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A Holomorphic Bundle Criterion for Compactness of Pseudoconcave Solvmanifolds

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Abstract. We prove that a pseudoconcave complex homogeneous space of a connected solvable linear algebraic group is necessarily compact. This resolves a central conjecture in the theory, showing that pseudoconcavity characterizes compactness for this large class of solvmanifolds. The proof combines new pluriharmonic obstructions for $\mathbb{C}^*$-bundles with the structure theory of solvable groups, demonstrating that any noncompact solvmanifold admits a nonconstant pluriharmonic function, contradicting pseudoconcavity. Our result unifies and extends all known partial classifications for this class of solvable linear algebraic groups, establishing pseudoconcavity as a definitive geometric property that forces compactness in the solvable setting.

AMS (MOS) Subject Classification Codes: 32M10, 32Q55, 32U05, 32L05
Key Words: Complex Solvmanifold, Pseudoconcavity, Pluriharmonic Function, Holomorphic Fibration.

1. Introduction

The study of complex homogeneous spaces $X = G/\Gamma$, where G is a complex Lie group and $\Gamma \subset G$ is a discrete subgroup, occupies a central place in complex geometry, forging deep connections between Lie theory, several complex variables, and the classification of complex manifolds [7, 11]. Among these, the solvmanifolds—where G is solvable—present a particularly rich and challenging class, whose geometric and function-theoretic properties are intimately tied to the algebraic structure of the group and the arithmetic of the lattice [1, 6].

A profound property that such a manifold can exhibit is pseudoconcavity, a notion introduced by Andreotti [2] to capture a weak form of "positivity" in the complement of a compact set. A complex manifold X is pseudoconcave if it contains a relatively compact open subset Z such that every point in the boundary of Z is the center of a holomorphic disk that lies predominantly inside Z. This property has profound analytic consequences; notably, pseudoconcave manifolds possess only constant holomorphic functions [7], a feature they share with compact complex manifolds. This immediately distinguishes them

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from Stein manifolds (see [5]) and suggests that pseudoconcavity represents an intermediate geometric condition that strongly constraints the global structure. Related aspects of holomorphic and analytic function theory have been actively studied in recent literature, including investigations of analytic and multivalently analytic function classes and their structural properties [9, 14].

The quest to classify pseudoconcave solvmanifolds has been a driving force in the field. In the abelian case, where $G = (\mathbb{C}^n, +)$, the homogeneous spaces are Cousin groups. The foundational work of Andreotti [1] showed that under certain conditions, pseudoconcavity forces such a group to be compact, i.e. a complex torus. This established a compelling dichotomy in the abelian setting.

For non-abelian solvable groups, the situation is significantly more intricate. The interplay between the semidirect product structure, the lattice embedding, and the resulting fibrations creates a complex geometric landscape. Partial results and extensive examples, including a complete classification in dimension two showing that non-abelian solvmanifolds are never pseudoconcave [4], have strongly supported the general conjecture that *a pseudoconcave solvmanifold must be compact*. However, a unified proof for general solvable groups has remained elusive, with previous approaches often relying on case-specific analyses or constructions of explicit examples for particular subgroups like the Borel groups [3, 8].

In this paper, we resolve this conjecture for the wide class of linear solvable algebraic groups. We prove that if $X = G/\Gamma$ is a pseudoconcave homogeneous space of a connected solvable linear algebraic group G, then X is compact.

The role of harmonic and pluriharmonic functions in complex and real analysis has been emphasized in various contexts, including recent studies on harmonic convexity and related inequalities [12]. Our strategy is structural and inductive. The proof hinges on a fundamental new obstruction to pseudoconcavity, which we develop here: the existence of a nonconstant pluriharmonic function on any holomorphic principal $\mathbb{C}^*$-bundle over a compact base, provided the bundle is flat (i.e., has vanishing real Chern class) or has torsion Chern class. A homogeneous analogue, where the bundle arises from a holomorphic character $\chi : G \to \mathbb{C}^*$ with unitary monodromy $\chi(\Gamma) \subset S^1$, yields the same obstruction (Theorem 3.6). Since pseudoconcave manifolds do not admit nonconstant pluriharmonic functions, such bundle structures are forbidden.

Armed with this tool, we analyze the structure of G/Γ. We consider the commutator subgroup G' and the Zariski closure A of $\Gamma \cap G'$. Through a sequence of Γ-equivariant holomorphic fibrations (which are open maps, hence preserve pseudoconcavity), and by applying the Lie-Kolchin theorem to linearized actions on associated Lie algebras, we systematically construct a one-dimensional, normal, Γ-invariant subgroup $B \subset G$. The final fibration $G/\Gamma \to G/B\Gamma$ has a base that is compact by induction. The fiber $B/(B \cap \Gamma)$ is either compact, forcing X itself to be compact, or non-compact (isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$), which allows us to apply our pluriharmonic obstruction and derive a contradiction. This resulting dichotomy forces the conclusion that the initial assumption of noncompactness is untenable.

