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Analysis and Refinement of Unified Integral Operators using a convexity parameter Via Increasing Functions

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Abstract. This research focuses on establishing bounds of unified integral operators which are used in the refinement of convex functions with the parameters $(\alpha,\, h - m)$, emphasizing their behavior under strictly monotone increasing functions. Conditions are identified for the existence of such bounds, with special attention to power, logarithmic and reciprocal functions within the refined $(\alpha,\, h - m)$-convex framework. The study refines existing inequalities across diverse convex function classes, contributing to the broader field of functional analysis and its applications.

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1. Introduction

The theory of convexity is a core topic in the fields of functional analysis, mathematical inequalities and optimization theory. Analysing convexity exposes a researcher to the existence and characteristics of numerous functions. There has been much published work involving convex functions and related operators. The study has proven to be extremely fruitful, leading to advancements in both theoretical and applied mathematics. The $(\alpha, h - m)$-convexity condition describes a special class of convex functions parameterized by α, a convexity parameter, h a function which defines the nature of the convexity condition and m a constant, a positive integer or a real number that indicates the shift or a boundary in

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the convexity condition. Convexity is closely related to the existence and properties of certain functions. Classical mathematical inequalities like Jensen's inequality, Hölder's and Minkowski's inequalities have all been significantly refined and generalized over the years by applying convex functions and other well-known classes of functions. For example the Hadamard inequality is generalized; via s-convex and (s, m)-convex [6, 1], m-convex, (α, m)-convex [2, 3] and (h, m)-convex [9] functions, the Jensen inequality is generalized in [5] via φ-convex functions. The Hadamard inequality is also generalized by using different kinds of fractional integrals. For example, for Riemann-Liouville fractional integrals the Hadamard inequality is studied in [11, 12, 13, 14, 15]. These generalizations have broad applications in fields such as optimization, economics, statistics, information theory, and machine learning, leading to more significant tools for handling many problems.
The strictly monotonically increasing functions often play an important role in bounds of inequalities and other optimization problems, especially when establishing relationships between various classes of functions.
Unified integral operators are a class of operators used to generalize or unify certain classes of integral operators.
This paper aims to explore the behavior of unified integral operators applied to refined $(\alpha, h-m)$-convex functions, focusing on establishing upper/lower bounds under monotonic conditions. We will investigate how the refined $(\alpha, h-m)$-convexity impacts the behavior of these functions under certain integral operators. Our results will refine existing inequalities that involve classes of functions related to convexity. The inequalities are applied to sub classes of functions ,that is, log-convex, power-convex and exponential-convex functions.

2. Notations and Preliminaries

In the following, we give some important definitions required for the understanding of the rest of the paper.

Definition 2.1. [10] *A function* $\bigvee : \mathbb{R} \to \mathbb{R}$ *is called convex if*

$$\bigvee(\tau u_1 + (1-\tau)u_2) \leq \tau \bigvee(u_1) + (1-\tau)\bigvee(u_2), \tag{2. 1}$$

holds.

Definition 2.2. [9] *Let h be a function from J to R and $(0,1)$ be a subset of J. A function $\bigvee$ defined on $[0, v]$ is said to be (α, h, m)-convex provided that each $u_1, u_2 \in [0, v] \subseteq \mathbb{R}$ and $\bigvee, h \geq 0$ we have*

$$\bigvee(\tau u_1 + m(1-\tau)u_2) \leq h(\tau^\alpha)\bigvee(u_1) + mh(1-\tau^\alpha)\bigvee(u_2), \tag{2. 2}$$

where $(\alpha, m) \in [0,1]^2$ and $\tau \in (0,1)$.

Definition 2.3. *We assume $h, J, \bigvee$ are similar to Definition 2.2. $\bigvee$ is a refined $(\alpha, h-m)$-convex function, if*

$$\bigvee(\tau u_1 + m(1-\tau)u_2) \leq h(\tau^\alpha)h(1-\tau^\alpha)\left(\bigvee(u_1) + m\bigvee(u_2)\right), \tag{2. 3}$$

is satisfied.

Definition 2.4. [16] *We assume $h, J, \bigvee$ are similar to Definition 2.2. Also assume $I, K \subset (0,\infty)$ are two different intervals in $\mathbb{R}$. $\bigvee$ is refined $(\alpha, h-m)$-convex function via strictly monotone function $\vartheta : K \to \mathbb{R}$, if*

$$\bigvee \circ\vartheta^{-1}(tu_1 + m(1-\tau)u_2) \leq h(\tau^\alpha)h(1-\tau^\alpha)\left(\bigvee \circ\vartheta^{-1}(u_1) + m\bigvee \circ\vartheta^{-1}(u_2)\right), \tag{2. 4}$$

holds provided $tu_1 + m(1-\tau)u_2 \in Im(\vartheta)$ for $u_1, u_2 \in Im(\vartheta)$, where $(\alpha, m) \in (0,1]^2$ and $\tau \in (0,1)$.

Definition 2.5. [16] *We assume $h, J, \bigvee$ are similar to Definition 2.2. Also assume $I, K \subset (0,\infty)$ are two different intervals in $\mathbb{R}$. $\bigvee$ is refined $(\alpha, h-m)$-convex function via strictly monotone function $\vartheta : K \to \mathbb{R}$, if*

$$\bigvee \circ\vartheta^{-1}(tu_1 + m(1-\tau)u_2) \leq h(\tau^\alpha)h(1-\tau^\alpha)\left(\bigvee \circ\vartheta^{-1}(u_1) + m\bigvee \circ\vartheta^{-1}(u_2)\right), \tag{2. 5}$$

holds provided $tu_1 + m(1-\tau)u_2 \in Im(\vartheta)$ for $u_1, u_2 \in Im(\vartheta)$, where $(\alpha, m) \in (0,1]^2$ and $\tau \in (0,1)$.

Recently, the study of integral operators has been widely employed to derive new results in mathematical inequalities, as evident from [3, 6, 8]. We now introduce the definition of integral operators that will be utilized in subsequent discussions.

