Punjab University Journal of Mathematics (2025), 57(08), 853-895
https://doi.org/10.52280/pujm.2025.57(08)04

Entropy-Driven Decision-Making for Cybersecurity Risk Assessment using PULq-ROFS and the CODAS Technique

Received
04 August, 2025

Revised
14 November, 2025
09 December, 2025

Accepted
11 December, 2025

Published Online
23 December, 2025

Uzma Ahmad
Institute of Mathematics,
University of the Punjab, New Campus, Lahore, Pakistan

Saira Hameed
Institute of Mathematics,
University of the Punjab, New Campus, Lahore, Pakistan

Rehman Khan
Institute of Mathematics,
University of the Punjab, New Campus, Lahore, Pakistan

*Ayesha Khan
Department of Mathematics, School of Science,
University of Management and Technology, Lahore, 54770, Pakistan

Abstract. Multi-criteria group decision-making (MCGDM) is a significant procedure because it facilitates and enhances decision-making (DM) by incorporating diverse variables and professional viewpoints, producing more adequate results. A set of flexibility to manage confusion and un-predictable information is improved by adding probability which makes choices more predictable. The present work suggests a technique that utilizes fuzziness to handle MCGDM issues that frequently arise in cybersecurity risk assessment. This strategy overcomes the fundamental difficulties in privacy and security information by using the probabilistic uncertain linguistic q-rung orthopair fuzzy set (PULq-ROFS). Comparedwith other fuzzy collections, such as statistical tentative, linguistically in-tuitionistic, and linguistically Euclidean imprecise collections that effec-tively include erratic and non-erraticproblems, the PULq-ROFS provides multiple characteristics. To advance this framework, we propose two new operators the PULq-ROF einstein weighted average (PULq-ROFEWA) and PULq-ROF Einstein order weighted average (PULq-ROFEOWA) that efficiently integrate the statistical lexical choice data. These operations introduce a creative concept within the PULq-ROF context. Furthermore, by applying the entropy technique, we determine the relative weights of parameters based on their informational contribution to the study.

**Corresponding Author:* ayeshakhan02oct@gmail.com

In addition, we employ the combinative distance-based assessment (CODAS) method to evaluate choices according to their distance from the least op-timal solution, thereby ensuring a more accurate and reliable decision-making procedure. The suggested PULq-ROF-CODAS technique effi-ciency is illustrated by its implementation in cybersecurity risk assess-ments, where managing ambiguities and communication judgment are critical. The results support the theoretical frameworks capacity to rank cybersecurity threats based on significance while maintaining agreement between specialist perspectives and collective evaluations, ultimately con-tributing to stronger and more well-planned cybersecurity mechanisms.

AMS (MOS) Subject Classification Codes: 03E72;94D05,90B50;90C31
Key Words: PULq-ROFS, PULq-ROFEWA operator, PULq-ROFEOWA operator, CODAS method.

1. Introduction

A crucial area of research that concentrates on locating, evaluating, and reducing possible safety hazards in the digital world is cybersecurity risk assessments. Controlling safety hazards is a major concern for enterprises due to a rising incidence of assaults and expanding modern backdrops. It is frequently difficult for conventional protection models to allow for statistical differences in specialist judgments, translational bias, and confusion. Examples of cyber attacks include antivirus assaults, fake emails, hacking, intrusions by employees, and advanced persistent threats (APTs). These threats often target critical records. To effectively investigate threats, enterprises need to adopt complex decision-making frameworks. These frameworks also help execute protection methods as attackers techniques evolve. Safety monitoring assessments protect private information. They also ensure client trust, organizational resilience, and regulatory compliance. Emerging technological hazards are unpredictable as well as complicated, therefore rendering it difficult for conventional safety frameworks to handle them. For this purpose, vulnerability identification techniques must incorporate artificial intelligence (AI) and deep learning (DL). More complex threat evaluation techniques, including probabilistic and fuzzy reasoning, can improve the effectiveness of vulnerability identification. The expansion of vulnerabilities caused by the increasing number of connected gadgets has made strict constraints on access and protection techniques mandatory. Furthermore, malicious acts by insiders are a permanent issue, emphasizing the significance of regular monitoring and safety education initiatives for employees. By adopting creative measures, enterprises can prevent potential cyber crises before they fully develop. Real-time threat assessment enables timely interventions. A strong cyber defense plan guarantees that companies can minimize operational interruptions and quickly bounce back from safety incidents. Effective processes for verification and frequent staff education are essential because hacking and social media manipulation assaults are still common.

The critical network is particularly targeted by advanced persistent threats (APTs), which highlights the significance of a multi-layered protection plan. Frequent safety inspections and hacking investigations are necessary for enterprises. These processes help identify system flaws and stay ahead of malicious crim-inals. The safety of information is made even stronger by computer system improvements like multiple logins and zero trust infrastructure. For successful control and elimination of cyber hazards, a well-organized crisis management plan is essential.

Due to the erratic character of human behavior and the complexity of situations, it is a tough challenge for specialists to represent their skills as contexts. Specialists cannot convey their opinions by providing accurate scores, regardless of whether the assessment data obtained from them is imprecise or insufficient. To capture the qualities of a qualitative assessment, Herrera and Martinez [12] developed linguistic term sets (LTSs). Therefore, specialists may use a unique linguistic term (LT) to present the assessment of knowledge enabled by the LT [14].

There are occasions when specialists are unable to clearly describe their studies with a single LT, and we face uncertainty between several LTs. For illustration, while determining the sustainability of a life insurance policy, a specialist may use any LT that is part of LTS, ρ={p_0 : very bad, p_1 : bad, p_2 : moderately bad, p_3 : moderate, p_4 : slightly good, p_5 : good, p_6 : very good} to indicate the assessment. If the specialist feels that the approach is indeed slightly good, it can be described as $\{p_4\}$. Also, if the specialist feels that the approach is indeed very good, it can be described as $\{p_6\}$.

Due to their expertise and the complexity of the decision-making (DM) environment [15], specialists may provide multiple evaluation levels for similar DM problems. Rodriguez et al.[24] presented a hesitating fuzzier linguistics terminology set (HFLTS) that allows specialists to indicate their opinions with multiple LTS. The HFLTS assigns identical values to each object, which could not be correct in practice. Although specialists may have different kinds of experience or choice for possible LTS, it can be concluded that the weights of linguistic evaluations cannot be ignored in actual DM concerns [1]. For example, the rating information includes probabilistic information in addition to LTS if an expert is 70 percent convinced that a life insurance plan is good and 30 percent convinced that such a plan is moderate.

To solve these issues, Pang et al. [22] developed probabilistic linguistic term sets (PLTSs) that contained a few potential LTS as well as related probability information. Those who make decisions apply PLTS [27] to represent their assessment data and as-sess difficulties. It provides data sets together with probabilities for all linguistic terms that might be used. The DMs can generate a variety of linguistic terms for an item (alter-native or attribute) by using PLTSs, as well as display actual probabilistic data of such a collection of values. We can obtain complete and reliable details well about DM opinions in this technique [22]. Suppose $\mathcal{N}$={m_{-2} : very bad, m_{-1} : bad, m_0 : general, m_1 : excellent, m_2 : very excellent}, and a specialist is asked to assess a students potential.

He/she may assert that 50 percent of the time the individuals potential is excellent and 40 percent of the time it is very excellent. Furthermore, it might be expressed as Capability (person)={excellent(0.5), very excellent(0.4)}, that is $\mathcal{O}(\rho) = \{m_1(0.5), m_2(0.4)\}$. Furthermore, it can explain that a group makes decisions. For example, let $\mathcal{M}$={n_{-2} : very little, n_{-1} : little, n_0 : general, n_1 : large, n_2 : very large}, and 100 clients are asked to evaluate the probability that a plan will succeed. If 56 clients perceive it to be large and 24 believe it to be general, while the remaining clients remain silent, this condition can be determined by the possibility of achievement (task) ={general(24 percent), large(56 percent)}, that is $\mathcal{Y}(\rho) = \{n_0(0.24), n_1(0.56)\}$. Furthermore, under consideration of PLTSs benefits and variety of uses, it acts as the essential assessment results. The two illustrations represent the PLTSs imprecise probabilistic ranges. The PLTS must generate an appropriate distribution of possibilities for data blending, the PLTS needs to be adjusted for normality [25].

In some real-world scenarios, DMs may lack precise or complete information. They may therefore present their assessments using uncertain linguistic concepts [31] within group decision-making (GDM) procedures. Moreover, such uncertain linguistic concepts are distinct from one another, and each linguistic expression also exists at a specific rate. Lin et al. [19] constructed on the basis of unclear linguistic concepts and PLTSs to develop the PULTS approach for dealing with inconsistent linguistic assessments in DM. To develop the popular level and the target variable, Xie et al. [30] developed the PUL preference relation (PULPR) and the normalized PULPR. Scoring each uncertain linguistic term demonstrates the richness of data in the presence of multiple ULTs. It also provides a probabilistic model for analysis. The related probabilities of UL variables in a PULTS might be considered as the concepts of appearing for the ULTs [28]. To express their analyzed data, few DMs can offer PULTSs. For example, a high-ranking DM utilizes the intensity of relaxation in a vehicle the way of given LTs A={$\mathfrak{a}_{-3}$: very poor, $\mathfrak{a}_{-2}$: poor, $\mathfrak{a}_{-1}$: slightly poor, $\mathfrak{a}_0$: moderate, $\mathfrak{a}_1$: slightly excellent, $\mathfrak{a}_2$: excellent, $\mathfrak{a}_3$: very excellent}. Consider that they believe their level of calm is between "slightly excellent and excellent, 30 percent are certain," and between "excellent and very excellent, 70 percent are certain.". Thus the PULTS $\{\langle[\mathfrak{a}_1,\ \mathfrak{a}_2],\ 0.3\rangle,\ \langle[\mathfrak{a}_2,\ \mathfrak{a}_3],\ 0.7\rangle\}$ can be utilized to describe a car's comfort level. Additionally, PULTS can indicate the assessment of texts with the entire group or a subgroup in GDM situations of significant size. For instance, out of the ten DMs judging the possibility x_1 over x_2 based on a specific set of linguistic concepts, three DMs respond among "slightly excellent and excellent," and seven DMs respond among "excellent and very excellent". B={$\mathfrak{a}_{-3}$: very poor, $\mathfrak{a}_{-2}$: poor, $\mathfrak{a}_{-1}$: slightly poor, $\mathfrak{a}_0$: moderate, $\mathfrak{a}_1$: slightly excellent, $\mathfrak{a}_2$: excellent, $\mathfrak{a}_3$: very excellent}. In this circumstance, x_1 prefers x_2, which is expressed using a PULTS as: $\{\langle[\mathfrak{a}_1, \mathfrak{a}_2], 0.3\rangle, \langle[\mathfrak{a}_2, \mathfrak{a}_3], 0.7\rangle\}$.

Earlier than 1965, even misconceptions in pure mathematics and probability concepts may partially address the requirement to control a specific type of ambiguity, notably unpredictability. The fuzzy criteria, including such tiny, immature, notably higher, etc., are not sufficiently explained by the probability theory. Zadeh [35] published his innovative work,"Fuzzy Sets" in 1965. Fuzzy sets (FSs) are now becoming essential building blocks for machine learning and the MCGDM process because they have better feasibility in the search for accessible information among difficulties. A collection of components with membership values falling between [0,1] is referred to as a FS. An element's or item's membership value is always expressed by a fuzzy set. If an item partially fulfills a specified condition, we must give it a membership value within the range [0,1]. Furthermore, real-world assessments reveal that events are becoming increasingly complicated, emphasizing that only a unique degree of membership is unable to capture the true nature of the items. However, the fuzzy set is unable to express the absence of membership value of a thing. To address these problems, Atanassov [6] proposed a generalized version of FS known as an intuitionistic fuzzy set, which represents both the membership degree (MD) of its elements as well as non-membership degree (NMD). Each MD and NMD value for IFS must fall within the range [0,1]. The earlier collections, including FS and IFS, operate on the exponential connection between MD and NMD with a condition that potentially the total of each value is required to be below or equivalent to one. However, if we take MD $\alpha = 0.7$ and NMD $\beta = 0.6$, then $\alpha+\beta > 1$; it exceeds the boundaries of these ranges, and FS and IFS are unable to resolve this type of issue. The Pythagorean fuzzy set (PyFS) represents a more extended type of FS that Yager [33] described. Considering the value of the quadratic sum of MD and NMD is equal to or less than 1, this idea has been extended from IFS to PyFS. He extended the collaboration from $\alpha+\beta \leq 1$ to $\alpha^2+\beta^2 \leq 1$, which expands the range and helps us to resolve the multiple input [2]. Whereas PyFSs are effective in their specific uses, q-rung orthopair fuzzy sets (q-ROFSs) utilize a more generalized form of expanded gap cover-up. To increase validity and allow researchers to generate their own opinions regarding MD and NMD values, Yager [34] proposed a generalized statement of IFS and PyFS called q-ROFS, defined as $\alpha^q+\beta^q \leq 1$, $q \geq 1$. For example, if MD $\alpha = 0.9$ and NMD $\beta = 0.8$, IFS and PyFS are unable to deal with this problem. However, Yager's q-ROFS can effectively handle these circumstances by changing the requirement such as $0.9^3 + 0.8^3 \leq 1$. Using the q-ROFS in this kind of condition is suitable, too. To manage the q-ROFS level, Akram et al. [3, 4] utilized Einstein aggregation operators (EAOs) like the Einstein weighted average and the Einstein ordered weighted average operators. Sheng [9] proposed the EAOs for q-ROFS and built the MCGDM method to handle challenges in DM. An essential approach in MCGDM is the CODAS approach, which was first presented by Ghorabaee et al.[10]. CODAS provides an advanced way of rating choices by combining Euclidean and taxicab distances, especially in conditions when conventional techniques might not be able to differentiate between choices with similar scores. Scholars have expanded CODAS into a variety of fuzzy contexts in recognition of the intricacies and inconsistencies present in practical DM processes. To solve supplier selection problems in production, Boltrk [7] created a Pythagorean fuzzy (PyF) CODAS approach that successfully captures the reluctance and inconsistency in experts ratings. The effectiveness of a triangular intuitionistic fuzzy (TIF) CODAS technique in combining economical and ecological requirements despite unpredictability was also demonstrated by Daami Remadi and Moalla Frikha [23] in their application in the choice of sustainable resources. The incorporation of CODAS into additional fuzzy approaches is an instance of additional improvements. Andukuri and Rao [5] used fuzzy CODAS to choose manufacturing machinery status tracking devices, demonstrating its ability to endure in scientific choices. Additionally, Kundaki and Arman [16] demonstrated the approach's flexibility in a variety of commercial circumstances by combining fuzzy CODAS with the IMF-SWARA technique to simplify the company's consultant choice process. The use of CODAS for economic evaluations is an additional instance of its flexibility. In their extensive review of MCGDM techniques, Baydas et al. [8] concluded that fuzzy CODAS was the best technique for assessing the economic performance information, especially when combined with highest leveling. For further study on linguistic DM and its applications, the readers are referred to [21, 13, 19, 31, 32].

1.1. **Significance of PULq-ROFS.** Conventional fuzzy and fuzzy intuitionistic frameworks struggle to capture the broad spectrum of individual thinking in DM. This difficulty persists even when answers are sophisticated but imperfect. The PULq-ROFS combines PUL details with the q-ROF principle. This approach overcomes the previous limitations and provides a more accurate and adaptable method to express ambiguity. In contrast to conventional methods, PULq-ROFS takes into consideration probabilistic linguistic hesitation along with the presence or absence of membership grades. The result is important for complicated everyday life DM instances, including safety assessments, medical treatment, and economic investment review. Furthermore, PULq-ROFSs capacity to control multidimensional doubt assures that it will continue to function well while dealing with complicated interactions and MCGDM problems. It enables specialists to express uncertain linguistic choices effectively. This results in a more rational and informed framework for DM. The models capacity to handle volatile and unforeseen circumstances is enhanced by including probability elements. This feature makes it suitable for situations with immediate and dynamic outcomes. In the end, PULq-ROFS is an effective tool for increasing choice quality, decreasing losing data, and facilitating more open and rational DM in challenging and unplanned circumstances. Researchers and scholars studied, expanded and then utilized different DM strategies in various disciplines [17, 11, 20, 26, 29, 18].

1.2. **Motivation of PULq-ROFS.** PULq-ROFS was developed to address the growing number of decision-making contexts where linguistic evaluations and cognitive abilities are critical. Most existing fuzzy algorithms perform poorly when handling highly subjective, uncertain, or imperfect information. Consequently, they yield suboptimal DM results. PULq-ROFS provides administrators with greater flexibility in areas where linguistic and human cognition are important. It employs the q-ROF framework to store additional information without imposing unnecessary constraints. The DM process gets even better with the addition of PUL phrases, which are especially useful in situations with competing requirements, contradictory choices, and unsure analyst judgments. PULq-ROFS essentially increases the depth of unreliable simulation, which in turn improves the stability and dependability of convoluted multi-criteria evaluations, consequently enabling DMs. By integrating linguistic uncertainty, probabilistic information, and q-rung flexibility, PULq-ROFS serves as an effective tool for addressing real-world problems where traditional imprecise frameworks are inadequate.

1.3. **Contribution.** The PULq-ROFS is an additional area of linguistic terms technique integrating the ideas of PUL and q-ROFS. Since the MD and NMD of PULq-ROFS are represented by a set of possible uncertain linguistic terms (ULTs) with their associated probabilities, PULq-ROFS can process considerably more information provided by experts. A sophisticated development of FS theory is PULq-ROFS that combines uncertain linguistic variables, probabilistic information, and the q-ROFS framework. The goal of PULq-ROFS is to manage the uncertainties and ambiguity in DM problems where linguistic variables are used and probabilistic information is available. The q-rung behavior efficiently covers larger amounts of ambiguity. Consequently, this composite set allows DMs to convey safety hazards utilizing linguistic phrases with probabilistic patterns. PULq-ROF Einstein weighted average (PULq-ROFEWA) operator is used to improve the accumulation of vulnerability threats. This assures that extremely important risk indicators have a higher influence on DM while preserving the unpredictable nature of expert ratings. Furthermore, to handle the fears with range of motion, the PULq-ROF Einstein ordered weighted average (PULq-ROFEWA) operator is used for aggregating safety hazards while taking DMs ideological choices into account. Subsequently, by combining information, the uniform distribution of safety-related information is evaluated by the entropy technique to determine the desired weights for factors related to security. The CODAS technique focuses on the most significant gaps by ranking safety hazards according to their Einstein-based distances from the negative ideal solution (NIS) after the weights have been provided. Cybersecurity specialists can perform precise and credible threat assessments while taking linguistic, fuzzy, and probabilistic difficulty into consideration due to this scientific technique. For cybercrime risk control, this framework provides a deliberate, adaptable, and effective methodology by combining PULq-ROFS with EAOs and MCDM approaches. A practical instance dealing with cybersecurity risk assessment is taken under consideration in order to show the success of the suggested method. Moreover, a comparison study is done to emphasize the technique's dependability and realistic characteristics.

The manuscript is organized as Sect.2 gives a summary of PULq-ROFSs and their association PULq-ROFS. Sect.3 demonstrates an aggregation procedures PULq-ROFEWA and PULq-ROFEOWA. The suggested DM technique, which is constructed on the CODAS technique for selecting choices and the Entropy technique for weighting requirements, is described in Sect.4. Sect.5 illustrates a numerical instance of cybersecurity risk assessment. A comparison assessment is provided in Sect.6 to verify the effectiveness of the suggested approach, and the research is concluded with important results and recommendations for the future in Sect.7.

2. PRELIMINARIES

Definition 2.1. [22, 21] *Suppose if* $\beth = \{\flat_\vartheta | \vartheta = -\Delta, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \Delta\}$ *represents a LTS; a PLTS can be defined as follows:*

$$\hbar_\flat(\rho) = \{\flat^\varepsilon(\rho^\varepsilon) | \flat^\varepsilon \in \beth, \rho^\varepsilon \geq 0, \varepsilon = 1, 2, \ldots, \#\hbar_\flat(\rho), \sum_{\varepsilon=1}^{\#\hbar_\flat(\rho)} \rho^{(\varepsilon)} \leq 1\}, \tag{2. 1}$$

where in the LT is $\flat^{(\varepsilon)}(\rho^{(\varepsilon)})$, $\flat^{(\varepsilon)}$ *is associated with the possibility* $\rho^{(\varepsilon)}$, *and* $\#\hbar_\flat(\rho)$ *denotes the total number of LTs in* $\hbar_\flat(\rho)$.

Definition 2.2. [13] *Suppose* $\mathcal{Z} = [\flat_\mathfrak{a}, \flat_\mathfrak{b}]$, *where* $\flat_\mathfrak{a}, \flat_\mathfrak{b} \in \beth_{[\Delta,-\Delta]}$, $\flat_\mathfrak{a}$ *and* $\flat_\mathfrak{b}$ *include the most significant and less restriction; we consequently define* $\mathcal{Z}$ *as the UL component.*

To effectively illustrate the DMs rigidity, Lin et al. [19] presented across an innovative concept called PULTS, which utilizes unclear linguistic parameters [31] and PLTSs.

Definition 2.3. [19] *A PULTS is described in the following way:*

$$\mathcal{Z}(\rho) = \{[\eta^{(\varepsilon)}, \gamma^{(\varepsilon)}](\rho^{(\varepsilon)})|\eta^{(\varepsilon)}, \gamma^{(\varepsilon)} \in \daleth, \rho^{(\varepsilon)} \geq 0, \varepsilon = 1, 2, \ldots, \#\mathcal{Z}(\rho), \sum_{\varepsilon=1}^{\#\mathcal{Z}(\rho)} \rho^{(\varepsilon)} \leq 1\}, \tag{2. 2}$$

where $[\eta^{(\varepsilon)}, \gamma^{(\varepsilon)}](\rho^{(\varepsilon)})$ *indicates the ULT* $[\eta^{(\varepsilon)}, \gamma^{(\varepsilon)}]$ *associated with the probability* $\rho^{(\varepsilon)}$ *and* $\eta^{(\varepsilon)}, \gamma^{(\varepsilon)}$ *are LTs,* $\eta^{(\varepsilon)} \leq \gamma^{(\varepsilon)}$ *and* $\#\mathcal{Z}(\rho)$ *is the cardinality of* $\mathcal{Z}(\rho)$.
It is suggested that no assessment of information is now available, if $\sum_{\varepsilon=1}^{\#\mathcal{Z}(\rho)} \rho^{(\varepsilon)} = 0$. *In the occurrence, if* $\sum_{\varepsilon=1}^{\#\mathcal{Z}(\rho)} \rho^{(\varepsilon)} = 1$, *ultimately all information regarding their semantic assessment is provided. In the circumstance in which* $0 < \sum_{\varepsilon=1}^{\#\mathcal{Z}(\rho)} \rho^{(\varepsilon)} < 1$, *it indicates only a portion of the linguistic evaluations are provided since some DMs are capable of providing the full analysis information as well as given some DMs are responsible for sending the information. It happens often in real-world MAGDM problems, and it is necessary to resolve unknown risk.*

Definition 2.4. [19] *Considering a PULTS* $\mathcal{Z}(\rho) = \{[\eta^{(\varepsilon)}, \gamma^{(\varepsilon)}](\rho^{(\varepsilon)})|\varepsilon = 1, 2, \ldots, \#\mathcal{Z}(\rho)\}$, *where* $\mathcal{Z}(\rho)$ *is referred to as a sequence of PULTS if any of its constituent parts are arranged in descending order. It includes two components* $\langle[\eta^{(\iota)}, \gamma^{(\iota)}](\rho^{(\iota)})\rangle$ *and* $\langle[\eta^{(\epsilon)}, \gamma^{(\epsilon)}](\rho^{(\epsilon)})\rangle$ *which are compared to the probability ratio of* $[\eta^{(\iota)} \times (\rho^{(\iota)}), \gamma^{(\iota)} \times (\rho^{(\iota)})]$ *and* $[\eta^{(\epsilon)} \times (\rho^{(\epsilon)}), \gamma^{(\epsilon)} \times (\rho^{(\epsilon)})]$, *respectively.*

In order to address the balance of unjustified error that makes PULTS's importance identical, PULTS are normalized. Two distinct processes are involved in PULTS normalization. Assessing a person's insufficient statistical understanding is the fundamental component, and adjusting a PULTS's efficiency serves computational reasons.