This result provides a unified framework that subsumes previous partial classifications and confirms the long-standing intuition that pseudoconcavity is a “compact phenomenon” in the realm of solvable complex geometry. It characterizes the compact solvmanifolds within this class by a purely complex-analytic property, closing a significant chapter in the theory of complex homogeneous spaces.

The paper is organized as follows: Section 2 recalls the definition of pseudoconcavity and establishes the key pluriharmonic obstructions for $\mathbb{C}^*$-bundles. Section 3 is dedicated to the proof of the main compactness theorem. The concluding section offers final remarks on the implications of our result.

2. Preliminaries and Key Obstructions

Definition 2.1 (see Andreotti–Huckleberry [2])**.** A connected complex manifold X is *pseudoconcave* if there exists a relatively compact open subset $Z \subset X$ such that for every point $p \in \mathrm{cl}(Z)$, there exists a holomorphic map $\psi : \Delta \to \mathrm{cl}(Z)$ (where Δ denotes the unit disk in $\mathbb{C}$) from the unit disk with $\psi(0) = p$ and $\psi(\partial\Delta) \subset Z$.

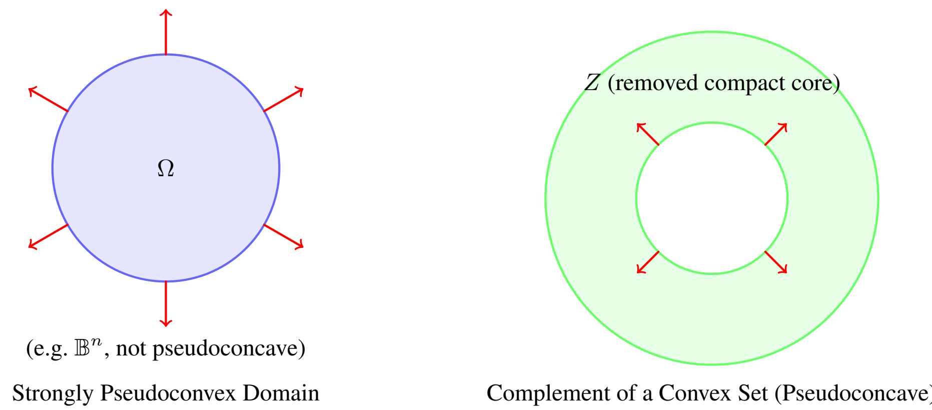
Fundamental Examples.

Example 2.1 (Stein Manifolds are Not Pseudoconcave). Every Stein manifold fails to be pseudoconcave. In particular, complex Euclidean space $\mathbb{C}^n$ and more generally, any domain of holomorphy in $\mathbb{C}^n$ cannot be pseudoconcave. The existence of many holomorphic functions and the Levi convexity properties [5] prevent the existence of a relatively compact set Z with the disk extension property from the boundary. For instance, take any bounded domain $D \subset \mathbb{C}^n$ with smooth strongly pseudoconvex boundary. At each boundary point, all analytic disks must point outward, violating the pseudoconcavity condition.

Example 2.2 (Complement of a Complex Hyperplane in Projective Space). Consider $X = \mathbb{P}^n \setminus H$ where H is a complex hyperplane. This space is pseudoconcave. Using homogeneous coordinates $[z_0 : \cdots : z_n]$ with $H = \{z_0 = 0\}$, take $Z = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n : |z_1|^2 + \cdots + |z_n|^2 \leq \epsilon |z_0|^2\}$ for small $\epsilon > 0$. Then Z is relatively compact in X and every boundary point admits analytic disks from the interior. This contrasts with the Stein manifold case and illustrates how removing a complex hypersurface from a compact manifold can create pseudoconcavity.

Example 2.3 (Non-Example). Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary that is strongly pseudoconvex. Then Ω is not pseudoconcave. In fact, at each boundary point, the Levi form is positive definite, meaning all analytic disks through boundary points must point outward. This is the geometric reason why such domains admit many holomorphic functions and are domains of holomorphy. This example illustrates a class of domains that fail to satisfy pseudoconcavity, and is therefore included as a non-example.

Remark 2.2. These examples illustrate the fundamental dichotomy: Stein manifolds (many holomorphic functions) are never pseudoconcave, while compact manifolds and their modifications (few holomorphic functions) typically are pseudoconcave. The pseudoconcave condition captures an intermediate geometric property that forces function-theoretic restrictions while allowing interesting non-compact geometries.



(A) Analytic disks point *outward* — domain is pseudoconvex.

(B) Analytic disks point *inward* — domain is pseudoconcave.

FIGURE 1. Geometric distinction between pseudoconvex and pseudoconcave domains. Pseudoconvex domains (left) admit outward-pointing analytic disks, while pseudoconcave domains (right), such as the complement of a compact set, admit inward-pointing analytic disks at the boundary.
(Figure 1 is a schematic illustration intended to convey geometric intuition and does not represent a specific example from the literature.)