Definition 2.6. *Assume $0 < u_1 < u_2$ and two real-valued functions $\bigvee, \chi$ over $[u_1, u_2]$. In this case, χ is strictly increasing and differentiable, but $\bigvee$ is integrable and positive. Assume additionally that all of the parameters are complex, positive values and that $\frac{\varrho}{x}$ is an increasing function on $[u_1, \infty)$. Next, for any $x \in [u_1, u_2]$, the right and left integral operators are defined as follows:*

$$({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u_1^+}\bigvee)(x,\theta;p) = \int_{u_1}^{x} H_x^y(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\chi;\varrho)\chi'(y)\bigvee(y)dy, \tag{2. 6}$$

$$({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u_2^-}\bigvee)(x,\theta;p) = \int_{x}^{u_2} H_y^x(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\chi;\varrho)\chi'(y)\bigvee(y)dy, \tag{2. 7}$$

where

$$H_x^y(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\chi;\varrho) = \frac{\varrho(\chi(x)-\chi(y))}{\chi(x)-\chi(y)}E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(\theta(\chi(x)-\chi(y))^\kappa;p). \tag{2. 8}$$

Here E is the extended generalized Mittag-Leffler function.

In section 2, Theorem 3.1 and Theorem 3.13 provide bounds of unified integral operators as upper bounds and a modulus inequality. Theorem 3.2 and Theorem 3.14 provide the refinements under certain conditions. Theorem 3.8 provides the Hadamard inequality for symmetric like functions which satisfy Lemma 3.6. Theorem 3.13 also refines Theorem 3.9. Section 3 consists of applications of results of Section 2 under specific conditions. Also, some examples are identified.

3. Main Results

Following notation will be used in our results :

$$\int_0^1 h\left(q_1\right)^{\alpha} h\left(1-q_1^{\alpha}\right) \chi'(x-u(x-u_1))du = P_x{}^{u_1}(q_1^{\alpha}; h, \chi).$$

Theorem 3.1. *Let $\bigvee, h, \vartheta$ be are similar to Definition 2.5. Also, let $\varpi, \xi, \gamma, \iota, p, \kappa, \varepsilon, \delta$ are real and satisfy the convergence conditions of unified Mittag-Leffler functions. Then the following inequality holds:*

$$\begin{aligned}&\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p)+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p)\\&\leq H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(x-u_1)P_x{}^{u_1}(q_1^{\alpha};h,\chi)\\&+H_{u_2}^{x}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(u_2-x)P_{u_2}^{x}(q_2^{\alpha};h,\chi).\end{aligned} \tag{3. 9}$$

Proof. For $\frac{\varrho}{x}$ and χ, also For $x\in(u_1,u_2)$ and $\tau\in[u_1,x)$ we write

$$P_x^{\tau}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\chi'(\tau)\leq P_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\chi;\varrho)\chi'(\tau). \tag{3. 10}$$

Using Definition 2.5 , we obtain

$$\begin{aligned}&\bigvee\circ\vartheta^{-1}(\tau)\leq h\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}h\left(1-\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}\right)\\&\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right).\end{aligned} \tag{3. 11}$$

Using (3. 10) and (3. 11), one obtains

$$\begin{aligned}&\int_{u_1}^{x}P_x^{\tau}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\chi'(\tau)\bigvee\circ\vartheta^{-1}(\tau)d\tau\\&\leq P_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\\&\int_{u_1}^{x}h\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}h\left(1-\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}\right)\chi'(\tau)d\tau,\end{aligned} \tag{3. 12}$$

$$\begin{aligned}&\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p)\leq P_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\,(x-u_1)\\&\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\int_0^1 h(q_1^{\alpha})h(1-q_1^{\alpha})\chi'(x-u(x-u_1))du,\end{aligned} \tag{3. 13}$$

from which we obtain

$$\begin{aligned}&\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p)\leq P_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\,(x-u_1)\\&\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(x-u_1)P_x^{u_1}(q_1^{\alpha};h,\chi).\end{aligned} \tag{3. 14}$$

For $x \in (u_1, u_2)$ and $\tau \in (x, u_2]$, we have:

$$H_{\tau}^{x}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho)\chi'(\tau) \leq P_{u_2}^{x}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho)\chi'(\tau), \tag{3. 15}$$

$$\begin{aligned} \bigvee \circ\vartheta^{-1}(\tau) \leq h\left(\frac{\tau - x}{u_2 - x}\right)^{\alpha} h\left(1 - \left(\frac{\tau - x}{u_2 - x}\right)^{\alpha}\right) \\ \left(\bigvee \circ\vartheta^{-1}(u_2) + m \bigvee \circ\vartheta^{-1}\left(\frac{x}{m}\right)\right). \end{aligned} \tag{3. 16}$$

Using (3. 15) and (3. 16), we obtain

$$\begin{aligned} &\int_{x}^{u_2} H_{\tau}^{x}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho)\chi'(\tau) \bigvee \circ\vartheta^{-1}(\tau)d\tau \leq P_{u_2}^{x}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho) \\ &\left(\bigvee \circ\vartheta^{-1}(u_2) + m \bigvee \circ\vartheta^{-1}\left(\frac{x}{m}\right)\right) \\ &\int_{x}^{u_2} h\left(\frac{\tau - x}{u_2 - x}\right)^{\alpha} h\left(1 - \left(\frac{\tau - x}{u_2 - x}\right)^{\alpha}\right) \chi'(\tau)d\tau, \end{aligned}$$

from which we obtain

$$\begin{aligned} &\left({}_{\chi}\mathbb{F}_{\varepsilon,\varpi,\xi,u_2^-}^{\varrho,\gamma,\delta,k,\iota} \bigvee \circ\vartheta^{-1}\right)(x, \theta; p) \\ &\leq P_{u_2}^{x}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho) \left(\bigvee \circ\vartheta^{-1}(u_2) + m \bigvee \circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(u_2 - x)P_{u_2}^{x}(q_2^{\alpha}; h, \chi). \end{aligned} \tag{3. 17}$$

(3. 14) and (3. 17) give us (3. 9).

□

The subsequent corollary provides an enhancement of the theorem 3.1.

