}-\hat{B}|(h(\hat{t}))^{\alpha k}\,{}_0d_{p,q}\hat{t}\\
+\delta\int_0^{1}\hat{t}|pq\hat{t}-\hat{B}|\,{}_0d_{p,q}\hat{t}\Big)^{\frac{1}{r_2}}\Big\}\Big].
\end{aligned}
$$

By using the evaluated (p,q)-integrals, we obtain the desired inequality (3. 30). □

Now, the validity of above theorem is presented graphical as under:



(A) L.H.S of inequality (3. 30) for $\hat{A} \in [0,100]$ (B) R.H.S of inequality (3. 30) for $\hat{A} \in [0,100]$

FIGURE 3. Graphiccall illustraion of inequality (3. 30).

Remark 3.17. *Figure 3 shows that that Theorem 3.16 holds unconditionally for the parameters described earlier for generalized* 2*-polynomial pre-invex in particular and hence in general for generalized* n*-polynomial pre-invexity whenever* $n \geq 2$*.*

Theorem 3.18. *Assume that* $f : Ķ \to \mathbb{R}$ *be continuous on* $Ķ$ (p,q)*-differentiable on the interior of* $Ķ$*. If* ${}_{\hat{e_1}}D_{p,q}f$ *is integrable such that* $-\infty < L_1 \leq {}_{\hat{e_1}}D_{p,q}f(t) \leq L_2 < +\infty$ *for all* $\hat{t} \in Ķ$ *and parameters* A, B*, the following inequality holds:*

$$\begin{aligned}&\left|\mathcal{S}_1 - \frac{\eta(\hat{e_2},\hat{e_1})(L_1+L_2)}{2}\Big(\frac{pq}{p+q} - \frac{\hat{A}+\hat{B}}{2}\Big)\right| \\ &\leq \frac{\eta(\hat{e_2},\hat{e_1})\,(L_2-L_1)}{2}\left[\frac{|\hat{B}-\hat{A}| + 2\mathcal{I}_5(q,p,\hat{t};\hat{B})}{2}\right]. \qquad (3.\ 34)\end{aligned}$$

Proof. By taking into account the Lemma3.4

$$
\begin{aligned}
&\mathcal{S}_1 - \frac{\eta(\hat{e_2},\hat{e_1})(L_1+L_2)}{2}\left(\frac{pq}{p+q} - \frac{\hat{A}+\hat{B}}{2}\right)\\
&\leq \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (pq\hat{t}-\hat{A})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\\
&\quad + \int_{\frac{1}{2}}^1 (pq\hat{t}-\hat{B})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\Bigg]\\
&\quad - \frac{\eta(\hat{e_2},\hat{e_1})(L_1+L_2)}{2}\left(\frac{pq}{p+q} - \frac{\hat{A}+\hat{B}}{2}\right)\\
&= \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (\hat{B}-\hat{A})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\\
&\quad + \int_0^1 \left(pq\hat{t}-\hat{B}\right)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\Bigg]\\
&\quad - \frac{\eta(\hat{e_2},\hat{e_1})(L_1+L_2)}{2}\left(\frac{pq}{p+q} - \frac{\hat{A}+\hat{B}}{2}\right)\\
&= \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (\hat{B}-\hat{A})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\\
&\quad + \int_0^1 (pq\hat{t}-\hat{B})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t}\Bigg]\\
&\quad - \frac{\eta(\hat{e_2},\hat{e_1})(L_1+L_2)}{2}\Bigg[\int_0^{\frac{1}{2}} (\hat{B}-\hat{A})\ {}_0d_{p,q}\hat{t} + \int_0^1 (pq\hat{t}-\hat{B})\ {}_0d_{p,q}\hat{t}\Bigg]\\
&= \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (\hat{B}-\hat{A})\left(\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})) - \frac{L_1+L_2}{2}\right)\ {}_0d_{p,q}\hat{t}\\
&\quad + \int_0^1 \left(pq\hat{t}-\hat{B}\right)\left(\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1}))\ {}_0d_{p,q}\hat{t} - \frac{L_1+L_2}{2}\right)\ {}_0d_{p,q}\hat{t}\Bigg].
\end{aligned}
$$