A fundamental consequence of the definition of **Pseudoconcave space** is the following function-theoretic restriction:

Proposition 2.1 ([7])**.** *A connected pseudoconcave complex manifold does not admit any nonconstant holomorphic functions.*

For our purposes, we require the following stronger version:

Proposition 2.2. *A connected pseudoconcave complex manifold does not admit any nonconstant pluriharmonic functions.*

Proof. Let X be a connected pseudoconcave complex manifold and suppose $u : X \to \mathbb{R}$ is a pluriharmonic function. Fix a relatively compact open set $Z \subset X$ as in the definition of pseudoconcavity; then $\overline{Z}$ is compact and u is continuous on $\overline{Z}$, hence attains a maximum value M on $\overline{Z}$. Let $p \in \overline{Z}$ satisfy $u(p) = M$. By pseudoconcavity there exists a holomorphic map $\psi : \Delta \to \overline{Z}$ from the unit disk $\Delta \subset \mathbb{C}$ with $\psi(0) = p$ and $\psi(\partial\Delta) \subset Z$.

The composition $h := u \circ \psi$ is a harmonic function on Δ which extends continuously to $\overline{\Delta}$. Since M is the maximum of u on $\overline{Z}$, we have $h(\zeta) \leq M$ for every $\zeta \in \overline{\Delta}$, and $h(0) = M$. The maximum principle for harmonic functions thus implies that h is constant on Δ, equal to M. Since the image of the holomorphic disk contains a nonempty open subset and pluriharmonic functions are real-analytic, constancy on this open set implies global constancy on the connected manifold. Consequently u equals M on the whole image $\psi(\Delta)$.

Since $\psi(\partial\Delta) \subset Z$ and ψ is nonconstant holomorphic, the image $\psi(\Delta)$ contains an open neighborhood of points in Z. In particular, u attains its maximum value M on a nonempty open subset of Z.

Now, because pluriharmonic functions are real-analytic, the set $\{x \in X : u(x) = M\}$ is real-analytic; containing a nonempty open subset it follows that $u \equiv M$ on the connected component of X containing that open set. As X is assumed connected, u is constant on all of X. Therefore no nonconstant pluriharmonic function can exist on X. □

Also there is an important result which will be used in fibrations to prove compactness.

Theorem 2.3 (Open Mapping Theorem for Pseudoconcave Spaces)**.** *Let $\pi : X \to Y$ be an open holomorphic map between complex spaces. If X is pseudoconcave, then Y is also pseudoconcave.*

Proof. For a proof, see [4, Theorem 3.1]. □

The following theorems provide the core obstructions that drive our main result.

3. Pseudoconcavity and Compactness for Solvable Homogeneous Manifolds

In this section we analyze the relation between pseudoconcavity and compactness (here compactness is understood in the usual topological sense) for complex solvmanifolds. We begin with general obstruction results showing that the presence of a $\mathbb{C}^*$–direction (or, equivalently, a nontrivial flat or torsion line bundle) always produces nonconstant pluriharmonic functions, thereby excluding pseudoconcavity. These results will then serve as the key tools in proving that a pseudoconcave solvmanifold must be compact.

3.1. **Pluriharmonic Obstructions to Pseudoconcavity.**

Theorem 3.1. *Let Y be a compact connected complex manifold and $L \to Y$ a holomorphic line bundle. Assume that L admits a Hermitian metric of zero curvature (equivalently the Chern form of L vanishes, so $c_1(L) = 0$ in $H^2(Y;\mathbb{R})$). Let X denote the total space of the associated holomorphic principal $\mathbb{C}^*$–bundle (the complement of the zero section in L). Then X admits a nonconstant pluriharmonic function and consequently X is not pseudoconcave.*

Proof. Choose a flat Hermitian metric h on L (possible by hypothesis). Locally on a coordinate neighborhood $U \subset Y$ trivializing L we identify

$$\pi^{-1}(U) \cong U \times \mathbb{C}^*,$$

and write a point as (y, z) with $y \in U$, $z \in \mathbb{C}^*$. The metric h is represented locally by a positive smooth function $\rho(y)$ with

$$\|(y, z)\|_h = |z|\,\rho(y).$$

Define a function $u : X \to \mathbb{R}$ by

$$u(x) = \log \|x\|_h.$$

On the local chart $\pi^{-1}(U) \cong U \times \mathbb{C}^*$ we have

$$u(y,z) = \log|z| + \phi(y), \qquad \phi(y) = \log\rho(y).$$

Because the metric is flat, $\partial\bar{\partial}\phi = 0$ on U. Moreover $\partial\bar{\partial}\log|z| = 0$ on $\mathbb{C}^*$ (since $\log|z|$ is locally the real part of a holomorphic branch of $\log z$). Hence

$$i\partial\bar{\partial}u = i\partial\bar{\partial}(\log|z|) + i\partial\bar{\partial}\phi = 0$$

on each trivializing chart, so u is pluriharmonic locally. The flatness of the metric ensures that these local pluriharmonic expressions glue to a globally defined pluriharmonic function on X (transition functions change $\log z$ only by additive constants since the metric is flat).