Corollary 3.2. *Subject to the conditions of Theorem 3.1 and taking* $0< h(\tau) <1$*, following result holds:*

$$\begin{aligned} &\left({}_{\chi}\mathbb{F}_{\kappa,\varpi,\xi,u_1+}^{\varrho,\gamma,\delta,k,\iota} \bigvee \circ\vartheta^{-1}\right)(x, \theta; p) + \left({}_{\chi}\mathbb{F}_{\varepsilon,\varpi,\xi,u_2-}^{\varrho,\gamma,\delta,k,\iota} \bigvee \circ\vartheta^{-1}\right)(x, \theta; p) \\ &\leq H_{x}^{u_1}(E_{\kappa,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho) \left(\bigvee \circ\vartheta^{-1}(u_1) + m \bigvee \circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(x - u_1){P_x}^{u_1}(q_1^{\alpha}; h, \chi) \\ &+ H_{u_2}^{x}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho) \left(\bigvee \circ\vartheta^{-1}(u_2) + m \bigvee \circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(u_2 - x)P_{u_2}^{x}(q_2^{\alpha}; h, \chi) \\ &\leq H_{x}^{u_1}(E_{\kappa,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho)(x - u_1) \left(\bigvee \circ\vartheta^{-1}(u_1)P_{x}^{u_1}(q_1^{\alpha}; h, \chi)\right. \\ &\left. + m \bigvee \circ\vartheta^{-1}\left(\frac{x}{m}\right) P_{x}^{u_1}(1 - q_1^{\alpha}; h, \chi)\right) + H_{u_2}^{x}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota}, \chi; \varrho)(u_2 - x) \\ &\left(\bigvee \circ\vartheta^{-1}(u_2)P_{u_2}^{x}(q_2^{\alpha}; h, \chi) + m \bigvee \circ\vartheta^{-1}\left(\frac{x}{m}\right) P_{u_2}^{x}(1 - q_2^{\alpha}; h, \chi)\right). \end{aligned} \tag{3. 18}$$

Corollary 3.3. *Complying with the terms of theorem 3.1, it follows that,*

$$\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p)+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p) \tag{3. 19}$$
$$\leq H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(x-u_1)\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)P_x{}^{u_1}(q_1^{\alpha};h,\chi)$$
$$+H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(u_2-x)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)P^{x}_{u_2}(q_2^{\alpha};h,\chi).$$

Corollary 3.4. *For $\alpha=1$, (3. 9) gives the refinement that follows.*

$$\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p)+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(x,\theta;p) \tag{3. 20}$$
$$\leq\Bigg(H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(\chi(x)-\chi(u_1))$$
$$+H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(\chi(u_2)-\chi(x))\Bigg)||h||_{\infty}^{2}$$
$$\leq\Bigg(H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(\chi(x)-\chi(u_1))$$
$$+H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(\chi(u_2)-\chi(x))\Bigg)||h||_{\infty}.$$

Remark 3.5. *(i) Using $\vartheta(x)=x$ in inequalities (3. 9), (3. 19), (3. 18) and (3. 20) one obtains* [16, Theorem 1], [16, Theorem 2], [16, Corollary 1] *and* [16, Corollary 2] *respectively.*

(ii) Using $\varrho(x)=\frac{x^{\frac{\alpha'}{k}}\Gamma(\alpha')}{k\Gamma_k(\alpha')}$, $\alpha'>k>0$ with $p=0=\varphi$ inequality (3. 9) gives result for refined (α, h-m) convex functions.
(iii) Using $\vartheta(x)=x=h(x)$ and $m=1=\alpha$ in inequalities (3. 9) and (3. 19) one obtains [4, Theorem 4] *and* [4, Corollary 1] *respectively.*
(iv) Using $\vartheta(x)=x^p$ in inequalities (3. 9), (3. 19), (3. 18) and (3. 20) one obtains [17, Theorem 1], [17, Corollary 1], [17, Theorem 2] *and* [17, Corollary 2] *respectively.*

The result of following lemma will be used in Theorem 3.8.

Lemma 3.6. *Let $\bigvee$ is similar to Definition 2.5. If $\bigvee\circ\vartheta^{-1}(x)=\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2-x}{m}\right)$, $x\in[u_1,u_2]$, then the following inequality holds:*

$$\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)\bigvee\circ\vartheta^{-1}(x). \tag{3. 21}$$

Proof. Utilizing Definition 2.5 for the function $\bigvee$, we can write

$$\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)$$
$$\left[\bigvee\circ\vartheta^{-1}\left(\frac{x-u_1}{u_2-u_1}u_2+\frac{u_2-x}{u_2-u_1}u_1\right)+m\bigvee\circ\vartheta^{-1}\left(\frac{\frac{x-u_1}{u_2-u_1}u_1+\frac{u_2-x}{u_2-u_1}u_2}{m}\right)\right]$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(\bigvee\circ\vartheta^{-1}(x)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2-x}{m}\right)\right).$$

Using $\bigvee\circ\vartheta^{-1}(x)=\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2-x}{m}\right)$ results in (3. 21). □

Remark 3.7. *(i) Using $\vartheta(x)=x$, (3. 21) gives* [16, Lemma 1].
(ii) Using $\vartheta(x)=x^p$, (3. 21) gives [17, Lemma 1].
(iii) Using $h(\tau)=\tau$, $m=\alpha=1$ as well as satisfying (i), (3. 21) gives [4, Lemma 1].

Theorem 3.8. *The following outcome is true under the assumptions of Theorem 3.1 and Lemma 3.6.*

$$\begin{aligned}&\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)}\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)\\&\left(\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}1\right)(u_1,\theta;p)+\left({}_\chi F^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}1\right)(u_2,\theta;p)\right)\\&\le\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(u_1,\theta;p)+\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(u_2,\theta;p)\\&\le(u_2-u_1)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right)\Big[H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)P^{u_1}_{u_2}(q_2^\alpha;h,\chi)\\&+H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)P^{u_1}_{u_2}(q_2^\alpha;h,\chi)\Big].\end{aligned}\tag{3. 22}$$

Proof. Using (2. 8) and function χ, following inequality holds:

$$H^{u_1}_{x}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\chi'(x)\le H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\chi'(x),\ x\in(u_1,u_2).\tag{3. 23}$$

Using Definition 2.5, we can write

$$\begin{aligned}&\bigvee\circ\vartheta^{-1}(x)\le h\left(\frac{x-u_1}{u_2-u_1}\right)^\alpha h\left(1-\left(\frac{x-u_1}{u_2-u_1}\right)^\alpha\right)\\&\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right).\end{aligned}\tag{3. 24}$$