Definition 2.5. [19] *Suppose a PULTS* $\mathcal{Z}(\rho)$ *with* $\sum_{\varepsilon=1}^{\#\mathcal{Z}(\rho)} \rho^{(\varepsilon)} < 1$, *consequently, the related PULTS* $\dot{\mathcal{Z}}(\rho)$ *can be described as:*

$$\dot{\mathcal{Z}}(\rho) = \{[\eta^{(\varepsilon)}, \gamma^{(\varepsilon)}](\dot{\rho}^{(\varepsilon)})|\varepsilon = 1, 2, \ldots, \#\mathcal{Z}(\rho)\}, \tag{2. 3}$$

where $\dot{\rho}^{(\varepsilon)} = \frac{\rho^{(\varepsilon)}}{\sum_{\varepsilon=1}^{\#\mathcal{Z}(\rho)} \rho^{(\varepsilon)}}$, *for all* $\varepsilon = 1, 2, ..., \#\mathcal{Z}(\rho)$.
The PULTS significance is typically inconsistent in real-world DM purposes, which causes substantial operational issues. Thus, inserting unclear linguistic concepts containing chance 0 in PULTS with just a few components, we are able to raise their cardinalities.

Definition 2.6. [19] *Suppose* $\mathcal{Z}_1(\rho)$ *and* $\mathcal{Z}_2(\rho)$ *be any two PULTS,* $\mathcal{Z}_1(\rho) = \{[\eta_1^{(\varepsilon)}, \gamma_1^{(\varepsilon)}](\rho_1^{(\varepsilon)})|\varepsilon = 1, 2, \ldots, \#\mathcal{Z}_1(\rho)\}$ *and* $\mathcal{Z}_2(\rho) = \{[\eta_2^{(\varepsilon)}, \gamma_2^{(\varepsilon)}](\rho_2^{(\varepsilon)})|\varepsilon = 1, 2, \ldots, \#\mathcal{Z}_2(\rho)\}$, *and let* $\#\mathcal{Z}_1(\rho)$ *and* $\#\mathcal{Z}_2(\rho)$ *be the number of linguistic terms in* $\mathcal{Z}_1(\rho)$ *and* $\mathcal{Z}_2(\rho)$, *respectively. If* $\#\mathcal{Z}_1(\rho) > \#\mathcal{Z}_2(\rho)$, *then we will add* $\#\mathcal{Z}_1(\rho) - \#\mathcal{Z}_2(\rho)$ *linguistic terms to* $\mathcal{Z}_2(\rho)$ *so that the amount* *of expressions in $\mathcal{Z}_1(\rho)$ and $\mathcal{Z}_2(\rho)$ are similar. The probability of each verbal phrase is zero and the fewest phrases in $\mathcal{Z}_2(\rho)$ are those that have been inserted.*

Let $\mathcal{Z}_1(\rho) = \{[\eta_1^{(\varepsilon)}, \gamma_1^{(\varepsilon)}](\rho_1^{(\varepsilon)})|\varepsilon = 1, 2, \ldots, \#\mathcal{Z}_1(\rho)\}$ and $\mathcal{Z}_2(\rho) = \{[\eta_2^{(\varepsilon)}, \gamma_2^{(\varepsilon)}](\rho_2^{(\varepsilon)})|\varepsilon = 1, 2, \ldots, \#\mathcal{Z}_2(\rho)\}$, afterward, the two phases listed below can be used to carry out the normalized procedure:

(1) *If $\sum_{\varepsilon=1}^{\#\mathcal{Z}(\rho)} \rho^{(\varepsilon)} < 1$ then by formula (2. 3), the value $\dot{\mathcal{Z}}_\iota(\rho), \iota = 1, 2$, can be calculated.*
(2) *If $\#\mathcal{Z}_1(\rho) \neq \#\mathcal{Z}_2(\rho)$, then based on the definition 2.6, it is necessary to include additional aspects in alongside those with fewer components.*

We refer to the resulting PULTS as the adjusted PULTS. The adjusted PULTS are also represented by $\mathcal{Z}_1(\rho)$ and $\mathcal{Z}_2(\rho)$ for visual purposes.

Definition 2.7. [17] *Given a conventional predetermined set χ, the expression for a q-ROFS, $\mathfrak{Q}$ formed on χ is as follows:*

$$\mathfrak{Q} = \{\langle \mathrm{x}, \sigma_{\mathfrak{Q}}(\mathrm{x}), \zeta_{\mathfrak{Q}}(\mathrm{x})\rangle | \mathrm{x} \in \chi\}, \tag{2. 4}$$

where $\sigma_{\mathfrak{Q}}(\mathrm{x})$ and $\zeta_{\mathfrak{Q}}(\mathrm{x})$ symbolize the MD and NMD of the component $\mathrm{x} \in \chi$ to the set $\mathfrak{Q}$, correspondingly, achieving $0 \leq \sigma_{\mathfrak{Q}}(\mathrm{x}), \zeta_{\mathfrak{Q}}(\mathfrak{x}) \leq 1$ and $(\sigma_{\mathfrak{Q}}(\mathrm{x}))^{\mathfrak{q}} + (\zeta_{\mathfrak{Q}}(\mathrm{x}))^{\mathfrak{q}} \leq 1, (\mathfrak{q} \geq 1)$. The amount of inconsistency is described as

$$\pi_{\mathfrak{Q}}(\mathrm{x}) = \sqrt[\mathfrak{q}]{(\sigma_{\mathfrak{Q}}(\mathrm{x}))^{\mathfrak{q}} + (\zeta_{\mathfrak{Q}}(\mathrm{x}))^{\mathfrak{q}} - (\sigma_{\mathfrak{Q}}(\mathrm{x}))^{\mathfrak{q}}(\zeta_{\mathfrak{Q}}(\mathrm{x}))^{\mathfrak{q}}}.$$

Liu and Wang [17] *known as the arranged combination $(\sigma_{\mathfrak{Q}}(\mathrm{x}), \zeta_{\mathfrak{Q}}(\mathrm{x}))$ a q-ROFN, which is represented as $\beta = (\sigma, \zeta)$.*

Definition 2.8. [32, 11] *Suppose $\beth = \{\flat_\vartheta | \vartheta = -\Delta, \ldots, -2, -1, 0, 1, 2, \ldots, \Delta\}$ be a LTS* [32]. *The corresponding details to ϖ that is obtained by using the transformation function* [11] *can be represented by the linguistic phrase $\flat_\vartheta$:*

$$\mathfrak{t} : [\flat_{-\Delta}, \flat_{\Delta}] \to [0, 1], \mathfrak{t}(\flat_\vartheta) = \frac{\vartheta + \Delta}{2\Delta} = \varpi. \tag{2. 5}$$

Additionally, data whose value is equal to the linguistic concepts $\flat_\vartheta$ can be expressed using ϖ, so that ϖ can be obtained using the transformation function $\mathfrak{t}^{-1}$:

$$\mathfrak{t}^{-1} : [0, 1] \to [\flat_{-\Delta}, \flat_{\Delta}], \mathfrak{t}^{-1} = \flat_{(2\varpi-1)\Delta} = \flat_\vartheta. \tag{2. 6}$$

We now employ the probabilistic fuzzy set (PULq-ROFS), which permits experts to display evaluation information in several linguistic and incorporates the possibility for each of them. The problem is figuring out the best way to apply PULq-ROFS criteria correctly when the corresponding probability models diverge.

Definition 2.9. [20]*Let $\mathfrak{Y} = \{\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_\epsilon\}$ refer to a uniform collection and $\beth = \{\flat_\vartheta | \vartheta = -\Delta, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \Delta\}$ be a LTS. Then a PULq-ROFS $\hbar(\rho)$ on $\mathfrak{Y}$ is given as:*

$$\hbar(\rho) = \{\langle \mathfrak{y}_\epsilon, \sigma(\hat{\rho})(\mathfrak{y}_\epsilon), \zeta(\tilde{\rho})(\mathfrak{y}_\epsilon)\rangle : \mathfrak{y}_\epsilon \in \mathfrak{Y}\}, \tag{2. 6}$$

where $\sigma(\hat{\rho})(\mathfrak{y}_\epsilon) = \{[\aleph_{\theta^{\epsilon(\mathfrak{g})}}, \mu_{\theta^{\epsilon(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}) : \aleph_{\theta^{\epsilon(\mathfrak{g})}}, \mu_{\theta^{\epsilon(\mathfrak{g})}} \in \beth_{[\Delta,-\Delta]}, \hat{\rho}^{(\mathfrak{g})} \geq 0, \sum_{\mathfrak{g}=1}^{\mathfrak{G}} \hat{\rho}^{(\mathfrak{g})} \leq 1\}$ *and* $\zeta(\tilde{\rho})(\mathfrak{y}_\epsilon) = \{[\pounds_{\Theta^{\epsilon(\mathrm{j})}}, \nu_{\Theta^{\epsilon(\mathrm{j})}}](\tilde{\rho}^{(\mathrm{j})}) : \pounds_{\Theta^{\epsilon(\mathrm{j})}}, \nu_{\Theta^{\epsilon(\mathrm{j})}} \in \beth_{[\Delta,-\Delta]}, \tilde{\rho}^{(\mathrm{j})} \geq 0, \sum_{\mathrm{j}=1}^{\mathfrak{J}} \tilde{\rho}^{(\mathrm{j})} \leq 1\}$ *indicate the affiliation and non-affiliation grade, appropriately, of* $\mathfrak{y}_\epsilon \in \mathfrak{Y}$*, with the corresponding chances are* $\hat{\rho}^{(\mathfrak{g})}$ *and* $\tilde{\rho}^{(\mathrm{j})}$*, respectively;* $\theta^{\epsilon(\mathfrak{g})}$ *and* $\Theta^{\epsilon(\mathrm{j})}$ *are the subscripts of the ULTs* $[\aleph_{\theta^{\epsilon(\mathfrak{g})}}, \mu_{\theta^{\epsilon(\mathfrak{g})}}]$ *and* $[\pounds_{\Theta^{\epsilon(\mathrm{j})}}, \nu_{\Theta^{\epsilon(\mathrm{j})}}]$*, respectively; fulfilling the criteria* $0 \leq (\max_{\mathfrak{g}=1}^{\mathfrak{G}} \theta^{\epsilon(\mathfrak{g})})^{\mathfrak{q}} + (\max_{\mathrm{j}=1}^{\mathfrak{J}} \Theta^{\epsilon(\mathrm{j})})^{\mathfrak{q}} \leq \Delta^{\mathfrak{q}}$ $(\mathfrak{q} \geq 1)$.

The set $\hbar(\rho)$ *minimizes the PULq-ROFN if it contains exclusively unique elements and we highlight this as* $\hbar(\rho) = \langle\{[\aleph_{\theta(\mathfrak{g})}, \mu_{\theta(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})})\}, \{[\pounds_{\Theta(\mathrm{j})}, \nu_{\Theta(\mathrm{j})}](\tilde{\rho}^{(\mathrm{j})})\}\rangle$ *where* $[\aleph_{\theta(\mathfrak{g})}, \mu_{\theta(\mathfrak{g})}]$, $[\pounds_{\Theta(\mathrm{j})}, \nu_{\Theta(\mathrm{j})}] \in \beth_{[\Delta,-\Delta]}$ *and* $\hat{\rho}^{(\mathfrak{g})}, \tilde{\rho}^{(\mathrm{j})} \geq 0$, $\sum_{\mathfrak{g}=1}^{\mathfrak{G}} \hat{\rho}^{(\mathfrak{g})} \leq 1$, $\sum_{\mathrm{j}=1}^{\mathfrak{J}} \tilde{\rho}^{(\mathrm{j})} \leq 1$.

Definition 2.10. [20] *Suppose* $\beth_{[\Delta,-\Delta]}$ *be a LTS for any adjusted PULq-ROFN* $\hbar(\rho) = \langle\{[\aleph_{\theta(\mathfrak{g})}, \mu_{\theta(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta(\mathrm{j})}, \nu_{\Theta(\mathrm{j})}](\tilde{\rho}^{(\mathrm{j})})\}\rangle$*, where* $\aleph_{\theta(\mathfrak{g})}, \mu_{\theta(\mathfrak{g})}, \pounds_{\Theta(\mathrm{j})}$*, and* $\nu_{\Theta(\mathrm{j})} \in \beth_{[\Delta,-\Delta]}$, $(\mathfrak{g} = 1, 2, 3 \ldots \mathfrak{G}; \mathrm{j} = 1, 2, 3 \ldots, \mathfrak{J})$*, the score function of* $\hbar(\rho)$ *is described as*

$$F(\hbar(\rho)) = \frac{\sum_{\mathfrak{g}=1}^{\#\mathfrak{G}_\theta} \left(\frac{\mathfrak{t}(\aleph_{\theta(\mathfrak{g})})\hat{\rho}^{(\mathfrak{g})} + \mathfrak{t}(\mu_{\theta(\mathfrak{g})})\hat{\rho}^{(\mathfrak{g})}}{2} \right)^{\mathfrak{q}}}{\sum_{\mathfrak{g}=1}^{\#\mathfrak{G}_\theta} \hat{\rho}^{(\mathfrak{g})}} - \frac{\sum_{\mathrm{j}=1}^{\#\mathfrak{J}_\Theta} \left(\frac{\mathfrak{t}(\pounds_{\Theta(\mathrm{j})})\tilde{\rho}^{(\mathrm{j})} + \mathfrak{t}(\nu_{\Theta(\mathrm{j})})\tilde{\rho}^{(\mathrm{j})}}{2} \right)^{\mathfrak{q}}}{\sum_{\mathrm{j}=1}^{\#\mathfrak{J}_\Theta} \tilde{\rho}^{(\mathrm{j})}}, \quad (2.\ 7)$$

where $\mathfrak{t}(\aleph_{\theta(\mathfrak{g})}), \mathfrak{t}(\mu_{\theta(\mathfrak{g})}), \mathfrak{t}(\pounds_{\Theta(\mathrm{j})})$*, and* $\mathfrak{t}(\nu_{\Theta(\mathrm{j})}) \in [0, 1]$, $\#\mathfrak{G}_\theta$ *and* $\#\mathfrak{J}_\Theta$ *indicate how many items there are in the related set, respectively. The standard deviation of* $\hbar(\rho)$ *is described as*

$$\beth(\hbar(\rho)) = \frac{\sqrt{\sum_{\mathfrak{g}=1}^{\#\mathfrak{G}_\theta} \left(\frac{\mathfrak{t}(\aleph_{\theta(\mathfrak{g})})\hat{\rho}^{(\mathfrak{g})} + \mathfrak{t}(\mu_{\theta(\mathfrak{g})})\hat{\rho}^{(\mathfrak{g})}}{2} - F(\hbar(\rho)) \right)^{\mathfrak{q}}}}{\sum_{\mathfrak{g}=1}^{\#\mathfrak{G}_\theta} \hat{\rho}^{(\mathfrak{g})}} + \frac{\sqrt{\sum_{\mathrm{j}=1}^{\#\mathfrak{J}_\Theta} \left(\frac{\mathfrak{t}(\pounds_{\Theta(\mathrm{j})})\tilde{\rho}^{(\mathrm{j})} + \mathfrak{t}(\nu_{\Theta(\mathrm{j})})\tilde{\rho}^{(\mathrm{j})}}{2} - F(\hbar(\rho)) \right)^{\mathfrak{q}}}}{\sum_{\mathrm{j}=1}^{\#\mathfrak{J}_\Theta} \tilde{\rho}^{(\mathrm{j})}}, \quad (2.\ 8)$$

where $\mathfrak{t}(\aleph_{\theta(\mathfrak{g})}), \mathfrak{t}(\mu_{\theta(\mathfrak{g})}), \mathfrak{t}(\pounds_{\Theta(\mathrm{j})})$*, and* $\mathfrak{t}(\nu_{\Theta(\mathrm{j})}) \in [0, 1]$, $\#\mathfrak{G}_\theta$ *and* $\#\mathfrak{J}_\Theta$ *indicate number of items there are in the related set, respectively.*

Definition 2.11. [20] *Suppose* $\hbar_{(1)}(\rho)$ *and* $\hbar_{(2)}(\rho)$ *be two PULq-ROFNs. Therefore, the arrangement of PULq-ROFNs juxtaposition appears as below:*

- **:** *(A) If* $F(\hbar^{(1)}(\rho)) \succ F(\hbar^{(2)}(\rho))$*, then* $\hbar^{(1)}(\rho) \succ \hbar^{(2)}(\rho)$.
- **:** *(B) If* $F(\hbar^{(1)}(\rho)) \prec F(\hbar^{(2)}(\rho))$*, then* $\hbar^{(1)}(\rho) \prec \hbar^{(2)}(\rho)$
- **:** *(C) If* $F(\hbar^{(1)}(\rho)) = F(\hbar^{(2)}(\rho))$*, then*
  - *(a) If* $\beth(\hbar^{(1)}(\rho)) \succ \beth(\hbar^{(2)}(\rho))$*, then* $\hbar^{(1)}(\rho) \prec \hbar^{(2)}(\rho)$.
  - *(b) If* $\beth(\hbar^{(1)}(\rho)) \prec \beth(\hbar^{(2)}(\rho))$*, then* $\hbar^{(1)}(\rho) \succ \hbar^{(2)}(\rho)$.
  - *(c) If* $\beth(\hbar^{(1)}(\rho)) = \beth(\hbar^{(2)}(\rho))$*, then* $\hbar^{(1)}(\rho) \approx \hbar^{(2)}(\rho)$.

Definition 2.12. [20] *Suppose* $\hbar^1(\rho) = \langle\{[\aleph_{\theta^{1(\mathfrak{g})}}, \mu_{\theta^{1(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{1(\mathfrak{j})}}, \nu_{\Theta^{1(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})})\}\rangle$ *and* $\hbar^2(\rho) = \langle\{[\aleph_{\theta^{2(\mathfrak{g})}}, \mu_{\theta^{2(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{2(\mathfrak{j})}}, \nu_{\Theta^{2(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})})\}\rangle$ $(\mathfrak{g} = 1,2,\ldots,\mathfrak{G}; \mathfrak{j} = 1,2,\ldots,\mathfrak{J})$ *include two customized PULq-ROFNs where* $\theta^{\epsilon(\mathfrak{g})}$ *and* $\Theta^{\epsilon(\mathfrak{j})}(\epsilon = 1,2)$ *include the relevant description of* $[\aleph_{\theta^{\epsilon(\mathfrak{g})}}, \mu_{\theta^{\epsilon(\mathfrak{g})}}]$ *and* $[\pounds_{\Theta^{\epsilon(\mathfrak{j})}}, \nu_{\Theta^{\epsilon(\mathfrak{j})}}](\epsilon = 1,2)\lambda > 0$, *following that, the key properties of PULq-ROFNs are outlined as follows:*

(1) $neg(\hbar^1(\rho)) = \left\langle \left\{ \left[\pounds_{\Theta^{1(\mathfrak{j})}}, \nu_{\Theta^{1(\mathfrak{j})}}\right](\tilde{\rho}^{(\mathfrak{j})}), \left[\aleph_{\theta^{1(\mathfrak{g})}}, \mu_{\theta^{1(\mathfrak{g})}}\right](\hat{\rho}^{(\mathfrak{g})}) \right\} \right\rangle.$

(2) $\hbar^1(\rho) \oplus \hbar^2(\rho) =$

$$\left(\begin{array}{c} \left(\aleph_{\sqrt[\mathfrak{q}]{(\theta^{1(\mathfrak{g})})^{\mathfrak{q}} + (\theta^{2(\mathfrak{g})})^{\mathfrak{q}} - (\frac{(\theta^{1(\mathfrak{g})})(\theta^{2(\mathfrak{g})})}{\Delta})^{\mathfrak{q}}}}, \mu_{\sqrt[\mathfrak{q}]{(\theta^{1(\mathfrak{g})})^{\mathfrak{q}} + (\theta^{2(\mathfrak{g})})^{\mathfrak{q}} - (\frac{(\theta^{1(\mathfrak{g})})(\theta^{2(\mathfrak{g})})}{\Delta})^{\mathfrak{q}}}}\right)(\hat{\rho}^{(\mathfrak{g})}), \\ \left(\pounds_{\frac{\Theta^{1(\mathfrak{j})}\Theta^{2(\mathfrak{j})}}{\Delta}}, \nu_{\frac{\Theta^{1(\mathfrak{j})}\Theta^{2(\mathfrak{j})}}{\Delta}}\right)(\tilde{\rho}^{(\mathfrak{j})}) \end{array}\right).$$

(3) $\hbar^1(\rho) \otimes \hbar^2(\rho) =$

$$\left(\begin{array}{c} \left(\aleph_{\frac{\theta^{1(\mathfrak{g})}\theta^{2(\mathfrak{g})}}{\Delta}}, \mu_{\frac{\theta^{1(\mathfrak{g})}\theta^{2(\mathfrak{g})}}{\Delta}}\right)(\hat{\rho}^{(\mathfrak{g})}), \\ \left(\pounds_{\sqrt[\mathfrak{q}]{(\Theta^{1(\mathfrak{j})})^{\mathfrak{q}} + (\Theta^{2(\mathfrak{j})})^{\mathfrak{q}} - (\frac{(\Theta^{1(\mathfrak{j})})(\Theta^{2(\mathfrak{j})})}{\Delta})^{\mathfrak{q}}}}, \nu_{\sqrt[\mathfrak{q}]{(\Theta^{1(\mathfrak{j})})^{\mathfrak{q}} + (\Theta^{2(\mathfrak{j})})^{\mathfrak{q}} - (\frac{(\Theta^{1(\mathfrak{j})})(\Theta^{2(\mathfrak{j})})}{\Delta})^{\mathfrak{q}}}}\right)(\tilde{\rho}^{(\mathfrak{j})}) \end{array}\right).$$