Taking modulus on both sides, we have

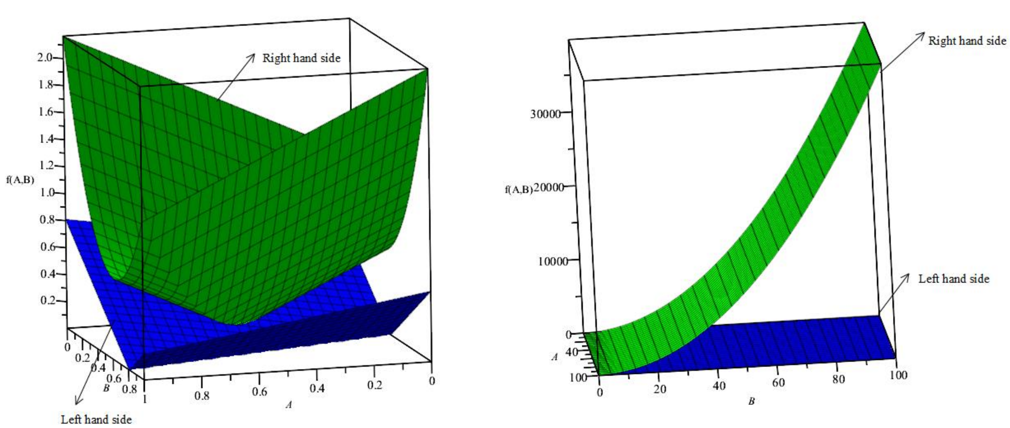
$$\begin{aligned}
&\left|\mathcal{S}_1 - \frac{\eta(\hat{e_2},\hat{e_1})(L_1+L_2)}{2}\left(\frac{pq}{p+q} - \frac{\hat{A}+\hat{B}}{2}\right)\right| \\
&\leq \eta(\hat{e_2},\hat{e_1})\Bigg[\int_0^{\frac{1}{2}} |\hat{B}-\hat{A}| \left| {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})) - \frac{L_1+L_2}{2}\right| {}_0d_{p,q}\hat{t} \\
&\quad + \int_0^{1} |pq\hat{t}-\hat{B}| \left| {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})) - \frac{L_1+L_2}{2}\right| {}_0d_{p,q}\hat{t}\Bigg].
\end{aligned}$$

Since $-\infty < L_1 \leq {}_{\hat{e_1}}D_{p,q}f(\hat{t}) \leq L_2 < \infty$, so, $L_1 - \frac{L_1+L_2}{2} \leq {}_{\hat{e_1}}D_{p,q}f(\hat{t}) - \frac{L_1+L_2}{2} \leq L_2 - \frac{L_1+L_2}{2}$,
this implies that

$$\left| {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})) - \frac{L_1+L_2}{2}\right| \leq \frac{L_2-L_1}{2}, \tag{3. 35}$$

using (3. 35) and evaluating the post-quantum integrals involved in above equation, we obtain the desired inequality (3. 34). □

Now, we check the validity of theorem under the limitations of parameters



(A) Here $\hat{A} \in [0,1]$ and $\hat{B} \in [0,1]$ (B) Here $\hat{A} \in [0,100]$ and $\hat{B} \in [0,100]$

FIGURE 4. Graphical illustration of inequality 3.18.

Remark 3.19. *From the above graphs, it is clear that the Theorem 3.18 holds for all non-negative real numbers as the values of parameters $\hat{A}$ and $\hat{B}$. However, it is pertinent to note that the geometrical growth starts in the right hand side whenever $\hat{A},\ \hat{B} \geq 1$.*

Next theorem involves the post-quantum integral inequality for the functions satisfying Lipschitz condition with certain Lipschitz number.

Theorem 3.20. *Assume that $f : Ķ \to \mathbb{R}$ be continuous on Ķ and (p, q)-differentiable on the interior of Ķ. If ${}_{\hat{e_1}}D_{p,q}f$ is integrable such that ${}_{\hat{e_1}}D_{p,q}f(t)$ satisfies the Lipschitz condition for all $\hat{t} \in Ķ$ with $\hat{L}$ as Lipschitz number. Then for parameters $A, B \in \mathbb{R}$, the following inequality holds:*

$$\left|\mathcal{S}_1 - \eta(\hat{e_2}, \hat{e_1})\left\{\frac{(\hat{B}-\hat{A})}{2}\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}) + \big(\frac{pq}{p+q} - \hat{B}\big)\ {}_{\hat{e_1}}D_{p,q}f\,(\hat{e_2})\right\}\right|$$
$$\leq \hat{L}\eta^2(\hat{e_2}, \hat{e_1})\left[\frac{|\hat{B}-\hat{A}|}{4(p+q)} + \mathcal{I}_{22}^{k}(\hat{t}, q, p; \hat{B})\right]. \tag{3. 36}$$

