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Energy Efficiency and Sustainability for Industrial Systems: DAM-LATEA Based IRTGA Approach

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Abstract. Industrial energy systems are under intense pressure to transition toward sustainable operations due to the rising energy demands and the adverse environmental impacts of fossil fuels dependency. The energy efficiency and sustainability are not just environmental imperatives but also key drivers of economic progress in global world. However, existing optimization approaches depend on crisp data, static expert judgments, and single-layer evaluation techniques which struggle to handle real world ambiguity. To meet these challenges, a novel hybrid decision making framework DAM-LATEA based Interval Rough Technique for Order Preference by Similarity to Ideal Solution and Interval Rough Grey Relational Analysis (IRTGA) is put forward. The incorporation of Dynamic Assessment Methodology (DAM) provides a flexible and evocative approach to handle uncertain and Lexical Assessment Temporal Evaluation Approach (LATEA) quantifies expert's judgments about the importance of criteria by preserving the hierarchy of expert's opinion weights. For selection of the best alternative, the Interval Rough Topsis and Grey Analysis (IRTGA) approach is implemented. The proposed DAM-LATEA-IRTGA framework significantly outperforms traditional MCDM approaches by shifting from static, crisp, and single-method decision-making to a dynamic, entropy-based objective weighting and uncertainty-aware intelligent decision system. The novelty of this work is to successfully addresses the uncertain, imprecise and incomplete information in sustainability-oriented optimization problems. The model integrates hybrid IR-TOPSIS and IR-GRA, combining distance-based and relational analysis for more stable and accurate rankings. In addition, the successful implementation of this study in the wool textile industry confirms the practical feasibility and industrial significance of the methodology for sustainable energy optimization and green industrial transformation. Comparative and sensitivity analyses

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confirm the model's robustness and adaptability across fluctuating decision scenarios in real-world.

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1. Introduction

Energy has become a cornerstone of modern civilization and a driving force of industrial production, transportation systems, and essential public services. However, increasing energy demands, driven by population growth, industrial expansion, and urbanization have accelerated the depletion of fossil fuels and intensified environmental degradation. Industrial energy consumption accounts for a significant share of global energy use, making it essential for factories to transition to renewable, efficient, and cost-effective energy systems. Industrial energy systems, particularly in energy-intensive sectors such as textile manufacturing, are under increasing pressure to achieve sustainability, cost efficiency, and carbon reduction simultaneously. Although renewable energy integration and optimization techniques have been widely explored, real industrial environments remain highly uncertain due to fluctuating energy demand, incomplete data, expert subjectivity, and dynamic operational conditions. In such contexts, conventional decision-making models often fail to provide reliable and adaptive solutions, making advanced intelligent frameworks essential for a practical energy transition. The open literature on multi-criteria decision making[12], [3], [13], focuses on crisp data and fixed environments; however, limited attention has been given to handling dynamic uncertainty in decision-making scenario.

Existing methodologies [11], [6] for industrial energy optimization primarily focus on the:

- Classical MCDM techniques [9], such as AHP, Fuzzy Logic, and TOPSIS
- Crisp or deterministic decision-making models
- Static expert-based weighting systems
- Cost and emission minimization under simplified assumptions

Despite significant progress in multi-criteria decision-making for energy optimization [1], [10], several critical research gaps still persist in the existing literature.

First, most conventional models exhibit limited capacity to handle uncertainty, as they are often unable to adequately represent the vagueness, ambiguity, and incomplete information inherent in real industrial environments. Second, many approaches rely on static decision assumptions, where expert judgments are considered fixed, thereby neglecting temporal variations in opinions and dynamic changes in industrial operating conditions. Integration of IRNs with entropy and temporal weighting mechanisms such as LATEA is rare. Third, there is a noticeable lack of fully integrated hybrid frameworks that simultaneously combine multiple uncertainty handling techniques such as rough sets, entropy-based weighting, and fuzzy MCDM models within a unified structure. MCDM has evolved from crisp models to hybrid approaches that integrates fuzzy sets, intuitionistic fuzzy sets (IFS), and rough sets. Fuzzy sets, introduced by Zadeh (1965), allow partial membership values to model imprecise data. Atanassov's IFS extended this by incorporating degrees of non-membership. Rough sets by Pawlak (1982) introduced boundary approximations to represent uncertainty in classification. However, these models often fall short in scenarios involving both hesitancy and interval-based vagueness. The emergence of IRNs addresses this gap by defining both lower and upper approximations in a bounded decision space. IRNs have demonstrated superiority in modeling real-world ambiguity in energy, medical, and environmental decision systems. Furthermore, a large proportion of existing studies suffer from insufficient real-world validation, as they are primarily tested on synthetic or hypothetical datasets rather than actual industrial systems. Finally, these models demonstrate weak adaptability to Industry 4.0 environments, as they are not fully aligned with the modern requirements of data-driven, intelligent, and adaptive decision-making systems.

To overcome these limitations, it is essential to develop:

- Robust framework capable of handling uncertainty, imprecision, and ambiguous information
- Reducing subjectivity in the decision-making approach
- Integrated multi-method decision-making systems that improve the stability and robustness of rankings.

  This study addresses the above gaps by proposing a novel hybrid DAM-LATEA-IRTGA framework that:
- Integrates Interval Rough Numbers (IRNs) to model uncertainty in a realistic and mathematically rigorous way
- Incorporates entropy-based objective weighting combined with temporal expert evaluation (LATEA)
- Develops a hybrid IR-TOPSIS and IR-GRA framework for enhanced ranking reliability and stability
- Provides a dynamic, adaptive, and uncertainty aware decision making structure suitable for industrial systems
- Validates the model through a real-world woolen textile factory as case study, ensuring practical relevance
- Supports sustainable energy transition, cost reduction, and carbon footprint minimization in industrial environments

Finally, the proposed framework advances industrial energy optimization by developing a unified uncertainty-aware, temporally adaptive, and hybrid MCDM framework that bridges the gap between theoretical decision models and real-world sustainable industrial energy systems.

This paper is structured as follows: Section 2 contains literature review. Section 3 and Section 4 outlines the case study with detailed definitions of alternatives and criteria. Section 5 introduces mathematical preliminaries on IRNs. Section 6 presents the proposed DAM-LATEA-IRTGA methodology. Section 7 provides a theoretical justification and methodology visualization. Section 8 contains application of the proposed methodology. Section 9 contains results and sensitivity analysis.

2. Literature Review

2.1. **Case-Based Literature on Energy Efficiency.** Research in energy efficiency has expanded across several industrial domains, using MCDM methods to evaluate alternative energy sources under complex criteria. For example, Meel et al. (2022) [8] used GRA-TOPSIS to optimize EDM parameters for magnesium alloys. Hasanzadeh et al. (2023) [4] investigated plastic waste gasification using GRA and TOPSIS to identify sustainable waste-to-energy strategies. Sanghvi et al. (2021) [12] compared GRA, TOPSIS, and fuzzy logic methods in machining applications. Other significant studies include Dai et al. (2024) [3], which incorporated game theory with GRA-TOPSIS to evaluate carrying capacities and Zhang and Duan (2024) [15], who examined water resource capacities via GRA-TOPSIS-CCDM. Zhang et al. (2023) [14] integrated TIFN-DEMATEL, EWM, GRA and TOPSIS to assess emergency evacuation capacities of Beijing metro stations, providing a robust and reliable framework for safety enhancements . Javeed et al. (2018) [5] studied the stability analysis and solutions of dynamical models. Sanghvi et al. (2021) [12] compared GRA, TOPSIS, and Fuzzy Logic in optimizing end milling of Inconel 825, introducing an 'Index of difference' parameter for large-scale comparisons. Bao et al. (2023)[2] evaluated black carrot juice sterilization using PCA, TOPSIS and GRA, optimizing sensory and functional properties for enhanced marketability. In [9] the concept of Picture hesitant fuzzy sets (PHFSs) was extended to picture type-2 hesitant fuzzy set (shortly PT2HFS) and the MCDM algorithm was defined to indicate effective of Vector similarity measures over PT2HFS. Ozlu et al. (2025) [11] addresses to Bipolar-Valued Complex Hesitant Fuzzy sets (BVCHFSs) to decode inconsistent, complexity data because of including bipolarity being opposite polar, complexity dividing membership value into two parts, hesitation degree including several membership values. Kumar and Mondal (2020) [7] optimized EDM parameters for M2 steel using Taguchi-based TOPSIS and GRA, highlighting precise parameter control for optimal material removal and surface finish .

3. Motivation

As societies transition toward cleaner energy, industrial decision-makers are challenged by incomplete data, fluctuating environmental conditions, and subjective human judgment. This uncertainty undermines the efficacy of traditional decision-support systems. So the proposed framework provides an optimal decision making by keep in view on :

- Dynamic assessment through DAM,
- Lexical assesment temporal evaluation via LATEA,
- Interval rough information representation,
- Temporal adaptability,
- Enhanced uncertainty handling.

This provides a resilient and interpretable mathematical structure that captures hesitation, imprecision, and conflicting expert opinions within a unified framework. In contexts like energy sustainability, such tools are indispensable.

4. Alternatives and Criteria

4.1. **Alternatives:** The study evaluates five energy sources: A1:Biomass energy, A2:Solar energy, A3: Wind energy, A4: Ocean energy, A5: Natural gas and Petroleum.

(1) **Biomass energy:** Biomass energy is an environment friendly source of energy, can be utilized in sectors with organic by-products by turning industrial waste into energy. Adoption of these renewable sources enables industries to comply with environmental regulations, reduce pollution, costs and their carbon footprint in environment.
(2) **Solar energy**: Solar energy is a viable solution for industrial applications through large scale photovoltaic (PV) systems and solar thermal technologies. The implementation cost at initial stage is high but the long-term energy savings and reduced carbon emissions can make it a sustainable alternative.
(3) **Wind energy**: Wind energy is also an another powerful option for supply of energy in industries, located in areas with strong wind currents. On-site wind farms or contracts with renewable energy providers, allow industries to fulfill their energy demands while reducing operational costs.
(4) **Ocean energy**: Utilizing ocean waves and currents is also a main source of energy. It is the third largest source of renewable energy.
(5) **Natural gas and petroleum**: Natural gas is a cleaner-burning fossil fuel used for electricity generation. o The other source is petroleum also known as crude oil refined into fuels such as gasoline, diesel and jet fuel, powering industries. Both of these sources are major contributors and prompting a global shift toward cleaner and sustainable energy alternatives.

4.2. **Criteria:** Six performance indicators are used to evaluate the performance of alternatives: C1: Economic viability, C2: Cost-effectiveness, C3: Environmental impact, C4: Lower emissions, C5: Scalability / Flexibility, C6: Regulatory compliance.