Using (3. 23) and (3. 24) we obtain

$$\begin{aligned}&\int_{u_1}^{u_2}H^{u_1}_{x}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\bigvee\circ\vartheta^{-1}(x)\chi'(x)dx\\&\le H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right)\\&\int_{x_1}^{u_2}h\left(\frac{x-u_1}{u_2-u_1}\right)^\alpha h\left(1-\left(\frac{x-u_1}{u_2-u_1}\right)^\alpha\right)\chi'(x)dx.\end{aligned}$$

Modifying the variable, we derive

$$\begin{aligned}&\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(u_1,\theta;p)\\&\le H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(u_2-u_1)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right)P^{u_1}_{u_2}(q_2^\alpha;h,\chi).\end{aligned}\tag{3. 25}$$

For $x\in(u_1,u_2)$ we can write

$$H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\chi'(x)\le H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\chi'(x).\tag{3. 26}$$

Using (3. 24) and (3. 26), we obtain

$$\int_{u_1}^{u_2} H_{u_2}^{x}(E_{\kappa,\varpi,\xi}^{\gamma,\delta,k,\iota},\chi;\varrho)\chi'(x)\bigvee\circ\vartheta^{-1}(x)dx$$
$$\leq H_{u_2}^{u_1}(E_{\kappa,\varpi,\xi}^{\gamma,\delta,k,\iota},\chi;\varrho)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right)$$
$$\int_{u_1}^{u_2} h\left(\frac{x-u_1}{u_2-u_1}\right)^{\alpha} h\left(1-\left(\frac{x-u_1}{u_2-u_1}\right)^{\alpha}\right)\chi'(x)dx.$$

Changing the variable, we obtain

$$\left({}_{\chi}\mathbb{F}_{\kappa,\varpi,\xi,u_1^+}^{\varrho,\gamma,\delta,k,\iota}\bigvee\circ\vartheta^{-1}\right)(u_2,\theta;p) \tag{3. 27}$$
$$\leq H_{u_2}^{u_1}(E_{\kappa,\varpi,\xi}^{\gamma,\delta,k,\iota},\chi;\varrho)(u_2-u_1)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right)P_{u_2}^{u_1}(q_2^{\alpha};h,\chi).$$

Now, using Lemma 3.6 we can write

$$\int_{u_1}^{u_2}\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)H_{x}^{u_1}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota},\chi;\varrho)\chi'(x)dx$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)\int_{u_1}^{u_2} H_{x}^{u_1}(E_{\varepsilon,\varpi,\xi}^{\gamma,\delta,k,\iota},\chi;\varrho)\chi'(x)\bigvee\circ\vartheta^{-1}(x)dx,$$

which by using (2. 7) reduces to following integral inequality

$$\frac{1}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)}\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)\left({}_{\chi}\mathbb{F}_{\varepsilon,\varpi,\xi,u_2^-}^{\varrho,\gamma,\delta,k,\iota}1\right)(u_1,\theta;p) \tag{3. 28}$$
$$\leq\left({}_{\chi}\mathbb{F}_{\varepsilon,\varpi,\xi,u_2^-}^{\varrho,\gamma,\delta,k,\iota}\bigvee\circ\vartheta^{-1}\right)(u_1,\theta;p).$$

Again using Lemma 3.6 we can write

$$\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)H_{u_2}^{x}(E_{\kappa,\varpi,\xi}^{\gamma,\delta,k,\iota},\chi;\varrho)\chi'(x)dx \tag{3. 29}$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)\int_{u_1}^{u_2} H_{u_2}^{x}(E_{\kappa,\varpi,\xi}^{\gamma,\delta,k,\iota},\chi;\varrho)\chi'(x)\bigvee\circ\vartheta^{-1}(x)dx,$$

which by using (2. 6) reduces to following integral inequality

$$\frac{1}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)}\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)\left({}_{\chi}\mathbb{F}_{\kappa,\varpi,\xi,u_1^+}^{\varrho,\gamma,\delta,k,\iota}1\right)(u_2,\theta;p) \tag{3. 30}$$
$$\leq\left({}_{\chi}\mathbb{F}_{\kappa,\varpi,\xi,u_1^+}^{\varrho,\gamma,\delta,k,\iota}\bigvee\circ\vartheta^{-1}\right)(u_2,\theta;p).$$

The inequality (3. 22) follows by using (3. 25), (3. 27), (3. 28) and (3. 30). $\square$

Corollary 3.9. *Assuming the conditions of Theorem 3.8 in addition with* $0< h(\tau)< 1$*,*

$$
\begin{aligned}
&\frac{1}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)}\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right) \\
&\left(\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}1\right)(u_1,\theta;p)+\left({}_{\chi}F^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}1\right)(u_2,\theta;p)\right) \\
&\leq \frac{1}{h\left(\frac{1}{2^{\alpha}}\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)}\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right) \\
&\left(\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}1\right)(u_1,\theta;p)+\left({}_{\chi}F^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}1\right)(u_2,\theta;p)\right) \\
&\leq \left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(u_1,\theta;p)+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(u_2,\theta;p) \\
&\leq (u_2-u_1)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right)\Big[H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)P^{u_1}_{u_2}(q_2^{\alpha};h,\chi) \\
&+H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)P^{u_1}_{u_2}(q_2^{\alpha};h,\chi)\Big] \\
&\leq (u_2-u_1)\left(H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)+H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\right) \\
&\left(\bigvee\circ\vartheta^{-1}(u_2)P^{u_1}_{u_2}(q_2^{\alpha};h,\chi)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)P^{u_1}_{u_2}(1-q_2^{\alpha};h,\chi)\right),
\end{aligned}
\tag{3. 31}
$$

is satisfied.

Corollary 3.10. *Taking* $\kappa=\varepsilon$ *in (3. 22) following inequality holds true:*

$$
\begin{aligned}
&\frac{1}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)}\bigvee\circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)\left(\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_2^-}1\right)(u_1,\theta;p)\right. \\
&\left.+\left({}_{\chi}F^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}1\right)(u_2,\theta;p)\right) \\
&\leq \left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(u_1,\theta;p)+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(u_2,\theta;p) \\
&\leq 2(u_2-u_1)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right)H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)P^{u_1}_{u_2}(q_2^{\alpha};h,\chi).
\end{aligned}
\tag{3. 32}
$$

The upcoming corollary produces the refinement of [16, Theorem 3].