Proof. Lemma 3.4 gives

$$\mathcal{S}_1 - \eta(\hat{e_2}, \hat{e_1})\left\{\frac{(\hat{B}-\hat{A})}{2}\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}) + \big(\frac{pq}{p+q} - \hat{B}\big)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_2})\right\}$$
$$= \eta(\hat{e_2}, \hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (pq\hat{t} - \hat{A})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2}, \hat{e_1}))\ {}_0d_{p,q}\hat{t}$$
$$+ \int_{\frac{1}{2}}^{1} (pq\hat{t} - \hat{B})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2}, \hat{e_1}))\ {}_0d_{p,q}\hat{t}\Bigg]$$
$$- \eta(\hat{e_2}, \hat{e_1})\left\{\frac{\hat{B}-\hat{A}}{2}\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}) + \big(\frac{pq}{p+q} - \hat{B}\big)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_2})\right\}$$
$$= \eta(\hat{e_2}, \hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (\hat{B} - \hat{A})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2}, \hat{e_1}))\ {}_0d_{p,q}\hat{t}$$
$$+ \int_0^{1} (pq\hat{t} - \hat{B})\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2}, \hat{e_1}))\ {}_0d_{p,q}\hat{t}\Bigg]$$
$$- \eta(\hat{e_2}, \hat{e_1})\left\{\frac{\hat{B}-\hat{A}}{2}\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}) + \big(\frac{pq}{p+q} - \hat{B}\big)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_2})\right\}$$
$$= \eta(\hat{e_2}, \hat{e_1})\Bigg[\int_0^{\frac{1}{2}} (\hat{B} - \hat{A})\left\{\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2}, \hat{e_1})) -\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1})\right\}\ {}_0d_{p,q}\hat{t}$$
$$+ \int_0^{1} (pq\hat{t} - \hat{B})\left\{\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1} + \hat{t}\eta(\hat{e_2}, \hat{e_1})) -\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_2})\right\}\ {}_0d_{p,q}\hat{t}\Bigg].$$

Using the modulus property

$$\Big|\mathcal{S}_1 - \eta(\hat{e_2},\hat{e_1})\Big\{\frac{(\hat{B}-\hat{A})}{2}\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}) + \big(\frac{pq}{p+q}-\hat{B}\big)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_2})\Big\}\Big|$$

$$\leq \eta(\hat{e_2},\hat{e_1})\Big|\int_0^{\frac{1}{2}} |\hat{B}-\hat{A}|\,\big|\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})) - {}_{\hat{e_1}}D_{p,q}f(\hat{e_1})\big|\ {}_0d_{p,q}\hat{t}$$

$$+\int_0^1 |pq\hat{t}-\hat{B}|\,\big|\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}+\hat{t}\eta(\hat{e_2},\hat{e_1})) - {}_{\hat{e_1}}D_{p,q}f\,(\hat{e_2})\big|\ {}_0d_{p,q}\hat{t}.$$

By using Lipschitz condition, we have

$$\Big|\mathcal{S}_1 - \eta(\hat{e_2},\hat{e_1})\Big\{\frac{(\hat{B}-\hat{A})}{2}\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_1}) + \big(\frac{pq}{p+q}-\hat{B}\big)\ {}_{\hat{e_1}}D_{p,q}f(\hat{e_2})\Big\}\Big|$$

$$\leq \hat{L}\eta(\hat{e_2},\hat{e_1})\Big[\frac{|\eta(\hat{e_2},\hat{e_1})|\,|\hat{B}-\hat{A}|}{4(p+q)} + \int_0^1 |pq\hat{t}-\hat{B}|\Big|\hat{t}-\frac{(\hat{e_2}-\hat{e_1})}{\eta(\hat{e_2},\hat{e_1})}\Big|\ {}_0d_{p,q}\hat{t}\Big].$$

Utilizing the (p,q)-integral, we obtain the desired inequality. □

Next, graphical illustration of inequality (3. 36) is presented as follows.

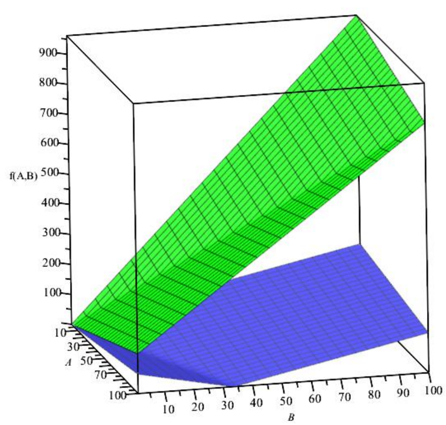


FIGURE 5. Here $\hat{A}\in[0,100]$ and $\hat{B}\in[0,100]$.