(4) $\lambda\hbar^1(\rho) =$

$$\left(\begin{array}{c} \left(\aleph_{\sqrt[\mathfrak{q}]{\Delta^{\mathfrak{q}} - \Delta^{\mathfrak{q}}(1 - \frac{(\theta^{1(\mathfrak{g})})^{\mathfrak{q}}}{\Delta^{\mathfrak{q}}})}}, \mu_{\sqrt[\mathfrak{q}]{\Delta^{\mathfrak{q}} - \Delta^{\mathfrak{q}}(1 - \frac{(\theta^{1(\mathfrak{g})})^{\mathfrak{q}}}{\Delta^{\mathfrak{q}}})}}\right)^{\lambda}(\hat{\rho}^{(\mathfrak{g})}), \\ \left(\pounds_{\Delta(\frac{\Theta^{1(\mathfrak{j})}}{\Delta})^{\lambda}}, \nu_{\Delta(\frac{\Theta^{1(\mathfrak{j})}}{\Delta})^{\lambda}}\right)(\tilde{\rho}^{(\mathfrak{j})}) \end{array}\right).$$

(5) $(\hbar^1(\rho))^{\lambda} =$

$$\left(\begin{array}{c} \left(\aleph_{\Delta(\frac{\theta^{1(\mathfrak{g})}}{\Delta})}, \mu_{\Delta(\frac{\theta^{1(\mathfrak{g})}}{\Delta})}\right)^{\lambda}(\hat{\rho}^{(\mathfrak{g})}), \\ \left(\pounds_{\sqrt[\mathfrak{q}]{\Delta^{\mathfrak{q}} - \Delta^{\mathfrak{q}}(1 - \frac{(\Theta^{1(\mathfrak{j})})^{\mathfrak{q}}}{\Delta^{\mathfrak{q}}})}}, \nu_{\sqrt[\mathfrak{q}]{\Delta^{\mathfrak{q}} - \Delta^{\mathfrak{q}}(1 - \frac{(\Theta^{1(\mathfrak{j})})^{\mathfrak{q}}}{\Delta^{\mathfrak{q}}})}}\right)^{\lambda}(\tilde{\rho}^{(\mathfrak{j})}) \end{array}\right).$$

Theorem 2.1. [20] *Let* $\hbar^1(\rho) = \langle\{[\aleph_{\theta^{1(\mathfrak{g})}}, \mu_{\theta^{1(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{1(\mathfrak{j})}}, \nu_{\Theta^{1(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})})\}\rangle$ *and* $\hbar^2(\rho) = \langle\{[\aleph_{\theta^{2(\mathfrak{g})}}, \mu_{\theta^{2(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{2(\mathfrak{j})}}, \nu_{\Theta^{2(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})})\}\rangle$ $(\mathfrak{g} = 1,2,\ldots,\mathfrak{G}; \mathfrak{j} = 1,2,\ldots,\mathfrak{J})$ *be any two modified PULq-ROFNs,* ξ, ξ_1, ξ_2, > 0, *then*

(1): $\hbar^1(\rho) \oplus \hbar^2(\rho) = \hbar^2(\rho) \oplus \hbar^1(\rho)$;
(2): $\hbar^1(\rho) \otimes \hbar^2(\rho) = \hbar^2(\rho) \otimes \hbar^1(\rho)$;
(3): $\xi(\hbar^1(\rho) \oplus \hbar^2(\rho)) = \xi\hbar^1(\rho) \oplus \xi\hbar^2(\rho)$;
(4): $\xi_1\hbar^1(\rho) \oplus \xi_2\hbar^1(\rho) = (\xi_1 + \xi_2)\hbar^1(\rho)$;
(5): $(\hbar^1(\rho))^{\xi_1} \otimes (\hbar^1(\rho))^{\xi_2} = (\hbar^1(\rho))^{\xi_1+\xi_2}$;

(6): $(\hbar^1(\rho))^{\xi} \otimes (\hbar^2(\rho))^{\xi} = (\hbar^1(\rho) \otimes \hbar^2(\rho))^{\xi}$.

Definition 2.13. [20] *Let* $\hbar^1(\rho) = \langle \{ [\aleph_{\theta^{1(\mathfrak{g})}}, \mu_{\theta^{1(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{1(\mathfrak{j})}}, \nu_{\Theta^{1(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})}) \} \rangle$ *and* $\hbar^2(\rho) = \langle \{ [\aleph_{\theta^{2(\mathfrak{g})}}, \mu_{\theta^{2(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{2(\mathfrak{j})}}, \nu_{\Theta^{2(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})}) \} \rangle$ $(\mathfrak{g} = 1, 2, \ldots, \mathfrak{G}; \mathfrak{j} = 1, 2, \ldots, \mathfrak{J})$ *consist of two modified PULq-ROFNs, then Hamming distance HD* $(\hbar^1(\rho), \hbar^2(\rho))$ *between* $\hbar^1(\rho)$ *and* $\hbar^2(\rho)$ *is described as follows:*

$$HD(\hbar^1(\rho), \hbar^2(\rho)) = \sqrt{\frac{\sum_{\mathfrak{g}=1}^{\#\mathfrak{G}_\theta} \left(|\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})\hat{\rho}^{(\mathfrak{g})} - \mathfrak{t}(\aleph_{\theta^{2(\mathfrak{g})}})\hat{\rho}^{(\mathfrak{g})}|^q + |\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})\hat{\rho}^{(\mathfrak{g})} - \mathfrak{t}(\mu_{\theta^{2(\mathfrak{g})}})\hat{\rho}^{(\mathfrak{g})}|^q \right)}{2\#\mathfrak{G}_\theta}} + \sqrt{\frac{\sum_{\mathfrak{j}=1}^{\#\mathfrak{J}_\Theta} \left(|\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})\tilde{\rho}^{(\mathfrak{j})} - \mathfrak{t}(\pounds_{\Theta^{2(\mathfrak{j})}})\tilde{\rho}^{(\mathfrak{j})}|^q + |\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})\tilde{\rho}^{(\mathfrak{j})} - \mathfrak{t}(\nu_{\Theta^{2(\mathfrak{j})}})\tilde{\rho}^{(\mathfrak{j})}|^q \right)}{2\#\mathfrak{J}_\Theta}}. \quad (2.\ 9)$$

3. PULq-ROF EINSTEIN AGGREGATION OPERATORS

The following section presents the Einstein aggregation operators for PULq-ROFNs. The fundamental Einstein laws of operation between two PULq-ROFNs are covered initially in the definition that follows.

Definition 3.1. *Let* $\hbar^1(\rho) = \langle \{ [\aleph_{\theta^{1(\mathfrak{g})}}, \mu_{\theta^{1(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{1(\mathfrak{j})}}, \nu_{\Theta^{1(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})}) \} \rangle$ *and* $\hbar^2(\rho) = \langle \{ [\aleph_{\theta^{2(\mathfrak{g})}}, \mu_{\theta^{2(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{2(\mathfrak{j})}}, \nu_{\Theta^{2(\mathfrak{j})}}](\tilde{\rho}^{(\mathfrak{j})}) \} \rangle$ $(\mathfrak{g} = 1, 2, \ldots, \mathfrak{G}; \mathfrak{j} = 1, 2, \ldots, \mathfrak{J})$ *consist of two modified PULq-ROFNs where* $\theta^{\epsilon(\mathfrak{g})}$ *and* $\Theta^{\epsilon(\mathfrak{j})}(\epsilon = 1, 2)$ *include the associated subtitle of* $[\aleph_{\theta^{\epsilon(\mathfrak{g})}}, \mu_{\theta^{\epsilon(\mathfrak{g})}}]$ *and* $[\pounds_{\Theta^{\epsilon(\mathfrak{j})}}, \nu_{\Theta^{\epsilon(\mathfrak{j})}}](\epsilon = 1, 2)\lambda > 0$*, then the PULq-ROF Einstein operations between* $\hbar^1(\rho)$ *and* $\hbar^2(\rho)$ *are:*

$$(1)\quad \hbar^1(\rho) \bigoplus \hbar^2(\rho) = \left(\begin{array}{c} \left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(\frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^q + \left(\frac{\mathfrak{t}(\aleph_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)^q}{1 + \left(\frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^q \left(\frac{\mathfrak{t}(\aleph_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)^q}} \right) \right), \mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(\frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^q + \left(\frac{\mathfrak{t}(\mu_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)^q}{1 + \left(\frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^q \left(\frac{\mathfrak{t}(\mu_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)^q}} \right) \right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\left(\frac{\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)\left(\frac{\mathfrak{t}(\pounds_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)}{\sqrt[q]{1 + \left(1 - \left(\frac{\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q\right)\left(1 - \left(\frac{\mathfrak{t}(\pounds_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)^q\right)}} \right) \right), \mathfrak{t}^{-1}\left(\Delta\left(\frac{\left(\frac{\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)\left(\frac{\mathfrak{t}(\nu_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)}{\sqrt[q]{1 + \left(1 - \left(\frac{\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q\right)\left(1 - \left(\frac{\mathfrak{t}(\nu_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)^q\right)}} \right) \right) \right] \left(\tilde{\rho}^{(\mathfrak{j})}\right) \end{array} \right).$$

$$(2)\quad \hbar^1(\rho) \bigotimes \hbar^2(\rho) = \left(\begin{array}{c} \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\left(\frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)\left(\frac{\mathfrak{t}(\aleph_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)}{\sqrt[q]{1 + \left(1 - \left(\frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^q\right)\left(1 - \left(\frac{\mathfrak{t}(\aleph_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)^q\right)}} \right) \right), \mathfrak{t}^{-1}\left(\Delta\left(\frac{\left(\frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)\left(\frac{\mathfrak{t}(\mu_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)}{\sqrt[q]{1 + \left(1 - \left(\frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^q\right)\left(1 - \left(\frac{\mathfrak{t}(\mu_{\theta^{2(\mathfrak{g})}})}{\Delta}\right)^q\right)}} \right) \right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right) \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(\frac{\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q + \left(\frac{\mathfrak{t}(\pounds_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)^q}{1 + \left(\frac{\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q \left(\frac{\mathfrak{t}(\pounds_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)^q}} \right) \right), \mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(\frac{\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q + \left(\frac{\mathfrak{t}(\nu_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)^q}{1 + \left(\frac{\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q \left(\frac{\mathfrak{t}(\nu_{\Theta^{2(\mathfrak{j})}})}{\Delta}\right)^q}} \right) \right) \right] \left(\tilde{\rho}^{(\mathfrak{j})}\right) \end{array} \right).$$

$$(3)\quad \lambda\hbar^1(\rho) = \left(\begin{array}{c} \left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1 + \frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda - \left(1 - \frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda}{\left(1 + \frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda + \left(1 - \frac{\mathfrak{t}(\aleph_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda}} \right) \right), \mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1 + \frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda - \left(1 - \frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda}{\left(1 + \frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda + \left(1 - \frac{\mathfrak{t}(\mu_{\theta^{1(\mathfrak{g})}})}{\Delta}\right)^\lambda}} \right) \right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^\lambda}{\sqrt[q]{\left(2 - \left(\frac{\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q\right)^\lambda + \left(\left(\frac{\mathfrak{t}(\pounds_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q\right)^\lambda}} \right) \right), \mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^\lambda}{\sqrt[q]{\left(2 - \left(\frac{\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q\right)^\lambda + \left(\left(\frac{\mathfrak{t}(\nu_{\Theta^{1(\mathfrak{j})}})}{\Delta}\right)^q\right)^\lambda}} \right) \right) \right] \left(\tilde{\rho}^{(\mathfrak{j})}\right) \end{array} \right).$$

(4) $$(\hbar^1(\rho))^\lambda = \left(\begin{array}{c} \left[\mathfrak{t}^{-1}\left(\Delta \left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\aleph_{\theta^1(\mathfrak{g})}\right)}{\Delta}\right)^\lambda}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\aleph_{\theta^1(\mathfrak{g})}\right)}{\Delta}\right)^q\right)^\lambda + \left(\left(\frac{\mathfrak{t}\left(\aleph_{\theta^1(\mathfrak{g})}\right)}{\Delta}\right)^q\right)^\lambda}} \right)\right), \mathfrak{t}^{-1}\left(\Delta \left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\mu_{\theta^1(\mathfrak{g})}\right)}{\Delta}\right)^\lambda}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\mu_{\theta^1(\mathfrak{g})}\right)}{\Delta}\right)^q\right)^\lambda + \left(\left(\frac{\mathfrak{t}\left(\mu_{\theta^1(\mathfrak{g})}\right)}{\Delta}\right)^q\right)^\lambda}} \right)\right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \left[\mathfrak{t}^{-1}\left(\Delta \left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\pounds_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda - \left(1-\frac{\mathfrak{t}\left(\pounds_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda}{\left(1+\frac{\mathfrak{t}\left(\pounds_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda + \left(1-\frac{\mathfrak{t}\left(\pounds_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda}} \right)\right), \mathfrak{t}^{-1}\left(\Delta \left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\nu_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda - \left(1-\frac{\mathfrak{t}\left(\nu_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda}{\left(1+\frac{\mathfrak{t}\left(\nu_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda + \left(1-\frac{\mathfrak{t}\left(\nu_{\Theta^1(\mathfrak{j})}\right)}{\Delta}\right)^\lambda}} \right)\right) \right] \left(\tilde{\rho}^{(\mathfrak{j})}\right) \end{array} \right).$$

We now introduce the PULq-ROFEWA operation and its associated basic characteristics:

Definition 3.2. *Assume* $\beth = \{\flat_\vartheta | \vartheta = -\Delta, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \Delta\}$ *is a LTS,*

$$\hbar^\epsilon(\rho) = \langle \{[\aleph_{\theta^\epsilon(\mathfrak{g})}, \mu_{\theta^\epsilon(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^\epsilon(\mathfrak{j})}, \nu_{\Theta^\epsilon(\mathfrak{j})}](\tilde{\rho}^{(\mathfrak{j})})\} \rangle$$

$(\epsilon = 1, 2, \ldots, \varsigma; \mathfrak{g} = 1, 2, \ldots, \mathfrak{G}; \mathfrak{j} = 1, 2, \ldots, \mathfrak{J})$ *be the modified PULq-ROFNs, then the PULq-ROFEWA operator is described as below:*

$$\textit{PULq-ROFEWA}(\hbar^1(\rho), \hbar^2(\rho), \ldots, \hbar^\varsigma(\rho)) = \bigoplus_{\epsilon=1}^{\varsigma} \mathcal{W}_\epsilon \hbar^\epsilon(\rho), \tag{3. 10}$$

where $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_\varsigma)^T$ *is the weight vector of* M_ϵ $(\epsilon = 1, 2, \ldots, \varsigma)$ *with* $\mathcal{W}_\epsilon \in [0, 1]$ *and* $\sum_{\epsilon=1}^{\varsigma} \mathcal{W}_\epsilon = 1$.

Theorem 3.1. *Considering an assortment of PULq-ROFNs*

$$\hbar^\epsilon(\rho) = \langle \{[\aleph_{\theta^\epsilon(\mathfrak{g})}, \mu_{\theta^\epsilon(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^\epsilon(\mathfrak{j})}, \nu_{\Theta^\epsilon(\mathfrak{j})}](\tilde{\rho}^{(\mathfrak{j})})\} \rangle$$

$(\epsilon = 1, 2, \ldots, \varsigma; \mathfrak{g} = 1, 2, \ldots, \mathfrak{G}; \mathfrak{j} = 1, 2, \ldots, \mathfrak{J})$ *having weight vector* $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_\varsigma)^T$ *with* $\mathcal{W}_\epsilon \in [0, 1]$ *and* $\sum_{\epsilon=1}^{\varsigma} \mathcal{W}_\epsilon = 1$. *Then*

$$\begin{aligned}
&\textit{PULq-ROFEWA}(\hbar^1(\rho), \hbar^2(\rho), \ldots, \hbar^\varsigma(\rho)) \\
&= \left(\begin{array}{l}
\left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right),\right. \\
\left. \mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon}(\mathfrak{g})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right)\right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\
\\
\left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\prod_{\epsilon=1}^{\varsigma}\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon}(\mathfrak{j})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\varsigma}\left(2-\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon}(\mathfrak{j})}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon}(\mathfrak{j})}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}}}\right)\right),\right. \\
\left. \mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\prod_{\epsilon=1}^{\varsigma}\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon}(\mathfrak{j})}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\varsigma}\left(2-\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon}(\mathfrak{j})}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon}(\mathfrak{j})}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}}}\right)\right)\right] \left(\tilde{\rho}^{(\mathfrak{j})}\right)
\end{array} \right)
\end{aligned} \tag{3.11}$$

Proof. We use induction method and definition 3.2 to prove this theorem. For $\varsigma = 2$, we have

$$\mathcal{W}_1\hbar^1(\rho) \bigoplus \mathcal{W}_2\hbar^2(\rho) =$$

$$\left(\begin{array}{c}\left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}-\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}}{\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}+\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}}}\right)\right),\right. \\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}-\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}}{\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}+\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}}}\right)\right)\right]\left(\hat{\rho}^{(\mathfrak{g})}\right) \\ \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{1(\mathrm{j})}}\right)}{\Delta}\right)^{\mathcal{W}_1}}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{1(\mathrm{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_1}+\left(\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{1(\mathrm{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_1}}}\right)\right),\right. \\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{1(\mathrm{j})}}\right)}{\Delta}\right)^{\mathcal{W}_1}}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{1(\mathrm{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_1}+\left(\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{1(\mathrm{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_1}}}\right)\right)\right]\left(\tilde{\rho}^{(\mathrm{j})}\right)\end{array}\right)\bigoplus$$

$$\left(\begin{array}{c}\left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{2(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_2}-\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{2(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_2}}{\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{2(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}+\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{2(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_2}}}\right)\right),\right. \\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{2(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_2}-\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{2(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_2}}{\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_1}+\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{2(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_2}}}\right)\right)\right]\left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{2(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_2}}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{2(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_2}+\left(\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{2(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_2}}}\right)\right),\right. \\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{2(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_1}}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{1(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_2}+\left(\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{2(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_2}}}\right)\right)\right]\left(\tilde{\rho}^{(\mathfrak{j})}\right)\end{array}\right).$$

$$
= \begin{pmatrix}
\left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{2}\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{2}\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{2}\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{2}\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right), \right. \\
\left. \mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{2}\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{2}\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{2}\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{2}\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\
\\
\left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\prod_{\epsilon=1}^{2}\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{2}\left(2-\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{2}\left(\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}}}\right)\right), \right. \\
\left. \mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\prod_{\epsilon=1}^{2}\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{2}\left(2-\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{2}\left(\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}}}\right)\right) \right] \left(\tilde{\rho}^{(\mathfrak{j})}\right)
\end{pmatrix}.
$$

Equation 3. 11 keeps for $\varsigma = 2$. Assume that Equation (3. 11) keeps for $\varsigma = \psi$.

$$\text{PUL}q\text{-ROFEWA}(\hbar^1(\rho), \hbar^2(\rho), \ldots, \hbar^{\psi}(\rho)) =$$

$$\left(\begin{array}{c}\left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\dfrac{\prod_{\epsilon=1}^{\psi}\left(1+\dfrac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{\psi}\left(1-\dfrac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\psi}\left(1+\dfrac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\psi}\left(1-\dfrac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right),\right. \\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\dfrac{\prod_{\epsilon=1}^{\psi}\left(1+\dfrac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{\psi}\left(1-\dfrac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\psi}\left(1+\dfrac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\psi}\left(1-\dfrac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right)\right]\left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\dfrac{\sqrt[q]{2}\prod_{\epsilon=1}^{\psi}\left(\dfrac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\psi}\left(2-\left(\dfrac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\psi}\left(\left(\dfrac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon}}}\right)\right),\right. \\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\dfrac{\sqrt[q]{2}\prod_{\epsilon=1}^{\psi}\left(\dfrac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\psi}\left(2-\left(\dfrac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\psi}\left(\left(\dfrac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon}}}\right)\right)\right]\left(\tilde{\rho}^{(\mathfrak{j})}\right)\end{array}\right).$$

For $\varsigma = \psi + 1$ by the assumption logic, we have

$$\text{PUL}q\text{-ROFEWA}(\hbar^1(\rho), \hbar^2(\rho), \ldots, \hbar^{\psi}(\rho), \hbar^{\psi+1}(\rho))$$

$$
= \left(\begin{array}{c}
\left[\mathsf{t}^{-1} \left(\Delta \left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{\psi} \left(1 + \frac{\mathsf{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon} - \prod_{\epsilon=1}^{\psi} \left(1 - \frac{\mathsf{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\psi} \left(1 + \frac{\mathsf{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon} + \prod_{\epsilon=1}^{\psi} \left(1 - \frac{\mathsf{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}} \right) \right), \right. \\
\left. \mathsf{t}^{-1} \left(\Delta \left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{\psi} \left(1 + \frac{\mathsf{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon} - \prod_{\epsilon=1}^{\psi} \left(1 - \frac{\mathsf{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\psi} \left(1 + \frac{\mathsf{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon} + \prod_{\epsilon=1}^{\psi} \left(1 - \frac{\mathsf{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_\epsilon}}} \right) \right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\
\\
\left[\mathsf{t}^{-1} \left(\Delta \left(\frac{\sqrt[q]{2} \prod_{\epsilon=1}^{\psi} \left(\frac{\mathsf{t}\left(\pounds_{\Theta^{\epsilon(\mathsf{j})}}\right)}{\Delta} \right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\psi} \left(2 - \left(\frac{\mathsf{t}\left(\pounds_{\Theta^{\epsilon(\mathsf{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon} + \prod_{\epsilon=1}^{\psi} \left(\left(\frac{\mathsf{t}\left(\pounds_{\Theta^{\epsilon(\mathsf{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon}}} \right) \right), \right. \\
\left. \mathsf{t}^{-1} \left(\Delta \left(\frac{\sqrt[q]{2} \prod_{\epsilon=1}^{\psi} \left(\frac{\mathsf{t}\left(\nu_{\Theta^{\epsilon(\mathsf{j})}}\right)}{\Delta} \right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\psi} \left(2 - \left(\frac{\mathsf{t}\left(\nu_{\Theta^{\epsilon(\mathsf{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon} + \prod_{\epsilon=1}^{\psi} \left(\left(\frac{\mathsf{t}\left(\nu_{\Theta^{\epsilon(\mathsf{j})}}\right)}{\Delta}\right)^q\right)^{\mathcal{W}_\epsilon}}} \right) \right) \right] \left(\tilde{\rho}^{(\mathsf{j})}\right)
\end{array} \right) \bigoplus
$$

$$\left(\begin{array}{c}\left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}-\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}}{\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}+\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}}}\right)\right),\right.\\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}-\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}}{\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}+\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\psi+1(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}}}\right)\right)\right]\left(\hat{\rho}^{(\mathfrak{g})}\right),\\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\psi+1(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\psi+1(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\psi+1}}+\left(\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\psi+1(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\psi+1}}}}\right)\right),\right.\\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\psi+1(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_{\psi+1}}}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\psi+1(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\psi+1}}+\left(\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\psi+1(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\psi+1}}}}\right)\right)\right]\left(\tilde{\rho}^{(\mathfrak{j})}\right)\end{array}\right).$$

$$=\left(\begin{array}{c}\left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{\psi+1}\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}-\prod_{\epsilon=1}^{\psi+1}\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\prod_{\epsilon=1}^{\psi+1}\left(1+\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}+\prod_{\epsilon=1}^{\psi+1}\left(1-\frac{\mathfrak{t}\left(\aleph_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}}\right)\right),\right.\\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\frac{\prod_{\epsilon=1}^{\psi+1}\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}-\prod_{\epsilon=1}^{\psi+1}\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\prod_{\epsilon=1}^{\psi+1}\left(1+\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}+\prod_{\epsilon=1}^{\psi+1}\left(1-\frac{\mathfrak{t}\left(\mu_{\theta^{\epsilon(\mathfrak{g})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}}\right)\right)\right]\left(\hat{\rho}^{(\mathfrak{g})}\right),\\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\prod_{\epsilon=1}^{\psi+1}\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\sqrt[q]{\prod_{\epsilon=1}^{\psi+1}\left(2-\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}}+\prod_{\epsilon=1}^{\psi+1}\left(\left(\frac{\mathfrak{t}\left(\pounds_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}}}}\right)\right),\right.\\ \left.\mathfrak{t}^{-1}\left(\Delta\left(\frac{\sqrt[q]{2}\prod_{\epsilon=1}^{\psi+1}\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\sqrt[q]{\prod_{\epsilon=1}^{\psi+1}\left(2-\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}}+\prod_{\epsilon=1}^{\psi+1}\left(\left(\frac{\mathfrak{t}\left(\nu_{\Theta^{\epsilon(\mathfrak{j})}}\right)}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}}}}\right)\right)\right]\left(\tilde{\rho}^{(\mathfrak{j})}\right)\end{array}\right).$$