Targets/Attributes	**Code**	**Description**
Economic viability	C1	Return on investment and long term economic sustainability.
Cost-Effectiveness	C2	The alternative must balance initial cost with long-term economic benefits, offering a reasonable payback period and reducing operational expenses.
Environmental Impact	C3	The optimal solution should reduce environmental harm by lowering pollution, conserving resources, and promoting sustainable practices.
Lower Emissions	C4	For cleaner and healthier environment the alternative should offer lower emissions and lower carbon footprints. The lower emissions are necessary for reducing air pollution and combating climate change.
Scalability & Flexibility	C5	The solution should accommodate future technological advancements, be scalable for larger operations, and adaptable to various environments.
Regulatory Compliance	C6	The alternative must comply with national and international legal frameworks on energy and environmental standards.

Each criterion is classified as either benefit or cost criteria, and weighted using entropy-based methods.

5. Notations and Preliminaries

5.1. Mathematical approach in problem solution.

Definition 5.2. *A tuple $\langle[\underline{r}_1,\underline{r}_2],[\overline{r}_1,\overline{r}_2]\rangle$ is said to be an IRN if $\underline{r}_1,\underline{r}_2,\overline{r}_1,\overline{r}_2 \in [-1,1]$, $\underline{r}_1 \le \underline{r}_2$, $\overline{r}_1 \le \overline{r}_2$.*

Definition 5.3. *Define the unit IRN as: **1** $= [\underline{1},\overline{1}] = [[1,1],[1,1]]$*

Notation 1. (1) *Above given IRN may also be denoted as follows:*

$$\langle[\underline{r}_1,\underline{r}_2],[\overline{r}_1,\overline{r}_2]\rangle = \langle\underline{r},\overline{r}\rangle = r$$

(2) *Subscripted IRN (say) r_k shall be denoted as: $r_k = \left\langle \underline{r}_1^k,\overline{r}_2^k\right\rangle = \left\langle \left[\underline{r}_1^k,\underline{r}_2^k\right],\left[\overline{r}_1^k,\overline{r}_2^k\right]\right\rangle$*

(3) *In case, r^{ij} denotes ij^{th} IRN entry of a matrix, we may also use following notational conventions*

$$r^{ij} = \left\langle \underline{r}^{ij},\overline{r}^{ij}\right\rangle = \left\langle \left[\underline{r}_1^{ij},\underline{r}_2^{ij}\right],\left[\overline{r}_1^{ij},\overline{r}_2^{ij}\right]\right\rangle$$

Definition 5.4. *The interval rough numbers are characterized by specific arithmetic operations that is different from those dealing with typical rough numbers. Given any two interval rough numbers*

$$r_1 = \langle\underline{r}_1,\overline{r}_1\rangle = \left\langle \left[\underline{r}_1^1,\underline{r}_2^1\right],\left[\overline{r}_1^1,\overline{r}_2^1\right]\right\rangle \text{ and } r_2 = \langle\underline{r}_2,\overline{r}_2\rangle = \left\langle \left[\underline{r}_1^2,\underline{r}_2^2\right],\left[\overline{r}_1^2,\overline{r}_2^2\right]\right\rangle$$

The following are the arithmetic laws defined on these interval rough numbers.

5.5. **Arithmetical laws on IRNs.** *The arithmetic operations on interval rough numbers are defined as follows:*

(1) *Addition of two interval rough numbers:*

$$\begin{aligned} r_1 \oplus r_2 &= \left\langle \left[\underline{r}_1^1,\underline{r}_2^1\right],\left[\overline{r}_1^1,\overline{r}_2^1\right]\right\rangle \\ &\oplus \left\langle \left[\underline{r}_1^2,\underline{r}_2^2\right],\left[\overline{r}_1^2,\overline{r}_2^2\right]\right\rangle \\ &= \left\langle \left[\underline{r}_1^1+\underline{r}_1^2,\underline{r}_2^1+\underline{r}_2^2\right],\left[\overline{r}_1^1+\overline{r}_1^2,\overline{r}_2^1+\overline{r}_2^2\right]\right\rangle \end{aligned}$$

(2) *Subtraction of two interval rough numbers:*

$$\begin{aligned} r_1 \ominus r_2 &= \left\langle \left[\underline{r}_1^1,\underline{r}_2^1\right],\left[\overline{r}_1^1,\overline{r}_2^1\right]\right\rangle \\ &\ominus \left\langle \left[\underline{r}_1^2,\underline{r}_2^2\right],\left[\overline{r}_1^2,\overline{r}_2^2\right]\right\rangle \\ &= \left\langle \left[\underline{r}_1^1-\underline{r}_1^2,\underline{r}_2^1-\underline{r}_2^2\right],\left[\overline{r}_1^1-\overline{r}_1^2,\overline{r}_2^1-\overline{r}_2^2\right]\right\rangle \end{aligned}$$

(3) *Multiplication of two interval rough numbers:*

$$\begin{aligned} r_1 \otimes r_2 &= \left\langle \left[\underline{r}_1^1,\underline{r}_2^1\right],\left[\overline{r}_1^1,\overline{r}_2^1\right]\right\rangle \\ &\otimes \left\langle \left[\underline{r}_1^2,\underline{r}_2^2\right],\left[\overline{r}_1^2,\overline{r}_2^2\right]\right\rangle \\ &= \left\langle \left[\underline{r}_1^1\underline{r}_1^2,\underline{r}_2^1\underline{r}_2^2\right],\left[\overline{r}_1^1\overline{r}_1^2,\overline{r}_2^1\overline{r}_2^2\right]\right\rangle \end{aligned}$$

(4) *Division of two interval rough numbers:*

$$\begin{aligned}\frac{r_1}{r_2} &= \frac{\left\langle \left[\underline{r}_1^1, \underline{r}_2^1\right], \left[\overline{r}_1^1, \overline{r}_2^1\right] \right\rangle}{\left\langle \left[\underline{r}_1^2, \underline{r}_2^2\right], \left[\overline{r}_1^2, \overline{r}_2^2\right] \right\rangle} \\ &= \left\langle \left[\frac{\underline{r}_1^1}{\underline{r}_1^2}, \frac{\underline{r}_2^1}{\underline{r}_2^2}\right], \left[\frac{\overline{r}_1^1}{\overline{r}_1^2}, \frac{\overline{r}_2^1}{\overline{r}_2^2}\right] \right\rangle\end{aligned}$$

Scalar multiplication of an interval rough number with a scalar k.

$$k \times r = k \times \langle \underline{r}_1, \overline{r}_1 \rangle = \left\langle \left[k \times \underline{r}_1^1, k \times \underline{r}_2^1\right], \left[k \times \overline{r}_1^1, k \times \overline{r}_2^1\right] \right\rangle$$

5.5.1. *Innovative Contribution to Mathematical Structure Of IRNs.*

(1) Power of an interval rough number is calculated as:

$$r_1^n = \langle \underline{r}_1, \overline{r}_1 \rangle^n = \left\langle \left[\underline{r}_1^1, \underline{r}_2^1\right], \left[\overline{r}_1^1, \overline{r}_2^1\right] \right\rangle^n = \left(\left[m_1' - R_1', m_1' + R_1'\right], \left[m_2' - R_2', m_2' + R_2'\right]\right)$$

where

$$m_1' = \left(\tfrac{\underline{r}_1^1 + \underline{r}_2^1}{2}\right)^n \;,\; m_2' = \left(\tfrac{\overline{r}_1^1 + \overline{r}_2^1}{2}\right)^n$$

$$R_1' = R_1 \left|n\right| \left(\frac{\underline{r}_2^{1\left(\frac{n-1}{2}\right)} + \underline{r}_1^{1\left(\frac{n-1}{2}\right)}}{2}\right) \;,\; R_2' = R_2 \left|n\right| \left(\frac{\overline{r}_2^{1\left(\frac{n-1}{2}\right)} + \overline{r}_1^{1\left(\frac{n-1}{2}\right)}}{2}\right)$$

$$R_1 = \tfrac{\underline{r}_2^1 - \underline{r}_1^{\,1}}{2} \,, \quad R_2 = \tfrac{\overline{r}_2^1 - \overline{r}_1^{\,1}}{2}$$

Based upon the above definitions and notations we can immediately prove following two theorems. Proofs of these theorems are straight forward, hence only statement (3) of second theorem 5.7 is proved below.

Theorem 5.6. *Let $r_1 = \langle \underline{r}_1, \overline{r}_1 \rangle$ and $r_2 = \langle \underline{r}_2, \overline{r}_2 \rangle$ be two interval rough numbers. Then the following properties hold:*

(1) ***Commutativity of Addition:*** $r_1 \oplus r_2 = r_2 \oplus r_1$.
(2) ***Associativity of Addition:*** $(r_1 \oplus r_2) \oplus r_3 = r_1 \oplus (r_2 \oplus r_3)$.
(3) ***Existence of Additive Identity:*** *There exists an interval rough number $\mathbf{0} = \langle [0,0], [0,0] \rangle$ such that:* $r_1 \oplus \mathbf{0} = r_1$.
(4) ***Existence of Additive Inverse:*** *For every interval rough number r_1, there exists an interval rough number $-r_1 = \langle -\underline{r}_1, -\overline{r}_1 \rangle$ such that:* $r_1 \oplus (-r_1) = \mathbf{0}$.
(5) ***Commutativity of Multiplication:*** $r_1 \otimes r_2 = r_2 \otimes r_1$.
(6) ***Associativity of Multiplication:*** $(r_1 \otimes r_2) \otimes r_3 = r_1 \otimes (r_2 \otimes r_3)$.
(7) ***Existence of Multiplicative Identity:*** *There exists an interval rough number $\mathbf{1} = \langle [1,1], [1,1] \rangle$ such that:* $r_1 \otimes \mathbf{1} = r_1$.
(8) ***Distributivity of Multiplication over Addition*** $r_1 \otimes (r_2 \oplus r_3) = (r_1 \otimes r_2) \oplus (r_1 \otimes r_3)$.
(9) ***Scalar Multiplication:*** *For any scalar $k \in \mathbb{R}$ and an interval rough number $r = \langle \underline{r}, \overline{r} \rangle$: $k \times r = \langle k \times \underline{r}, k \times \overline{r} \rangle$.*
(10) ***Power of an Interval Rough Number:*** *For any interval rough number $r_1 = \langle \underline{r}_1, \overline{r}_1 \rangle$ and a positive integer n:*

$$r_1^n = \left\langle \left(\frac{\underline{r}_1^1 + \underline{r}_2^1}{2}\right)^n - R_1', \left(\frac{\overline{r}_1^1 + \overline{r}_2^1}{2}\right)^n + R_2' \right\rangle$$

where R_1' and R_2' are defined as in the previous definitions.