Corollary 3.11. *For $\alpha = 1$, (3. 22) gives the refinement that follows..*

$$\frac{1}{h^2\left(\frac{1}{2}\right)(m+1)} \bigvee \circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right) \tag{3. 33}$$
$$\left(\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u_2^-}1\right)(u_1,\theta;p) + \left({}_\chi F^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}1\right)(u_2,\theta;p)\right)$$
$$\leq \frac{1}{h\left(\frac{1}{2}\right)(m+1)} \bigvee \circ\vartheta^{-1}\left(\frac{u_1+u_2}{2}\right)$$
$$\left(\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u_2^-}1\right)(u_1,\theta;p) + \left({}_\chi F^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}1\right)(u_2,\theta;p)\right)$$
$$\leq \left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u_2^-}\bigvee\circ\vartheta^{-1}\right)(u_1,\theta;p) + \left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}\right)(u_2,\theta;p)$$
$$\leq 2\left(\bigvee\circ\vartheta^{-1}(u_2) + m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right) H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\chi(u_2)-\chi(u_1)\right)||h||^2_\infty$$
$$\leq 2\left(\bigvee\circ\vartheta^{-1}(u_2) + m\bigvee\circ\vartheta^{-1}\left(\frac{u_1}{m}\right)\right) H^{u_1}_{u_2}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\chi(u_2)-\chi(u_1)\right)||h||_\infty.$$

Remark 3.12. *(i) Using $\vartheta(x) = x$ inequality (3. 22) gives* [16, Theorem 3].
(ii) Using $\vartheta(x) = x$ inequality (3. 31) gives [16, Theorem 4].
(iii) Using $\vartheta(x) = x$ inequality (3. 32) gives [16, Corollary 3].
(iv) Using $\vartheta(x) = x$ inequality (3. 33) gives [16, Corollary 4].
(v) Using $\vartheta(x) = x^p$ inequality (3. 22) gives [17, Theorem 3].
(vi) Using $\vartheta(x) = x^p$ inequality (3. 32) gives [17, Corollary 4].
(vii) Using $\vartheta(x) = x^p$ inequality (3. 31) gives [17, Theorem 4].
(viii) Using $\vartheta(x) = x^p$ inequality (3. 33) gives [17, Corollary 5].
(ix) Using $h(\tau) = \tau$ and $m = 1 = \alpha$ as well as satisfying (i) inequality (3. 22) gives [4, Theorem 5].
(x) Using $h(\tau) = \tau$ and $m = 1 = \alpha$ as well as satisfying (i) inequality (3. 32) gives [4, Corollary 2].

Theorem 3.13. *Assume $\bigvee, \chi$ are two functions which are differentiable and $|\bigvee\circ\vartheta'^{-1}|$ is the refinement for the monotone increasing function ϑ and χ be strictly increasing over $[u_1, u_2]$ and differentiable over $[u_1, u_2]$. All other conditions are similar to Theorem 3.5. Then the following inequality holds:*

$$\left|\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}*\chi\right)(x,\theta;p) + \left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}*\chi\right)(x,\theta;p)\right| \tag{3. 34}$$
$$\leq H^{u_1}_x(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(x-u_1)\left(|\bigvee\circ\vartheta'^{-1}(u_1)| + m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right) P_x{}^{u_1}(q_1^\alpha;h,\chi)$$
$$+ H^x_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(u_2-x)\left(|\bigvee\circ\vartheta'^{-1}(u_2)| + m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right) P^x_{u_2}(q_2^\alpha;h,\chi),$$

where

$$\left({}_\chi\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}*\chi\right)(x,\theta;p) = \int_{u_1}^{x} H^\tau_x(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\chi'(\tau)\bigvee\circ\vartheta'^{-1}(\tau)d\tau, \tag{3. 35}$$

$$\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}*\chi\right)(x,\theta;p)=\int_x^{u_2}H_\tau^x(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\chi'(\tau)\bigvee\circ\vartheta'^{-1}(\tau)d\tau. \tag{3. 36}$$

Proof. Using Definition 2.5 for the absolute function $|\bigvee\circ\vartheta'^{-1}|$ over $[u_1,u_2]$ we can write

$$|\bigvee\circ\vartheta'^{-1}(\tau)|\leq h\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}h\left(1-\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}\right) \tag{3. 37}$$
$$\left(|\bigvee\circ\vartheta'^{-1}(u_1)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right),$$

$$-h\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}h\left(1-\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}\right)\left(|\bigvee\circ\vartheta'^{-1}(u_1)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right) \tag{3. 38}$$
$$\leq\bigvee\circ\vartheta'^{-1}(\tau)\leq$$
$$h\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}h\left(1-\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}\right)\left(|\bigvee\circ\vartheta'^{-1}(u_1)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right).$$

Using (3. 10) and (3. 38) we obtain

$$\int_{x_1}^{x}H_x^{\tau}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\chi'(\tau)\bigvee\circ\vartheta'^{-1}(\tau)|d\tau \tag{3. 39}$$
$$\leq H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(|\bigvee\circ\vartheta'^{-1}(u_1)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)$$
$$\int_{u_1}^{x}h\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}h\left(1-\left(\frac{x-\tau}{x-u_1}\right)^{\alpha}\right)\chi'(\tau)d\tau,$$

which further gives

$$\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}*\Delta\right)(x,\theta;p)\leq \tag{3. 40}$$
$$H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(x-u_1)\left(\left|\bigvee\circ\vartheta'^{-1}(u_1)\right|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)P_x{}^{u_1}(q_1^{\alpha};h,\chi).$$

Also from (3. 10) and (3. 38) we have

$$\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}*\Delta\right)(x,\theta;p)\geq \tag{3. 41}$$
$$-H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(x-u_1)\left(\left|\bigvee\circ\vartheta'^{-1}(u_1)\right|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)P_x{}^{u_1}(q_1^{\alpha};h,\chi).$$