The $\mathbb{C}^*$–action on X satisfies for $t \in \mathbb{C}^*$ and $x \in X$

$$u(t \cdot x) = \log\|t \cdot x\|_h = \log|t| + \log\|x\|_h = \log|t| + u(x).$$

As t ranges over $\mathbb{C}^*$ the quantity $\log|t|$ attains all real values, so u is unbounded along fibers and therefore nonconstant.

A connected pseudoconcave complex manifold does not admit any nonconstant pluriharmonic functions. Since X admits the nonconstant pluriharmonic function u, it cannot be pseudoconcave. □

Theorem 3.1 should be understood as an obstruction result: it asserts that the presence of such a bundle structure excludes pseudoconcavity.

The following corollary is a direct special case of Theorem 3.1. Indeed, if the holomorphic line bundle $L \to Y$ is trivial, then the associated principal $\mathbb{C}^*$–bundle is globally biholomorphic to the product $Y \times \mathbb{C}^*$. In this case, the pluriharmonic function constructed in Theorem 3.1 reduces to $u(y,z) = \log|z|$, which is clearly nonconstant. Therefore, the total space cannot be pseudoconcave, yielding the stated conclusion.

Corollary 3.2. *If the holomorphic line bundle $L \to Y$ is topologically (or holomorphically) trivial, then the total space $X \cong Y \times \mathbb{C}^*$ admits the pluriharmonic function $u(y,z) = \log|z|$ and hence is not pseudoconcave.*

Proposition 3.2 (Flat Chern Class Obstruction)**.** *Let $L \to Y$ be a holomorphic line bundle over a compact connected complex manifold Y. If $c_1(L) = 0$ in $H^2(Y;\mathbb{R})$, then the total space*

$$X = L \setminus s_0(Y),$$

where $s_0 : Y \to L$ denotes the zero section, admits a nonconstant pluriharmonic function. In particular, X is not pseudoconcave.

Proof. The hypothesis $c_1(L) = 0$ implies that L admits a Hermitian metric h with zero curvature (a flat Hermitian metric). Define

$$u : X \longrightarrow \mathbb{R}, \qquad u(x) = \log\|x\|_h.$$

Locally on a trivializing chart $U \subset Y$, write points of $\pi^{-1}(U) \subset X$ as $(y,z) \in U \times \mathbb{C}^*$. Then $\|(y,z)\|_h = |z|\,\rho(y)$ for a positive smooth function ρ on U, and

$$u(y,z) = \log|z| + \phi(y), \qquad \phi = \log\rho.$$

Flatness of the metric gives $\partial\bar{\partial}\phi = 0$, hence $i\partial\bar{\partial}u = 0$. The function u is nonconstant (it varies along the $\mathbb{C}^*$–fibers). Thus X is not pseudoconcave. □

Remark 3.3. Proposition 3.2 is essentially the same result as Theorem 3.1, restated in terms of Chern classes. For a holomorphic line bundle, having a flat Hermitian metric is equivalent to the condition $c_1(L) = 0$ in $H^2(Y;\mathbb{R})$. We state it separately to emphasize this special case in cohomological language.

For instance, let Y be a compact complex torus and let $L = Y \times \mathbb{C}$ be the trivial holomorphic line bundle. The bundle L admits a flat Hermitian metric given by $\|(y,z)\| = |z|$, whose curvature form vanishes identically. Consequently, the real Chern class $c_1(L)$ vanishes in $H^2(Y;\mathbb{R})$. The associated principal $\mathbb{C}^*$–bundle is $Y \times \mathbb{C}^*$, and the function $u(y,z) = \log|z|$ provides an explicit nonconstant pluriharmonic function. This example illustrates the equivalence described in Remark 3.3.

Proposition 3.3 (Torsion Chern Class Obstruction)**.** *Let $L \to Y$ be a holomorphic line bundle over a compact connected complex manifold Y. Assume $c_1(L) \in H^2(Y;\mathbb{Z})$ is torsion. Let*

$$X = L \setminus s_0(Y),$$

where $s_0 : Y \to L$ denotes the zero section. Then X is the associated holomorphic principal $\mathbb{C}^*$–bundle and admits a nonconstant pluriharmonic function; in particular X is not pseudoconcave.

Proof. Let $m \geq 1$ be such that $m \cdot c_1(L) = 0$. There exists a finite unramified cover $\pi : \widetilde{Y} \to Y$ for which the pulled-back bundle $\widetilde{L} = \pi^* L$ is topologically trivial (hence admits a flat Hermitian metric). Let

$$\widetilde{X} = \widetilde{L} \setminus s_0(\widetilde{Y}) \cong \widetilde{Y} \times \mathbb{C}^*.$$

As in Proposition 3.2, the function

$$\widetilde{u}(y,z) = \log|z|$$

is globally pluriharmonic and nonconstant on $\widetilde{X}$, so $\widetilde{X}$ is not pseudoconcave.