Theorem 5.7. *Let $r = \langle \underline{r}, \overline{r} \rangle$ be an interval rough number and $n \in \mathbb{N}$. Then the following hold:*

(1) *If $r_1 \leq r_2$, then: $r_1 \oplus r_3 \leq r_2 \oplus r_3, \quad \forall r_3$.*
(2) *If $r_1 \leq r_2$, then: $r_1 \otimes r_3 \leq r_2 \otimes r_3, \quad \forall r_3 \geq \mathbf{0}$.*
(3) *For any $n \in \mathbb{N}$: $(r_1 \oplus r_2)^n = \sum_{k=0}^{n} \binom{n}{k} r_1^k \otimes r_2^{n-k}$.*

Proof. We need to prove that for any $n \in \mathbb{N}$:

$$(r_1 \oplus r_2)^n = \sum_{k=0}^{n} \binom{n}{k} r_1^k \otimes r_2^{n-k}$$

Base Case: $n = 1$ For $n = 1$, the left-hand side of the equation is: $(r_1 \oplus r_2)^1 = r_1 \oplus r_2$
The right-hand side of the equation is:

$$\sum_{k=0}^{1} \binom{1}{k} r_1^k \otimes r_2^{1-k} = \binom{1}{0} r_1^0 \otimes r_2^1 + \binom{1}{1} r_1^1 \otimes r_2^0$$

Since $\binom{1}{0} = 1$ and $\binom{1}{1} = 1$, and $r_1^0 = r_2^0 = 1$, this simplifies to: $r_2 + r_1 = r_1 \oplus r_2$
Thus, the base case holds true.
Inductive Step: Assume that the formula holds for some $n = m \in \mathbb{N}$, i.e.,

$$(r_1 \oplus r_2)^m = \sum_{k=0}^{m} \binom{m}{k} r_1^k \otimes r_2^{m-k}$$

We need to show that the formula also holds for $n = m + 1$.
Consider:

$$(r_1 \oplus r_2)^{m+1} = (r_1 \oplus r_2)^m \otimes (r_1 \oplus r_2)$$

Using the inductive hypothesis, we substitute:

$$(r_1 \oplus r_2)^{m+1} = \left(\sum_{k=0}^{m} \binom{m}{k} r_1^k \otimes r_2^{m-k} \right) \otimes (r_1 \oplus r_2)$$

Expanding the product using the distributive property:

$$(r_1 \oplus r_2)^{m+1} = \sum_{k=0}^{m} \binom{m}{k} \left(r_1^k \otimes r_2^{m-k} \right) \otimes (r_1 \oplus r_2)$$

Now, distribute $r_1 \oplus r_2$:

$$(r_1 \oplus r_2)^{m+1} = \sum_{k=0}^{m} \binom{m}{k} \left(r_1^k \otimes r_1 \otimes r_2^{m-k} + r_1^k \otimes r_2^{m-k} \otimes r_2 \right)$$

Simplifying each term, this becomes:

$$(r_1 \oplus r_2)^{m+1} = \sum_{k=0}^{m} \binom{m}{k} \left(r_1^{k+1} \otimes r_2^{m-k} + r_1^k \otimes r_2^{m+1-k} \right)$$

We can re-index the summation terms to combine them:

$$(r_1 \oplus r_2)^{m+1} = \sum_{k=1}^{m+1} \binom{m}{k-1} r_1^k \otimes r_2^{m+1-k} + \sum_{k=0}^{m} \binom{m}{k} r_1^k \otimes r_2^{m+1-k}$$

Combine the two sums using Pascal's identity $\binom{m}{k-1} + \binom{m}{k} = \binom{m+1}{k}$:

$$(r_1 \oplus r_2)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} r_1^k \otimes r_2^{m+1-k}$$

This completes the inductive step. By mathematical induction, the given identity holds for all $n \in \mathbb{N}$.

$\square$

6. DAM-LATEA-IRTGA METHODOLOGY

The methodology includes:

- Constructing the interval rough decision matrix.
  The case of energy efficiency and sustainability consists of m alternatives $A_i, i = 1, 2, .., m$ that are evaluated on the basis of n criteria $C_j, j = 1, 2, .., n$ by k decision makers $DM_1, DM_2, ..., DM_k$. The decision makers utilized the linguistic terms to express their opinions and their opinions are rated numerically as:

Linguistic terms	Symbols	Rating
Extremely Poor	EP	0.1
Very Poor	VP	0.2
Poor	P	0.3
Slightly Poor	SP	0.4
Fair	F	0.5
Slightly Good	SG	0.6
Good	G	0.7
Very Good	VG	0.8
Extremely Good	EG	0.9

  The numerical opinions of decision makers $DM_1, DM_2, ..., DM_k$ in favor of an alternative are taken as membership values and reservations as non membership values. The membership values are also termed as lower bounds and non-membership values as upper bounds.
  These interval numbers which are basically judgments of decision makers $DM_1, DM_2, ..., DM_k$ construct the decision matrix D as:

$$\begin{array}{c} \\ D_1 \\ D_2 \\ \vdots \\ \\ D_k \end{array} \begin{bmatrix} C_1 & C_2 & \cdots & C_n \\ \langle [\underline{r}^{11}_{1k}, \underline{r}^{11}_{2k}], [\overline{r}^{11}_{1k}, \overline{r}^{11}_{2k}] \rangle & \langle [\underline{r}^{12}_{1k}, \underline{r}^{12}_{2k}], [\overline{r}^{12}_{1k}, \overline{r}^{12}_{2k}] \rangle & \cdots & \langle [\underline{r}^{1n}_{1k}, \underline{r}^{1n}_{2k}], [\overline{r}^{1n}_{1k}, \overline{r}^{1n}_{2k}] \rangle \\ \langle [\underline{r}^{21}_{1k}, \underline{r}^{21}_{2k}], [\overline{r}^{21}_{1k}, \overline{r}^{21}_{2k}] \rangle & \langle [\underline{r}^{22}_{1k}, \underline{r}^{22}_{2k}], [\overline{r}^{22}_{1k}, \overline{r}^{22}_{2k}] \rangle & \cdots & \langle [\underline{r}^{2n}_{1k}, \underline{r}^{2n}_{2k}], [\overline{r}^{2n}_{1k}, \overline{r}^{2n}_{2k}] \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \\ \langle [\underline{r}^{m1}_{1k}, \underline{r}^{m1}_{2k}], [\overline{r}^{m1}_{1k}, \overline{r}^{m1}_{2k}] \rangle & \langle [\underline{r}^{m2}_{1k}, \underline{r}^{m2}_{2k}], [\overline{r}^{m2}_{1k}, \overline{r}^{m2}_{2k}] \rangle & \cdots & \langle [\underline{r}^{mn}_{1k}, \underline{r}^{mn}_{2k}], [\overline{r}^{mn}_{1k}, \overline{r}^{mn}_{2k}] \rangle \end{bmatrix} \tag{6. 1}$$

$\underline{r}^{ij} = \left[\underline{r}_{1k}^{11}, \underline{r}_{2k}^{11}\right]$ represents membership values and $\overline{r}^{ij} = \left[\overline{r}_{1k}^{11}, \overline{r}_{2k}^{11}\right]$ non membership values.

- Boundary Region of an IRN is the difference between membership values and non membership values. Matrix Bd(D) computes this boundary for each r^{ij} in

$$Bd(D) = \left[b^{ij}\right] = \left[1 - \underline{r}^{ij} - \overline{r}^{ij}\right] = \left[1 - \underline{r}_{2k}^{11} - \overline{r}_{2k}^{11}, 1 - \underline{r}_{1k}^{11} - \overline{r}_{1k}^{11}\right] \tag{6. 2}$$

- Computing entropy weights.
  Entropy represented as E is the disorder or randomness in a system; essentially, it quantifies the degree of uncertainty or unpredictability within a system. Here it is computed as:

$$E = \frac{\left[\underline{r}_1^{ij} \wedge \overline{r}_1^{ij}, \underline{r}_2^{ij} \wedge \overline{r}_2^{ij}\right] \oplus \left[1 - \underline{r}_2^{ij} \wedge 1 - \underline{r}_1^{ij}, 1 - \overline{r}_2^{ij} \wedge 1 - \overline{r}_1^{ij}\right]}{\left[\underline{r}_1^{ij} \vee \overline{r}_1^{ij}, \underline{r}_2^{ij} \vee \overline{r}_2^{ij}\right] \oplus \left[1 - \underline{r}_2^{ij} \vee 1 - \underline{r}_1^{ij}, 1 - \overline{r}_2^{ij} \vee 1 - \overline{r}_1^{ij}\right]} \tag{6. 3}$$

  Entropy matrix is normalized to obtain a dimensionless form of the data .

$$NormE = \left[\eta^{ij}\right]_{m\times n} = \leftrightarrows \left[\frac{e_1^{ij}}{\bigvee_{k=1}^{m}\left(e_1^{kj}\right)}, \frac{e_2^{ij}}{\bigvee_{k=1}^{m}\left(e_2^{kj}\right)}\right] \tag{6. 4}$$

- Applying LATEA for time-based expert input evaluation.
  The decision-making experts have different prestige, decision power and risk preferences. To value the opinions of decision makers in different times the duration of their skills or experiences are also valued.

Linguistic terms	Symbols	Scale
Beginner	Bg	$\frac{1}{5}$
Small experienced	Se	$\frac{2}{5}$
Low skilled	Ls	$\frac{3}{5}$
Moderately Skilled	Ms	$\frac{4}{5}$
Experienced and Skilled	Es	1

TABLE 2. Linguistic terms and scaling

- The criteria weights reflect their relative importance in view of decision makers. After assessment by DMs the entropy weights are aggregated to obtain criteria weights.

$$B^j = \sum_{\substack{k=j\\ i=1}}^{m} \eta^{ik} \tag{6. 5}$$

$$T = \sum_{j=1}^{n} B^j = \left[\sum_{j=1}^{n} B_1^j, \sum_{j=1}^{n} B_2^j\right] \tag{6. 6}$$

$$W^j = \left[w_1^j, w_2^j\right] = \frac{\left[1 - B_2^j, 1 - B_1^j\right]}{[n,n] - \left[\sum_{j=1}^n B_1^j, \sum_{j=1}^n B_2^j\right]} \tag{6. 7}$$

The w_1^j are lower weights and w_2^j are upper weights of IRN criteria weights.