Hence, Equation (3. 11) keeps for every positive integers $\varsigma \geq 1$. □

Proposition 3.1. *Suppose*

$$\hbar^{\epsilon}(\rho) = \langle\{[\aleph_{\theta^{\epsilon}(\mathfrak{g})}, \mu_{\theta^{\epsilon}(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{\epsilon}(\mathfrak{j})}, \nu_{\Theta^{\epsilon}(\mathfrak{j})}](\tilde{\rho}^{(\mathfrak{j})})\}\rangle$$

$(\epsilon = 1, 2, \ldots, \varsigma; \mathfrak{g} = 1, 2, \ldots, \mathfrak{G}; \mathfrak{j} = 1, 2, \ldots, \mathfrak{J})$ *be the collection of PULq-ROFNs with weight vector* $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{\varsigma})^T$*, such that* $\mathcal{W}_{\epsilon} \in [0, 1]$ *and* $\sum_{\epsilon=1}^{\varsigma} \mathcal{W}_{\epsilon} = 1$*. Therefore, certain characteristics apply to the PULq-ROFEWA operator:*

(1) *(Idempotency) If all* $\hbar^{\epsilon}(\rho) = ([\aleph_{\theta^{\epsilon}(\mathfrak{g})}, \mu_{\theta^{\epsilon}(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})})), [\pounds_{\Theta^{\epsilon}(\mathfrak{j})}, \nu_{\Theta^{\epsilon}(\mathfrak{j})}](\tilde{\rho}^{(\mathfrak{j})})(\epsilon = 1, 2, \ldots, \varsigma)$ *are equal, for all* ϵ*, then*

$$\textit{PULq-ROFEWA}(\hbar^{1}(\rho), \hbar^{2}(\rho), \ldots, \hbar^{\varsigma}(\rho)) = \hbar(\rho).$$

Proof. Suppose $\hbar^{\epsilon}(\rho)$=$\langle\{[\aleph_{\theta^{\epsilon}(\mathfrak{g})}, \mu_{\theta^{\epsilon}(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})}),\ [\pounds_{\Theta^{\epsilon}(\mathfrak{j})}, \nu_{\Theta^{\epsilon}(\mathfrak{j})}](\tilde{\rho}^{(\mathfrak{j})})\}\rangle$ is a collection of PULq-ROFNs such that $\hbar^{\epsilon}(\rho) = \hbar(\rho)$ for all $(\epsilon = 1, 2, \ldots, \varsigma)$, $\mathcal{W}_{\epsilon} \in [0, 1]$ and from Equation 3. 11 , we get

$$\text{PUL}q\text{-ROFEWA}(\hbar^{1}(\rho), \hbar^{2}(\rho), \ldots, \hbar^{\varsigma}(\rho)) =$$

$$= \left(\begin{array}{c} \left[\mathfrak{t}^{-1}\left(\Delta \sqrt[q]{\dfrac{\prod_{\epsilon=1}^{\varsigma}\left(1+\dfrac{\mathfrak{t}(\aleph_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}} - \prod_{\epsilon=1}^{\varsigma}\left(1-\dfrac{\mathfrak{t}(\aleph_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\prod_{\epsilon=1}^{\varsigma}\left(1+\dfrac{\mathfrak{t}(\aleph_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}} + \prod_{\epsilon=1}^{\varsigma}\left(1-\dfrac{\mathfrak{t}(\aleph_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}} \right), \right. \\ \left. \mathfrak{t}^{-1}\left(\Delta \sqrt[q]{\dfrac{\prod_{\epsilon=1}^{\varsigma}\left(1+\dfrac{\mathfrak{t}(\mu_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}} - \prod_{\epsilon=1}^{\varsigma}\left(1-\dfrac{\mathfrak{t}(\mu_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\prod_{\epsilon=1}^{\varsigma}\left(1+\dfrac{\mathfrak{t}(\mu_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}} + \prod_{\epsilon=1}^{\varsigma}\left(1-\dfrac{\mathfrak{t}(\mu_{\theta^{\epsilon}(\mathfrak{g})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}} \right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \left[\mathfrak{t}^{-1}\left(\Delta \dfrac{\sqrt[q]{2}\prod_{\epsilon=1}^{\varsigma}\left(\dfrac{\mathfrak{t}(\pounds_{\Theta^{\epsilon}(\mathfrak{j})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\sqrt[q]{\prod_{\epsilon=1}^{\varsigma}\left(2-\left(\dfrac{\mathfrak{t}(\pounds_{\Theta^{\epsilon}(\mathfrak{j})})}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}} + \prod_{\epsilon=1}^{\varsigma}\left(\left(\dfrac{\mathfrak{t}(\pounds_{\Theta^{\epsilon}(\mathfrak{j})})}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}}}} \right), \right. \\ \left. \mathfrak{t}^{-1}\left(\Delta \dfrac{\sqrt[q]{2}\prod_{\epsilon=1}^{\varsigma}\left(\dfrac{\mathfrak{t}(\nu_{\Theta^{\epsilon}(\mathfrak{j})})}{\Delta}\right)^{\mathcal{W}_{\epsilon}}}{\sqrt[q]{\prod_{\epsilon=1}^{\varsigma}\left(2-\left(\dfrac{\mathfrak{t}(\nu_{\Theta^{\epsilon}(\mathfrak{j})})}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}} + \prod_{\epsilon=1}^{\varsigma}\left(\left(\dfrac{\mathfrak{t}(\nu_{\Theta^{\epsilon}(\mathfrak{j})})}{\Delta}\right)^{q}\right)^{\mathcal{W}_{\epsilon}}}} \right) \right] \left(\tilde{\rho}^{(\mathfrak{j})}\right) \end{array} \right)$$

$$= \left(\begin{array}{l} \left[\mathfrak{t}^{-1}\left(\Delta \sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}(\aleph_{\theta(\mathfrak{g})})}{\Delta}\right)-\left(1-\frac{\mathfrak{t}(\aleph_{\theta(\mathfrak{g})})}{\Delta}\right)}{\left(1+\frac{\mathfrak{t}(\aleph_{\theta(\mathfrak{g})})}{\Delta}\right)+\left(1-\frac{\mathfrak{t}(\aleph_{\theta(\mathfrak{g})})}{\Delta}\right)}} \right), \right. \\ \left. \mathfrak{t}^{-1}\left(\Delta \sqrt[q]{\frac{\left(1+\frac{\mathfrak{t}(\mu_{\theta(\mathfrak{g})})}{\Delta}\right)-\left(1-\frac{\mathfrak{t}(\mu_{\theta(\mathfrak{g})})}{\Delta}\right)}{\left(1+\frac{\mathfrak{t}(\mu_{\theta(\mathfrak{g})})}{\Delta}\right)+\left(1-\frac{\mathfrak{t}(\mu_{\theta(\mathfrak{g})})}{\Delta}\right)}} \right) \right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \left[\mathfrak{t}^{-1}\left(\Delta \frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}(\pounds_{\Theta(\mathfrak{j})})}{\Delta}\right)}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}(\pounds_{\Theta(\mathfrak{j})})}{\Delta}\right)^{q}\right)+\left(\frac{\mathfrak{t}(\pounds_{\Theta(\mathfrak{j})})}{\Delta}\right)^{q}}} \right), \right. \\ \left. \mathfrak{t}^{-1}\left(\Delta \frac{\sqrt[q]{2}\left(\frac{\mathfrak{t}(\nu_{\Theta(\mathfrak{j})})}{\Delta}\right)}{\sqrt[q]{\left(2-\left(\frac{\mathfrak{t}(\nu_{\Theta(\mathfrak{j})})}{\Delta}\right)^{q}\right)+\left(\frac{\mathfrak{t}(\nu_{\Theta(\mathfrak{j})})}{\Delta}\right)^{q}}} \right) \right] \left(\tilde{\rho}^{(\mathfrak{j})}\right) \end{array} \right)$$

$$= \left(\begin{array}{l} \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\mathfrak{t}\left(\aleph_{\theta(\mathfrak{g})}\right)}{\Delta}\right)\right), \mathfrak{t}^{-1}\left(\Delta\left(\frac{\mathfrak{t}\left(\mu_{\theta(\mathfrak{g})}\right)}{\Delta}\right)\right)\right] \left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\frac{\mathfrak{t}\left(\pounds_{\Theta(\mathfrak{j})}\right)}{\Delta}\right)\right), \mathfrak{t}^{-1}\left(\Delta\left(\frac{\mathfrak{t}\left(\nu_{\Theta(\mathfrak{j})}\right)}{\Delta}\right)\right)\right] \left(\tilde{\rho}^{(\mathfrak{j})}\right) \end{array} \right)$$

$$= \left(\left[\aleph_{\theta(\mathfrak{g})}, \mu_{\theta(\mathfrak{g})}\right]\left(\hat{\rho}^{(\mathfrak{g})}\right), \left[\pounds_{\Theta(\mathfrak{j})}, \nu_{\Theta(\mathfrak{j})}\right]\left(\tilde{\rho}^{(\mathfrak{j})}\right) \right)$$

$$= \hbar(\rho).$$

□

(2) *(Monotonicity) Let* $\hbar^{\epsilon}(\rho) = (\hbar^{1}(\rho), \hbar^{2}(\rho), \ldots, \hbar^{\varsigma}(\rho))$ *and* $\hbar'^{\epsilon}(\rho') = (\hbar'^{1}(\rho'), \hbar'^{2}(\rho'), \ldots, \hbar'^{\varsigma}(\rho'))$ *be two collections of adjusted PULq-ROFNs, for all* ϵ, $\aleph_{\theta^{\epsilon}(\mathfrak{f})} < \aleph_{\theta'^{\epsilon}(\mathfrak{f})}$, $\mu_{\theta^{\epsilon}(\mathfrak{f})} < \mu_{\theta'^{\epsilon}(\mathfrak{f})}$, $\pounds_{\Theta^{\epsilon}(\mathfrak{h})} > \pounds_{\Theta'^{\epsilon}(\mathfrak{h})}$ *and* $\nu_{\Theta^{\epsilon}(\mathfrak{h})} > \nu_{\Theta'^{\epsilon}(\mathfrak{h})}$, *then*

$$\text{PULq-ROFEWA}(\hbar^{1}(\rho), \hbar^{2}(\rho), \ldots, \hbar^{\varsigma}(\rho)) < \text{PULq-ROFEWA}(\hbar'^{1}(\rho'), \hbar'^{2}(\rho'), \ldots, \hbar'^{\varsigma}(\rho')).$$

(3) *(Boundedness) Let* $\aleph_{\theta^{\epsilon}(+)} = \max_{\mathfrak{g}=1}^{\mathfrak{G}} \aleph_{\theta^{\epsilon}(\mathfrak{g})}$, $\mu_{\theta^{\epsilon}(+)} = \max_{\mathfrak{g}=1}^{\mathfrak{G}} \mu_{\theta^{\epsilon}(\mathfrak{g})}$, $\aleph_{\theta^{\epsilon}(-)} = \min_{\mathfrak{g}=1}^{\mathfrak{G}} \aleph_{\theta^{\epsilon}(\mathfrak{g})}$, $\mu_{\theta^{\epsilon}(-)} = \min_{\mathfrak{g}=1}^{\mathfrak{G}} \mu_{\theta^{\epsilon}(\mathfrak{g})}$, $\pounds_{\Theta^{\epsilon}(+)} = \max_{\mathfrak{j}=1}^{\mathfrak{J}} \pounds_{\Theta^{\epsilon}(\mathfrak{j})}$, $\nu_{\Theta^{\epsilon}(+)} = \max_{\mathfrak{j}=1}^{\mathfrak{J}} \nu_{\Theta^{\epsilon}(\mathfrak{j})}$, $\pounds_{\Theta^{\epsilon}(-)} =$

$\min_{j=1}^{\mathfrak{J}} \pounds_{\Theta^{\epsilon(j)}}$, $\nu_{\Theta^{\epsilon(-)}} = \min_{j=1}^{\mathfrak{J}} \nu_{\Theta^{\epsilon(j)}}$, *then*

$$(([\aleph_{\theta^{\epsilon(-)}}, \mu_{\theta^{\epsilon(-)}}](\hat{\rho}^{(\mathfrak{g})})), ([\pounds_{\Theta^{\epsilon(+)}}, \nu_{\Theta^{\epsilon(+)}}](\tilde{\rho}^{(j)}))) \leq \text{PULq-ROFEWA}(\hbar^1(\rho), \hbar^2(\rho), \ldots, \hbar^{\varsigma}(\rho)) \leq (([\aleph_{\theta^{\epsilon(+)}}, \mu_{\theta^{\epsilon(+)}}](\hat{\rho}^{(\mathfrak{g})})), ([\pounds_{\Theta^{\epsilon(-)}}, \nu_{\Theta^{\epsilon(-)}}](\tilde{\rho}^{(j)}))).$$

Definition 3.3. *The PULq-ROFEOWA operator is a mapping $\mathfrak{H}^{\varsigma} \to \mathfrak{H}$ such that: for each collection of PULq-ROFNs, $\hbar^{\epsilon}(\rho)=\langle\{[\aleph_{\theta^{\epsilon(\mathfrak{g})}}, \mu_{\theta^{\epsilon(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}),\ [\pounds_{\Theta^{\epsilon(j)}}, \nu_{\Theta^{\epsilon(j)}}](\tilde{\rho}^{(j)})\}\rangle$ $(\epsilon = 1, 2, \ldots, \varsigma; \mathfrak{g} = 1, 2, \ldots, \mathfrak{G}; j = 1, 2, \ldots, \mathfrak{J})$*

$$PULq-ROFEOWA(\hbar^1(\rho), \hbar^2(\rho), \ldots, \hbar^{\varsigma}(\rho)) = \bigoplus_{k=1}^{n} \mathcal{W}_{\epsilon} \hbar^{\alpha(\epsilon)}(\rho), \tag{3. 12}$$

where $\alpha(\epsilon)$ is such that $\hbar^{\alpha(\epsilon-1)}(\rho) \geq \hbar^{\alpha(\epsilon)}(\rho)$ for all ϵ, $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{\varsigma})^T$ is the weight vector $\hbar^{\epsilon}(\epsilon = 1, 2, \ldots, \varsigma)$ with $\mathcal{W}_{\epsilon} \in [0, 1]$ and $\sum_{\epsilon=1}^{\varsigma} \mathcal{W}_{\epsilon} = 1$.

Theorem 3.2. *Consider a collection of PULq-ROFNs,*

$$\hbar^{\epsilon}(\rho) = \langle\{[\aleph_{\theta^{\epsilon(\mathfrak{g})}}, \mu_{\theta^{\epsilon(\mathfrak{g})}}](\hat{\rho}^{(\mathfrak{g})}), [\pounds_{\Theta^{\epsilon(j)}}, \nu_{\Theta^{\epsilon(j)}}](\tilde{\rho}^{(j)})\}\rangle$$

$(\epsilon = 1, 2, \ldots, \varsigma; \mathfrak{g} = 1, 2, \ldots, \mathfrak{G}; j = 1, 2, \ldots, \mathfrak{J})$ having weight vector $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{\varsigma})^T$ with $\mathcal{W}_{\epsilon} \in [0, 1]$ and $\sum_{\epsilon=1}^{\varsigma} \mathcal{W}_{\epsilon} = 1$. Then

$$\text{PULq-ROFEOWA}(\hbar^1(\rho), \hbar^2(\rho), \ldots, \hbar^{\varsigma}(\rho)) =$$

$$\left(\begin{array}{c} \left[\mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\dfrac{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}(\aleph_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}(\aleph_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}(\aleph_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}(\aleph_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right)\right], \\ \mathfrak{t}^{-1}\left(\Delta\left(\sqrt[q]{\dfrac{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}(\mu_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}-\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}(\mu_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\prod_{\epsilon=1}^{\varsigma}\left(1+\frac{\mathfrak{t}(\mu_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(1-\frac{\mathfrak{t}(\mu_{\theta^{\alpha(\epsilon)(\mathfrak{g})}})}{\Delta}\right)^{\mathcal{W}_\epsilon}}}\right)\right)\left(\hat{\rho}^{(\mathfrak{g})}\right), \\ \left[\mathfrak{t}^{-1}\left(\Delta\left(\dfrac{\sqrt[q]{2}\prod_{\epsilon=1}^{\varsigma}\left(\frac{\mathfrak{t}(\pounds_{\Theta^{\alpha(\epsilon)(j)}})}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\varsigma}\left(2-\left(\frac{\mathfrak{t}(\pounds_{\Theta^{\alpha(\epsilon)(j)}})}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(\left(\frac{\mathfrak{t}(\pounds_{\Theta^{\alpha(\epsilon)(j)}})}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}}}\right)\right)\right., \\ \left. \mathfrak{t}^{-1}\left(\Delta\left(\dfrac{\sqrt[q]{2}\prod_{\epsilon=1}^{\varsigma}\left(\frac{\mathfrak{t}(\nu_{\Theta^{\alpha(\epsilon)(j)}})}{\Delta}\right)^{\mathcal{W}_\epsilon}}{\sqrt[q]{\prod_{\epsilon=1}^{\varsigma}\left(2-\left(\frac{\mathfrak{t}(\nu_{\Theta^{\alpha(\epsilon)(j)}})}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}+\prod_{\epsilon=1}^{\varsigma}\left(\left(\frac{\mathfrak{t}(\nu_{\Theta^{\alpha(\epsilon)(j)}})}{\Delta}\right)^{q}\right)^{\mathcal{W}_\epsilon}}}\right)\right)\left(\tilde{\rho}^{(j)}\right)\right] \end{array} \right). \tag{3. 13}$$

This proof is similar to the proof of Theorem 3.1.