Let F denote the finite deck group of π. The F–action on $\widetilde{Y}$ lifts holomorphically to $\widetilde{X}$; each $\sigma \in F$ acts by a biholomorphism of $\widetilde{X}$, and therefore $\widetilde{u} \circ \sigma$ is pluriharmonic. Set

$$\overline{u} = \frac{1}{|F|} \sum_{\sigma \in F} \widetilde{u} \circ \sigma.$$

Then $\overline{u}$ is F–invariant and pluriharmonic, hence descends to a pluriharmonic function u on $X = \widetilde{X}/F$. The function u is nonconstant because the fiberwise unbounded variation of $\widetilde{u}$ cannot be annihilated by finite averaging. Therefore X admits a nonconstant pluriharmonic function and hence is not pseudoconcave. □

Example 3.1 (A 4×4 Borel Group Example with Flat Line Bundle). Let $G = B(4,\mathbb{C})$ be the Borel subgroup of 4×4 invertible upper-triangular complex matrices. Consider the discrete subgroup $\Gamma \subset G$ defined by:

$$\Gamma = \left\{ \begin{pmatrix} e^{2\pi i k_1} & m_{12} & m_{13} & m_{14} \\ 0 & e^{2\pi i k_2} & m_{23} & m_{24} \\ 0 & 0 & e^{2\pi i k_3} & m_{34} \\ 0 & 0 & 0 & e^{2\pi i k_4} \end{pmatrix} : k_j \in \mathbb{Z},\ m_{ij} \in \mathbb{Z}[i],\ \sum_{j=1}^{4} k_j = 0 \right\}.$$

Let $Y = G/U\Gamma$, where U is the unipotent radical of G. Then Y is a compact complex manifold, and the natural projection $\pi : X = G/\Gamma \to Y$ makes X the total space of a holomorphic principal $\mathbb{C}^*$-bundle associated to a line bundle $L \to Y$.

The condition $\sum k_j = 0$ ensures that the character $\chi : G \to \mathbb{C}^*$ defined by $\chi(g) = \det(g)$ satisfies $\chi(\Gamma) = \{1\}$, which implies that $c_1(L) = 0$ in $H^2(Y;\mathbb{R})$. By Proposition 3.2, X admits the nonconstant pluriharmonic function $u(g\Gamma) = \log|\det(g)|$ and is therefore not pseudoconcave.

To further illustrate the obstruction principles, we examine a case with torsion Chern class:

Example 3.2 (A 4×4 Example with Torsion Chern Class). Let $G = B(4,\mathbb{C})$ and define the discrete subgroup:

$$\Gamma = \left\{ \begin{array}{l} \begin{pmatrix} e^{2\pi i k_1/2} & m_{12} & m_{13} & m_{14} \\ 0 & e^{2\pi i k_2/2} & m_{23} & m_{24} \\ 0 & 0 & e^{2\pi i k_3/2} & m_{34} \\ 0 & 0 & 0 & e^{2\pi i k_4/2} \end{pmatrix} : k_j \in \mathbb{Z}, \quad m_{ij} \in \mathbb{Z}[i], \\ \hfill \sum_{j=1}^{4} k_j \equiv 0 \pmod 2 \end{array} \right\}.$$

Let $Y = G/U\Gamma$ and $X = G/\Gamma$ as before. The associated line bundle $L \to Y$ now satisfies $2c_1(L) = 0$ in $H^2(Y;\mathbb{Z})$, so $c_1(L)$ is 2-torsion.

To see this explicitly, consider the double cover $\widetilde{Y} = G/U\widetilde{\Gamma}$ where:

$$\widetilde{\Gamma} = \left\{ \begin{pmatrix} e^{2\pi i k_1} & m_{12} & m_{13} & m_{14} \\ 0 & e^{2\pi i k_2} & m_{23} & m_{24} \\ 0 & 0 & e^{2\pi i k_3} & m_{34} \\ 0 & 0 & 0 & e^{2\pi i k_4} \end{pmatrix} : k_j \in \mathbb{Z},\ m_{ij} \in \mathbb{Z}[i],\ \sum_{j=1}^{4} k_j = 0 \right\}.$$

On $\widetilde{Y}$, the pulled-back bundle is flat, and we can construct the pluriharmonic function $\tilde{u}(g\widetilde{\Gamma}) = \log|\det(g)|$. Averaging over the $\mathbb{Z}/2\mathbb{Z}$ deck transformation group as in Proposition 3.3 yields a nonconstant pluriharmonic function on X, showing that X is not pseudoconcave.

Remark 3.4. These examples demonstrate how algebraic conditions on the lattice Γ translate directly into topological properties of the associated line bundle (flat or torsion Chern class), which in turn yield analytic obstructions to pseudoconcavity through the construction of nonconstant pluriharmonic functions. The 4×4 case is particularly interesting as it provides non-trivial examples where the base Y is a compact solvmanifold of significant complexity, yet the bundle structure still forces the existence of the obstruction.