- The weighted matrices constructed by extending the formula
$\lambda I = \left(1 - (1-\mu)^\lambda, v^\lambda\right)$ for interval rough weights, the weighted decision matrix D^*l and D^*u are calculated for lower weights wl and upper weights wu respectively:

$$\begin{gathered}
(D_1)^{uv} = \left[\left[1 - \underline{r}_2^{ij}, 1 - \underline{r}_1^{ij}\right], \left[\overline{r}_1^{ij}, \overline{r}_2^{ij}\right]\right] \\
D_{l_1}^* = \left[\left(d_{l_1}^*\right)^{ij}\right] = \left(\left[\left[1 - \underline{r}_2^{ij}, 1 - \underline{r}_1^{ij}\right], \left[\overline{r}_1^{ij}, \overline{r}_2^{ij}\right]\right]\right)^{w_1^j} \\
D_l^* = \left[\left[s^{ij}, t^{ij}, u^{ij}\right]\right]_{m \times n} \\
s^{ij} = \left[1 - \underline{\left(d_{l_1}^*\right)}_2^{ij}, \underline{\left(d_{l_1}^*\right)}_1^{ij}\right] \\
t^{ij} = \overline{\left(d_{l_1}^*\right)}^{ij} \\
u^{ij} = \left[1 - \left(1 - \underline{\left(d_{l_1}^*\right)}_1^{ij}\right) - \overline{\left(d_{l_1}^*\right)}_2^{ij}, 1 - \left(1 - \underline{\left(d_{l_1}^*\right)}_2^{ij}\right) - \overline{\left(d_{l_1}^*\right)}_1^{ij}\right]
\end{gathered} \tag{6. 8}$$

Now following same steps for calculating weighted decision matrix D^*u for upper weights wu.

$$\begin{gathered}
D_{u_1}^* = \left[\left(d_{u_1}^*\right)^{ij}\right] = \left(\left[\left[1 - \underline{r}_2^{ij}, 1 - \underline{r}_1^{ij}\right], \left[\overline{r}_1^{ij}, \overline{r}_2^{ij}\right]\right]\right)^{w_2^j} \\
D_u^* = \left[\left[p^{ij}, q^{ij}, v^{ij}\right]\right]_{m \times n} \\
p^{ij} = \left[1 - \underline{\left(d_{u_1}^*\right)}_2^{ij}, \underline{\left(d_{u_1}^*\right)}_1^{ij}\right] \\
q^{ij} = \overline{\left(d_{u_1}^*\right)}^{ij} \\
v^{ij} = \left[1 - \left(1 - \underline{\left(d_{u_1}^*\right)}_1^{ij}\right) - \overline{\left(d_{u_1}^*\right)}_2^{ij}, 1 - \left(1 - \underline{\left(d_{u_1}^*\right)}_2^{ij}\right) - \overline{\left(d_{u_1}^*\right)}_1^{ij}\right]
\end{gathered} \tag{6. 9}$$

- Implementing IR-TOPSIS method
The assessment criteria are divided into benefit criteria and loss criteria by using IR-Topsis method.
For benefit criteria we take maximum value of membership and minimum value of non-membership of the alternatives for positive ideal solution and vice versa for negative ideal solution. Consider $C_B = C_p, ...C_s$ as benefit criteria and $C_L = C_u, ...C_x$ as loss criteria.

$$^+D_{l_{A1}}^* = \left[\bigvee_{\substack{k=1...m \\ j=p...s}} \underline{\left(d_l^*\right)}_1^{kj}, \bigwedge_{\substack{k=1...m \\ j=p...s}} \overline{\left(d_l^*\right)}_1^{kj}, \bigwedge_{\substack{k=1...m \\ j=u...x}} \underline{\left(d_l^*\right)}_1^{kj}, \bigvee_{\substack{k=1...m \\ j=u...x}} \overline{\left(d_l^*\right)}_1^{kj}\right] \tag{6. 10}$$

$$ {}^{+}D^{*}_{l_{uA1}} = \left[\bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_l^{*})}_2^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_l^{*})}_2^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=u\ldots x}} \underline{(d_l^{*})}_2^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=u\ldots x}} \overline{(d_l^{*})}_2^{kj} \right] \tag{6. 11} $$

$$ bd^{+}D^{*}_{l_{luA1}} = \left[\mathbf{1} - \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_l^{*})}_1^{kj} - \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_l^{*})}_1^{kj}, \mathbf{1} - \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_l^{*})}_2^{kj} - \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_l^{*})}_2^{kj} \right] \tag{6. 12} $$

$$ {}^{-}D^{*}_{l_{lA1}} = \left[\bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_l^{*})}_1^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_l^{*})}_1^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=u\ldots x}} \underline{(d_l^{*})}_1^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=u\ldots x}} \overline{(d_l^{*})}_1^{kj} \right] \tag{6. 13} $$

$$ {}^{-}D^{*}_{l_{uA1}} = \left[\bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_l^{*})}_2^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_l^{*})}_2^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=u\ldots x}} \underline{(d_l^{*})}_2^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=u\ldots x}} \overline{(d_l^{*})}_2^{kj} \right] \tag{6. 14} $$

$$ bd^{-}D^{*}_{l_{luA1}} = \left[\mathbf{1} - \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_l^{*})}_1^{kj} - \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_l^{*})}_1^{kj}, \mathbf{1} - \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_l^{*})}_2^{kj} - \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_l^{*})}_2^{kj} \right] \tag{6. 15} $$

$$ {}^{+}D^{*}_{u_{lA2}} = \left[\bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_u^{*})}_1^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_u^{*})}_1^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=u\ldots x}} \underline{(d_u^{*})}_1^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=u\ldots x}} \overline{(d_u^{*})}_1^{kj} \right] \tag{6. 16} $$

$$ {}^{+}D^{*}_{u_{uA2}} = \left[\bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_u^{*})}_2^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_u^{*})}_2^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=u\ldots x}} \underline{(d_u^{*})}_2^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=u\ldots x}} \overline{(d_u^{*})}_2^{kj} \right] \tag{6. 17} $$

$$ bd^{+}D^{*}_{u_{luA2}} = \left[\mathbf{1} - \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_u^{*})}_1^{kj} - \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_u^{*})}_1^{kj}, \mathbf{1} - \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_u^{*})}_2^{kj} - \bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_u^{*})}_2^{kj} \right] \tag{6. 18} $$

$$ {}^{-}D^{*}_{u_{lA2}} = \left[\bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}} \underline{(d_u^{*})}_1^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}} \overline{(d_u^{*})}_1^{kj}, \bigvee_{\substack{k=1\ldots m\\ j=u\ldots x}} \underline{(d_u^{*})}_1^{kj}, \bigwedge_{\substack{k=1\ldots m\\ j=u\ldots x}} \overline{(d_u^{*})}_1^{kj} \right] \tag{6. 19} $$

$$^{-}D^{*}_{u_{uA2}}=\left[\bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}}\underline{(d_u^*)}_2^{kj},\ \bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}}\overline{(d_u^*)}_2^{kj},\ \bigvee_{\substack{k=1\ldots m\\ j=u\ldots x}}\underline{(d_u^*)}_2^{kj},\ \bigwedge_{\substack{k=1\ldots m\\ j=u\ldots x}}\overline{(d_u^*)}_2^{kj}\right] \tag{6. 20}$$

$$bd^{-}D^{*}_{u_{luA2}}=\left[\mathbf{1}-\bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}}\underline{(d_u^*)}_1^{kj}-\bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}}\overline{(d_u^*)}_1^{kj},\ \mathbf{1}-\bigvee_{\substack{k=1\ldots m\\ j=p\ldots s}}\underline{(d_u^*)}_2^{kj}-\bigwedge_{\substack{k=1\ldots m\\ j=p\ldots s}}\overline{(d_u^*)}_2^{kj}\right] \tag{6. 21}$$

- Calculating the distance measured from positive ideal solutions and negative ideal solutions:
  The optimal alternative has lowest distance from positive ideal alternative and farthest distance from the negative alternative, vice versa for pessimist alternative.

$$^{+}S_1=\left[\left(\frac{1}{2}\left(\left(\underline{(^{+}d^*_{l_{lA1}})}^{1j}-\underline{(d_l^*)}_1^{ij}\right)^2+\left(\overline{(^{+}d^*_{l_{lA1}})}^{1j}-\overline{(d_l^*)}_1^{ij}\right)^2+\left(bd\underline{(^{+}d^*_{l_{luA1}})}^{1j}-bd\,(d_l^*)_1^{ij}\right)^2\right)\right)^{\frac{1}{2}},\left(\frac{1}{2}\left(\left(\underline{(^{+}d^*_{l_{uA1}})}^{1j}-\underline{(d_l^*)}_2^{ij}\right)^2+\left(\overline{(^{+}d^*_{l_{uA1}})}^{1j}-\overline{(d_l^*)}_2^{ij}\right)^2+\left(bd\overline{(^{+}d^*_{l_{luA1}})}^{1j}-bd\,(d_l^*)_2^{ij}\right)^2\right)\right)^{\frac{1}{2}}\right] \tag{6. 22}$$

$$^{-}S_1=\left[\left(\frac{1}{2}\left(\left(\underline{(^{-}d^*_{l_{lA1}})}^{1j}-\underline{(d_l^*)}_1^{ij}\right)^2+\left(\overline{(^{-}d^*_{l_{lA1}})}^{1j}-\overline{(d_l^*)}_1^{ij}\right)^2+\left(bd\underline{(^{-}d^*_{l_{luA1}})}^{1j}-bd\,(d_l^*)_1^{ij}\right)^2\right)\right)^{\frac{1}{2}},\left(\frac{1}{2}\left(\left(\underline{(^{-}d^*_{l_{uA1}})}^{1j}-\underline{(d_l^*)}_2^{ij}\right)^2+\left(\overline{(^{-}d^*_{l_{uA1}})}^{1j}-\overline{(d_l^*)}_2^{ij}\right)^2+\left(bd\overline{(^{-}d^*_{l_{luA1}})}^{1j}-bd\,(d_l^*)_2^{ij}\right)^2\right)\right)^{\frac{1}{2}}\right] \tag{6. 23}$$

$$^{+}S_2=\left(\frac{1}{2}\left(\left(\underline{(^{+}d^*_{u_{lA2}})}^{1j}-\underline{(d_u^*)}_1^{ij}\right)^2+\left(\overline{(^{+}d^*_{u_{lA2}})}^{1j}-\overline{(d_u^*)}_1^{ij}\right)^2+\left(bd\underline{(^{+}d^*_{u_{luA2}})}^{1j}-bd\,(d_u^*)_1^{ij}\right)^2\right)\right)^{\frac{1}{2}},\left(\frac{1}{2}\left(\left(\underline{(^{+}d^*_{u_{uA2}})}^{1j}-\underline{(d_u^*)}_2^{ij}\right)^2+\overline{(^{+}d^*_{u_{uA2}})}^{1j}-\overline{(d_u^*)}_2^{ij}\right)^2+\left(bd\overline{(^{+}d^*_{u_{luA2}})}^{1j}-bd\,(d_u^*)_2^{ij}\right)^2\right)^{\frac{1}{2}} \tag{6. 24}$$

$$^{-}S_2=\left[\left(\frac{1}{2}\left(\left(\underline{(^{-}d^*_{u_{lA2}})}^{1j}-\underline{(d_u^*)}_1^{ij}\right)^2+\left(\overline{(^{-}d^*_{u_{lA2}})}^{1j}-\overline{(d_u^*)}_1^{ij}\right)^2+\left(bd\underline{(^{-}d^*_{u_{luA2}})}^{1j}-bd\,(d_u^*)_1^{ij}\right)^2\right)\right)^{\frac{1}{2}},\left(\frac{1}{2}\left(\left(\underline{(^{-}d^*_{u_{uA2}})}^{1j}-\underline{(d_u^*)}_2^{ij}\right)^2+\left(\overline{(^{-}d^*_{u_{uA2}})}^{1j}-\overline{(d_u^*)}_2^{ij}\right)^2+\left(bd\overline{(^{-}d^*_{u_{luA2}})}^{1j}-bd\,(d_u^*)_2^{ij}\right)^2\right)\right)^{\frac{1}{2}}\right] \tag{6. 25}$$