Theorem 3.21. *Assume that $f : Ķ \to \mathbb{R}$ be continuous, (p, q)-integrable and $P_n^{(\alpha mh)-\delta}-$prinvex on Ķ. If $g : Ķ \to \mathbb{R}$ is non-negative integrable function having the symmetric property about $m\sigma = \left(\frac{2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})}{2}\right)$. Then, the following inequality is satisfied:*

$$\begin{aligned}
& f\left(\frac{2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})}{2}\right)\int_{\hat{e_1}}^{\hat{e_1}+\eta(\hat{e_2},\hat{e_1})} g(\sigma)\ {}_{\hat{e_1}}d_{p,q}\sigma \\
& \leq \frac{1}{n}\sum_{k=1}^{n}(1-(h(\frac{1}{2}))^{\alpha k})\int_{\hat{e_1}}^{\hat{e_1}+\eta(\hat{e_2},\hat{e_1})} f(\sigma)g(\sigma)\ {}_{\hat{e_1}}d_{p,q}\sigma \\
& \quad + \frac{m^2}{n}\sum_{k=1}^{n}(h(\frac{1}{2}))^{\alpha k}\int_{\frac{\hat{e_1}}{m}}^{\frac{\hat{e_1}}{m}+\frac{\eta(\hat{e_2},\hat{e_1})}{m}} f(\sigma)g(m\sigma)\ {}_{\hat{e_1}}d_{p,q}\sigma \\
& \quad + \delta\int_{\hat{e_1}}^{\hat{e_1}+\eta(\hat{e_2},\hat{e_1})} g(\sigma)\ {}_{\hat{e_1}}d_{p,q}\sigma. \qquad (3.\ 37)
\end{aligned}$$

Proof. Since f is $P_n^{(\alpha mh)-\delta}-$prinvex on Ķ, so for $\hat{u_1},\hat{u_2} \in$ Ķ, we have

$$f\left(\frac{2\hat{u_1}+\eta(\hat{u_2},\hat{u_1})}{2}\right) \leq \frac{1}{n}\sum_{k=1}^{n}(1-(h(\frac{1}{2}))^{\alpha k})f(u_1) + \frac{m}{n}\sum_{k=1}^{n}(h(\frac{1}{2}))^{\alpha k}f(\frac{u_2}{m}) + \delta.$$

For any non-negative function g with the choice of $u_1 = \hat{e_1} + \left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})$, $u_2 = \hat{e_1} + \left(\frac{1+\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})$ and taking into account the condition $'\mathrm{C}'$, we have

$$\begin{aligned}
& f\left(\frac{2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})}{2}\right)g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right) \\
& \leq \frac{1}{n}\sum_{k=1}^{n}(1-(h(\frac{1}{2}))^{\alpha k})f\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right)g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right) \\
& \quad + \frac{m}{n}\sum_{k=1}^{n}(h(\frac{1}{2}))^{\alpha k}f\left(\frac{\hat{e_1}}{m}+\left(\frac{1+\hat{t}}{2m}\right)\eta(\hat{e_2},\hat{e_1})\right)g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right) \\
& \quad + \delta\, g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right).
\end{aligned}$$

Taking (p,q)-integral w.r.t $\hat{t}$ on the interval $[-1,1]$

$$f\left(\frac{2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})}{2}\right)\int_{-1}^{1} g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right)\ {}_{\hat{e_1}}d_{p,q}\hat{t}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}(1-(h(\frac{1}{2}))^{\alpha k})$$
$$\times\int_{-1}^{1} f\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right) g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right)\ {}_{\hat{e_1}}d_{p,q}\hat{t}$$
$$+\frac{m}{n}\sum_{k=1}^{n}(h(\frac{1}{2}))^{\alpha k}$$
$$\times\int_{-1}^{1} f\left(\frac{\hat{e_1}}{m}+\left(\frac{1+\hat{t}}{2m}\right)\eta(\hat{e_2},\hat{e_1})\right) g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right)\ {}_{\hat{e_1}}d_{p,q}\hat{t}$$
$$+\delta\int_{-1}^{1} g\left(\hat{e_1}+\left(\frac{1-\hat{t}}{2}\right)\eta(\hat{e_2},\hat{e_1})\right)\ {}_{\hat{e_1}}d_{p,q}\hat{t}.$$