Proposition 3.2. *Let* $\hbar^{\epsilon}(\rho)=\langle\{[\aleph_{\theta^{\epsilon}(\mathfrak{g})},\mu_{\theta^{\epsilon}(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})}),\ [\pounds_{\Theta^{\epsilon}(\mathrm{j})},\nu_{\Theta^{\epsilon}(\mathrm{j})}](\tilde{\rho}^{(\mathrm{j})})\}\rangle$ $(\epsilon=1,2,\dots,\varsigma;\mathfrak{g}=1,2,\dots,\mathfrak{G};\mathrm{j}=1,2,\dots,\mathfrak{J})$ *be the collection of PULq-ROFNs with weight vector* $\mathcal{W}=(\mathcal{W}_1,\mathcal{W}_2,\dots,\mathcal{W}_{\varsigma})^T$*, such that* $\mathcal{W}_{\epsilon}\in[0,1]$ *and* $\sum_{\epsilon=1}^{\varsigma}\mathcal{W}_{\epsilon}=1$*. Therefore, certain characteristics apply to the PULq-ROFEOWA operator:*

(1) *(Idempotency) If all* $\hbar^{\epsilon}(\rho)=([\aleph_{\theta^{\epsilon}(\mathfrak{g})},\mu_{\theta^{\epsilon}(\mathfrak{g})}](\hat{\rho}^{(\mathfrak{g})})),[\pounds_{\Theta^{\epsilon}(\mathrm{j})},\nu_{\Theta^{\epsilon}(\mathrm{j})}](\tilde{\rho}^{(\mathrm{j})})(\epsilon=1,2,\dots,\varsigma)$ *are equal, for all* ϵ*, then*

$$\textit{PULq-ROFEOWA}(\hbar^1(\rho),\hbar^2(\rho),\dots,\hbar^{\varsigma}(\rho))=\hbar(\rho).$$

(2) *(Monotonicity) Let* $\hbar^{\epsilon}(\rho)=(\hbar^1(\rho),\hbar^2(\rho),\dots,\hbar^{\varsigma}(\rho))$ *and* $\hbar'^{\epsilon}(\rho')=(\hbar'^1(\rho'),\hbar'^2(\rho'),\dots,\hbar'^{\varsigma}(\rho'))$ *be two collections of adjusted PULq-ROFNs, for all* ϵ, $\aleph_{\theta^{\epsilon}(\mathfrak{g})}<\aleph_{\theta'^{\epsilon}(\mathfrak{g})}$, $\mu_{\theta^{\epsilon}(\mathfrak{g})}<\mu_{\theta'^{\epsilon}(\mathfrak{g})}$, $\pounds_{\Theta^{\epsilon}(\mathrm{j})}>\pounds_{\Theta'^{\epsilon}(\mathrm{j})}$ *and*
$\nu_{\Theta^{\epsilon}(\mathrm{j})}>\nu_{\Theta'^{\epsilon}(\mathrm{j})}$ *then*

$$\textit{PULq-ROFEOWA}(\hbar^1(\rho),\hbar^2(\rho),\dots,\hbar^{\varsigma}(\rho))<\textit{PULq-ROFEOWA}(\hbar'^1(\rho'),\hbar'^2(\rho'),\dots,\hbar'^{\varsigma}(\rho')).$$

(3) *(Boundedness) Let* $\aleph_{\theta^{\epsilon}(+)}=\max_{\mathfrak{g}=1}^{\mathfrak{G}}\aleph_{\theta^{\epsilon}(\mathfrak{g})}$, $\mu_{\theta^{\epsilon}(+)}=\max_{\mathfrak{g}=1}^{\mathfrak{G}}\mu_{\theta^{\epsilon}(\mathfrak{g})}$, $\aleph_{\theta^{\epsilon}(-)}=\min_{\mathfrak{g}=1}^{\mathfrak{G}}\aleph_{\theta^{\epsilon}(\mathfrak{g})}$, $\mu_{\theta^{\epsilon}(-)}=\min_{\mathfrak{g}=1}^{\mathfrak{G}}\mu_{\theta^{\epsilon}(\mathfrak{g})}$, $\pounds_{\Theta^{\epsilon}(+)}=\max_{\mathrm{j}=1}^{\mathfrak{J}}\pounds_{\Theta^{\epsilon}(\mathrm{j})}$, $\nu_{\Theta^{\epsilon}(+)}=\max_{\mathrm{j}=1}^{\mathfrak{J}}\nu_{\Theta^{\epsilon}(\mathrm{j})}$, $\pounds_{\Theta^{\epsilon}(-)}=\min_{\mathrm{j}=1}^{\mathfrak{J}}\pounds_{\Theta^{\epsilon}(\mathrm{j})}$, $\nu_{\Theta^{\epsilon}(-)}=\min_{\mathrm{j}=1}^{\mathfrak{J}}\nu_{\Theta^{\epsilon}(\mathrm{j})}$*, then*

$$\begin{aligned}(([\aleph_{\theta^{\epsilon}(-)},\mu_{\theta^{\epsilon}(-)}](\hat{\rho}^{(\mathfrak{g})})),([\pounds_{\Theta^{\epsilon}(+)},\nu_{\Theta^{\epsilon}(+)}](\tilde{\rho}^{(\mathrm{j})})))&\leq \textit{PULq-ROFEOWA}(\hbar^1(\rho),\hbar^2(\rho),\dots,\hbar^{\varsigma}(\rho))\\&\leq(([\aleph_{\theta^{\epsilon}(+)},\mu_{\theta^{\epsilon}(+)}](\hat{\rho}^{(\mathfrak{g})})),([\pounds_{\Theta^{\epsilon}(-)},\nu_{\Theta^{\epsilon}(-)}](\tilde{\rho}^{(\mathrm{j})}))).\end{aligned}$$

4. PULq-RPF-CODAS METHOD

The CODAS technique for the PULq-ROF context will be developed in this part to tackle the problems associated with MCGDM. Utilizing the CODAS approach expression in a PULq-ROF context, our goal is to identify the optimal option. The judgments are fused using additional Einstein aggregation procedures. Additionally, the weights for the requirement are generated using the entropy measurement. The complex MCGDM challenges can be solved using the PULq-ROF-CODAS approach.
The most prevalent determining variables are the surroundings, evaluations of achievement, information filtering, and implementation generation. There are a group of specialists, a number of feasible solutions and a set of requirements for every MCGDM problems. Selecting the optimal choice from an assortment of ψ prospective choices or alternative $\mathcal{N}=\{\mathcal{N}_1,\mathcal{N}_2,\dots,\mathcal{N}_{\psi}\}$, $(\iota=1,2,\dots,\psi)$ that will be examined depending on ς criteria $\mathcal{M}=\{\mathcal{M}_1,\mathcal{M}_2,\dots,\mathcal{M}_{\varsigma}\}$, $(\iota=1,2,\dots,\psi)$ a group of τ DM experts, in order to evaluate the different choices, $\mathcal{D}=\{\mathcal{D}_1,\mathcal{D}_2,\dots,\mathcal{D}_{\mathfrak{e}}\}$, $\tau=1,2,\dots,\mathfrak{e}$ are permitted.

Step 1: *Development od linguistic-based ideas.* The behavior of different options throughout the DM procedure must be explained using suitable syntax. To assess the choices or possibilities offered in MCGDM problems, decision specialists typically utilize linguistic words. The specialists in this approach first determine the linguistic terms and their corresponding PULq-ROFNs for the MCGDM process.

Step 2: *Specialists self-construct matrix structures.* In this phase, every specialist develops a ranking framework and provides feedback in the form of PULq-ROFNs. Suppose the assessment of the matrix structures provided by the specialists are $\mathbb{R} = [\hbar^{\tau}_{\iota\epsilon}]_{\psi\times\varsigma}$. The judgment framework of the τth specialist should be described below:

$$\mathbb{R} = [\hbar^{\tau}_{\iota\epsilon}]_{\psi\times\varsigma} = \begin{bmatrix} \hbar^{\tau}_{11} & \hbar^{\tau}_{12} & \dots & \hbar^{\tau}_{1\varsigma} \\ \hbar^{\tau}_{21} & \hbar^{\tau}_{22} & \dots & \hbar^{\tau}_{2\varsigma} \\ \vdots & \vdots & \vdots & \vdots \\ \hbar^{\tau}_{\psi 1} & \hbar^{\tau}_{\psi 2} & \dots & \hbar^{\tau}_{\psi\varsigma} \end{bmatrix}. \tag{4. 14}$$

Every element in the assessment matrix $\mathbb{R} = [\hbar^{\tau}_{\iota\epsilon}]_{\psi\times\varsigma}$ $(\tau = 1, 2, \dots, \mathfrak{e})$ indicates a PULq-ROFN as $\hbar^{\tau}_{\iota\epsilon} = \langle[\aleph^{\tau}_{\theta_{\iota\epsilon}}, \mu^{\tau}_{\theta_{\iota\epsilon}}](\hat{\rho}^{\tau}_{\iota\epsilon}), [\pounds^{\tau}_{\Theta_{\iota\epsilon}}, \nu^{\tau}_{\Theta_{\iota\epsilon}}](\tilde{\rho}^{\tau}_{\iota\epsilon})\rangle$, where $[\aleph^{\tau}_{\theta_{\iota\epsilon}}, \mu^{\tau}_{\theta_{\iota\epsilon}}]$, $[\pounds^{\tau}_{\Theta_{\iota\epsilon}}, \nu^{\tau}_{\Theta_{\iota\epsilon}}]$ are the grades of affiliation and non-affiliation, respectively.

Step 3: *Independent normalization of matrices.* Every single matrix is adjusted using a particular criteria:

$$\hbar_{\iota\epsilon} = \begin{cases} \langle[\aleph^{\tau}_{\theta_{\iota\epsilon}}, \mu^{\tau}_{\theta_{\iota\epsilon}}](\hat{\rho}^{\tau}_{\iota\epsilon}), [\pounds^{\tau}_{\Theta_{\iota\epsilon}}, \nu^{\tau}_{\Theta_{\iota\epsilon}}](\tilde{\rho}^{\tau}_{\iota\epsilon})\rangle, & \text{for benefit type criteria;} \\ \langle[\pounds^{\tau}_{\Theta_{\iota\epsilon}}, \nu^{\tau}_{\Theta_{\iota\epsilon}}](\tilde{\rho}^{\tau}_{\iota\epsilon}), [\aleph^{\tau}_{\theta_{\iota\epsilon}}, \mu^{\tau}_{\theta_{\iota\epsilon}}](\hat{\rho}^{\tau}_{\iota\epsilon})\rangle, & \text{for cost type criteria.} \end{cases} \tag{4. 15}$$

Step 4: *Computing a unified matrix.* To produce a matrix of aggregated assessments, $\mathbb{G} = (\hbar^{\tau}_{\iota\epsilon})_{\psi\times\varsigma}$, it is necessary to sum up each expert's rating matrix sequentially. The result is achieved by using the PULq-ROFEWA operator. **Step 5:** *Weighting the criterion.* By using the entropy approach, the weights of the criteria are determined. The main steps are as follows:

Step 1∗*: Compute aggregated scores.* The scores corresponding to the combined aggregated matrix $\mathbb{G}$, as presented in **Step 4**, are computed using the formula presented in Equation 4. 16 .

$$\mathfrak{S}(\hbar_{\iota\epsilon}) = \left(\Delta + \left(\left(\frac{\mathfrak{t}\,(\aleph_{\theta_{\iota\epsilon}})}{\Delta}\right)^{q} + \left(\frac{\mathfrak{t}\,(\mu_{\theta_{\iota\epsilon}})}{\Delta}\right)^{q}\right)(\hat{\rho}_{\iota\epsilon}) - \left(\left(\frac{\mathfrak{t}\,(\pounds_{\Theta_{\iota\epsilon}})}{\Delta}\right)^{q} + \left(\frac{\mathfrak{t}\,(\nu_{\Theta_{\iota\epsilon}})}{\Delta}\right)^{q}\right)(\hat{\rho}_{\iota\epsilon})\right). \tag{4. 16}$$

Step 2∗*: Calculate projection values.* Equation 4. 17 is used at this step to compute the projection values $\mathcal{P}_{ij}$ for each criteria:

$$\mathfrak{P}_{\iota\epsilon} = \frac{\mathfrak{S}(\hbar_{\iota\epsilon})}{\sum_{\iota=1}^{\psi} \mathfrak{S}(\hbar_{\iota\epsilon})}. \tag{4. 17}$$

Step $3*$*: Determine entropy scores.* Entropy values $\mathcal{E}_j$ for each attribute are derived using the projection values:

$$\mathfrak{E}_\epsilon = \frac{-1}{\log(\psi)} \sum_{\iota=1}^{\psi} \mathfrak{P}_{\iota\epsilon} \log(\mathfrak{P}_{\iota\epsilon}). \tag{4. 18}$$

Step $4*$*: Evaluate the divergence scores.* Based on entropy values, the formula below derives the divergence degree $\mathfrak{d}_\epsilon$ for each attribute, including its built-in variation strength:

$$\mathfrak{d}_\epsilon = 1 - \mathfrak{E}_\epsilon. \tag{4. 19}$$

Step $5*$*: Identify relative weights.* Equation 4. 20 serves as the basis for determining the weights of all criteria:

$$\mathcal{W}_\epsilon = \frac{\mathfrak{d}_\epsilon}{\sum\limits_{\epsilon=1}^{\varsigma} \mathfrak{d}_\epsilon}, \tag{4. 20}$$

where $\sum\limits_{\epsilon=1}^{\varsigma} \mathcal{W}_\epsilon = 1$.

Step 6: *Evaluating the weighted unified matrix.* By applying Equation 4. 21 , the weighted aggregated matrix is formulated through the integration of weights $\mathcal{W}_\epsilon$ and matrix $\mathbb{G}$:

$$\mathfrak{b}_{\iota\epsilon} = \mathcal{W}_\epsilon \mathbb{G} = \left(\left[\mathfrak{t}^{-1} \left(\Delta \left(1 - \left(1 - \left(\frac{\mathfrak{t}(\aleph_{\theta_{\iota\epsilon}})}{\Delta} \right)^q \right)^{\mathcal{W}_\epsilon} \right)^{\frac{1}{q}} \right), \mathfrak{t}^{-1} \left(\Delta \left(1 - \left(1 - \left(\frac{\mathfrak{t}(\mu_{\theta_{\iota\epsilon}})}{\Delta} \right)^q \right)^{\mathcal{W}_\epsilon} \right)^{\frac{1}{q}} \right) \right] (\hat{\rho}_{\iota\epsilon}), \left[\mathfrak{t}^{-1} \left(\Delta \left(\frac{\mathfrak{t}(\pounds_{\Theta_{\iota\epsilon}})}{\Delta} \right)^{\mathcal{W}_\epsilon} \right), \mathfrak{t}^{-1} \left(\Delta \left(\frac{\mathfrak{t}(\nu_{\Theta_{\iota\epsilon}})}{\Delta} \right)^{\mathcal{W}_\epsilon} \right) \right] (\tilde{\rho}_{\iota\epsilon}) \right). \tag{4. 21}$$

Step 7: *Score evaluation and NIS determination.* The scores from the weighted matrix are computed and negative ideal solution (NIS) values are determined for each criteria reflecting the minimum desired outcomes:

$$NIS = [NIS_\epsilon]_{1 \times \varsigma}; \tag{4. 22}$$

$$NIS_\epsilon = \min_\iota \mathfrak{S}(\mathfrak{b}_{\iota\epsilon}). \tag{4. 23}$$

Step 8: *Computation of Euclidean and Hamming distances.* Using the equations below, Hamming distances $\mathcal{HD}_\iota$ and Euclidean distances $\mathcal{ED}_\iota$ between the weighted matrix and NIS are determined.

$$\mathcal{HD}_\iota = \sum_{\epsilon=1}^{\varsigma} \mathcal{HD}(\mathfrak{b}_{\iota\epsilon}, NIS_\epsilon), \tag{4. 24}$$

$$\mathcal{ED}_\iota = \sum_{\epsilon=1}^{\varsigma} \mathcal{ED}(\mathfrak{b}_{\iota\epsilon}, NIS_\epsilon). \tag{4. 25}$$

Let $\mathcal{A}_1 = \langle \{ [\aleph_{\theta^1}, \mu_{\theta^1}](\hat{\rho}^1), [\pounds_{\Theta^1}, \nu_{\Theta^1}](\tilde{\rho}^1) \} \rangle$ and and $\mathcal{A}_2 = \langle \{ [\aleph_{\theta^2}, \mu_{\theta^2}](\hat{\rho}^2), [\pounds_{\Theta^2}, \nu_{\Theta^2}](\tilde{\rho}^2) \} \rangle$ be PULq-RPFNs. Then the values of $\mathcal{HD}$ and $\mathcal{ED}$ between two PULq-RPFNs A_1 and A_2 can be computed as:

$$\mathcal{HD}(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2\Delta} \left(\left| \mathfrak{t}(\aleph_{\theta^1})(\hat{\rho}^1) - \mathfrak{t}(\aleph_{\theta^2})(\hat{\rho}^2) \right| + \left| \mathfrak{t}(\mu_{\theta^1})(\hat{\rho}^1) - \mathfrak{t}(\mu_{\theta^2})(\hat{\rho}^2) \right| + \left| \mathfrak{t}(\pounds_{\Theta^1})(\tilde{\rho}^1) - \mathfrak{t}(\pounds_{\Theta^2})(\tilde{\rho}^2) \right| + \left| \mathfrak{t}(\nu_{\Theta^1})(\tilde{\rho}^1) - \mathfrak{t}(\nu_{\Theta^2})(\tilde{\rho}^2) \right| \right), \tag{4. 26}$$

$$\mathcal{ED}(\mathcal{A}_1, \mathcal{A}_2) = \frac{\Delta}{2} \left(\left| \left(\frac{\mathfrak{t}(\aleph_{\theta^1})(\hat{\rho}^1) - \mathfrak{t}(\aleph_{\theta^2})(\hat{\rho}^2)}{\Delta} \right)^q \right| + \left| \left(\frac{\mathfrak{t}(\mu_{\theta^1})(\hat{\rho}^1) - \mathfrak{t}(\mu_{\theta^2})(\hat{\rho}^2)}{\Delta} \right)^q \right| + \left| \left(\frac{\mathfrak{t}(\pounds_{\Theta^1})(\tilde{\rho}^1) - \mathfrak{t}(\pounds_{\Theta^2})(\tilde{\rho}^2)}{\Delta} \right)^q \right| + \left| \left(\frac{\mathfrak{t}(\nu_{\Theta^1})(\tilde{\rho}^1) - \mathfrak{t}(\nu_{\Theta^2})(\tilde{\rho}^2)}{\Delta} \right)^q \right| \right). \tag{4. 27}$$

Step 9: $\mathcal{RA}$ *matrix calculation.* Construction of relative assessment $\mathcal{RA}$ matrix is carried out in this step:

$$\mathcal{RA} = [c_{\iota\tau}]_{\psi\times\psi}; \tag{4. 28}$$

$$\mathfrak{c}_{\iota\tau} = (\mathcal{ED}_\iota - \mathcal{ED}_\tau) + \big(g(\mathcal{ED}_\iota - \mathcal{ED}_\tau) \times (\mathcal{HD}_\iota - \mathcal{HD}_\tau)\big);\ \tau = 1, 2, 3, \ldots, \psi. \tag{4. 29}$$

Here the function $\mathfrak{g}$ is expressed as follows:

$$\mathfrak{g}(\eta) = \begin{cases} 1, & |\eta| \geq \varrho; \\ 0, & |\eta| < \varrho, \end{cases} \tag{4. 30}$$

where $\varrho \in [0.01, 0.05]$ as suggested by specialists. In current analysis, $\varrho = 0.02$.

Step 10: *Rank determination.* Using Equation 4. 31 , the average result is computed:

$$\mathcal{AS}_\iota = \sum_{\epsilon=1}^{\varsigma} c_{\iota\epsilon}. \tag{4. 31}$$

Finally, the alternatives are arranged based on the $\mathcal{AS}_\iota$ values. The alternative having highest $\mathcal{AS}_\iota$ value will be regarded the best one.

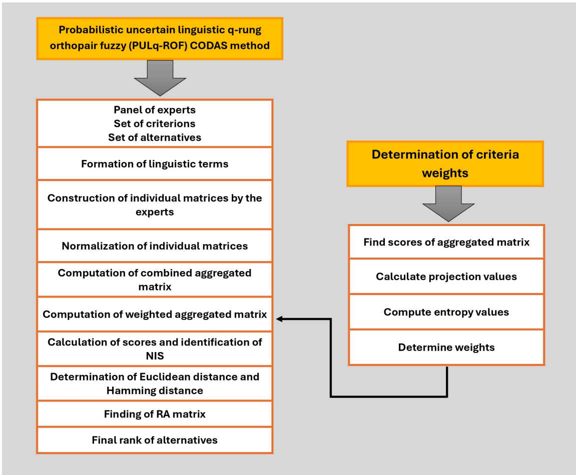

FIGURE 1. PULq-ROF-CODAS method with entropy-based criteria weight determination

5. NUMERICAL ILLUSTRATION

The endurance, ongoing operations, and economical viability of contemporary businesses are all affected directly by the crucial task of tackling cybersecurity hazards. Selecting reliable and long-lasting security measures is essential to protecting business assets and guaranteeing long-term expansion. The companies must regularly assess their outside attacker threats and their own safety precautions, create appropriate cybersecurity plans, and set up efficient criteria for assessment based on risk consequences and probability in the current continuously shifting threat environment. To build a robust and flexible cybersecurity structure, a thorough evaluation methodology is necessary that extends outside conventional cost-cutting or risk-minimizing tactics. However, the unforeseen consequences

of changing hazards, divergent expert opinions, and the challenge of establishing exact assessment requirements due to unidentified assault channels, unexpected faults, and disparate levels of experience sometimes interfere with cybersecurity decision-making.
To address these issues, this work employs an MCGDM technique that integrates the PULq-ROFS.The methodology makes it possible to combine real-world data with expert knowledge to assess cybersecurity solutions in a context of uncertainty. A case study is conducted using real organizational data collected from a multinational IT services firm that recently underwent a comprehensive cybersecurity audit. In this study, a panel of four domain experts $\mathcal{D}_\tau(\tau = 1, 2, 3, 4)$ assessed seven cybersecurity strategies $\mathcal{N}_\iota(\iota = 1, 2, 3, 4, 5, 6, 7)$, including intrusion detection systems, zero trust architecture, and endpoint protection, based on four key criteria $\mathcal{M}_\epsilon(\epsilon = 1, 2, 3, 4)$: implementation cost, threat detection capability, scalability, and compliance alignment. The use of real audit data validates the applicability of the proposed method and demonstrates how PULq-ROFS can effectively handle linguistic uncertainty and probabilistic judgments in cybersecurity decision-making. In addition to confirming the importance and resilience of the PULq-ROFS-based MCGDM model, the outcomes offer useful information that can be applied to comprehensive cybersecurity assessment.

TABLE 1. **Four essential features and their description**

Attributes	**Description**
Threat Detection Efficiency ($\mathcal{M}_1$)	The solution's effectiveness in identifying and mitigating evolving cyber threats based on real-time security data.
Implementation Cost ($\mathcal{M}_2$)	The financial investment required for deploying and sustaining the cybersecurity solution within the organization's existing infrastructure.
Scalability ($\mathcal{M}_3$)	The ability of the security system to seamlessly expand alongside organizational growth and increased digital operations.
Compliance Alignment ($\mathcal{M}_4$)	The degree to which the security strategy meets industry regulations and international cybersecurity standards relevant to the firm's operating environment.

TABLE 2. **Seven safety measures and their description**

Alternatives	**Description**
Intrusion Detection System (IDS) ($\mathcal{N}_1$)	It monitors network or system activities for malicious actions and raises alerts in real time.
Zero Trust Architecture (ZTA) ($\mathcal{N}_2$)	It enforces strict access control by verifying every user and device attempting to access the network regardless of their location.
Endpoint Protection Platform (EPP) ($\mathcal{N}_3$)	It provides centralized security for devices like laptop, mobiles and servers by detecting, blocking and remediating threats at the endpoint level.
Security Information and Event Management (SIEM) ($\mathcal{N}_4$)	It collects and analyzes security event data to detect anomalies and provide actionable insights for incident response.
Multi-Factor Authentication (MFA) ($\mathcal{N}_5$)	It strengths user verification by requiring multiple forms of authentication to access sensitive systems or data.
Cloud Access Security Broker (CASB) ($\mathcal{N}_6$)	It acts as a security control point between cloud service users and providers to enforce security policies and monitor data.
Network Firewall with Deep Packet Inspection (DPI) ($\mathcal{N}_7$)	It filters incoming and outgoing network traffic based on rules while inspecting packet content to detect sophisticated threats.