- Using IR-GRA to calculate relational closeness.

$$^{+}\gamma D_l^*=\leftrightarrows\left[\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{+}s_1)}^{kj}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{+}s_1)}^{kj}}{\underline{(^{+}s_1)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{+}s_1)}^{kj}},\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{+}s_1)}^{kj}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{+}s_1)}^{kj}}{\overline{(^{+}s_1)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{+}s_1)}^{kj}}\right] \tag{6. 26}$$

$$^{-}\gamma D_l^*=\leftrightarrows\left[\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{-}s_1)}^{kj}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{-}s_1)}^{kj}}{\underline{(^{-}s_1)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{-}s_1)}^{kj}},\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{-}s_1)}^{kj}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{-}s_1)}^{kj}}{\overline{(^{-}s_1)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{-}s_1)}^{kj}}\right] \tag{6. 27}$$

$$^{+}\gamma D_u^*=\leftrightarrows\left[\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{+}s_2)}^{kj}+\rho\cdot\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{+}s_2)}^{kj}}{\underline{(^{+}s_2)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{+}s_2)}^{kj}},\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{+}s_2)}^{kj}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{+}s_2)}^{kj}}{\overline{(^{+}s_2)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{+}s_2)}^{kj}}\right] \tag{6. 28}$$

$$^{-}\gamma D_u^*=\leftrightarrows\left[\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{-}s_2)}^{kj}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{-}s_2)}^{kj}}{\underline{(^{-}s_2)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\underline{(^{-}s_2)}^{kj}},\frac{\bigwedge_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{-}s_2)}^{kj}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{-}s_2)}^{kj}}{\overline{(^{-}s_2)}^{ij}+\rho\cdot\bigvee_{\substack{k=1\ldots m\\ j=1\ldots n}}\overline{(^{-}s_2)}^{kj}}\right] \tag{6. 29}$$

- Calculating the IR-GRA relational grade of alternatives: According to degree of similarity between two alternatives, the closer the distance between the actual and the ideal object, the greater the correlation between them.

$$^{+}\tau^{+}\gamma D_l^* = \left[\sum_{k=1}^{n}\left(\underline{\left(^{+}\gamma d_l^*\right)}^{ik}\cdot(w_1)^k\right), \sum_{k=1}^{n}\left(\overline{\left(^{+}\gamma d_l^*\right)}^{ik}\cdot(w_1)^k\right)\right] \tag{6. 30}$$

$$^{-}\tau^{-}\gamma D_l^* = \left[\sum_{k=1}^{n}\left(\underline{\left(^{-}\gamma d_l^*\right)}^{ik}\cdot(w_1)^k\right), \sum_{k=1}^{n}\left(\overline{\left(^{-}\gamma d_l^*\right)}^{ik}\cdot(w_1)^k\right)\right] \tag{6. 31}$$

$$^{+}\tau^{+}\gamma D_u^* = \left[\sum_{k=1}^{n}\left(\underline{\left(^{+}\gamma d_u^*\right)}^{ik}\cdot(w_2)^k\right), \sum_{k=1}^{n}\left(\overline{\left(^{+}\gamma d_u^*\right)}^{ik}\cdot(w_2)^k\right)\right] \tag{6. 32}$$

$$^{-}\tau^{-}\gamma D_u^* = \left[\sum_{k=1}^{n}\left(\underline{\left(^{-}\gamma d_u^*\right)}^{ik}\cdot(w_2)^k\right), \sum_{k=1}^{n}\left(\overline{\left(^{-}\gamma d_u^*\right)}^{ik}\cdot(w_2)^k\right)\right] \tag{6. 33}$$

- Calculating the relative Grey relational grade with regard to positive ideal solutions:
  $\phi\gamma D^*l$ is the relational grade for the decision matrix D^*l

$$\phi\gamma D_l^* = \left[\frac{^{+}\tau^{+}\gamma\underline{(d_l^*)}^{ij}}{^{+}\tau^{+}\gamma\underline{(d_l^*)}^{ij} +^{-}\tau^{-}\gamma\underline{(d_l^*)}^{ij}}, \frac{^{+}\tau^{+}\gamma\overline{(d_l^*)}^{ij}}{^{+}\tau^{+}\gamma\overline{(d_l^*)}^{ij} +^{-}\tau^{-}\gamma\overline{(d_l^*)}^{ij}}\right] \tag{6. 34}$$

$\phi\gamma D^*u$ is the relational grade for the decision matrix D^*u

$$\phi\gamma D_u^* = \left[\frac{^{+}\tau^{+}\gamma\underline{(d_u^*)}^{ij}}{^{+}\tau^{+}\gamma\underline{(d_u^*)}^{ij} +^{-}\tau^{-}\gamma\underline{(d_u^*)}^{ij}}, \frac{^{+}\tau^{+}\gamma\overline{(d_u^*)}^{ij}}{^{+}\tau^{+}\gamma\overline{(d_u^*)}^{ij} +^{-}\tau^{-}\gamma\overline{(d_u^*)}^{ij}}\right] \tag{6. 35}$$

ϕ is the matrix of overall grade of each alternative.

$$\phi = convert\left(\left[\left[op\underline{(\phi\gamma d_l^*)^i}\right], \left[op\underline{(\phi\gamma d_u^*)^i}\right]\right], i = 1, .., m\right)$$

- Ranking alternatives based on final decision scores.
  (1) Ranking based on mid-points:

$$\phi = \left(\left[\frac{\underline{(\phi\gamma d_l^*)}^{ij} + \overline{(\phi\gamma d_u^*)}^{ij}}{2}\right],\ i = 1, .., m\right) \tag{6. 36}$$

  (2) Ranking based on coverage:

$$\phi = \left(\left[\frac{\overline{(\phi\gamma d_u^*)}^{ij} - \underline{(\phi\gamma d_l^*)}^{ij}}{2}\right],\ i = 1, .., m\right) \tag{6. 37}$$

7. Visualizing the DAM-LATEA-IRTGA Methodology

The decision-making process begins with the construction of an IRN decision matrix, which captures the lower and upper bounds of criteria values for each alternative. As the proposed methodology evaluates alternatives by considering both upper and lower bounds of criteria values, enables decision-makers to assess performance under varying conditions. The boundary region, computed using the $Bd(D)$ matrix, quantifies the variation between the lower and upper approximations, offering insights into the inherent uncertainty and variability in the data. This step establishes a solid foundation for analyzing the data while preserving information about ambiguity.

Entropy, as a measure of disorder or randomness, is employed to quantify the degree of uncertainty within the system. Normalization of the entropy matrix ensures a dimensionless representation of data, making it suitable for comparison across criteria. The weighting process accounts for the relative importance of criteria while incorporating experts opinions. By considering varying weights for experts, the approach reflects differences in expertise, decision power, and risk preferences, ensuring a balanced and realistic assessment.

The weighted decision matrices, D^*l and D^*u, are constructed using Atanassov extended formula. These matrices account for both lower and upper weights, providing a flexible representation of preferences under uncertain conditions. This step ensures that the analysis accommodates varying perspectives, enabling robust decision-making.

To determine ideal solutions, the interval rough TOPSIS method classifies criteria into benefit and cost types.

The positive ideal solutions maximize membership values and minimize non-membership values. Conversely, negative ideal solutions adopt opposite characteristics. This classification simplifies the process of evaluating alternatives against desired objectives.

Distances from positive and negative ideal solutions are computed to assess the relative closeness of alternatives to optimal solutions. Incorporating interval rough Grey Relational Analysis further refines the evaluation by calculating relational grades that measure the similarity between alternatives and ideal solutions. The higher relational grades indicate closer proximity to optimal performance, enabling a clear differentiation between alternatives.

The final ranking of alternatives is based on their relative Grey relational grades and mid-point calculations, ensuring a transparent and consistent evaluation framework. By ranking alternatives based on mid-point scores, the approach balances optimism and pessimism, thereby mitigating the effects of extreme values.

Overall, this methodology demonstrates its capability to handle uncertainty, integrate expert preferences, and combine multiple decision-making techniques into a unified framework. It provides decision-makers with a structured and intuitive approach to evaluating alternatives, making it suitable for complex and dynamic decision-making scenarios.

The methodology is pictorially depicted in 1

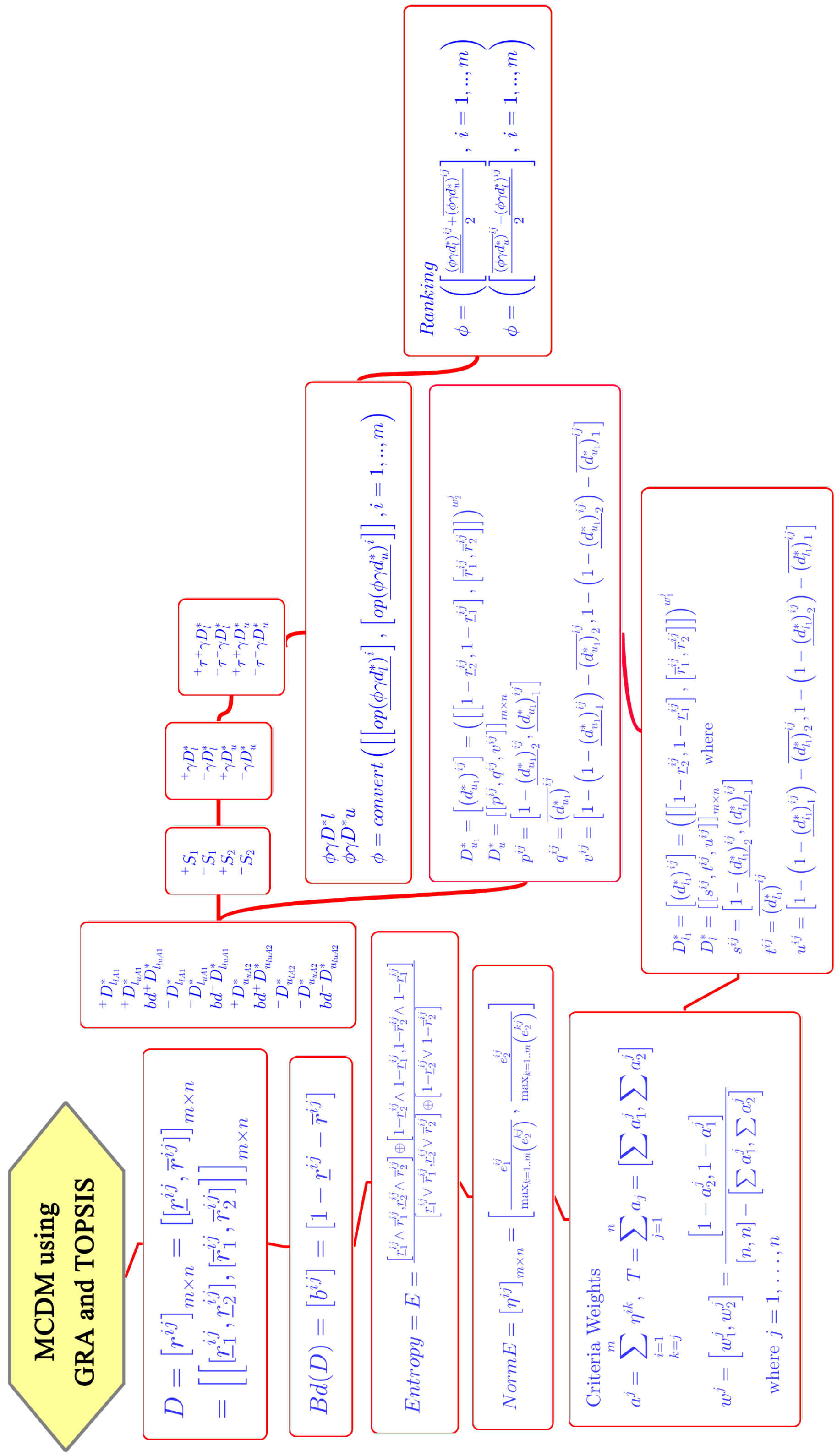


FIGURE 1. Framework of the proposed methodology for alternative ranking

8. APPLICATION

The factory's specific electricity usage, thermal energy consumption and air emissions were examined to identify opportunities for energy reduction and improved sustainability. The objective was to select the most suitable energy source to enhance production efficiency and minimize environmental impacts.