Remark 3.5 (Non-Torsion Case). If $c_1(L)$ is non-torsion (equivalently nonzero in real cohomology), the previous constructions fail, and pseudoconcavity must be analyzed using additional geometric or representation-theoretic arguments.

The preceding line–bundle results admit a natural homogeneous analogue in the setting of complex solvable Lie groups. When a nontrivial holomorphic character $\chi : G \to \mathbb{C}^*$ exists whose lattice image $\chi(\Gamma)$ is contained in S^1, the associated principal $\mathbb{C}^*$–bundle inherits a flat structure in the same way as a line bundle with vanishing or torsion Chern class. This yields the following solvable analogue of the $\mathbb{C}^*$–bundle obstruction.

Theorem 3.6 (Solvable $\mathbb{C}^*$–Bundle Obstruction)**.** *Let G be a connected complex solvable Lie group and let*

$$\chi : G \longrightarrow \mathbb{C}^*$$

be a nontrivial holomorphic character (group homomorphism). Let $\Gamma \subset G$ be a discrete cocompact subgroup (a lattice) such that

$$\chi(\Gamma) \subset S^1 = \{z \in \mathbb{C}^* : |z| = 1\}.$$

Set $X = G/\Gamma$ and $H = \ker \chi$. Then $Y := G/H\Gamma$ is compact and the natural map

$$\pi : X = G/\Gamma \longrightarrow Y = G/H\Gamma$$

exhibits X as the holomorphic principal $\mathbb{C}^*$–bundle induced by χ. Define

$$u : X \longrightarrow \mathbb{R}, \qquad u(g\Gamma) = \log |\chi(g)|.$$

Then u is a well-defined, nonconstant pluriharmonic function on X. In particular X is not pseudoconcave.

Proof. For any $g \in G$ and $\gamma \in \Gamma$ we have

$$|\chi(g\gamma)| = |\chi(g)\chi(\gamma)| = |\chi(g)| \cdot |\chi(\gamma)| = |\chi(g)|,$$

because $\chi(\gamma) \in S^1$ implies $|\chi(\gamma)| = 1$. Hence the assignment $u(g\Gamma) = \log |\chi(g)|$ is independent of the representative of the coset and defines a globally well-defined real-valued function on $X = G/\Gamma$.

Since χ is holomorphic and nowhere vanishing, locally on G we may choose a holomorphic logarithm of χ: on each simply connected open set $U \subset G$ there exists a holomorphic function F satisfying $\chi|_U = e^F$. On such a set,

$$u = \log |\chi| = \Re(F),$$

so u is pluriharmonic on U. The difference of two local logarithms is an additive constant multiple of $2\pi i$, hence their real parts coincide; the local pluriharmonic expressions glue to a global pluriharmonic function on G. The invariance $u(g\gamma) = u(g)$ ensures that u descends to a globally defined pluriharmonic function on $X = G/\Gamma$.

The character χ is nontrivial, so there exists $g \in G$ such that $|\chi(g)| \neq 1$. Then

$$u(g\Gamma) = \log |\chi(g)| \neq 0 = u(e\Gamma),$$

showing that u is nonconstant. Equivalently, along the fiberwise $\mathbb{C}^*$–action we have

$$u(t \cdot x) = \log |t| + u(x), \qquad t \in \mathbb{C}^*,\ x \in X,$$

so u varies unboundedly along each fiber.

Consequently, u is a nonconstant pluriharmonic function on X. By Proposition 2.2, a pseudoconcave complex manifold cannot admit such a function, and hence X is not pseudoconcave.

□

Remark 3.7. This theorem can also be viewed as a special case of the flat and torsion line–bundle obstructions (Propositions 3.2 and 3.3) when the $\mathbb{C}^*$–bundle arises from a holomorphic character $\chi : G \to \mathbb{C}^*$ with unitary monodromy $\chi(\Gamma) \subset S^1$.

From the obstruction results established above, it follows that a complex manifold supporting a holomorphic principal $\mathbb{C}^*$–bundle with unitary monodromy (equivalently, with flat or torsion Chern class) necessarily admits a nonconstant pluriharmonic function and therefore fails to be pseudoconcave. In the context of complex solvmanifolds, this observation becomes decisive: any noncompact fiber direction of type $\mathbb{C}$ or $\mathbb{C}^*$ forces the existence of such a function, contradicting pseudoconcavity.

3.4. **Compactness of Pseudoconcave Solvmanifolds.** We now apply the above obstruction principle to obtain the compactness theorem for pseudoconcave solvmanifolds.

Theorem 3.8 (Compactness of Pseudoconcave Solvmanifolds)**.** *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a connected solvable linear algebraic group and let $\Gamma \subset G$ be a discrete subgroup. Consider the complex homogeneous space*

$$X = G/\Gamma.$$

If X is pseudoconcave, then X is compact.

Proof. Assume for contradiction that $X = G/\Gamma$ is pseudoconcave but noncompact. We shall derive a contradiction by successive reduction through the algebraic structure of G.