Further calculation reveals that

$$f\left(\frac{2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})}{2}\right)\int_{\hat{e_1}}^{\hat{e_1}+\eta(\hat{e_2},\hat{e_1})} g(\sigma)\ {}_{\hat{e_1}}d_{p,q}\sigma$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}(1-(h(\frac{1}{2}))^{\alpha k})\int_{\hat{e_1}}^{\hat{e_1}+\eta(\hat{e_2},\hat{e_1})} f(\sigma)g(\sigma)\ {}_{\hat{e_1}}d_{p,q}\sigma$$
$$+\frac{m^2}{n}\sum_{k=1}^{n}(h(\frac{1}{2}))^{\alpha k}\int_{\frac{\hat{e_1}}{m}}^{\frac{\hat{e_1}}{m}+\frac{\eta(\hat{e_2},\hat{e_1})}{m}} f(\sigma)g(2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})-\sigma m)\ {}_{\hat{e_1}}d_{p,q}\sigma$$
$$+\delta\int_{\hat{e_1}}^{\hat{e_1}+\eta(\hat{e_2},\hat{e_1})} g(\sigma)\ {}_{\hat{e_1}}d_{p,q}\sigma.$$

since g is symmetric about $m\sigma=\left(\frac{2\hat{e_1}+\eta(\hat{e_2},\hat{e_1})}{2}\right)$, so we obtain the desired inequality (3. 37). □

4. APPLICATIONS OF RESULTS

Let us recall some useful means of real numbers $\hat{e_1},\hat{e_2}$.
For arithmetic means, we use $\mathcal{A}:\mathcal{A}(\hat{e_1},\hat{e_2})=\frac{\hat{e_1}+\hat{e_2}}{2}$.
For geometric means, we use $\mathcal{G}:\mathcal{G}(\hat{e_1},\hat{e_2})=\sqrt{\hat{e_1}\hat{e_2}}$.

For generalized log-mean, we use $\pounds_\rho(\hat{e_1},\hat{e_2}) = \left(\frac{\hat{e_2}^{\rho+1}-\hat{e_1}^{\rho+1}}{(\rho+1)(\hat{e_2}-\hat{e_1})}\right)^{\frac{1}{\rho}}$, $\rho \in \mathbb{R} - \{-1, 0\}$.
For log-mean, we use $\pounds(\hat{e_1},\hat{e_2}) = \frac{\hat{e_2}-\hat{e_1}}{\ln\hat{e_2}-\ln\hat{e_1}}$.

Proposition 4.1. *Assume that $\hat{A}$,$\hat{B} \in \mathbb{R}$ are such that $\hat{A}$+$\hat{B} = p$ for $\hat{A},\hat{B} \in [0,1]$. Then for $\beta > 1$, we have*

$$\begin{aligned}
&\Big|2\hat{A}\mathcal{A}(\hat{e_1}^{\beta},\hat{e_2}^{\beta}) + (p-2\hat{A})\mathcal{A}^{\beta}\left(2\hat{e_1},\eta(\hat{e_2},\hat{e_1})\right) - \mathcal{S}_\beta(\hat{e_1},\hat{e_2};p,q)\Big| \\
&\leq \beta\eta(\hat{e_2},\hat{e_1})\big[\frac{1}{n}\sum_{k=1}^{n}\big\{|\hat{B}-\hat{A}|\mathcal{I}_1^k(\hat{t},h;\alpha) + \mathcal{I}_2^k(\hat{t},h,\alpha;\hat{B})\big\}\hat{e_1}^{\beta-1} \\
&+ \frac{m}{n}\sum_{k=1}^{n}\big\{|\hat{B}-\hat{A}|\mathcal{I}_3^k(\hat{t},h;\alpha) + \mathcal{I}_4^k(\hat{t},h,\alpha;\hat{B})\big\} + \big\{\frac{|\hat{B}-\hat{A}|}{2} + \mathcal{I}_5^k(q,p;\hat{B})\big\}\delta\big].
\end{aligned} \tag{4. 38}$$

where

$$\mathcal{S}_\beta(\hat{e_1},\hat{e_2};p,q) := p(p-q)\sum_{r_1=0}^{\infty}\frac{q^{r_1}}{p^{r_1+1}}\left(\hat{e_1} + \eta(\hat{e_2},\hat{e_1})\frac{q^{r_1}}{p^{r_1+1}}\right)^{\beta}. \tag{4. 39}$$