8.1. **Assessment Framework:** To perform a systematic analysis of the data, the approach Interval Rough Topsis and Grey Analysis (IRTGA) is implemented as:

(1) The opinions of decision makers are taken in linguistic form and then converted numerically in the interval $[0, 1]$. The Table 3 presents a tabulated compilation of experts response .

Alternative	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.59	0.62	0.58	0.59	0.63	0.59
A_2	0.57	0.59	0.62	0.56	0.61	0.61
A_3	0.61	0.63	0.58	0.62	0.65	0.58
A_4	0.60	0.63	0.60	0.62	0.63	0.60
A_5	0.62	0.61	0.60	0.62	0.63	0.62

TABLE 3. Alternative Evaluation Based on Criteria

The numerical form of experts judgments are converted into interval rough numbers by calculating membership and non-membership approximations of these numbers.
The opinions of decision makers, numerically in the form of interval rough numbers constructed the decision matrix 4. For the purpose of simplification, we have taken five "different possible courses of action" (alternatives) and six "tools for evaluating and comparing alternatives.

TABLE 4. Decision matrix

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	$[[0.586, 0.662], [0.182, 0.274]]$	$[[0.608, 0.656], [0.197, 0.209]]$	$[[0.585, 0.616], [0.189, 0.205]]$	$[[0.584, 0.660], [0.184, 0.276]]$	$[[0.610, 0.654], [0.199, 0.207]]$	$[[0.587, 0.618], [0.187, 0.203]]$
A_2	$[[0.548, 0.642], [0.211, 0.296]]$	$[[0.573, 0.626], [0.181, 0.197]]$	$[[0.605, 0.631], [0.195, 0.210]]$	$[[0.545, 0.642], [0.211, 0.299]]$	$[[0.570, 0.626], [0.184, 0.197]]$	$[[0.608, 0.631], [0.195, 0.213]]$
A_3	$[[0.612, 0.619], [0.143, 0.237]]$	$[[0.626, 0.681], [0.187, 0.202]]$	$[[0.573, 0.605], [0.181, 0.199]]$	$[[0.610, 0.619], [0.148, 0.249]]$	$[[0.626, 0.684], [0.184, 0.202]]$	$[[0.576, 0.605], [0.181, 0.202]]$
A_4	$[[0.592, 0.637], [0.133, 0.211]]$	$[[0.596, 0.643], [0.193, 0.207]]$	$[[0.596, 0.625], [0.199, 0.211]]$	$[[0.590, 0.637], [0.133, 0.213]]$	$[[0.594, 0.643], [0.193, 0.210]]$	$[[0.596, 0.630], [0.204, 0.211]]$
A_5	$[[0.590, 0.637], [0.133, 0.213]]$	$[[0.598, 0.643], [0.193, 0.206]]$	$[[0.594, 0.625], [0.199, 0.213]]$	$[[0.591, 0.639], [0.131, 0.212]]$	$[[0.590, 0.643], [0.193, 0.214]]$	$[[0.593, 0.630], [0.204, 0.214]]$

(2) To highlight the points where the uncertainty is highly expected in MCDM data the boundary points by using equation (6. 2) are calculated (see table 5. These points provide more optimal results to capture uncertainty:

TABLE 5. Values capturing uncertainty in decision-making

$$Bd(D) = \begin{bmatrix} [0.064, 0.232] & [0.135, 0.195] & [0.179, 0.226] & [0.064, 0.232] & [0.139, 0.191] & [0.179, 0.226] \\ [0.062, 0.241] & [0.177, 0.246] & [0.159, 0.200] & [0.059, 0.244] & [0.177, 0.246] & [0.156, 0.197] \\ [0.144, 0.245] & [0.117, 0.187] & [0.196, 0.246] & [0.132, 0.242] & [0.114, 0.190] & [0.193, 0.243] \\ [0.152, 0.275] & [0.150, 0.211] & [0.164, 0.205] & [0.150, 0.277] & [0.147, 0.213] & [0.159, 0.200] \\ [0.150, 0.277] & [0.151, 0.209] & [0.162, 0.207] & [0.149, 0.278] & [0.143, 0.217] & [0.156, 0.203] \end{bmatrix}$$

(3) Interval rough entropy values are calculated by equation (6. 3) which are shown in table(6) reduces subjectivity and give unbiased results to determine the weight of each criterion based on data dispersion.

TABLE 6. Entropy based evaluation of information content in criteria

$$E = \begin{bmatrix} [0.520, 0.676] & [0.541, 0.685] & [0.573, 0.699] & [0.524, 0.676] & [0.545, 0.685] & [0.569, 0.699] \\ [0.569, 0.699] & [0.555, 0.694] & [0.564, 0.694] & [0.569, 0.699] & [0.558, 0.694] & [0.564, 0.694] \\ [0.524, 0.676] & [0.506, 0.671] & [0.576, 0.704] & [0.529, 0.680] & [0.500, 0.667] & [0.576, 0.704] \\ [0.496, 0.667] & [0.550, 0.690] & [0.574, 0.699] & [0.496, 0.667] & [0.550, 0.690] & [0.574, 0.699] \\ [0.496, 0.667] & [0.550, 0.690] & [0.574, 0.699] & [0.492, 0.662] & [0.550, 0.690] & [0.574, 0.699] \end{bmatrix}$$

The data obtained from different sources consists different units, to get smooth data the entropy values are normalized.
The normalized entropy matrix is given in table 7 :

TABLE 7. Normalized entropy values of evaluation criteria

$$NormE = \begin{bmatrix} [0.914, 0.967] & [0.975, 0.987] & [0.993, 0.995] & [0.921, 0.967] & [0.977, 0.987] & [0.988, 0.993] \\ [1.000, 1.000] & [1.000, 1.000] & [0.979, 0.986] & [1.000, 1.000] & [1.000, 1.000] & [0.979, 0.986] \\ [0.921, 0.967] & [0.912, 0.967] & [1.000, 1.000] & [0.930, 0.973] & [0.896, 0.961] & [1.000, 1.000] \\ [0.872, 0.954] & [0.991, 0.994] & [0.993, 0.997] & [0.872, 0.954] & [0.986, 0.994] & [0.993, 0.997] \\ [0.872, 0.954] & [0.991, 0.994] & [0.993, 0.997] & [0.865, 0.947] & [0.986, 0.994] & [0.993, 0.997] \end{bmatrix}$$

(4) Keeping in view the importance of different criteria IRN weights are assigned to each criteria and the values are calculated by equation (6. 5):
$B^i = [[4.57, 4.85], [4.87, 4.95], [4.96, 4.97], [4.58, 4.84], [4.85, 4.94], [4.96, 4.97]]$
by using equation (6. 6) we get,
$T = \sum_{j=1}^{n} B^j = [28.8, 29.4]$
The entropy weights are assigned as:
$w^l = [0.153, 0.165, 0.169, 0.153, 0.165, 0.169]$
$w^u = [0.169, 0.173, 0.174, 0.168, 0.173, 0.174]$

(5) To see the impact of assigned weight to each criteria for lower boundary wl and upper boundary wu the weighted decision matrix D_l^* and D_u^* are calculated respectively. $(D_1)^{uv}$ is basically the form of $(1 - (1 - \mu_{Aij}), v_{Aij})$
Thus we obtain $(D_1)^{uv}$ as given in matrix 16(see page 24)
$\underline{D_l^*}$ is the form of $(1 - (1 - \mu_{Aij})^{wl}, v_{Aij}^{wl})$
the table is given as Table17(see page 24).
$\underline{D_u^*}$ is the form of $(1 - (1 - \mu_{Aij})^{wu}, v_{Aij}^{wu})$
the table is given as Table18(see page 24).
And D_l^* is the form of:

$$\left(1 - (1 - \mu_{Aij})^{wl}, (v_{Aij})^{wl}, 1 - \left(1 - (1 - \mu_{Aij})^{wl}\right) - (v_{Aij})^{wl}\right)$$

shown in Table19. D_u^* is the form of

$$(1 - (1 - \mu_{Aij})^{wu}, (v_{Aij})^{wu}, 1 - (1 - (1 - \mu_{Aij})^{wu}) - (v_{Aij})^{wu})$$

shown in Table20. (see page both tables at $page$ 25)

(6) Dividing criteria into beneficial $C_B = \{C_1, C_4, C_5, C_6, C_8, C_9\}$ and non-beneficial criteria $C_L = \{C_2, C_3, C_7\}$ and (by using equations (6. 10), (6. 11),(6. 12) respectively) the lower ${}^{+}D^{*}_{l_{lA1}}$, upper ${}^{+}D^{*}_{l_{uA1}}$ and boundary values $bd^{+}D^{*}_{l_{luA1}}$ of the matrix D^{*}_{l} are calculated for positive ideal solutions.

$${}^{+}D^{*}_{l_{lA1}} = \begin{bmatrix} [0.135, 0.751] & [0.133, 0.766] & [0.146, 0.752] & [0.117, 0.800] & [0.154, 0.758] & [0.148, 0.753] \end{bmatrix}$$
$${}^{+}D^{*}_{l_{uA1}} = \begin{bmatrix} [0.148, 0.777] & [0.147, 0.770] & [0.154, 0.758] & [0.137, 0.824] & [0.168, 0.762] & [0.154, 0.761] \end{bmatrix}$$
$$bd^{+}D^{*}_{l_{luA1}} = \begin{bmatrix} [0.075, 0.114] & [0.083, 0.101] & [0.088, 0.102] & [0.039, 0.083] & [0.070, 0.088] & [0.085, 0.099] \end{bmatrix}$$