Let $\Lambda := \Gamma \cap G'$ where G' denotes the commutator subgroup of G, and let A be the Zariski closure of Λ in G. Since Γ normalizes Λ, it also normalizes A, and we obtain a Γ–equivariant holomorphic fibration

$$\pi_1 : X \longrightarrow X_1 := G/A\Gamma.$$

The map π_1 is holomorphic and open. By the open mapping theorem for pseudoconcave spaces (Theorem 2.3), X_1 is also pseudoconcave. By induction on $\dim G$, we may assume that X_1 is compact. (The base cases of dimension one and two have already been established: in those cases, every pseudoconcave solvmanifold is compact (see [4]).)

If $\dim A = 0$, then Λ is discrete and X fibers holomorphically over a compact base X_1 with compact fiber, so X itself is compact. Hence we may suppose $\dim A > 0$.

By the structure theory of solvable linear algebraic groups (see Borel [3]), the unipotent subgroup $A \subset G'$ contains a canonical connected abelian central subgroup $H \subset A$ which is invariant under the action of Γ. Passing to the quotient by H gives another holomorphic fibration

$$\pi_2 : X \longrightarrow X_2 := G/H\Gamma.$$

The map π_2 is open, so X_2 is pseudoconcave (see [4] for details, where the Open Mapping Theorem for pseudoconcave spaces is established) and therefore compact by induction.

Next, consider the induced action of Γ on the Lie algebra $\mathfrak{h}$ of H. Let S denote the Zariski closure of Γ in G. By **Malcev's theorem on lattices in solvable groups** [13], the closure S is a connected solvable linear algebraic subgroup of $\mathrm{GL}(n,\mathbb{C})$. Applying the **Lie–Kolchin theorem** (see [3, Cor. 10.5], originally due to Kolchin [10]), every finite-dimensional representation of a connected solvable linear algebraic group admits a complete invariant flag. In particular, the representation of S on $\mathfrak{h}$ possesses a nonzero S–invariant line $L \subset \mathfrak{h}$. Since $\Gamma \subset S$, the same line L is also Γ–invariant. Exponentiating L yields a one-dimensional connected Γ–invariant subgroup $B \subset H$, which is normal in G.

This gives a final holomorphic fibration

$$\pi_3 : X \longrightarrow X_3 := G/B\Gamma$$

with fiber $B/(B \cap \Gamma)$. Two possibilities occur.

If $B/(B \cap \Gamma)$ is compact (for instance, an elliptic curve), then the fiber is compact, and since X_3 is compact by induction, the total space X must also be compact.

If $B/(B \cap \Gamma)$ is noncompact, then B is isomorphic either to $\mathbb{C}$ or to $\mathbb{C}^*$. In this case, the conjugation action of G on the normal subgroup B induces a nontrivial holomorphic character $\chi : G \to \mathbb{C}^*$. Since B is central in H, we have $H \subset \ker \chi$. Moreover, the condition that Γ preserves the discrete subgroup $B \cap \Gamma$ forces $\chi(\Gamma) \subset S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$.

Consequently, the fibration $X \to X_3$ is the holomorphic principal $\mathbb{C}^*$–bundle induced by χ. By the solvable $\mathbb{C}^*$–bundle obstruction (Theorem 3.6), the total space X then admits a nonconstant pluriharmonic function, contradicting the pseudoconcavity assumption.

In both cases we obtain a contradiction with the hypothesis that X is pseudoconcave and noncompact. Therefore X must be compact. □

Example 3.3 (Non-Example: A Non-Compact Solvmanifold). Consider the semidirect product $G = \mathbb{C} \ltimes \mathbb{C}^2$ with group law

$$(a; z_1, z_2)(a'; z_1', z_2') = (a + a'; z_1 + e^a z_1', z_2 + e^{-a} z_2').$$

Let Γ be the discrete subgroup generated by:

$$\begin{aligned}
&\gamma_1 = (1; 0, 0),\\
&\gamma_2 = (i; 0, 0), where\ i\ denotes\ the\ imaginary\ unit,\\
&\gamma_3 = (0; 1, 0),\\
&\gamma_4 = (0; 0, 1).
\end{aligned}$$

Then $X = G/\Gamma$ is a non-compact solvmanifold. The holomorphic character $\chi : G \to \mathbb{C}^*$ defined by $\chi(a; z_1, z_2) = e^a$ satisfies $\chi(\Gamma) \subset S^1$, since $\chi(\gamma_1) = e^1$, $\chi(\gamma_2) = e^i$ both have modulus 1. By Theorem 3.6, X admits a nonconstant pluriharmonic function and therefore cannot be pseudoconcave. This example illustrates how the main theorem operates: the non-compactness of X forces the existence of a $\mathbb{C}^*$-direction that obstructs pseudoconcavity.