5.1. **Implementation stages.** Consider $\Delta = 5$ and $q = 3$ as given. The approach is carried out in the subsequent steps:

Step 1. The assessments provided by four experts for the alternatives based on the criteria they defined and expressed as normalized benefit type values in the form of PULq-ROFSs are presented in Tables 3, 4, 5 and 6.
Step 2. PULq-ROFEWA operator is employed to consolidate individual evaluations into an aggregated matrix. Table 7 displays the aggregated scores of the alternatives relative to each criteria.
Step 3. This stage involves calculating the weights of criteria using entropy measure .
*Step 1**. The calculation of the combined aggregated matrix scores is initiated using Equation 4. 16 along with the data outlined in the Table 8.
*Step 2**. The projection values are computed by applying Equation 4. 17 to the data provided in Table 9.
*Step 3**. The results of entropy computation through Equation 4. 18 are consolidated in Table 10.
*Step 4**. Based on Equation 4. 19 , the divergence values corresponding to each criteria are presented in Table 11.
*Step 5**. Equation 4. 20 is employed to compute the criteria weights as shown in Table 12.
Step 6. Using Equation 4. 21 , the components of the weighted aggregated matrix are outlined in Table 13.
Step 7. To determine the NIS, the scores from Table 8 are initially computed. Subsequently, the NIS values are identified using criteria 4. 22 and 4. 23 as shown in Table 14.
Step 8. The values of $\mathcal{HD}_i$ (4. 24) and $\mathcal{ED}_i$ (4. 25) are computed as given below:

$$\mathcal{ED}_1 = 0.1179,\ \mathcal{ED}_2 = 0.035,\ \mathcal{ED}_3 = 0.2317,\ \mathcal{ED}_4 = 0.1046,$$
$$\mathcal{ED}_5 = 0.095,\ \mathcal{ED}_6 = 0.2557,\ \mathcal{ED}_7 = 0.138.$$
$$\mathcal{HD}_1 = 1.2377,\ \mathcal{HD}_2 = 0.9509,\ \mathcal{HD}_3 = 1.352,\ \mathcal{HD}_4 = 1.1597,$$
$$\mathcal{HD}_5 = 0.9455,\ \mathcal{HD}_6 = 1.5747,\ \mathcal{HD}_7 = 1.3425.$$

Step 9. The values of $\mathcal{RA}$ matrix with their sum are given in Table 17.
Step 10. The best option is shown in Table 18 as $\mathcal{N}_6$.

$$\mathcal{N}_6 > \mathcal{N}_3 > \mathcal{N}_7 > \mathcal{N}_1 > \mathcal{N}_4 > \mathcal{N}_5 > \mathcal{N}_2$$

TABLE 3. **Probabilistic uncertain linguistic q-rung orthopair fuzzy decision matrix provided by $\mathcal{D}_1$.**

Alternatives	$\mathcal{M}_1$
$\mathcal{N}_1$	$\langle\{[\aleph_{-5},\mu_{-4}](0.1),[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.5)\},\{[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.2),[\pounds_{2},\nu_{3}](0.4)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-1},\mu_{0}](0.3),[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.4)\},\{[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.8)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{1},\mu_{2}](0.3),[\aleph_{2},\mu_{3}](0.6),[\aleph_{3},\mu_{4}](0.1)\},\{[\pounds_{-5},\nu_{-4}](0.2),[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.2),[\aleph_{-1},\mu_{0}](0.5)\},\{[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.4),[\pounds_{0},\nu_{1}](0.2)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-2},\mu_{-1}](0.6),[\aleph_{-1},\mu_{0}](0.1),[\aleph_{0},\mu_{1}](0.3)\},\{[\pounds_{-1},\nu_{0}](0.5),[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.1)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{0},\mu_{1}](0.2),[\aleph_{1},\mu_{2}](0.4),[\aleph_{2},\mu_{3}](0.4)\},\{[\pounds_{-3},\nu_{-2}](0.2),[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.4)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.4)\},\{[\pounds_{1},\nu_{2}](0.6),[\pounds_{2},\nu_{3}](0.2),[\pounds_{3},\nu_{4}](0.2)\}\rangle$

Alternatives	$\mathcal{M}_2$
$\mathcal{N}_1$	$\langle\{[\aleph_{-1},\mu_{0}](0.5),[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.2)\},\{[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.2),[\pounds_{-2},\nu_{-1}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.5),[\aleph_{0},\mu_{1}](0.2)\},\{[\pounds_{-1},\nu_{0}](0.4),[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.2)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-3},\mu_{-2}](0.1),[\aleph_{-2},\mu_{-1}](0.2),[\aleph_{-1},\mu_{0}](0.7)\},\{[\pounds_{-2},\nu_{-1}](0.1),[\pounds_{-1},\nu_{0}](0.1),[\pounds_{0},\nu_{1}](0.8)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.4)\},\{[\pounds_{0},\nu_{1}](0.3),[\pounds_{1},\nu_{2}](0.4),[\pounds_{2},\nu_{3}](0.3)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-5},\mu_{-4}](0.5),[\aleph_{-4},\mu_{-3}](0.2),[\aleph_{-3},\mu_{-2}](0.3)\},\{[\pounds_{1},\nu_{2}](0.1),[\pounds_{2},\nu_{3}](0.2),[\pounds_{3},\nu_{4}](0.7)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.3),[\aleph_{2},\mu_{3}](0.4)\},\{[\pounds_{-3},\nu_{-2}](0.2),[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_{0}](0.5)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{1},\mu_{2}](0.1),[\aleph_{2},\mu_{3}](0.4),[\aleph_{3},\mu_{4}](0.5)\},\{[\pounds_{-5},\nu_{-4}](0.3),[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.6)\}\rangle$

Alternatives	$\mathcal{M}_3$
$\mathcal{N}_1$	$\langle\{[\aleph_{0},\mu_{1}](0.1),[\aleph_{1},\mu_{2}](0.4),[\aleph_{2},\mu_{3}](0.5)\},\{[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.2),[\pounds_{-1},\nu_{0}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-1},\mu_{0}](0.4),[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.3)\},\{[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_{0}](0.3),[\pounds_{0},\nu_{1}](0.4)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.1),[\aleph_{-2},\mu_{-1}](0.8)\},\{[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4),[\pounds_{2},\nu_{3}](0.2)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{2},\mu_{3}](0.7),[\aleph_{3},\mu_{4}](0.2),[\aleph_{4},\mu_{5}](0.1)\},\{[\pounds_{-5},\nu_{-4}](0.8),[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.1)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-2},\mu_{-1}](0.4),[\aleph_{-1},\mu_{0}](0.4),[\aleph_{0},\mu_{1}](0.2)\},\{[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-3},\mu_{-2}](0.1),[\aleph_{-2},\mu_{-1}](0.1),[\aleph_{-1},\mu_{0}](0.8)\},\{[\pounds_{-1},\nu_{0}](0.5),[\pounds_{0},\nu_{1}](0.3),[\pounds_{1},\nu_{2}](0.2)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{1},\mu_{2}](0.2),[\aleph_{2},\mu_{3}](0.4),[\aleph_{3},\mu_{4}](0.4)\},\{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.2)\}\rangle$

Alternatives	$\mathcal{M}_4$
$\mathcal{N}_1$	$\langle\{[\aleph_{-1},\mu_{0}](0.2),[\aleph_{0},\mu_{1}](0.2),[\aleph_{1},\mu_{2}](0.6)\},\{[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.5),[\pounds_{0},\nu_{1}](0.1)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-2},\mu_{-1}](0.5),[\aleph_{-1},\mu_{0}](0.3),[\aleph_{0},\mu_{1}](0.2)\},\{[\pounds_{-1},\nu_{0}](0.3),[\pounds_{0},\nu_{1}](0.2),[\pounds_{1},\nu_{2}](0.5)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.6),[\aleph_{2},\mu_{3}](0.1)\},\{[\pounds_{-3},\nu_{-2}](0.2),[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{1},\mu_{2}](0.5),[\aleph_{2},\mu_{3}](0.2),[\aleph_{3},\mu_{4}](0.3)\},\{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.5)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{2},\mu_{3}](0.4),[\aleph_{3},\mu_{4}](0.4),[\aleph_{4},\mu_{5}](0.2)\},\{[\pounds_{-5},\nu_{-4}](0.3),[\pounds_{-4},\nu_{-3}](0.3),[\pounds_{-3},\nu_{-2}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-5},\mu_{-4}](0.1),[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.8)\},\{[\pounds_{1},\nu_{2}](0.5),[\pounds_{2},\nu_{3}](0.2),[\pounds_{3},\nu_{4}](0.3)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-3},\mu_{-2}](0.2),[\aleph_{-2},\mu_{-1}](0.4),[\aleph_{-1},\mu_{0}](0.4)\},\{[\pounds_{0},\nu_{1}](0.1),[\pounds_{1},\nu_{2}](0.1),[\pounds_{2},\nu_{3}](0.8)\}\rangle$

TABLE 4. **Probabilistic uncertain linguistic q-rung orthopair fuzzy decision matrix provided by $\mathcal{D}_2$.**

Alternatives	$\mathcal{M}_1$
$\mathcal{N}_1$	$\langle\{[\aleph_{-5},\mu_{-4}](0.1),[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.5)\},\{[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.2),[\pounds_{-1},\nu_{0}](0.4)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.4)\},\{[\pounds_{-5},\nu_{-4}](0.1),[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.8)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.6),[\aleph_{-2},\mu_{-1}](0.1)\},\{[\pounds_{0},\nu_{1}](0.2),[\pounds_{1},\nu_{2}](0.4),[\pounds_{2},\nu_{3}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.2),[\aleph_{2},\mu_{3}](0.5)\},\{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.2)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-2},\mu_{-1}](0.6),[\aleph_{-1},\mu_{0}](0.1),[\aleph_{0},\mu_{1}](0.3)\},\{[\pounds_{-1},\nu_{0}](0.5),[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.1)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-1},\mu_{0}](0.2),[\aleph_{0},\mu_{1}](0.4),[\aleph_{1},\mu_{2}](0.4)\},\{[\pounds_{-2},\nu_{-1}](0.2),[\pounds_{-1},\nu_{0}](0.4),[\pounds_{0},\nu_{1}](0.4)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.4)\},\{[\pounds_{-1},\nu_{0}](0.6),[\pounds_{0},\nu_{1}](0.2),[\pounds_{1},\nu_{2}](0.2)\}\rangle$

Alternatives	$\mathcal{M}_2$
$\mathcal{N}_1$	$\langle\{[\aleph_{-3},\mu_{-2}](0.5),[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.2)\},\{[\pounds_{-5},\nu_{-4}](0.1),[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-5},\mu_{-4}](0.3),[\aleph_{-4},\mu_{-3}](0.5),[\aleph_{-3},\mu_{-2}](0.2)\},\{[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.2)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{0},\mu_{1}](0.1),[\aleph_{1},\mu_{2}](0.2),[\aleph_{2},\mu_{3}](0.7)\},\{[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.8)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.4)\},\{[\pounds_{0},\nu_{1}](0.3),[\pounds_{1},\nu_{2}](0.4),[\pounds_{2},\nu_{3}](0.3)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-1},\mu_{0}](0.5),[\aleph_{0},\mu_{1}](0.2),[\aleph_{1},\mu_{2}](0.3)\},\{[\pounds_{-2},\nu_{-1}](0.1),[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.7)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.3),[\aleph_{0},\mu_{1}](0.4)\},\{[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.3),[\pounds_{1},\nu_{2}](0.5)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.4),[\aleph_{-2},\mu_{-1}](0.5)\},\{[\pounds_{-1},\nu_{0}](0.3),[\pounds_{0},\nu_{1}](0.1),[\pounds_{1},\nu_{2}](0.6)\}\rangle$

Alternatives	$\mathcal{M}_3$
$\mathcal{N}_1$	$\langle\{[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.4),[\aleph_{-2},\mu_{-1}](0.5)\},\{[\pounds_{0},\nu_{1}](0.1),[\pounds_{1},\nu_{2}](0.2),[\pounds_{2},\nu_{3}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{0},\mu_{1}](0.4),[\aleph_{1},\mu_{2}](0.3),[\aleph_{2},\mu_{3}](0.3)\},\{[\pounds_{-4},\nu_{-3}](0.3),[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.4)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-5},\mu_{-4}](0.1),[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.8)\},\{[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.2)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-3},\mu_{-2}](0.7),[\aleph_{-2},\mu_{-1}](0.2),[\aleph_{-1},\mu_{0}](0.1)\},\{[\pounds_{-5},\nu_{-4}](0.8),[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.1)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.4),[\aleph_{-2},\mu_{-1}](0.2)\},\{[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-1},\mu_{0}](0.1),[\aleph_{0},\mu_{1}](0.1),[\aleph_{1},\mu_{2}](0.8)\},\{[\pounds_{-2},\nu_{-1}](0.5),[\pounds_{-1},\nu_{0}](0.3),[\pounds_{0},\nu_{1}](0.2)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-2},\mu_{-1}](0.2),[\aleph_{-1},\mu_{0}](0.4),[\aleph_{0},\mu_{1}](0.4)\},\{[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4),[\pounds_{2},\nu_{3}](0.2)\}\rangle$

Alternatives	$\mathcal{M}_4$
$\mathcal{N}_1$	$\langle\{[\aleph_{0},\mu_{1}](0.2),[\aleph_{1},\mu_{2}](0.2),[\aleph_{2},\mu_{3}](0.6)\},\{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.5),[\pounds_{-2},\nu_{-1}](0.1)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-1},\mu_{0}](0.5),[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.2)\},\{[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.5)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.6),[\aleph_{0},\mu_{1}](0.1)\},\{[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-5},\mu_{-4}](0.5),[\aleph_{-4},\mu_{-3}](0.2),[\aleph_{-3},\mu_{-2}](0.3)\},\{[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.1),[\pounds_{-1},\nu_{0}](0.5)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.4),[\aleph_{-2},\mu_{-1}](0.2)\},\{[\pounds_{0},\nu_{1}](0.3),[\pounds_{1},\nu_{2}](0.3),[\pounds_{2},\nu_{3}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-3},\mu_{-2}](0.1),[\aleph_{-2},\mu_{-1}](0.1),[\aleph_{-1},\mu_{0}](0.8)\},\{[\pounds_{-5},\nu_{-4}](0.5),[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.3)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-1},\mu_{0}](0.2),[\aleph_{0},\mu_{1}](0.4),[\aleph_{1},\mu_{2}](0.4)\},\{[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.8)\}\rangle$

TABLE 5. **Probabilistic uncertain linguistic q-rung orthopair fuzzy decision matrix provided by $\mathcal{D}_3$.**

Alternatives	$\mathcal{M}_1$
$\mathcal{N}_1$	$\langle\{[\aleph_{-1},\mu_0](0.1),[\aleph_0,\mu_1](0.4),[\aleph_1,\mu_2](0.5)\}, \{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.2),[\pounds_{-2},\nu_{-1}](0.4)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_0,\mu_1](0.3),[\aleph_1,\mu_2](0.3),[\aleph_2,\mu_3](0.4)\}, \{[\pounds_{-1},\nu_0](0.1),[\pounds_0,\nu_1](0.1),[\pounds_1,\nu_2](0.8)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.6),[\aleph_{-1},\mu_0](0.1)\}, \{[\pounds_{-2},\nu_{-1}](0.2),[\pounds_{-1},\nu_0](0.4),[\pounds_0,\nu_1](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_0](0.2),[\aleph_0,\mu_1](0.5)\}, \{[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_0](0.2)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_1,\mu_2](0.6),[\aleph_2,\mu_3](0.1),[\aleph_3,\mu_4](0.3)\}, \{[\pounds_{-5},\nu_{-4}](0.5),[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.1)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-5},\mu_{-4}](0.2),[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.4)\}, \{[\pounds_1,\nu_2](0.2),[\pounds_2,\nu_3](0.4),[\pounds_3,\nu_4](0.4)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.4)\}, \{[\pounds_0,\nu_1](0.6),[\pounds_1,\nu_2](0.2),[\pounds_2,\nu_3](0.2)\}\rangle$

Alternatives	$\mathcal{M}_2$
$\mathcal{N}_1$	$\langle\{[\aleph_0,\mu_1](0.5),[\aleph_1,\mu_2](0.3),[\aleph_2,\mu_3](0.2)\}, \{[\pounds_{-1},\nu_0](0.1),[\pounds_0,\nu_1](0.2),[\pounds_1,\nu_2](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-1},\mu_0](0.3),[\aleph_0,\mu_1](0.5),[\aleph_1,\mu_2](0.2)\}, \{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.2)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-2},\mu_{-1}](0.1),[\aleph_{-1},\mu_0](0.2),[\aleph_0,\mu_1](0.7)\}, \{[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.1),[\pounds_{-1},\nu_0](0.8)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_0](0.4)\}, \{[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_0](0.4),[\pounds_0,\nu_1](0.3)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_1,\mu_2](0.5),[\aleph_2,\mu_3](0.2),[\aleph_3,\mu_4](0.3)\}, \{[\pounds_{-5},\nu_{-4}](0.1),[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.7)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.4)\}, \{[\pounds_0,\nu_1](0.2),[\pounds_1,\nu_2](0.3),[\pounds_2,\nu_3](0.5)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-5},\mu_{-4}](0.1),[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.5)\}, \{[\pounds_1,\nu_2](0.3),[\pounds_2,\nu_3](0.1),[\pounds_3,\nu_4](0.6)\}\rangle$

Alternatives	$\mathcal{M}_3$
$\mathcal{N}_1$	$\langle\{[\aleph_{-3},\mu_{-2}](0.1),[\aleph_{-2},\mu_{-1}](0.4),[\aleph_{-1},\mu_0](0.5)\}, \{[\pounds_{-2},\nu_{-1}](0.1),[\pounds_{-1},\nu_0](0.2),[\pounds_0,\nu_1](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-2},\mu_{-1}](0.4),[\aleph_{-1},\mu_0](0.3),[\aleph_0,\mu_1](0.3)\}, \{[\pounds_{-3},\nu_{-2}](0.3),[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_0](0.4)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-1},\mu_0](0.1),[\aleph_0,\mu_1](0.1),[\aleph_1,\mu_2](0.8)\}, \{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.2)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_0,\mu_1](0.7),[\aleph_1,\mu_2](0.2),[\aleph_2,\mu_3](0.1)\}, \{[\pounds_{-1},\nu_0](0.8),[\pounds_0,\nu_1](0.1),[\pounds_1,\nu_2](0.1)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.4),[\aleph_{-2},\mu_{-1}](0.2)\}, \{[\pounds_0,\nu_1](0.2),[\pounds_1,\nu_2](0.4),[\pounds_2,\nu_3](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-5},\mu_{-4}](0.1),[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.8)\}, \{[\pounds_1,\nu_2](0.5),[\pounds_2,\nu_3](0.3),[\pounds_3,\nu_4](0.2)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_1,\mu_2](0.2),[\aleph_2,\mu_3](0.4),[\aleph_3,\mu_4](0.4)\}, \{[\pounds_{-5},\nu_{-4}](0.4),[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.2)\}\rangle$

Alternatives	$\mathcal{M}_4$
$\mathcal{N}_1$	$\langle\{[\aleph_{-2},\mu_{-1}](0.2),[\aleph_{-1},\mu_0](0.2),[\aleph_0,\mu_1](0.6)\}, \{[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.5),[\pounds_{-1},\nu_0](0.1)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_1,\mu_2](0.5),[\aleph_2,\mu_3](0.3),[\aleph_3,\mu_4](0.2)\}, \{[\pounds_{-5},\nu_{-4}](0.3),[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.5)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-5},\mu_{-4}](0.3),[\aleph_{-4},\mu_{-3}](0.6),[\aleph_{-3},\mu_{-2}](0.1)\}, \{[\pounds_1,\nu_2](0.2),[\pounds_2,\nu_3](0.4),[\pounds_3,\nu_4](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-1},\mu_0](0.5),[\aleph_0,\mu_1](0.2),[\aleph_1,\mu_2](0.3)\}, \{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.5)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_0,\mu_1](0.4),[\aleph_1,\mu_2](0.4),[\aleph_2,\mu_3](0.2)\}, \{[\pounds_{-1},\nu_0](0.3),[\pounds_0,\nu_1](0.3),[\pounds_1,\nu_2](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-3},\mu_{-2}](0.1),[\aleph_{-2},\mu_{-1}](0.1),[\aleph_{-1},\mu_0](0.8)\}, \{[\pounds_{-2},\nu_{-1}](0.5),[\pounds_{-1},\nu_0](0.2),[\pounds_0,\nu_1](0.3)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-4},\mu_{-3}](0.2),[\aleph_{-3},\mu_{-2}](0.4),[\aleph_{-2},\mu_{-1}](0.4)\}, \{[\pounds_0,\nu_1](0.1),[\pounds_1,\nu_2](0.1),[\pounds_2,\nu_3](0.8)\}\rangle$

TABLE 6. **Probabilistic uncertain linguistic q-rung orthopair fuzzy decision matrix provided by $\mathcal{D}_4$.**

Alternatives	$\mathcal{M}_1$
$\mathcal{N}_1$	$\langle\{[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.4),[\aleph_{-2},\mu_{-1}](0.5)\},\{[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.2),[\pounds_{2},\nu_{3}](0.4)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-5},\mu_{-4}](0.3),[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.4)\},\{[\pounds_{1},\nu_{2}](0.1),[\pounds_{2},\nu_{3}](0.1),[\pounds_{3},\nu_{4}](0.8)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{1},\mu_{2}](0.3),[\aleph_{2},\mu_{3}](0.6),[\aleph_{3},\mu_{4}](0.1)\},\{[\pounds_{-5},\nu_{-4}](0.2),[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.2),[\aleph_{0},\mu_{1}](0.5)\},\{[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.2)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-3},\mu_{-2}](0.6),[\aleph_{-2},\mu_{-1}](0.1),[\aleph_{-1},\mu_{0}](0.3)\},\{[\pounds_{-2},\nu_{-1}](0.5),[\pounds_{-1},\nu_{0}](0.4),[\pounds_{0},\nu_{1}](0.1)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{0},\mu_{1}](0.2),[\aleph_{1},\mu_{2}](0.4),[\aleph_{2},\mu_{3}](0.4)\},\{[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-1},\mu_{0}](0.3),[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.4)\},\{[\pounds_{-4},\nu_{-3}](0.6),[\pounds_{-3},\nu_{-2}](0.2),[\pounds_{-2},\nu_{-1}](0.2)\}\rangle$

Alternatives	$\mathcal{M}_2$
$\mathcal{N}_1$	$\langle\{[\aleph_{-5},\mu_{-4}](0.5),[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.2)\},\{[\pounds_{1},\nu_{2}](0.1),[\pounds_{2},\nu_{3}](0.2),[\pounds_{3},\nu_{4}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.5),[\aleph_{-2},\mu_{-1}](0.2)\},\{[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4),[\pounds_{2},\nu_{3}](0.2)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{1},\mu_{2}](0.1),[\aleph_{2},\mu_{3}](0.2),[\aleph_{3},\mu_{4}](0.7)\},\{[\pounds_{-5},\nu_{-4}](0.1),[\pounds_{-4},\nu_{-3}](0.1),[\pounds_{-3},\nu_{-2}](0.8)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-3},\mu_{-2}](0.3),[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.4)\},\{[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_{0}](0.4),[\pounds_{0},\nu_{1}](0.3)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-2},\mu_{-1}](0.5),[\aleph_{-1},\mu_{0}](0.2),[\aleph_{0},\mu_{1}](0.3)\},\{[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.2),[\pounds_{-1},\nu_{0}](0.7)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-1},\mu_{0}](0.3),[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.4)\},\{[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.3),[\pounds_{-2},\nu_{-1}](0.5)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{0},\mu_{1}](0.1),[\aleph_{1},\mu_{2}](0.4),[\aleph_{2},\mu_{3}](0.5)\},\{[\pounds_{-1},\nu_{0}](0.3),[\pounds_{0},\nu_{1}](0.1),[\pounds_{1},\nu_{2}](0.6)\}\rangle$