Proof. Let $f(\sigma) = \sigma^\beta, \beta \geq 2$ in Theorem 3.7,then the required inequality (4. 38) is achieved. $\square$

Proposition 4.2. *Assume that $\hat{A}$,$\hat{B} \in \mathbb{R}$ are such that $\hat{A}$+$\hat{B} = p$ for $\hat{A}, \hat{B} \in [0,1]$. Then for $r > 1$, we have*

$$\begin{aligned}
&\left|(2\hat{A})\mathcal{A}(\hat{e_1}^{-1}, \hat{e_2}^{-1}) + (p-2\hat{A})\mathcal{A}^{-1}\left(2\hat{e_1}, \eta(\hat{e_2}, \hat{e_1})\right) - \mathcal{S}_{-1}(\hat{e_1}, \hat{e_2}; p, q)\right| \\
&\leq \eta(\hat{e_2}, \hat{e_1})\Bigg[2|\hat{B}-\hat{A}|\Bigg\{\Big(\frac{p}{4(p+q)}\Big)^{1-\frac{1}{r_2}}\Bigg((\hat{e_1}^{-2})^{r_2}\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{12}^{k}(\hat{t}, q, p, h; \alpha) \\
&+ \left(\pounds_{-2}^{-2}\left(\hat{e_1} + q\eta(\hat{e_2}, \hat{e_1}), \hat{e_1} + p\eta(\hat{e_2}, \hat{e_1})\right)\right)^{r_2}\frac{m}{n}\sum_{k=1}^{n}\mathcal{I}_{13}^{k}\left(\hat{t}, q, p, h; \alpha\right) + \frac{p\delta}{4(p+q)}\Bigg)^{\frac{1}{r_2}} \\
&+ \Big(\frac{1}{4(p+q)}\Big)^{1-\frac{1}{r_2}}\Bigg((\hat{e_1}^{-2})^{r_2}\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{14}^{k}\left(\hat{t}, q, p, h; \alpha\right) \\
&+ \left(\pounds_{-2}^{-2}\left(\hat{e_1} + q\eta(\hat{e_2}, \hat{e_1}), \hat{e_1} + p\eta(\hat{e_2}, \hat{e_1})\right)\right)^{r_2}\frac{m}{n}\sum_{k=1}^{n}\mathcal{I}_{15}^{k}\left(\hat{t}, q, p, h; \alpha\right) + \frac{\delta}{4(p+q)}\Bigg)^{\frac{1}{r_2}} \\
&+ \left(\mathcal{I}_{16}^{k}(\hat{t}, q, p; \hat{B})\right)^{1-\frac{1}{r_2}}\Bigg((\hat{e_1}^{-2})^{r_2}\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{17}^{k}(\hat{t}, q, p, h\alpha; \hat{B}) \\
&+ \left(\pounds_{-2}^{-2}\left(\hat{e_1} + q\eta(\hat{e_2}, \hat{e_1}), \hat{e_1} + p\eta(\hat{e_2}, \hat{e_1})\right)\right)^{r_2}\frac{m}{n}\sum_{k=1}^{n}\mathcal{I}_{18}^{k}\left(\hat{t}, q, p, h\alpha; \hat{B}\right) \\
&+ \delta\mathcal{I}_{16}^{k}(\hat{t}, q, p; \hat{B})\Bigg)^{\frac{1}{r_2}} + \left(\mathcal{I}_{19}^{k}(q, p; \hat{B})\right)^{1-\frac{1}{r_2}}\Bigg((\hat{e_1}^{-2})^{r_2}\frac{1}{n}\sum_{k=1}^{n}\mathcal{I}_{20}^{k}(\hat{t}, q, p, \alpha; \hat{B}) \\
&+ \left(\pounds_{-2}^{-2}\left(\hat{e_1} + q\eta(\hat{e_2}, \hat{e_1}), \hat{e_1} + p\eta(\hat{e_2}, \hat{e_1})\right)\right)^{r_2}\frac{m}{n}\sum_{k=1}^{n}\mathcal{I}_{21}^{k}(\hat{t}, q, p, \alpha; \hat{B}) \\
&+ \delta\mathcal{I}_{19}^{k}(q, p; \hat{B})\Bigg)^{\frac{1}{r_2}}\Bigg\}\Bigg]. \qquad (4.\ 40)
\end{aligned}$$

Proof. By setting $f(\sigma) = \sigma^{-1}$ in Theorem3.16, the required inequality (4. 40) is achieved. □