Example 3.4 (A 4×4 Borel Group with $\mathbb{C}^*$-Obstruction). Let $G = B(4,\mathbb{C})$ be the subgroup of 4×4 invertible upper-triangular matrices. Consider the discrete subgroup:

$$\Gamma = \left\{ \begin{pmatrix} e^{2\pi i k_1} & m_{12} & m_{13} & m_{14} \\ 0 & e^{2\pi i k_2} & m_{23} & m_{24} \\ 0 & 0 & e^{2\pi i k_3} & m_{34} \\ 0 & 0 & 0 & e^{2\pi i k_4} \end{pmatrix} : \begin{array}{l} k_j \in \mathbb{Z},\ \sum k_j = 0, \\ m_{ij} \in \mathbb{Z}[i] \end{array} \right\}.$$

The commutator subgroup is:

$$G' = U = \left\{ \begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} : u_{ij} \in \mathbb{C} \right\},$$

and $\Lambda = \Gamma \cap G'$ consists of unipotent matrices with Gaussian integer entries. The Zariski closure A equals G'.

Following the structure theory, we find the canonical central subgroup:

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & u_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : u_{14} \in \mathbb{C} \right\} \cong \mathbb{C}.$$

The Γ-action on $\mathfrak{h}$ yields the 1-dimensional invariant subgroup:

$$B = H.$$

Now, $B \cap \Gamma$ consists of matrices with $u_{14} \in \mathbb{Z}[i]$, so $B/(B \cap \Gamma) \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is compact. However, let us examine the alternative scenario that would occur if the lattice were chosen differently.

Suppose instead we take:

$$\Gamma' = \left\{ \begin{pmatrix} e^{2\pi i k_1} & m_{12} & m_{13} & m_{14} \\ 0 & 1 & m_{23} & m_{24} \\ 0 & 0 & 1 & m_{34} \\ 0 & 0 & 0 & e^{-2\pi i k_1} \end{pmatrix} : k_1 \in \mathbb{Z},\ m_{ij} \in \mathbb{Z}[i] \right\}.$$

Then the same procedure yields $B \cong \mathbb{C}^*$ with $B/(B \cap \Gamma') \cong \mathbb{C}^*$, which is non-compact. The character $\chi : G \to \mathbb{C}^*$ given by $\chi(g) = g_{11}/g_{44}$ satisfies $\chi(\Gamma') \subset S^1$, and by Theorem 3.6, $X' = G/\Gamma'$ admits a nonconstant pluriharmonic function and cannot be pseudoconcave. This demonstrates how the theorem's mechanism detects non-pseudoconcavity through the $\mathbb{C}^*$-bundle obstruction.

Remark 3.9. This example illustrates the delicate interplay between the algebraic structure of the lattice Γ and the resulting geometry of $X = G/\Gamma$. In the first case with Γ, the invariant subgroup B is isomorphic to $\mathbb{C}$, and $B/(B \cap \Gamma)$ is compact (an elliptic curve), placing us in the first case of Theorem 3.8. However, with the modified lattice Γ', the invariant subgroup becomes $B \cong \mathbb{C}^*$ with non-compact quotient, activating the $\mathbb{C}^*$-bundle obstruction. The explicit character $\chi(g) = g_{11}/g_{44}$ with $\chi(\Gamma') \subset S^1$ provides the nonconstant pluriharmonic function that prevents pseudoconcavity. This demonstrates how the theorem's proof mechanism effectively distinguishes between compact and non-compact scenarios through the structure of the lattice.

4. CONCLUSION

These results are significant for the classification of complex homogeneous spaces and for understanding the interaction between complex geometry and Lie group structure. Moreover, the results developed above reveal a coherent picture linking the structure of complex solvmanifolds with the analytic property of pseudoconcavity. The key ingredients are the pluriharmonic obstructions provided by holomorphic principal $\mathbb{C}^*$–bundles and their flat or torsion Chern classes. Whenever a solvmanifold admits such a noncompact fiber direction, a nonconstant pluriharmonic function can be explicitly constructed, contradicting pseudoconcavity.

Combining these analytic obstructions with the algebraic structure theory of solvable linear algebraic groups leads to a precise dichotomy:

- The existence of a $\mathbb{C}^*$–fiber or a nontrivial holomorphic character implies the presence of a nonconstant pluriharmonic function.
- Pseudoconcave complex manifolds, on the other hand, admit only constant holomorphic (and hence pluriharmonic) functions.

Consequently, a pseudoconcave solvmanifold $X = G/\Gamma$ cannot contain any noncompact $\mathbb{C}$ or $\mathbb{C}^*$–direction. The only remaining possibility is that X is compact. This completes the proof of the compactness phenomenon for pseudoconcave solvmanifolds. Several directions for further research naturally arise from this work. One potential extension is to investigate whether analogous compactness results hold for pseudoconcave homogeneous spaces of solvable complex Lie groups that are not linear algebraic. Another direction is to study related Levi-type conditions weaker than pseudoconcavity and to determine whether similar obstruction phenomena occur in those settings. These questions may further clarify the role of complex-analytic conditions in the classification of homogeneous complex manifolds.

DECLARATION

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