Alternatives	$\mathcal{M}_3$
$\mathcal{N}_1$	$\langle\{[\aleph_{1},\mu_{2}](0.1),[\aleph_{2},\mu_{3}](0.4),[\aleph_{3},\mu_{4}](0.5)\},\{[\pounds_{-5},\nu_{-4}](0.1),[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-5},\mu_{-4}](0.4),[\aleph_{-4},\mu_{-3}](0.3),[\aleph_{-3},\mu_{-2}](0.3)\},\{[\pounds_{1},\nu_{2}](0.3),[\pounds_{2},\nu_{3}](0.4),[\pounds_{3},\nu_{4}](0.4)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{-4},\mu_{-3}](0.1),[\aleph_{-3},\mu_{-2}](0.1),[\aleph_{-2},\mu_{-1}](0.8)\},\{[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4),[\pounds_{2},\nu_{3}](0.2)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{0},\mu_{1}](0.7),[\aleph_{1},\mu_{2}](0.2),[\aleph_{2},\mu_{3}](0.1)\},\{[\pounds_{-1},\nu_{0}](0.8),[\pounds_{0},\nu_{1}](0.1),[\pounds_{1},\nu_{2}](0.1)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-1},\mu_{0}](0.4),[\aleph_{0},\mu_{1}](0.4),[\aleph_{1},\mu_{2}](0.2)\},\{[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.4),[\pounds_{-2},\nu_{-1}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{-2},\mu_{-1}](0.1),[\aleph_{-1},\mu_{0}](0.1),[\aleph_{0},\mu_{1}](0.8)\},\{[\pounds_{-3},\nu_{-2}](0.5),[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_{0}](0.2)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-3},\mu_{-2}](0.2),[\aleph_{-2},\mu_{-1}](0.4),[\aleph_{-1},\mu_{0}](0.4)\},\{[\pounds_{-2},\nu_{-1}](0.4),[\pounds_{-1},\nu_{0}](0.4),[\pounds_{0},\nu_{1}](0.2)\}\rangle$

Alternatives	$\mathcal{M}_4$
$\mathcal{N}_1$	$\langle\{[\aleph_{-4},\mu_{-3}](0.2),[\aleph_{-3},\mu_{-2}](0.2),[\aleph_{-2},\mu_{-1}](0.6)\},\{[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.5),[\pounds_{2},\nu_{3}](0.1)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{-3},\mu_{-2}](0.5),[\aleph_{-2},\mu_{-1}](0.3),[\aleph_{-1},\mu_{0}](0.2)\},\{[\pounds_{-2},\nu_{-1}](0.3),[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.5)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{0},\mu_{1}](0.3),[\aleph_{1},\mu_{2}](0.6),[\aleph_{2},\mu_{3}](0.1)\},\{[\pounds_{-1},\nu_{0}](0.2),[\pounds_{0},\nu_{1}](0.4),[\pounds_{1},\nu_{2}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{-1},\mu_{0}](0.5),[\aleph_{0},\mu_{1}](0.2),[\aleph_{1},\mu_{2}](0.3)\},\{[\pounds_{-4},\nu_{-3}](0.4),[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.5)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{-5},\mu_{-4}](0.4),[\aleph_{-4},\mu_{-3}](0.4),[\aleph_{-3},\mu_{-2}](0.2)\},\{[\pounds_{1},\nu_{2}](0.3),[\pounds_{2},\nu_{3}](0.4),[\pounds_{3},\nu_{4}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{1},\mu_{2}](0.1),[\aleph_{2},\mu_{3}](0.1),[\aleph_{3},\mu_{4}](0.8)\},\{[\pounds_{-5},\nu_{-4}](0.5),[\pounds_{-4},\nu_{-3}](0.2),[\pounds_{-3},\nu_{-2}](0.3)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{-2},\mu_{-1}](0.2),[\aleph_{-1},\mu_{0}](0.4),[\aleph_{0},\mu_{1}](0.4)\},\{[\pounds_{-3},\nu_{-2}](0.1),[\pounds_{-2},\nu_{-1}](0.1),[\pounds_{-1},\nu_{0}](0.8)\}\rangle$

TABLE 7. **Combined aggregated matrix in the form of PULq-ROFNs**

Alternatives	$\mathcal{M}_1$
$\mathcal{N}_1$	$\langle\{[\aleph_{18.20},\mu_{23.15}](0.1),[\aleph_{23.15},\mu_{26.70}](0.4),[\aleph_{26.70},\mu_{29.50}](0.5)\}$, $\{[\pounds_{-4.99},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.9976}](0.2),[\pounds_{-4.9976},\nu_{-4.9949}](0.4)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{25.05},\mu_{28.18}](0.3),[\aleph_{28.18},\mu_{30.71}](0.3),[\aleph_{30.71},\mu_{32.81}](0.4)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9988}](0.1),[\pounds_{-4.9988},\nu_{-4.997}](0.8)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{28.17},\mu_{30.71}](0.3),[\aleph_{30.71},\mu_{32.81}](0.6),[\aleph_{32.81},\mu_{34.59}](0.1)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.2),[\pounds_{-4.999},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.9983}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{26.70},\mu_{29.49}](0.3),[\aleph_{29.49},\mu_{31.79}](0.2),[\aleph_{31.79},\mu_{33.72}](0.5)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.9990}](0.4),[\pounds_{-4.9990},\nu_{-4.9976}](0.2)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{27.46},\mu_{30.12}](0.6),[\aleph_{30.12},\mu_{32.31}](0.1),[\aleph_{32.31},\mu_{34.16}](0.3)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.5),[\pounds_{-4.999},\nu_{-4.9986}](0.4),[\pounds_{-4.9986},\nu_{-4.9965}](0.1)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{27.46},\mu_{30.12}](0.2),[\aleph_{30.12},\mu_{32.32}](0.4),[\aleph_{32.32},\mu_{34.17}](0.4)\}$, $\{[\pounds_{-4.999},\nu_{-4.9983}](0.2),[\pounds_{-4.9983},\nu_{-4.9962}](0.4),[\pounds_{-4.9962},\nu_{-4.9924}](0.4)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{20.92},\mu_{25.04}](0.3),[\aleph_{25.04},\mu_{28.17}](0.3),[\aleph_{28.17},\mu_{30.70}](0.4)\}$, $\{[\pounds_{-4.999},\nu_{-4.9983}](0.6),[\pounds_{-4.9983},\nu_{-4.9959}](0.2),[\pounds_{-4.9959},\nu_{-4.9919}](0.2)\}\rangle$
Alternatives	$\mathcal{M}_2$
$\mathcal{N}_1$	$\langle\{[\aleph_{25.05},\mu_{28.18}](0.5),[\aleph_{28.18},\mu_{30.71}](0.3),[\aleph_{30.71},\mu_{32.81}](0.2)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9988}](0.2),[\pounds_{-4.9988},\nu_{-4.9969}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{22.08},\mu_{25.90}](0.3),[\aleph_{25.90},\mu_{28.85}](0.5),[\aleph_{28.85},\mu_{31.26}](0.2)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.9979}](0.4),[\pounds_{-4.9979},\nu_{-4.9955}](0.2)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{28.85},\mu_{31.26}](0.1),[\aleph_{31.26},\mu_{33.27}](0.2),[\aleph_{33.27},\mu_{34.98}](0.7)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9985}](0.8)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{19.63},\mu_{24.12}](0.3),[\aleph_{24.12},\mu_{27.45}](0.3),[\aleph_{27.45},\mu_{30.11}](0.4)\}$, $\{[\pounds_{-4.999},\nu_{-4.9977}](0.3),[\pounds_{-4.9977},\nu_{-4.9951}](0.4),[\pounds_{-4.9951},\nu_{-4.9907}](0.3)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{26.71},\mu_{29.51}](0.5),[\aleph_{29.51},\mu_{31.81}](0.2),[\aleph_{31.81},\mu_{33.73}](0.3)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9987}](0.2),[\pounds_{-4.9987},\nu_{-4.9968}](0.7)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{26.70},\mu_{29.50}](0.3),[\aleph_{29.50},\mu_{31.80}](0.3),[\aleph_{31.80},\mu_{33.72}](0.4)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.2),[\pounds_{-4.999},\nu_{-4.9979}](0.3),[\pounds_{-4.9979},\nu_{-4.9955}](0.5)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{25.91},\mu_{28.87}](0.1),[\aleph_{28.87},\mu_{31.28}](0.4),[\aleph_{31.28},\mu_{33.29}](0.5)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.3),[\pounds_{-4.999},\nu_{-4.9977}](0.1),[\pounds_{-4.9977},\nu_{-4.9947}](0.6)\}\rangle$
Alternatives	$\mathcal{M}_3$
$\mathcal{N}_1$	$\langle\{[\aleph_{27.47},\mu_{30.12}](0.1),[\aleph_{30.12},\mu_{32.32}](0.4),[\aleph_{32.32},\mu_{34.17}](0.5)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9988}](0.2),[\pounds_{-4.9988},\nu_{-4.9971}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{25.90},\mu_{28.86}](0.4),[\aleph_{28.86},\mu_{31.26}](0.3),[\aleph_{31.26},\mu_{33.28}](0.3)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.3),[\pounds_{-4.999},\nu_{-4.9980}](0.3),[\pounds_{-4.9980},\nu_{-4.9956}](0.4)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{19.63},\mu_{24.13}](0.1),[\aleph_{24.13},\mu_{27.46}](0.1),[\aleph_{27.46},\mu_{30.12}](0.8)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.9976}](0.4),[\pounds_{-4.9976},\nu_{-4.9949}](0.2)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{30.70},\mu_{32.80}](0.7),[\aleph_{32.80},\mu_{34.58}](0.2),[\aleph_{34.58},\mu_{36.09}](0.1)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.8),[\pounds_{-4.999},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9982}](0.1)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{23.14},\mu_{26.70}](0.4),[\aleph_{26.70},\mu_{29.50}](0.4),[\aleph_{29.50},\mu_{31.79}](0.2)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.2),[\pounds_{-4.999},\nu_{-4.9984}](0.4),[\pounds_{-4.9984},\nu_{-4.9964}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{23.14},\mu_{26.70}](0.1),[\aleph_{26.70},\mu_{29.50}](0.1),[\aleph_{29.50},\mu_{31.79}](0.8)\}$, $\{[\pounds_{-4.999},\nu_{-4.9983}](0.5),[\pounds_{-4.9983},\nu_{-4.9962}](0.3),[\pounds_{-4.9962},\nu_{-4.9924}](0.2)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{29.50},\mu_{31.80}](0.2),[\aleph_{31.80},\mu_{33.73}](0.4),[\aleph_{33.73},\mu_{35.37}](0.4)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.9976}](0.2)\}\rangle$
Alternatives	$\mathcal{M}_4$
$\mathcal{N}_1$	$\langle\{[\aleph_{26.70},\mu_{29.50}](0.2),[\aleph_{29.50},\mu_{31.80}](0.2),[\aleph_{31.80},\mu_{33.72}](0.6)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.9983}](0.5),[\pounds_{-4.9983},\nu_{-4.9961}](0.1)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{28.17},\mu_{30.70}](0.5),[\aleph_{30.70},\mu_{32.80}](0.3),[\aleph_{32.80},\mu_{34.58}](0.2)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.3),[\pounds_{-4.999},\nu_{-4.9987}](0.2),[\pounds_{-4.9987},\nu_{-4.9969}](0.5)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{26.71},\mu_{29.51}](0.3),[\aleph_{29.51},\mu_{31.80}](0.6),[\aleph_{31.80},\mu_{33.73}](0.1)\}$, $\{[\pounds_{-4.999},\nu_{-4.9978}](0.2),[\pounds_{-4.9978},\nu_{-4.9953}](0.4),[\pounds_{-4.9953},\nu_{-4.9911}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{27.47},\mu_{30.12}](0.5),[\aleph_{30.12},\mu_{32.32}](0.2),[\aleph_{32.32},\mu_{34.17}](0.3)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.4),[\pounds_{-4.999},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9987}](0.5)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{26.73},\mu_{29.53}](0.4),[\aleph_{29.53},\mu_{31.82}](0.4),[\aleph_{31.82},\mu_{33.75}](0.2)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.3),[\pounds_{-4.999},\nu_{-4.9973}](0.3),[\pounds_{-4.9973},\nu_{-4.9939}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{24.14},\mu_{27.47}](0.1),[\aleph_{27.47},\mu_{30.13}](0.1),[\aleph_{30.13},\mu_{32.32}](0.8)\}$, $\{[\pounds_{-5},\nu_{-4.999}](0.5),[\pounds_{-4.999},\nu_{-4.999}](0.2),[\pounds_{-4.999},\nu_{-4.9981}](0.3)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{24.12},\mu_{27.45}](0.2),[\aleph_{27.45},\mu_{30.11}](0.4),[\aleph_{30.11},\mu_{32.30}](0.4)\}$, $\{[\pounds_{-4.999},\nu_{-4.999}](0.1),[\pounds_{-4.999},\nu_{-4.9976}](0.1),[\pounds_{-4.9976},\nu_{-4.9948}](0.8)\}\rangle$

TABLE 8. **PULq-ROF score of aggregated matrix**

Alternatives	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$
$\mathcal{N}_1$	5.4928	5.6110	5.8153	5.780
$\mathcal{N}_2$	5.668	5.5736	5.6753	5.7524
$\mathcal{N}_3$	5.7678	5.9069	5.5750	5.6980
$\mathcal{N}_4$	5.7516	5.4842	5.8480	5.7317
$\mathcal{N}_5$	5.7171	5.6966	5.5520	5.6984
$\mathcal{N}_6$	5.7876	5.7384	5.6831	5.7194
$\mathcal{N}_7$	5.5219	5.7474	5.8872	5.6466

TABLE 9. **PULq-ROF projection values**

Alternatives	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$
$\mathcal{N}_1$	0.13833	0.14113	0.14525	0.14440
$\mathcal{N}_2$	0.14275	0.14019	0.14176	0.14371
$\mathcal{N}_3$	0.14526	0.14857	0.13925	0.14236
$\mathcal{N}_4$	0.14485	0.13794	0.14607	0.14320
$\mathcal{N}_5$	0.14398	0.14328	0.13868	0.14236
$\mathcal{N}_6$	0.14576	0.14433	0.14195	0.14289
$\mathcal{N}_7$	0.13907	0.14456	0.14705	0.14107

TABLE 10. **PULq-ROF entropy values**

Alternatives	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$
$\mathfrak{E}$	0.999902	0.999870	0.999885	0.999983

TABLE 11. **PULq-ROF divergence values**

Alternatives	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$
$\mathfrak{d}$	0.000098	0.00013	0.000115	0.000017

TABLE 12. **PULq-ROF criteria weights**

Alternatives	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$
$\mathcal{W}$	0.2722	0.3139	0.3194	0.0472

TABLE 13. **Combined weighted aggregated matrix in the form of PULq-ROFNs**

Alternatives	$\mathcal{M}_1$
$\mathcal{N}_1$	$\langle\{[\aleph_{10.2259},\mu_{13.674}](0.1),[\aleph_{13.674},\mu_{16.267}](0.4),[\aleph_{16.267},\mu_{18.418}](0.5)\},$ $\{[\pounds_{-2.37},\nu_{-2.37}](0.4),[\pounds_{-2.37},\nu_{-1.6660}](0.2),[\pounds_{-1.6660},\nu_{-0.9250}](0.4)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{15.046},\mu_{17.39}](0.3),[\aleph_{17.39},\mu_{19.386}](0.3),[\aleph_{19.386},\mu_{21.139}](0.4)\},$ $\{[\pounds_{-5},\nu_{-2.37}](0.1),[\pounds_{-2.37},\nu_{-2.236}](0.1),[\pounds_{-2.236},\nu_{-1.4540}](0.8)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{17.382},\mu_{19.386}](0.3),[\aleph_{19.386},\mu_{21.139}](0.6),[\aleph_{21.139},\mu_{22.716}](0.1)\},$ $\{[\pounds_{-5},\nu_{-2.37}](0.2),[\pounds_{-2.37},\nu_{-2.37}](0.4),[\pounds_{-2.37},\nu_{-1.9620}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{16.267},\mu_{18.41}](0.3),[\aleph_{18.41},\mu_{20.275}](0.2),[\aleph_{20.275},\mu_{21.933}](0.5)\},$ $\{[\pounds_{-2.37},\nu_{-2.37}](0.4),[\pounds_{-2.37},\nu_{-2.37}](0.4),[\pounds_{-2.37},\nu_{-1.663}](0.2)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{16.84},\mu_{18.91}](0.6),[\aleph_{18.91},\mu_{20.712}](0.1),[\aleph_{20.712},\mu_{22.326}](0.3)\},$ $\{[\pounds_{-5},\nu_{-2.37}](0.5),[\pounds_{-2.37},\nu_{-2.1181}](0.4),[\pounds_{-2.1181},\nu_{-1.302}](0.1)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{16.84},\mu_{18.91}](0.2),[\aleph_{18.91},\mu_{20.721}](0.4),[\aleph_{20.721},\mu_{22.335}](0.4)\},$ $\{[\pounds_{-2.37},\nu_{-1.962}](0.2),[\pounds_{-1.962},\nu_{-1.218}](0.4),[\pounds_{-1.218},\nu_{-0.433}](0.4)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{12.102},\mu_{15.039}](0.3),[\aleph_{15.039},\mu_{17.382}](0.3),[\aleph_{17.382},\mu_{19.378}](0.4)\},$ $\{[\pounds_{-2.37},\nu_{-1.962}](0.6),[\pounds_{-1.962},\nu_{-1.139}](0.2),[\pounds_{-1.139},\nu_{-0.368}](0.2)\}\rangle$
Alternatives	$\mathcal{M}_2$
$\mathcal{N}_1$	$\langle\{[\aleph_{15.986},\mu_{18.424}](0.5),[\aleph_{18.424},\mu_{20.494}](0.3),[\aleph_{20.494},\mu_{22.306}](0.2)\},$ $\{[\pounds_{-5},\nu_{-3.325}](0.1),[\pounds_{-3.325},\nu_{-3.227}](0.2),[\pounds_{-3.227},\nu_{-2.611}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{13.765},\mu_{16.637}](0.3),[\aleph_{16.637},\mu_{18.962}](0.5),[\aleph_{18.962},\mu_{20.959}](0.2)\},$ $\{[\pounds_{-3.325},\nu_{-3.325}](0.4),[\pounds_{-3.325},\nu_{-2.886}](0.4),[\pounds_{-2.886},\nu_{-2.315}](0.2)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{18.962},\mu_{20.959}](0.1),[\aleph_{20.959},\mu_{22.717}](0.2),[\aleph_{22.717},\mu_{24.301}](0.7)\},$ $\{[\pounds_{-5},\nu_{-3.325}](0.1),[\pounds_{-3.325},\nu_{-3.325}](0.1),[\pounds_{-3.325},\nu_{-3.098}](0.8)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{11.981},\mu_{15.282}](0.3),[\aleph_{15.282},\mu_{17.845}](0.3),[\aleph_{17.845},\mu_{19.993}](0.4)\},$ $\{[\pounds_{-3.325},\nu_{-2.825}](0.3),[\pounds_{-2.825},\nu_{-2.242}](0.4),[\pounds_{-2.242},\nu_{-1.628}](0.3)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{17.265},\mu_{19.499}](0.5),[\aleph_{19.499},\mu_{21.431}](0.2),[\aleph_{21.431},\mu_{23.134}](0.3)\},$ $\{[\pounds_{-5},\nu_{-3.325}](0.1),[\pounds_{-3.325},\nu_{-3.181}](0.2),[\pounds_{-3.181},\nu_{-2.587}](0.7)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{17.257},\mu_{19.491}](0.3),[\aleph_{19.491},\mu_{21.422}](0.3),[\aleph_{21.422},\mu_{23.125}](0.4)\},$ $\{[\pounds_{-3.325},\nu_{-3.325}](0.2),[\pounds_{-3.325},\nu_{-2.886}](0.3),[\pounds_{-2.886},\nu_{-2.315}](0.5)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{16.645},\mu_{18.978}](0.1),[\aleph_{18.978},\mu_{20.976}](0.4),[\aleph_{20.976},\mu_{22.735}](0.5)\},$ $\{[\pounds_{-5},\nu_{-3.325}](0.3),[\pounds_{-3.325},\nu_{-2.825}](0.1),[\pounds_{-2.825},\nu_{-2.173}](0.6)\}\rangle$
Alternatives	$\mathcal{M}_3$
$\mathcal{N}_1$	$\langle\{[\aleph_{17.986},\mu_{20.137}](0.1),[\aleph_{20.137},\mu_{22.017}](0.4),[\aleph_{22.017},\mu_{23.688}](0.5)\},$ $\{[\pounds_{-5},\nu_{-3.422}](0.1),[\pounds_{-3.422},\nu_{-3.328}](0.2),[\pounds_{-3.328},\nu_{-2.783}](0.7)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{16.757},\mu_{19.101}](0.4),[\aleph_{19.101},\mu_{21.099}](0.3),[\aleph_{21.099},\mu_{22.872}](0.3)\},$ $\{[\pounds_{-3.422},\nu_{-3.422}](0.3),[\pounds_{-3.422},\nu_{-3.031}](0.3),[\pounds_{-3.031},\nu_{-2.467}](0.4)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{12.078},\mu_{15.404}](0.1),[\aleph_{15.404},\mu_{17.979}](0.1),[\aleph_{17.979},\mu_{20.137}](0.8)\},$ $\{[\pounds_{-3.422},\nu_{-3.422}](0.4),[\pounds_{-3.422},\nu_{-2.913}](0.4),[\pounds_{-2.913},\nu_{-2.345}](0.2)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{20.623},\mu_{22.442}](0.7),[\aleph_{22.442},\mu_{24.072}](0.2),[\aleph_{24.072},\mu_{25.54}](0.1)\},$ $\{[\pounds_{-5},\nu_{-3.422}](0.8),[\pounds_{-3.422},\nu_{-3.422}](0.1),[\pounds_{-3.422},\nu_{-3.096}](0.1)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{14.658},\mu_{17.38}](0.4),[\aleph_{17.38},\mu_{19.624}](0.4),[\aleph_{19.624},\mu_{21.555}](0.2)\},$ $\{[\pounds_{-3.422},\nu_{-3.422}](0.2),[\pounds_{-3.422},\nu_{-3.166}](0.4),[\pounds_{-3.166},\nu_{-2.624}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{14.659},\mu_{17.38}](0.1),[\aleph_{17.38},\mu_{19.624}](0.1),[\aleph_{19.624},\mu_{21.555}](0.8)\},$ $\{[\pounds_{-3.422},\nu_{-3.131}](0.5),[\pounds_{-3.131},\nu_{-2.583}](0.3),[\pounds_{-2.583},\nu_{-1.984}](0.2)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{19.624},\mu_{21.564}](0.2),[\aleph_{21.564},\mu_{23.282}](0.4),[\aleph_{23.282},\mu_{24.829}](0.4)\},$ $\{[\pounds_{-5},\nu_{-3.422}](0.4),[\pounds_{-3.422},\nu_{-3.422}](0.4),[\pounds_{-3.422},\nu_{-2.913}](0.2)\}\rangle$
Alternatives	$\mathcal{M}_4$
$\mathcal{N}_1$	$\langle\{[\aleph_{6.99},\mu_{8.253}](0.2),[\aleph_{8.253},\mu_{9.367}](0.2),[\aleph_{9.367},\mu_{10.369}](0.6)\},$ $\{[\pounds_{25.004},\nu_{25.004}](0.4),[\pounds_{25.004},\nu_{25.765}](0.5),[\pounds_{25.765},\nu_{26.995}](0.1)\}\rangle$
$\mathcal{N}_2$	$\langle\{[\aleph_{7.641},\mu_{8.824}](0.5),[\aleph_{8.824},\mu_{9.879}](0.3),[\aleph_{9.879},\mu_{10.845}](0.2)\},$ $\{[\pounds_{-5},\nu_{25.004}](0.3),[\pounds_{25.004},\nu_{25.378}](0.2),[\pounds_{25.378},\nu_{26.65}](0.5)\}\rangle$
$\mathcal{N}_3$	$\langle\{[\aleph_{6.994},\mu_{8.258}](0.3),[\aleph_{8.258},\mu_{9.367}](0.6),[\aleph_{9.367},\mu_{10.374}](0.1)\},$ $\{[\pounds_{25.004},\nu_{26.142}](0.2),[\pounds_{26.142},\nu_{27.278}](0.4),[\pounds_{27.278},\nu_{28.265}](0.4)\}\rangle$
$\mathcal{N}_4$	$\langle\{[\aleph_{7.329},\mu_{8.545}](0.5),[\aleph_{8.545},\mu_{9.631}](0.2),[\aleph_{9.631},\mu_{10.616}](0.3)\},$ $\{[\pounds_{25.004},\nu_{25.004}](0.4),[\pounds_{25.004},\nu_{25.004}](0.1),[\pounds_{25.004},\nu_{25.378}](0.5)\}\rangle$
$\mathcal{N}_5$	$\langle\{[\aleph_{7.003},\mu_{8.267}](0.4),[\aleph_{8.267},\mu_{9.377}](0.4),[\aleph_{9.377},\mu_{10.385}](0.2)\},$ $\{[\pounds_{-5},\nu_{25.004}](0.3),[\pounds_{25.004},\nu_{26.444}](0.3),[\pounds_{26.444},\nu_{27.677}](0.4)\}\rangle$
$\mathcal{N}_6$	$\langle\{[\aleph_{5.899},\mu_{7.329}](0.1),[\aleph_{7.329},\mu_{8.55}](0.1),[\aleph_{8.55},\mu_{9.631}](0.8)\},$ $\{[\pounds_{-5},\nu_{25.004}](0.5),[\pounds_{25.004},\nu_{25.004}](0.2),[\pounds_{25.004},\nu_{25.927}](0.3)\}\rangle$
$\mathcal{N}_7$	$\langle\{[\aleph_{5.891},\mu_{7.32}](0.2),[\aleph_{7.32},\mu_{8.541}](0.4),[\aleph_{8.541},\mu_{9.62}](0.4)\},$ $\{[\pounds_{25.004},\nu_{25.004}](0.1),[\pounds_{25.004},\nu_{26.27}](0.1),[\pounds_{26.27},\nu_{27.432}](0.8)\}\rangle$