Proposition 4.3. *Assume that $\hat{A}$,$\hat{B} \in \mathbb{R}$ are such that $\hat{A}$+$\hat{B} = p$ for $\hat{A}, \hat{B} \in [0,1]$. Then*

$$\begin{aligned}
&\Bigg|(2\hat{A})\mathcal{A}(e^{\hat{e_1}}, e^{\hat{e_2}}) + (p-2\hat{A})\mathcal{G}\left(e^{\hat{e_1}}, e^{\frac{\eta(\hat{e_2}, \hat{e_1})}{2}}\right) - \mathcal{S}_1(\hat{e_1}, \hat{e_2}; p, q) \\
&\qquad + \frac{\eta(\hat{e_2}, \hat{e_1})(L_1 + L_2)p(p-q)}{4(p+q)}\Bigg| \\
&\leq \frac{\eta(\hat{e_2}, \hat{e_1})\left(L_2 - L_1\right)}{2}\left[\frac{|\hat{B}-\hat{A}| + 2\mathcal{I}_5(q, p, \hat{t}; \hat{B})}{2}\right]. \qquad (4.\ 41)
\end{aligned}$$

Proof. Taking $f(\sigma) = e^{\sigma}$ in Theorem 3.18 leads to the desired inequality. □

5. Conclusion

Post-quantum calculus is one of the active area of recent investigation because of its numerous applications in quantum cryptography, modeling problems on geometric series, hyper geometric series and quantum physics. Non-convex and non-differentiable objective functions or constraints are present in many real-world optimization problems in applied sciences, engineering, and economics, yet they nevertheless display some "generalized convex-like" behavior. This study involves the analysis of non-convex functions by introducing a new calss of n-polynomial pre-invex function. We show that every generalized 1-polynomial pre-invex function is pre-invex and every non-negative pre-invex function is 2-polynomial convex under the condition mentioned. At start, we establish a multi parameter identity for (p, q)-integrals to extend the (p, q)-estimates of inequalities (2. 11) and (2. 13) via newly defined n-polynomial pre-invexity. Furthermore, we analyze inequalities (3. 36) and (3. 37) for functions satisfying Lipschitz condition and symmetricity respectively. In order to justify the results, we utilize the graphical representation with the help of concrete examples. To focus on utilization of the presented results, two new applications via inequalities (4. 38) and (4. 40). For interested readers, there are few new areas to investigate, for instance, (p, q)-fractional integrals, fractional integrals with variable power, Simpson's, Owstroski's, fink's, Trapezoidal type, stochastic process and classic Hermite-Hadamard type inequalities by considering newly defined classes of polynomial pre-invex functions in (3. 14) to (3. 18).

Conflict of interest. The authors declare that they have no competing interests.

Author contributions. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Funding. No external funding was available for this research.

References

[1] A. Ben-Israel, B. Mond, *What is invexity?*, J. Austral. Math. Soc. Ser. E **28** (1986) 1–9.
[2] R. Chakrabarti, R. Jagannathan, *A (p, q)-oscillator realization of two-parameter quantum algebras*, J. Phys. A, Math. Gen. **24** (1991) L711.
[3] F. X. Chen, W. G. Yang, *Some new Chebyshev type quantum integral inequalities on finite intervals*, J. Comput. Anal. Appl. **21** No. 3 (2016) 417–426.
[4] W. T. Cheng, W. H. Zhang, Q. B. Cai, *(p, q)-Gamma operators which preserve χ^2*, J. Inequal. Appl. **108** (2019).
[5] B. D. Craven, *Invex functions and constrained local minima*, Bull. Austral. Math. Soc. **24** (1981) 357–366.
[6] B. D. Craven, B. M. Glover, *Invex function and duality*, J. Austral. Math. Soc. Ser. A **39** (1985) 1–20.
[7] T. Du, C. Luo, B. Yu, *Certain quantum estimates on the parametrized integral inequalities and their applications*, J. Math. Inequal. **15** No. 1 (2021) 201–228.
[8] T. Ernst, *A Comprehensive Treatment of q-Calculus*, Springer, Basel, 2012.
[9] H. Gauchman, *Integral inequalities in q-calculus*, Comput. Math. Appl. **47** (2004) 281–300.
[10] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981) 545–550.
[11] F. H. Jackson, *On a q-definite integrals*, Q. J. Pure Appl. Math. **41** (1910) 193–203.
[12] F. H. Jackson, *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edin. **46** (1908) 253–281.
[13] R. Jagannathan, K. S. Rao, *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, arXiv:math/0602613 [math.NT].
[14] V. Kac, P. Cheung, *Quantum Calculus*, Springer, Berlin, New York, 2002.