Utilizing refined $(\alpha,h-m)$-convexity of $|\bigvee\circ\vartheta'^{-1}|$ over $[u_1,u_2]$ one more time we can write

$$|\bigvee\circ\vartheta'^{-1}(\tau)|\leq h\left(\frac{\tau-x}{u_2-x}\right)^{\alpha}h\left(1-\left(\frac{\tau-x}{u_2-x}\right)^{\alpha}\right) \tag{3. 42}$$
$$\left(m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|+\left|\bigvee\circ\vartheta'^{-1}(u_2)\right|\right)$$

$$h\left(\frac{\tau-x}{u_2-x}\right)^{\alpha} h\left(1-\left(\frac{\tau-x}{u_2-x}\right)^{\alpha}\right)\left(m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|+\left|\bigvee\circ\vartheta'^{-1}(u_2)\right|\right) \quad (3.\ 43)$$
$$\leq \bigvee\circ\vartheta'^{-1}(\tau)$$
$$\leq h\left(\frac{\tau-x}{u_2-x}\right)^{\alpha} h\left(1-\left(\frac{\tau-x}{u_2-x}\right)^{\alpha}\right)\left(m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|+\left|\bigvee\circ\vartheta'^{-1}(u_2)\right|\right),$$

Using (3. 15) and (3. 43) we obtain

$$\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}*\Delta\right)(x,\theta;p)\leq \quad (3.\ 44)$$
$$H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(u_2-x)\left(|\bigvee\circ\vartheta'^{-1}(u_2)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)P^{x}_{u_2}(q_2^{\alpha};h,\chi).$$

Also from (3. 15) and (3. 43) we have

$$\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}*\Delta\right)(x,\theta;p)\geq \quad (3.\ 45)$$
$$-H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(u_2-x)\left(|\bigvee\circ\vartheta'^{-1}(u_2)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)P^{x}_{u_2}(q_2^{\alpha};h,\chi).$$

The inequality (3. 34) follows by using (3. 40), (3. 41), (3. 44) and (3. 45).

□

Next result presents the refinement of Theorem 3.13.

Corollary 3.14. *Assuming the conditions of Theorem 3.13 in addition with* $0< h(\tau)< 1$*,*

$$\left|\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\circ\vartheta^{-1}*\chi\right)(x,\theta;p)+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\circ\vartheta^{-1}*\chi\right)(x,\theta;p)\right| \quad (3.\ 46)$$
$$\leq P^{u_1}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(x-u_1)\left(|\bigvee\circ\vartheta'^{-1}(u_1)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right){P_x}^{u_1}(q_1^{\alpha};h,\chi)$$
$$+H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(u_2-x)\left(|\bigvee\circ\vartheta'^{-1}(u_2)|+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)P^{x}_{u_2}(q_2^{\alpha};h,\chi)$$
$$\leq H^{u_1}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(x-u_1)\left(|\bigvee\circ\vartheta'^{-1}(u_1)|{P_x}^{u_1}(q_1^{\alpha};h,\chi)\right.$$
$$\left.+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|{P_x}^{u_1}(1-q_1^{\alpha};h,\chi)\right)+H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(u_2-x)$$
$$\left(|\bigvee\circ\vartheta'^{-1}(u_2)|P^{x}_{u_2}(q_2^{\alpha};h,\chi)+m\left|\bigvee\circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|P^{x}_{u_2}(1-q_2^{\alpha};h,\chi)\right),$$

is satisfied.

Corollary 3.15. *For $\kappa = \varepsilon$, (3. 34) shows the following result:*

$$\begin{aligned} &\left|\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+} \bigvee \circ\vartheta^{-1} * \chi\right)(x,\theta;p) + \left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_2^-} \bigvee \circ\vartheta^{-1} * \chi\right)(x,\theta;p)\right| \\ &\leq H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(x-u_1)\left(|\bigvee \circ\vartheta'^{-1}(u_1)| + m\left|\bigvee \circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right) \\ &P_x{}^{u_1}(q_1^{\alpha};h,\chi) + H_{u_2}^{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(u_2-x)\left(|\bigvee \circ\vartheta'^{-1}(u_2)|\right. \\ &\left.+m\left|\bigvee \circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right) P_{u_2}^{x}(q_2^{\alpha};h,\chi). \end{aligned} \tag{3. 47}$$

The upcoming corollary produces the refinement of [16, Theorem 5].

Corollary 3.16. *For $\alpha = 1$, (3. 34) gives the refinement that follows.*

$$\begin{aligned} &\left|\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+} \bigvee \circ\vartheta^{-1} * \chi\right)(x,\theta;p) + \left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-} \bigvee \circ\vartheta^{-1} * \chi\right)(x,\theta;p)\right| \\ &\leq \Bigg[H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(\chi(x)-\chi(u_1))\left(|\bigvee \circ\vartheta'^{-1}(u_1)| + m\left|\bigvee \circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right) \\ &+H_{u_2}^{x}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(\chi(u_2)-\chi(x))\left(\left|\bigvee \circ\vartheta'^{-1}(u_2)\right| + m\left|\bigvee \circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)\Bigg]\|h\|_{\infty}^2 \\ &\leq \Bigg[H_x^{u_1}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)(\chi(x)-\chi(u_1))\left(|\bigvee \circ\vartheta'^{-1}(u_1)| + m\left|\bigvee \circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right) \\ &+H_{u_2}^{x}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)(\chi(u_2)-\chi(x))\left(\left|\bigvee \circ\vartheta'^{-1}(u_2)\right| + m\left|\bigvee \circ\vartheta'^{-1}\left(\frac{x}{m}\right)\right|\right)\Bigg]\|h\|_{\infty}. \end{aligned} \tag{3. 48}$$

Remark 3.17.

(i) *Using $\vartheta(x) = x$ inequality (3. 34) gives* [16, Theorem 5].

(ii) *Using $\vartheta(x) = x^p$ inequality (3. 34) gives* [17, Theorem 5].

(iii) *Using $\vartheta(x) = x$ inequality (3. 46) gives* [16, Theorem 6].

(iv) *Using $\vartheta(x) = x^p$ inequality (3. 46) gives* [17, Theorem 6].

(v) *Using $\vartheta(x) = x$ inequality (3. 47) gives* [16, Corollary 5].

(vi) *Using $\vartheta(x) = x$ inequality (3. 48) gives* [16, Corollary 6].

(vii) *Using $\vartheta(x) = x^p$ inequality (3. 47) gives* [17, Corollary 7].

(viii) *Using $\vartheta(x) = x^p$ inequality (3. 48) gives* [17, Corollary 8].

(ix) *Using $h(\tau) = \tau$ and $m = 1 = \alpha$ as well as satisfying (i) inequality (3. 34) gives* [4, Theorem 6].