TABLE 14. **PULq-ROF values of negative ideal solution matrix**

Alternatives	$\mathcal{M}_1$
$\mathcal{NSI}_1$	$\langle\{[1.5226, 1.8674](0.1), [1.8674, 2.1267](0.4), [2.1267, 2.3418](0.5)\}, \{[0.2630, 0.2630](0.4), [0.2630, 0.3334](0.2), [0.3334, 0.4075](0.4)\}\rangle$
Alternatives	$\mathcal{M}_2$
$\mathcal{NSI}_2$	$\langle\{[1.6981, 2.0282](0.3), [2.0282, 2.2845](0.3), [2.2845, 2.4993](0.4)\}, \{[0.1675, 0.2175](0.3), [0.2175, 0.2758](0.4), [0.2758, 0.3372](0.3)\}\rangle$
Alternatives	$\mathcal{M}_3$
$\mathcal{NSI}_3$	$\langle\{[1.9659, 2.2380](0.4), [2.2380, 2.4624](0.4), [2.4624, 2.6555](0.2)\}, \{[0.1578, 0.1578](0.2), [0.1578, 0.1834](0.4), [0.1834, 0.2376](0.4)\}\rangle$
Alternatives	$\mathcal{M}_4$
$\mathcal{NSI}_4$	$\langle\{[1.1994, 1.3258](0.3), [1.3258, 1.4367](0.6), [1.4367, 1.5374](0.1)\}, \{[3.0004, 3.1142](0.2), [3.1142, 3.2278](0.4), [3.2278, 3.3265](0.4)\}\rangle$

TABLE 15. **PULq-ROF Hammy distance**

Alternatives	$\mathcal{M}_1$
$\mathcal{HD}_1$	1.2377
$\mathcal{HD}_2$	0.9509
$\mathcal{HD}_3$	1.352
$\mathcal{HD}_4$	1.1597
$\mathcal{HD}_5$	0.9455
$\mathcal{HD}_6$	1.5747
$\mathcal{HD}_7$	1.3425

TABLE 16. **PULq-ROF Euclidean distance**

Alternatives	$\mathcal{M}_1$
$\mathcal{ED}_1$	0.1179
$\mathcal{ED}_2$	0.035
$\mathcal{ED}_3$	0.2317
$\mathcal{ED}_4$	0.1046
$\mathcal{ED}_5$	0.095
$\mathcal{ED}_6$	0.2557
$\mathcal{ED}_7$	0.138

TABLE 17. **PULq-ROF RA matrix**

Alternatives	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\mathcal{N}_5$	$\mathcal{N}_6$	$\mathcal{N}_7$
$\mathcal{N}_1$	0	0.3697	-0.2281	0.0133	0.3151	-0.4748	-0.1249
$\mathcal{N}_2$	-0.3697	0	-0.5978	-0.2784	-0.0546	-0.8445	-0.4946
$\mathcal{N}_3$	0.2281	0.5978	0	0.3194	0.5432	-0.2467	0.1032
$\mathcal{N}_4$	-0.0133	0.2784	-0.3194	0	0.0096	-0.5661	-0.2162
$\mathcal{N}_5$	-0.3151	0.0546	-0.5432	-0.0096	0	-0.7899	-0.44
$\mathcal{N}_6$	0.4748	0.8445	0.2467	0.5661	0.7899	0	0.3499
$\mathcal{N}_7$	0.1249	0.4946	-0.1032	0.2162	0.44	-0.3499	0

TABLE 18. **PULq-ROF final average solution**

Alternatives	AS_i
$\mathcal{N}_1$	-0.1297
$\mathcal{N}_2$	-2.6396
$\mathcal{N}_3$	1.5449
$\mathcal{N}_4$	-0.8270
$\mathcal{N}_5$	-2.0432
$\mathcal{N}_6$	3.2719
$\mathcal{N}_7$	0.8226

6. COMPARATIVE ANALYSIS

(1) In comparatively simple decision contexts, PUL-TOPSIS provides an intuitive framework for recognizing, evaluating, and selecting the most suitable alternative among numerous options. Even so, its ability to interpret convoluted, imprecise, or reluctant data is restricted. Although it makes extensive use of keeping away from perfect answers, it rejects the vague and contradictory assessments that DMs commonly face in real-life scenarios. It is capable of recognizing the best options, but it is unable to convey how much it prefers or believes in those choices. On the other hand, PULq-ROF-CODAS incorporates probabilistic uncertain linguistic data, which enables DMs to more precisely represent resistance, confusion, and inconsistency. This method assists in emphasizing more accurate variations among comparable alternatives, particularly when such variations are not particularly apparent. It also employs a strong distance-based process that is more conscious of contradiction in everyday situations. Consequently, PULq-ROF-CODAS produces rankings that are both scientifically valid and linguistically significant. In comparison with TOPSIS simple approach, it more accurately depicts an individual element of uncertainties.

(2) In organized circumstances, the traditional CODAS technique, which ranks objects based on both beneficial and detrimental measurements, is effective. Meanwhile, it excludes the vagueness of individual judgments and professional perspectives. In contrast, PULq-ROF-CODAS is far more descriptive since it expands on this basis by incorporating the capability to deal with linguistic phrases, statistical doubts,

and imprecise confusion. This improvement makes it possible for the approach to more accurately simulate the intricate structures of everyday life. Additionally, because of its sophisticated aggregation operators, PULq-ROF-CODAS is able to depict the amount of desire with much better consistency. As a result, rankings become more reliable and sophisticated. PULq-ROF-CODAS provides a far more comprehensive evaluation in instances in which human perception is crucial and frequently erratic. It turns the identical framework of CODAS into a more adaptable, perceptive, and cognitive instrument that reacts more effectively to speculative and reluctant choice issues.

(3) In an attempt to reconcile competing requirements, VIKOR is a useful technique that puts an emphasis on acceptable ideas. If the best alternatives are inconsistent throughout all dimensions, such a compromise may occasionally lead to their rating being inferior. It has a tendency to balance out the variations between options, which may not always be the best choice to undertake in circumstances involving competition or large values. On the other hand, PULq-ROF-CODAS doesn't reduce the effectiveness of powerful substitutes. Dominance is emphasized, and alternatives that perform noticeably better than others, as well as those that perform differently throughout some criteria, are given greater weight. Furthermore, by integrating probabilistic linguistic values, PULq-ROF-CODAS provides greater understanding into disorientation, rendering the ultimate choice closer to human thinking in difficult and reluctant situations. In contrast to VIKOR, it offers a clear and strong rating instead of presuming that teamwork is necessary. This renders it more appropriate for choices that need assurance and accuracy, especially when rewarding exceptional options is the aim instead of reaching an equitable solution.

TABLE 19. **Using various techniques to rank findings according to the PULEWA operator**

Methods	Ranking
PUL-TOPSIS [26]	$\mathcal{N}_3 > \mathcal{N}_7 > \mathcal{N}_6 > \mathcal{N}_4 > \mathcal{N}_1 > \mathcal{N}_5 > \mathcal{N}_2$
PUL-CODAS [29]	$\mathcal{N}_3 > \mathcal{N}_7 > \mathcal{N}_6 > \mathcal{N}_4 > \mathcal{N}_1 > \mathcal{N}_5 > \mathcal{N}_2$
PULq-ROF-VIKOR [20]	$\mathcal{N}_6 > \mathcal{N}_7 > \mathcal{N}_3 > \mathcal{N}_4 > \mathcal{N}_1 > \mathcal{N}_2 > \mathcal{N}_5$
PUL-EDAS [18]	$\mathcal{N}_6 > \mathcal{N}_3 > \mathcal{N}_7 > \mathcal{N}_1 > \mathcal{N}_4 > \mathcal{N}_5 > \mathcal{N}_2$
PULq-ROF-CODAS	$\mathcal{N}_6 > \mathcal{N}_3 > \mathcal{N}_7 > \mathcal{N}_1 > \mathcal{N}_4 > \mathcal{N}_5 > \mathcal{N}_2$

(4) The effectiveness and compatibility of PUL-EDAS in rating choices according to divergence from an average solution are well known. In particular, it is simple to calculate and understand, providing a moderate level of choice difference. Although when the circumstance calls for more complex choices or when there is a lot of confusion in the choice area, this kind of equilibrium turns into a constraint. The genuine value of exceptional alternatives may be obscured by EDAS's tendency to distort the line between high and average options. As an alternative, PULq-ROF-CODAS adds an exceptionally high-quality focus to the DM procedure. It takes probability-based reluctance and language misunderstanding into consideration while capturing even the smallest deviations in performance. This makes evaluations more important and transparent. In addition, whereas PULq-ROF-CODAS

performs well in certain situations, EDAS may struggle dealing with sophisticated or other types of information. It effectively transforms subjective opinions into an accurate statistical foundation. Consequently, it is particularly suited for common issues when figuring out the best option of action requires both comprehension and credibility.

The PULq-ROF-CODAS method stands apart as one influential tool, and it is adaptable among the compared techniques. It integrates most of the strengths within customary decision-making frameworks in a masterful way. It conquers the central faults of frameworks. Its ability to handle various linguistic expressions, uncertainty, and hesitation, as well as probabilistic beliefs, makes it a true reflection of how human experts think and also decide under real-world complexity. PULq-ROF-CODAS brings depth, as well as clarity and intelligence, in situations where other methods oversimplify or lack any representation of ambiguity. It ranks each alternative option, and it justifies each of them with reasoning within a context. This method gives decision-makers exact, assured, and discerning outcomes, especially in cases where stakes are extreme and information is distant from flawless. Its superior performance in sufficient aspects makes it virtually not only a preferred method but also quite a necessary one for modern decision analysis.

7. Conclusion

In this article, we employed PULq-ROFSs, an expanded form of PULTSs and q-ROFSs, to appropriately manage uncertainty and imprecision within detailed DM issues. Standard aggregation operators of PULq-ROF information often rely on basic algebraic operations as well as lack the ability to model interrelationships among criteria. For resolution of such limitations, we have proposed dual aggregation operators PULq-ROFEWA (Einstein Weighted Average) and PULq-ROFEOWA (Einstein Ordered Weighted Average) within the PULq-ROF environment. These operators are examined and validated through several properties. These properties include monotonicity, boundedness, and idempotency. In order to evaluate the performance of the approach proposed, we applied the CODAS method in the PULq-ROF framework so as to rank cybersecurity risk alternatives. The entropy method has been employed for objective determination of criteria weights; expert opinions are integrated for incorporation of subjectivity, thus improving decision accuracy. A case study for practical use in cybersecurity risk assessment has been done so as to show the applicability, also effectiveness, of the method that was developed. Further comparative analysis established the superiority and robustness of the proposed approach for management of real-world cybersecurity decision problems under uncertainty. The integration of such Einstein aggregation operators with such PULq-ROFSs indeed adds one new dimension to fuzzy MCDM for catching non-linear relationships among criteria in a more effective way. This enhancement enables DMs to achieve more realistic as well as adaptable evaluations in highly sensitive domains such as cybersecurity. Furthermore, the flexibility within the proposed framework allows for its extension into various other fields, such as into healthcare, into supply chain management, as well as into ecological risk assessment. This research virtually closes one specific theoretical gap, joining certain probabilistic linguistic models and Einstein operations, barely studied in prior work. This framework herein helps cybersecurity experts prioritize threats as well as vulnerabilities under vague and uncertain information. It also offers unto various policy makers a tool providing decision-support of great value. This is for people seeking firm data plans in dire situations. Future studies will integrate this model with hybrid MCDM and machine learning in real-time data systems for dynamic risk monitoring and assessment.

7.1. **Future Work.** Building off the existing research opens up many flexible and relevant gaps that can be pursued. One such opening can be the combination of PULq-ROFS approaches with other MCDM methods like TOPSIS, MOORA, or EDAS that create hybrid structures for dealing with different instances of DM problems. Moreover, the problem might be more effectively managed by shifting risk in cybersecurity through immediate information and adaptive balancing algorithm infusion responsive structures. In addition, using neural networks or cognitive computing can strengthen the highly vulnerable areas through the proposed PULq-ROFEWA and PULq-ROFEOWA operators, enabling controlled and immediate guidance on decision-making. Incorporating this methodology, future research might add computable decision support systems in the form of software and web applications usable by organizations or by security professionals. In addition, other modifications of the theory, such as the application of the Einstein-based aggregation method onto other fuzzy models like spherical fuzzy sets, complex intuitionistic fuzzy sets, or even within neutrosophic environments, could be made to expand the boundaries of the framework. Other interdisciplinary cross-domains include healthcare diagnostics, risk assessment in disasters, planning for smart cities, and evaluating risks in finance, which warrant exploration as well. Lastly, implementing the model with larger and more regionally diverse data sets would further contribute to the richness of the insights as well as the generalizability of the outlined approach.

Credit authorship contribution statement

Uzma Ahmad: Concept, Design, Analysis and Writing of the manuscript. **Saira Hameed:** Concept, Design, Analysis and Writing of the manuscript. **Rehman Khan:** Concept, Design, Analysis and Writing of the manuscript. **Ayesha Khan:** Concept, Design, Analysis and Writing of the manuscript.

Declaration of competing interest

The authors declare no conflicts of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Funding

There is no specific funding for this research article.

REFERENCES

[1] U. Ahmad, A. Khan, S. Shahzadi, Extended ELECTRE I method for decision-making based on 2-tuple linguistic q-rung picture fuzzy sets, *Soft Comput*. (2024).

[2] M. Akram, S. Zahid, Group decision-making with Pythagorean fuzzy rough numbers for evaluating design concepts, *Granul. Comput*. **8** (2023) 11211148.

[3] M. Akram, G. Shahzadi, X. Peng, Extension of Einstein geometric operators to MADM under q-rung orthopair fuzzy information, *Granul. Comput.* **6** No. 4, (2021) 779795.

[4] M. Akram, S. Naz, S. A. Edalatpanah, R. Mehreen, Group decision-making under linguistic q-rung orthopair fuzzy Einstein models, *Soft Comput*. **25** (2021) 1030910334.

[5] R. Andukuri, C. M. Rao, Application of fuzzy CODAS for selecting condition monitoring equipment, *Oper. Res. Forum* **5** (2024) 88.

[6] K. Atanassov,Review and New Results on Intuitionistic Fuzzy Sets, Mathematical Foundations of Artificial Intelligence Seminar, Sofia, 1988, Preprint IM-MFAIS1-88. Reprinted: Int. J. Bioautomation, 2016, 20(S1), S7-S16

[7] E. Bolturk, Pythagorean fuzzy CODAS and its application to supplier selection, *J. Enterprise Inf. Manag*. **31** No.4, (2018) 550564.

[8] M. Baydas, M. Yilmaz, φZ. Jovic, φZ. Stevic, S. E. G. Özuyar, A. Özçil, A Comprehensive MCDM assessment for economic data: success analysis of maximum normalization, CODAS and fuzzy approaches, *Financial Innov*. **10** (2024) 105.

[9] W. S. Du, A further investigation on q-rung orthopair fuzzy Einstein aggregation operators, *J. Intell. Fuzzy Syst*. **41** No.6, (2021) 66556673.

[10] M. K. Ghorabaee, E. K. Zavadskas, L. Olfat, Z. Turskis, Multi-criteria inventory classification using a new method of evaluation based on distance from average solution EDAS, *Informatica* **26** No.3, (2015) 435451.

[11] X. Gou, Z. Xu, H. Liao, MCDM based on Bonferroni means with hesitant fuzzy linguistic information, *Soft Comput*. **21** (2017) 65156529.

[12] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, *IEEE Trans. Fuzzy Syst*. **8** (2000) 746752.

[13] F. Herrera, E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, *Fuzzy Sets Syst*. **115** (2000) 6782.

[14] A. Khan, U. Ahmad, A. Farooq, M. M. A. Al-Shamiri, Combinative distance-based assessment method for decision-making with 2-tuple linguistic q-rung picture fuzzy sets, *AIMS Math*. **8** No.6, (2023) 1383013874.

[15] A. Khan, U. Ahmad, S. Shahzadi, A new decision analysis based on 2-tuple linguistic q-rung picture fuzzy ITARAVIKOR method, *Soft Comput*. (2023).

[16] N. Kundakci, K. Arman, A Novel Combined fuzzy MCDM approach based on IMF-SWARA and F-CODAS for consulting firm selection, *Ege Acad. Rev.* **23** No.4,(2023) 639652.

[17] P. Liu, P. Wang, Some q-rung orthopair fuzzy aggregation operators and applications in MADM, *Intell. Syst*. **33** No.4,(2018) 259280.

[18] P. Liu, X. Wang, Y. Fu, P. Wang, Graph model for conflict resolution based on the combination of probabilistic uncertain linguistic and EDAS method, *Inf. Sci.* **660** (2024) 120116.

[19] M. Lin, Z. Xu, Y. Zhai, Z. Yao, MAGDM under probabilistic uncertain linguistic environment, *J. Oper. Res. Soc*. **69** No.2,(2018) 157170.

[20] S. Naz, M. M. ul Hassan, A. Mehmood, G. P. Espitia, S. A. Butt, Enhancing industrial robot selection through a hybrid novel approach: integrating CRITIC-VIKOR method with probabilistic uncertain linguistic q-ROFS, *Artif. Intell. Rev*. **58** (2024) 59.

[21] R. X. Nie, J. Q. Wang, Prospect theory-based consistency recovery for multiplicative probabilistic linguistic preference relations in Managing Group Decision Making, *Arab. J. Sci. Eng*. **45** (2020) 21132130.

[22] Q. Pang, H. Wang, Z. Xu, Probabilistic linguistic term sets in multi-attribute group decision making, *Inf. Sci*. **369** (2016) 128143.

[23] F. D. Remadi, H. M. Frikha, The triangular intuitionistic fuzzy numbers CODAS method for green material selection, *Int. J. Oper. Res*. **46** No.3,(2023) 398415.

[24] R. M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, *IEEE Trans. Fuzzy Syst*. **20** No.1, (2011) 109119.

[25] W. Su, D. Luo, C. Zhang, S. Zeng, Evaluation of online learning platforms based on probabilistic linguistic term sets with self-confidence MAGDM, *Expert Syst. Appl*. **208** (2022) 118153.

[26] J. Sun, Y. Liu, J. Xu, F. Zhu, N. Wang, Probabilistic uncertain linguistic TOPSISBWM for resilient supplier selection, *Int. J. Fuzzy Syst*. **26** (2024) 9921015.

[27] X. Tian, Z. Xu, J. Gu, F. Herrera, A consensus process based on regret theory with probabilistic linguistic term sets and its application in venture capital, *Inf. Sci*. **562** (2021) 347369.

[28] G. Wei, Y. He, F. Lei, J. Wu, C. Wei, MABAC method for MAGDM with probabilistic uncertain linguistic information, J. Intell. Fuzzy Syst. **39** No.3,(2020) 33153327.

[29] C. Wei, J. Wu, Y. Guo, G. Wei, Green supplier selection based on CODAS in probabilistic uncertain linguistic environment, *Technol. Econ. Dev. Econ*. **27** (2021) 530549.

[30] W. Xie, Z. Ren, Z. Xu, H. Wang, The consensus of probabilistic uncertain linguistic preference relations and its application on the virtual reality industry, *Knowl.-Based Syst.* **162** (2018) 1428.

[31] Z. Xu, Uncertain linguistic aggregation operators based approach to MAGDM under uncertain linguistic environment, *Inf. Sci*. **168** (2004) 171184.

[32] Z. Xu, Deviation measures of linguistic preference relations in GDM, *Omega* **33** (2005) 249254.

[33] R. R. Yager, Pythagorean fuzzy subsets, *Proc. IFSA/NAFIPS* (2013) 5761.

[34] R. R. Yager, Generalized orthopair fuzzy sets, *IEEE Trans. Fuzzy Syst*. **25** (2016) 12221230.

[35] L. A. Zadeh, Fuzzy algorithms, *Inf. Control* **12** (1968) 94102.