[15] M. A. Latif, M. Shoaib, *Hermite–Hadamard type integral inequalities for differentiable m-pre-invex and (α, m)-pre-invex functions*, J. Egypt. Math. Soc. **23** No. 2 (2015) 236–241.
[16] X. Liu, K. N. Aslam, M.S. Saleem, Sh. Ren, *A generalized version of polynomial convex functions and some interesting inequalities with applications*, Journal of Mathematics **2023** No. 1 (2023) 10 pages.
[17] W. J. Liu, H. F. Zhuang, *Some quantum estimates of Hermite–Hadamard inequalities for convex functions*, J. Appl. Anal. Comput. **7** No.2 (2017) 501–522.
[18] S. Maden, H. Kadakal, M. Kadakal, İ. İscan, *Some new integral inequalities for n-times differentiable convex functions*, J. Nonlinear Sci. Appl. **10** No. 12 (2017) 6141–6148.
[19] M. Matloka, *On some new inequalities for differentiable (h_1, h_2)-pre-invex functions on the co-ordinates*, Mathematics and Statistics **2** (2014), 1, 6–14.
[20] G. V. Milovanović, V. Gupta, N. Malik, *(p, q)-beta functions and applications in approximation*, Bol. Soc. Mat. Mexicana (2016), `arXiv:1602.06307v2 [math.CA]`.
[21] S. R. Mohan, S. K. Neogy, *On invex sets and pre-invex functions*, J. Math. Anal. Appl. **189** (1995) 901–908.
[22] T. Nawaz, M. A. Memon, K. Jacob, *Hermite?Hadamard-type inequalities for product of functions by using convex functions*, Journal of Mathematics **2021** No. 1 (2021) 9 pages.
[23] M. A. Noor, *Hermite-Hadamard integral inequalities for log-pre-invex functions*, J. Math. Anal. Approx. Theory **2** (2007) 126–131.
[24] M. A. Noor, K. I. Noor, M. U. Awan, J. Li, *On Hermite?Hadamard inequalities for h-preinvex functions*, Filomat **28** No. 7 (2014) 1463–1474.
[25] R. Pini, *Invexity and generalized convexity*, Optimization **22** (1991) 513–525.
[26] M. Raees, A. Kashuri, M. U. Awan, M. Anwar, *Some new post-quantum integral inequalities involving multi-parameter and their applications*, Math. Meth. Appl. Sci. (2022) 1–23. `10.1002/mma.8151`.
[27] P. N. Sadjang, *On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas*, `arXiv:1309.3934 [math.QA]`.
[28] M. Z. Sarikaya, H. Bozkurt, N. Alp, *On Hermite?Hadamard type integral inequalities for pre-invex and log-pre-invex functions*, `arXiv:1203.4759v1`.
[29] D. F. Sofonea, *Some properties in q-calculus*, Gen. Math. **16** (2008) 47–54.
[30] D. F. Sofonea, *Numerical analysis and q-calculus. I*, Octogon **11** (2003) 151–156.
[31] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inf. Sci. **5** (2011) 390–444.
[32] H. M. Srivastava, J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Amsterdam (2012).
[33] W. Sudsutad, S. K. Ntouyas, J. Tariboon, *Quantum integral inequalities for convex functions*, J. Math. Inequal. **9** No. 3 (2015) 781–793.
[34] J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Diff. Equ. **282** (2013) 1–19.
[35] J. Tariboon, S. K. Ntouyas, *Quantum integral inequalities on finite intervals*, J. Inequal. Appl. **121** (2014).
[36] M. Z. Sarikaya, H. Bozkurt, N. Alp, *On Hermite?Hadamard type integral inequalities for pre-invex and log-pre-invex functions*, `arXiv:1203.4759v1`.
[37] M. Tunç, E. Göv, *(p,q)-integral inequalities*, RGMIA Res. Rep. Coll. **19** (2016), 1–13.
[38] M. Tunç, E. Göv, S. Balgeçti, *Simpson type quantum integral inequalities for convex functions*, Miskolc Math. Notes **19** No. 1 (2018) 649–664.
[39] T. Weir, B. Mond, *Preinvex functions in multiple objective optimization*, J. Math. Anal. Appl. **136** (1998) 29–38.
[40] X .M. Yang, D. Li, *On properties of pre-invex functions*, J. Math. Anal. Appl. **256** (2001) 229–241.