By using equations (6. 13), (6. 14), (6. 15) respectively, the lower ${}^{-}D^{*}_{l_{lA1}}$, upper ${}^{-}D^{*}_{l_{uA1}}$ and boundary values, $bd^{-}D^{*}_{l_{luA1}}$ of the matrix D^{*}_{l} for negative ideal solutions are calculated.

$${}^{-}D^{*}_{l_{lA1}} = \begin{bmatrix} [0.118, 0.799] & [0.153, 0.757] & [0.135, 0.764] & [0.135, 0.751] & [0.133, 0.768] & [0.137, 0.766] \end{bmatrix}$$
$${}^{-}D^{*}_{l_{uA1}} = \begin{bmatrix} [0.137, 0.823] & [0.167, 0.763] & [0.143, 0.768] & [0.147, 0.777] & [0.147, 0.772] & [0.143, 0.770] \end{bmatrix}$$
$$bd^{-}D^{*}_{l_{luA1}} = \begin{bmatrix} [0.040, 0.083] & [0.070, 0.090] & [0.089, 0.101] & [0.076, 0.114] & [0.081, 0.099] & [0.087, 0.097] \end{bmatrix}$$

The lower, upper and boundary values ${}^{+}D^{*}_{u_{lA2}}$, ${}^{+}D^{*}_{u_{uA2}}$, $bd^{+}D^{*}_{u_{luA2}}$ of the matrix D^{*}_{u} for positive ideal solutions are calculated by using equations (6. 16), (6. 17), (6. 18)

$${}^{+}D^{*}_{u_{lA2}} = \begin{bmatrix} [0.148, 0.729] & [0.139, 0.756] & [0.150, 0.746] & [0.128, 0.782] & [0.160, 0.748] & [0.152, 0.746] \end{bmatrix}$$
$${}^{+}D^{*}_{u_{uA2}} = \begin{bmatrix} [0.162, 0.757] & [0.153, 0.760] & [0.158, 0.752] & [0.149, 0.808] & [0.176, 0.752] & [0.158, 0.754] \end{bmatrix}$$
$$bd^{+}D^{*}_{u_{luA2}} = \begin{bmatrix} [0.081, 0.123] & [0.087, 0.105] & [0.090, 0.104] & [0.043, 0.090] & [0.072, 0.092] & [0.088, 0.102] \end{bmatrix}$$

The lower, upper and boundary values ${}^{-}D^{*}_{u_{lA2}}$, ${}^{-}D^{*}_{u_{uA2}}$,$bd^{-}D^{*}_{u_{luA2}}$ of the matrix D^{*}_{u} for negative ideal solutions calculated by equations (6. 19), (6. 20) , (6. 21).

$${}^{-}D^{*}_{u_{lA2}} = \begin{bmatrix} [0.130, 0.780] & [0.160, 0.747] & [0.139, 0.758] & [0.147, 0.730] & [0.139, 0.758] & [0.140, 0.760] \end{bmatrix}$$
$${}^{-}D^{*}_{u_{uA2}} = \begin{bmatrix} [0.150, 0.806] & [0.174, 0.753] & [0.147, 0.762] & [0.161, 0.758] & [0.153, 0.764] & [0.148, 0.764] \end{bmatrix}$$
$$bd^{-}D^{*}_{u_{luA2}} = \begin{bmatrix} [0.044, 0.090] & [0.073, 0.093] & [0.091, 0.103] & [0.081, 0.123] & [0.083, 0.103] & [0.088, 0.100] \end{bmatrix}$$

(7) To see the distance between elements of weighted decision matrices and elements of ideal solution matrices, the ${}^{+}S1$, ${}^{-}S1$,${}^{+}S2$,${}^{-}S2$ are calculated (by using equations: 6. 22), (6. 23), (6. 24), (6. 25).
The ${}^{+}S_1$ Table:8 is the metric distance between the elements of the matrices D^{*}_{l} and positive ideal solutions of D^{*}_{l}.

TABLE 8. Metric distance between the elements of $D^{*}l$ and positive ideal solutions of $D^{*}l$

$${}^{+}S1 = \begin{bmatrix} [0.0342, 0.0316] & [0.0120, 0.0120] & [0.0067, 0.00557] & [0.0131, 0.0111] & [0.009, 0.0091] & [0.0060, 0.0045] \\ [0.0448, 0.0396] & [0.0080, 0.0080] & [0.0080, 0.0080] & [0.0014, 0.0014] & [0.0210, 0.0210] & [0.0070, 0.0070] \\ [0.0073, 0.0144] & [0.0180, 0.0180] & [0.0110, 0.0110] & [0.0336, 0.0228] & [0.0022, 0.0022] & [0.0110, 0.0110] \\ [0.0057, 0.0057] & [0.0076, 0.0075] & [0.0100, 0.0097] & [0.0474, 0.0418] & [0.0113, 0.0130] & [0.0102, 0.0081] \\ [0.0066, 0.0066] & [0.0076, 0.0075] & [0.0100, 0.0097] & [0.0465, 0.0420] & [0.0125, 0.0136] & [0.0112, 0.0081] \end{bmatrix}$$

The ${}^{-}S_1$ is the metric distance between the elements of the matrix D_l^* and negative ideal solutions of D_l^* is Table 9.

TABLE 9. Metric distance between the elements of D^*l and negative ideal solutions of D^*l

$$
{}^{-}S1 = \begin{bmatrix}
[0.0130, 0.0116] & [0.0081, 0.0081] & [0.0060, 0.0060] & [0.0352, 0.0321] & [0.0112, 0.0121] & [0.0077, 0.0082] \\
[0.0021, 0.0021] & [0.0200, 0.0200] & [0.0098, 0.0098] & [0.0460, 0.0401] & [0.0100, 0.0100] & [0.0102, 0.0091] \\
[0.0370, 0.0270] & [0.0031, 0.0031] & [0.0110, 0.0110] & [0.0123, 0.0185] & [0.0189, 0.0178] & [0.0112, 0.0112] \\
[0.0459, 0.0410] & [0.0108, 0.0103] & [0.0080, 0.0080] & [0.0060, 0.0060] & [0.0070, 0.0055] & [0.0070, 0.0081] \\
[0.0463, 0.0414] & [0.0108, 0.0103] & [0.0080, 0.0080] & [0.0050, 0.0050] & [0.0061, 0.0051] & [0.0072, 0.0072]
\end{bmatrix}
$$

The ${}^{+}S_2$ (matrix 10) is the metric distance between the elements of the matrix D_u^* and positive ideal solutions of D_u^*.

TABLE 10. Metric distance between the elements of D^*u and positive ideal solutions of D^*u

$$
{}^{+}S2 = \begin{bmatrix}
[0.0362, 0.0332] & [0.0130, 0.0130] & [0.0060, 0.0060] & [0.0140, 0.0126] & [0.0086, 0.0107] & [0.0069, 0.0052] \\
[0.0480, 0.0422] & [0.0080, 0.0080] & [0.0080, 0.0080] & [0.0021, 0.0021] & [0.0220, 0.0220] & [0.0080, 0.0080] \\
[0.0072, 0.0148] & [0.0195, 0.0184] & [0.0110, 0.0110] & [0.0358, 0.0241] & [0.0022, 0.0022] & [0.0110, 0.0110] \\
[0.0060, 0.0060] & [0.0081, 0.0070] & [0.0100, 0.0097] & [0.0499, 0.0449] & [0.0110, 0.0134] & [0.0111, 0.0091] \\
[0.0070, 0.0070] & [0.0086, 0.0085] & [0.0100, 0.0097] & [0.0493, 0.0443] & [0.0117, 0.0148] & [0.0122, 0.0095]
\end{bmatrix}
$$

The ${}^{-}S_2$ (see matrix 11) is metric distance between the elements of the matrices D_u^* and negative ideal solutions of D_u^*.

TABLE 11. Metric distance between the elements of D^*u and negative ideal solutions of D^*u

$$
{}^{-}S2 = \begin{bmatrix}
[0.0141, 0.0123] & [0.0081, 0.0081] & [0.0064, 0.0058] & [0.0373, 0.0341] & [0.0122, 0.0131] & [0.0081, 0.0076] \\
[0.0028, 0.0028] & [0.0210, 0.0210] & [0.0104, 0.0093] & [0.0487, 0.0429] & [0.0110, 0.0110] & [0.0104, 0.0088] \\
[0.0401, 0.0296] & [0.0030, 0.0030] & [0.0110, 0.0110] & [0.0123, 0.0201] & [0.0189, 0.0194] & [0.0122, 0.0122] \\
[0.0485, 0.0439] & [0.0110, 0.0116] & [0.0080, 0.0080] & [0.0070, 0.0070] & [0.0072, 0.0068] & [0.0070, 0.0081] \\
[0.0488, 0.0444] & [0.0108, 0.0103] & [0.0080, 0.0080] & [0.0051, 0.0051] & [0.0071, 0.0060] & [0.0080, 0.0080]
\end{bmatrix}
$$

(8) On the basis of metric distances the closeness of each alternative to ideal solution is calculated by equation: 6. 26 .
The ${}^{+}\gamma D_l^*$ positive Grey relational coefficient matrix 12 for weighted decision matrix D_l^* is calculated from the elements of positive metric distance matrix ${}^{+}S_1$.

TABLE 12. Positive Grey relational coefficient matrix for weighted decision matrix D^*l

$$
{}^{+}\gamma D_l^* = \begin{bmatrix}
[0.496, 0.496] & [0.790, 0.795] & [1.000, 1.000] & [0.682, 0.698] & [0.645, 0.645] & [1.000, 1.000] \\
[0.418, 0.429] & [0.971, 0.977] & [0.816, 0.897] & [1.000, 1.000] & [0.403, 0.403] & [0.802, 0.921] \\
[0.746, 0.943] & [0.615, 0.619] & [0.673, 0.739] & [0.438, 0.511] & [1.000, 1.000] & [0.612, 0.705] \\
[1.000, 1.000] & [1.000, 1.000] & [0.730, 0.787] & [0.353, 0.357] & [0.540, 0.583] & [0.737, 0.740] \\
[0.962, 0.966] & [1.000, 1.000] & [0.730, 0.787] & [0.356, 0.358] & [0.527, 0.552] & [0.696, 0.737]
\end{bmatrix}
$$

The ${}^{-}\gamma D_l^*$ negative Grey relational coefficient matrix 13 by using equation (6. 27), for weighted decision matrix D_l^* is calculated from the elements of negative metric distance matrix ${}^{-}S_1$. The ${}^{+}\gamma D_u^*$ positive Grey relational coefficient matrix

TABLE 13. Negative Grey relational coefficient matrix for weighted decision matrix D^*l

$$ {}^{-}\gamma D_l^* = \begin{bmatrix} [0.699, 0.706] & [0.725, 0.725] & [1.000, 1.000] & [0.480, 0.481] & [0.667, 0.757] & [0.935, 0.940] \\ [1.000, 1.000] & [0.440, 0.440] & [0.753, 0.753] & [0.406, 0.416] & [0.741, 0.804] & [0.797, 0.878] \\ [0.420, 0.478] & [1.000, 1.000] & [0.703, 0.703] & [0.649, 0.793] & [0.524, 0.549] & [0.750, 0.768] \\ [0.366, 0.370] & [0.635, 0.650] & [0.859, 0.859] & [0.958, 0.962] & [0.951, 0.966] & [0.935, 1.000] \\ [0.364, 0.367] & [0.635, 0.650] & [0.859, 0.859] & [1.000, 1.000] & [1.000, 1.000] & [0.977, 1.000] \end{bmatrix} $$

for weighted decision matrix D_u^* is calculated by using equation (6. 28) from the elements of positive metric distance matrix ${}^{+}S_2$ (Table 14).