(x) *Using the conditions of (ix) inequality (3. 47) gives* [4, Corollary 3].

4. SOME SPECIAL CASES FOR REFINED CONVEX FUNCTIONS VIA STRICTLY INCREASING FUNCTIONS

In this section, some results for refined $(\alpha, h-m)-$convex function for a strictly monotone function using the results of previous section are presented. The given results are the refinements of the results in [17]. Throughout this section we assume that $p' = 0 = \varphi$.

Proposition 4.1. *By adhering the assumptions of Theorem 3.1,*

$$\begin{aligned}&\Gamma(\varpi)(({}_{\chi}^{\varpi}I_{u_1^+}\bigvee\circ\vartheta^{-1})(x)+({}_{\chi}^{\varpi}I_{u_2^-}\bigvee\circ\vartheta^{-1})(x)) \qquad (4.\ 49)\\&\leq(\chi(x)-\chi(u_1))^{\varpi-1}\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(x-u_1)P_x{}^{u_1}(q_1^{\alpha};h,\chi)\\&+(\chi(u_2)-\chi(x))^{\varpi-1}\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(u_2-x)P_{u_2}^{x}(q_2^{\alpha};h,\chi),\end{aligned}$$

is obtained.

Proof. Using $\varrho(\tau) = \tau^{\varpi}, \varpi > 0$ and following the proof of Theorem 3.1, (4. 49) is obtained. □

Proposition 4.2. *Theorem 3.1 yields the following inequality:*

$$\begin{aligned}&\Gamma(\varpi)\left(({}_{u_1^+}I_{\varrho}\bigvee\circ\vartheta^{-1})(x)+({}_{u_2^-}I_{\varrho}\bigvee\circ\vartheta^{-1})(x)\right) \qquad (4.\ 50)\\&\leq\varrho(x-u_1)\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\int_0^1 h\,(u)^{\alpha}\,h\,(1-q_1^{\alpha})\,du\\&+\varrho(u_2-x)\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\int_0^1 h(v^{\alpha})h\,(1-v^{\alpha})\,dv.\end{aligned}$$

Proof. Using $\chi = I$ we obtain the result. □

Corollary 4.3. *For $\varrho(\tau) = \frac{\Gamma(\varpi)\tau^{\frac{\varpi}{k}}}{k\Gamma_k(\varpi)}$, unified integral operators reduce to the following bound.*

$$\begin{aligned}&k\Gamma_k(\varpi)\left[({}_{\chi}^{\varpi}I_{u_1^+}^{k}\bigvee\circ\vartheta^{-1})(x)+({}_{\chi}^{\varpi}I_{u_2^-}^{k}\bigvee\circ\vartheta^{-1})(x)\right] \qquad (4.\ 51)\\&\leq(\chi(x)-\chi(u_1))^{\frac{\varpi}{k}-1}\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(x-u_1)P_x{}^{u_1}(q_1^{\alpha};h,\chi)\\&+(\chi(u_2)-\chi(x))^{\frac{\varpi}{k}-1}\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)(u_2-x)P_{u_2}^{x}(q_2^{\alpha};h,\chi).\end{aligned}$$

Corollary 4.4. *Applying $\varrho(\tau) = \tau^{\varpi}$ and χ as identity function for $\varpi \geq 1$, (2. 6) and (2. 7) give fractional integral ${}^{\varpi}I_{u_1^+}\bigvee\circ\vartheta^{-1}(x)$ and ${}^{\varpi}I_{u_2^-}\bigvee\circ\vartheta^{-1}(x)$ defined in* [7]*, which satisfy the following upper bound.*

$$\begin{aligned}&\Gamma(\varpi)\left(({}^{\varpi}I_{u_1^+}\bigvee\circ\vartheta^{-1})(x)+({}^{\varpi}I_{u_2^-}\bigvee\circ\vartheta^{-1})(x)\right)\\&\leq(x-u_1)^{\varpi}\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\int_0^1 h\,(u)^{\alpha}\,h\,(1-q_1^{\alpha})\,du\\&+(u_2-x)^{\varpi}\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\int_0^1 h\,(v)^{\alpha}\,h\,(1-q_2^{\alpha})\,dv.\end{aligned}$$

Corollary 4.5. *Applying* $\varrho(\tau)=\frac{\Gamma(\varpi)\tau^{\frac{\varpi}{k}}}{k\Gamma_k(\varpi)}$ *and* χ *as identity function, (2. 6) and (2. 7) satisfy the following upper bound.*

$$\begin{aligned}
&({}^{\varpi}I^{k}_{u_1^+}\bigvee\circ\vartheta^{-1})(x)+({}^{\varpi}I^{k}_{u_2^-}\bigvee\circ\vartheta^{-1})(x) \qquad (4.\ 52)\\
&\leq\frac{1}{k\Gamma_k(\varpi)}\Big[(x-u_1)^{\frac{\varpi}{k}}\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\\
&\int_0^1 h\left(u\right)^{\alpha}h\left(1-q_1^{\alpha}\right)du+(u_2-x)^{\frac{\varpi}{k}}\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\\
&\int_0^1 h\left(q_2^{\alpha}\right)h\left(1-q_2^{\alpha}\right)dv\Big].
\end{aligned}$$

Corollary 4.6. *For* $k=1$ *in Corollary 4.5 the following upper bound for Riemann-Liouville fractional integral is satisfied.*

$$\begin{aligned}
&({}^{\varpi}I_{u_1^+}\bigvee\circ\vartheta^{-1})(x)+({}^{\varpi}I_{u_2^-}\bigvee\circ\vartheta^{-1})(x) \qquad (4.\ 53)\\
&\leq\frac{1}{\Gamma(\varpi)}\Big[(x-u_1)^{\varpi}\left(\bigvee\circ\vartheta^{-1}(u_1)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\\
&\int_0^1 h\left(u\right)^{\alpha}h\left(1-q_1^{\alpha}\right)du+(u_2-x)^{\varpi}\left(\bigvee\circ\vartheta^{-1}(u_2)+m\bigvee\circ\vartheta^{-1}\left(\frac{x}{m}\right)\right)\\
&\int_0^1 h\left(q_2^{\alpha}\right)h\left(1-q_2^{\alpha}\right)dv\Big].
\end{aligned}$$

4.7. **Examples.**

Example 4.8. *For* $\vartheta(x)=x^p$*, we obtain the following inequality for Theorem 3.1.*

$$\begin{aligned}
&\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1+}\bigvee\right)(x,\theta;p')+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2-}\bigvee\right)(x,\theta;p') \qquad (4.\ 54)\\
&\leq H^{u_1}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\bigvee(u_1)^{\frac{1}{p}}+m\bigvee\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)(x-u_1){P_x}^{u_1}(q_1^{\alpha};h,\chi)\\
&+H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\left(\bigvee(u_2)^{\frac{1}{p}}+m\bigvee\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)(u_2-x)P^{x}_{u_2}(q_2^{\alpha};h,\chi).
\end{aligned}$$

Example 4.9. *For* $\vartheta(x)=ln(x)$*, we obtain the following inequality for Theorem 3.1.*

$$\begin{aligned}
&\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1+}\bigvee\right)(x,\theta;p')+\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2-}\bigvee\right)(x,\theta;p') \qquad (4.\ 55)\\
&\leq H^{u_1}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\bigvee(e^{u_1})+m\bigvee\left(e^{\frac{x}{m}}\right)\right)(x-u_1){P_x}^{u_1}(q_1^{\alpha};h,\chi)\\
&+H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\left(\bigvee(e^{u_2})+m\bigvee\left(e^{\frac{x}{m}}\right)\right)(u_2-x)P^{x}_{u_2}(q_2^{\alpha};h,\chi).
\end{aligned}$$

Example 4.10. *For $\vartheta(x) = \frac{1}{x}$, we obtain the following inequality for Theorem 3.1.*

$$\begin{aligned}
&\left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi,u_1^+}\bigvee\right)(x,\theta;p') + \left({}_{\chi}\mathbb{F}^{\varrho,\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi,u_2^-}\bigvee\right)(x,\theta;p') \qquad (4.\ 56)\\
&\leq H^{u_1}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\varpi,\xi},\chi;\varrho)\left(\bigvee\left(\frac{1}{u_1}\right) + m\bigvee\left(\frac{m}{x}\right)\right)(x-u_1){P_x}^{u_1}(q_1^{\alpha};h,\chi)\\
&+ H^{x}_{u_2}(E^{\gamma,\delta,k,\iota}_{\varepsilon,\varpi,\xi},\chi;\varrho)\left(\bigvee\left(\frac{1}{u_2}\right) + m\bigvee\left(\frac{m}{x}\right)\right)(u_2-x)P^{x}_{u_2}(q_2^{\alpha};h,\chi).
\end{aligned}$$

The reader may obtain similar examples for other theorems.

5. Conclusion

This article deals with the bounds of unified integral operator for refined $(\alpha, h-m)$-convex functions considering the effect of a strictly monotone increasing function ϑ. The results so obtained are linked with the previously published results, which consequently serve as refinements in particular cases. One can formulate bounds of fractional integral operators of several kinds for some new classes of functions deducible from Definition 2.5. This could provide tools for establishing bounds for several types of operators, which can be applied to new function classes.

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All the authors contributed equally for this article.

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References

[1] G. A. Anastassiou, *Generalized fractional Hermite-Hadamard inequalities involving m-convexity and (s, m)-convexity*, Ser. Math. Inform. **28**, No. 2 (2013) 107-126.

[2] M. K. Bakula, M. E. Ozdemir, J. Pecarivc, *Hadamard type inequalities for m-convex and (α, m)-convex functions*, JIPAM **9**, No. 4 (2008) 12pp.

[3] M. Bombardelli, S. Varosanec, *Properties of (α, m)-convex functions related to the Hermite-Hadamard-Fejer inequalities*, Comput. Math. Appl. **58**, No. 9 (2009) 1869-1877.

[4] G. Farid, M. Zahra, *Some integral inequalities involving Mittag-Leffler functions for tgs-convex functions*, Comp and Math Methods **5**, (2021) 80-93.

[5] M. E. Gordji, M. R. Delavar, M. De la Sen, *On φ-convex functions*, J. Math. Inequal **10**, No. 1 (2016) 173-183.

[6] S. Hussain, M. I. Bhatti, M. Iqbal, *Hadamard-type inequalities for s-convex functions*, J. Math **41**, (2009) 51-60.

[7] A. A. Kilbas, H. M. Srivastava, J. J Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2006.

[8] M. E. Özdemir, M. Avc, E. Set, *On some inequalities of Hermite-Hadamard type via m-convexity*, Appl. Math. Lett **23**, (2010) 1065-1070.

[9] M. E. Özdemir, A. O. Akdemri, E. Set, *On (h, m)-convexity and Hadamard type inequalities*, Transylv. J. Math. Mech **8**, No. 1 (2016) 51-58.

[10] A. W. Roberts, D. E. Varberg, *Convex functions*, Academic Press, New York, 1973. NY, USA.

[11] M. Z. Sarikaya, M. Dahmani, M. E. Kiris, F. Ahmad, *(k, s)-Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat **45**, No. 1 (2016) 77-89. doi:10.15672/HJMS.20164512484.

[12] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, *Hermite-Hadamard inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model **57**, No. 9 (2013) 2403-2407.

[13] M. Z. Sarikaya, H. Yildirim, *On Hermite-Hadamard type inequalities for RL fractional integrals*, Miskolc Math. Notes **17**, No. 2 (2017) 1049-1059.

[14] E. Set, B. Celik, *Fractional Hermite-Hadamard type inequalities for quasi-convex functions*, Ordu Univ. J. Sci. Tech **6**, No. 1 (2016) 137-149.

[15] J. Tariboon, S. K. Ntouyas, W. Sudsutad, *Some new RL fractional integral inequalities*, Int. J. Math. Math. Sci **2014**, Article ID 869434, 6 pp.

[16] M. Zahra, M. Ashraf, G. Farid, K. Nonlapon, *Some new kinds of fractional integral inequalities via refined $(\alpha, h-m)$-convex function*, Math. Prob. Eng., **2021**, Article ID 8331092, 15 pages.

[17] M. Zahra, M. Ashraf, G. Farid, K. Nonlapon, *Inequalities for unified integral operators of generalized refined convex functions*, Aims. Math **7**, No. 4 (2022) 6218-6223.