TABLE 14. Positive Grey relational coefficient matrix for weighted decision matrix D^*u

$$ {}^{+}\gamma D_u^* = \begin{bmatrix} [0.500, 0.501] & [0.730, 0.781] & [1.0000, 1.000] & [0.695, 0.700] & [0.608, 0.670] & [1.000, 1.000] \\ [0.418, 0.430] & [0.936, 1.000] & [0.859, 0.859] & [1.000, 1.000] & [0.400, 0.400] & [0.794, 0.915] \\ [0.758, 0.965] & [0.587, 0.610] & [0.703, 0.703] & [0.446, 0.527] & [1.000, 1.000] & [0.655, 0.760] \\ [1.000, 1.000] & [0.994, 1.000] & [0.748, 0.763] & [0.362, 0.364] & [0.541, 0.600] & [0.735, 0.756] \\ [0.965, 0.968] & [0.915, 0.967] & [0.748, 0.763] & [0.365, 0.367] & [0.512, 0.581] & [0.710, 0.720] \end{bmatrix} $$

The negative Grey relational coefficient matrix ${}^{-}\gamma D_u^*$ (15) by equation (6. 29) for weighted decision matrix D_u^* is calculated from the elements of negative metric distance matrix ${}^{-}S_2$.

TABLE 15. Negative Grey relational coefficient matrix for weighted decision matrix D^*u

$$^{-}\gamma D_u^* = \begin{bmatrix} [0.706, 0.725] & [0.722, 0.722] & [1.000, 1.000] & [0.477, 0.478] & [0.693, 0.769] & [0.930, 1.000] \\ [1.000, 1.000] & [0.429, 0.429] & [0.755, 0.764] & [0.404, 0.412] & [0.763, 0.814] & [0.800, 0.913] \\ [0.422, 0.483] & [1.000, 1.000] & [0.685, 0.727] & [0.639, 0.804] & [0.543, 0.584] & [0.721, 0.749] \\ [0.373, 0.378] & [0.611, 0.628] & [0.837, 0.889] & [0.930, 0.936] & [0.952, 0.994] & [0.965, 1.000] \\ [0.372, 0.375] & [0.634, 0.649] & [0.837, 0.889] & [1.000, 1.000] & [1.000, 1.000] & [0.930, 0.965] \end{bmatrix}$$

(9) To check correlation between different alternatives,the Grey relational coefficients for lower values and upper values $^{+}\tau^{+}\gamma D_l^*$,$^{-}\tau^{-}\gamma D_l^*$ and $^{+}\tau^{+}\gamma D_u^*$,$^{-}\tau^{-}\gamma D_u^*$ are calculated by using equations, (6. 30), (6. 31), (6. 32), (6. 33) respectively. $^{+}\tau^{+}\gamma D_l^*$ is the positive Grey relational grade matrix calculated from positive Grey relational coefficient matrix $^{+}\gamma D_l^*$.

$$^{+}\tau^{+}\gamma D_l^* = \begin{bmatrix} [0.754, 0.758] \\ [0.718, 0.754] \\ [0.664, 0.733] \\ [0.709, 0.727] \\ [0.695, 0.717] \end{bmatrix}$$

The $^{-}\tau^{-}\gamma D_l^*$ is the negative Grey relational grade matrix calculated from negative Grey relational coefficient matrix $^{-}\gamma D_l^*$.

$$^{-}\tau^{-}\gamma D_l^* = \begin{bmatrix} [0.737, 0.755] \\ [0.672, 0.698] \\ [0.661, 0.699] \\ [0.768, 0.784] \\ [0.789, 0.795] \end{bmatrix}$$

The $^{+}\tau^{+}\gamma D_u^*$ is the positive Grey relational grade matrix calculated from positive Grey relational coefficient matrix $^{+}\gamma D_u^*$.

$$^{+}\tau^{+}\gamma D_u^* = \begin{bmatrix} [0.780, 0.802] \\ [0.757, 0.791] \\ [0.714, 0.785] \\ [0.754, 0.772] \\ [0.725, 0.752] \end{bmatrix}$$

The $^{-}\tau^{-}\gamma D_u^*$ is the negative Grey relational grade matrix calculated from negative Grey relational coefficient matrix $^{-}\gamma D_u^*$.

$$^{-}\tau^{-}\gamma D_u^* = \begin{bmatrix} [0.780, 0.809] \\ [0.713, 0.745] \\ [0.689, 0.747] \\ [0.804, 0.831] \\ [0.822, 0.839] \end{bmatrix}$$

The values of $\phi\gamma D_l^*$ are calculated by equation (6. 34) which is the relational grade for the Grey relational matrices ${}^{+}\tau^{+}\gamma D_l^*$ and ${}^{-}\tau^{-}\gamma D_l^*$.

$$\phi\gamma D_l^* = \begin{bmatrix} [0.506, 0.502] \\ [0.517, 0.520] \\ [0.503, 0.513] \\ [0.479, 0.481] \\ [0.470, 0.475] \end{bmatrix}$$

The values of $\phi\gamma D_u^*$ are calculated by equation (6. 35) which is the relational grade for the Grey relational matrices ${}^{+}\tau^{+}\gamma D_u^*$ and ${}^{-}\tau^{-}\gamma D_u^*$.

$$\phi\gamma D_u^* = \begin{bmatrix} [0.513, 0.514] \\ [0.529, 0.531] \\ [0.517, 0.530] \\ [0.496, 0.495] \\ [0.480, 0.485] \end{bmatrix}$$

(10) Ranking based on mid-points:

Bu using the values of $\phi\gamma D_l^*$ and $\phi\gamma D_u^*$ the matrix ϕ is obtained:

$$\phi = \begin{bmatrix} [0.506, 0.502] & [0.513, 0.514] \\ [0.517, 0.520] & [0.529, 0.531] \\ [0.503, 0.513] & [0.517, 0.530] \\ [0.479, 0.481] & [0.496, 0.495] \\ [0.470, 0.475] & [0.480, 0.485] \end{bmatrix}$$

By using formula:

$$\frac{(UpperBound + LowerBound)}{2}$$

the final ranking matrix is obtained.

$$Ranking = \begin{bmatrix} 0.5100 \\ 0.5240 \\ 0.5165 \\ 0.4870 \\ 0.4775 \end{bmatrix}$$

TABLE 16. Evaluation matrix of decision makers judgment

$$(D_1)^{uv} = \begin{bmatrix} [[0.338, 0.414], [0.182, 0.274]] & [[0.344, 0.392], [0.197, 0.209]] & [[0.384, 0.415], [0.189, 0.205]] & [[0.340, 0.416], [0.184, 0.276]] & [[0.346, 0.390], [0.199, 0.207]] & [[0.382, 0.413], [0.187, 0.203]] \\ [[0.358, 0.452], [0.211, 0.296]] & [[0.374, 0.427], [0.181, 0.197]] & [[0.369, 0.395], [0.195, 0.210]] & [[0.358, 0.455], [0.211, 0.299]] & [[0.374, 0.430], [0.184, 0.197]] & [[0.369, 0.392], [0.195, 0.213]] \\ [[0.381, 0.388], [0.143, 0.237]] & [[0.319, 0.374], [0.187, 0.202]] & [[0.395, 0.427], [0.181, 0.199]] & [[0.381, 0.390], [0.148, 0.249]] & [[0.316, 0.374], [0.184, 0.202]] & [[0.395, 0.424], [0.181, 0.202]] \\ [[0.363, 0.408], [0.133, 0.211]] & [[0.357, 0.404], [0.193, 0.207]] & [[0.375, 0.404], [0.199, 0.211]] & [[0.363, 0.410], [0.133, 0.213]] & [[0.357, 0.406], [0.193, 0.210]] & [[0.370, 0.404], [0.204, 0.211]] \\ [[0.363, 0.410], [0.133, 0.213]] & [[0.357, 0.402], [0.193, 0.206]] & [[0.375, 0.406], [0.199, 0.213]] & [[0.361, 0.409], [0.131, 0.212]] & [[0.357, 0.410], [0.193, 0.214]] & [[0.370, 0.407], [0.204, 0.214]] \end{bmatrix}$$

TABLE 17. Pre-weighted decision matrix for lower approximation weights

$$\underline{D_l^*} = \begin{bmatrix} [[0.852, 0.870], [0.785, 0.811]] & [[0.842, 0.854], [0.766, 0.770]] & [[0.853, 0.861], [0.757, 0.761]] & [[0.853, 0.871], [0.786, 0.812]] & [[0.842, 0.854], [0.768, 0.770]] & [[0.851, 0.859], [0.756, 0.762]] \\ [[0.860, 0.882], [0.799, 0.823]] & [[0.853, 0.867], [0.757, 0.763]] & [[0.846, 0.854], [0.761, 0.765]] & [[0.861, 0.883], [0.800, 0.824]] & [[0.853, 0.867], [0.758, 0.762]] & [[0.846, 0.852], [0.761, 0.767]] \\ [[0.863, 0.865], [0.761, 0.791]] & [[0.833, 0.847], [0.761, 0.765]] & [[0.857, 0.865], [0.752, 0.758]] & [[0.863, 0.865], [0.766, 0.796]] & [[0.832, 0.846], [0.759, 0.765]] & [[0.857, 0.863], [0.753, 0.761]] \\ [[0.859, 0.869], [0.751, 0.777]] & [[0.846, 0.858], [0.765, 0.769]] & [[0.849, 0.857], [0.764, 0.768]] & [[0.860, 0.870], [0.751, 0.777]] & [[0.847, 0.859], [0.765, 0.771]] & [[0.848, 0.856], [0.766, 0.768]] \\ [[0.860, 0.870], [0.751, 0.777]] & [[0.846, 0.858], [0.765, 0.769]] & [[0.849, 0.857], [0.764, 0.768]] & [[0.858, 0.870], [0.751, 0.777]] & [[0.847, 0.861], [0.766, 0.772]] & [[0.848, 0.858], [0.766, 0.770]] \end{bmatrix}$$

TABLE 18. Pre-weighted decision matrix for upper approximation weights

$$\underline{D_u^*} = \begin{bmatrix} [[0.838, 0.858], [0.764, 0.794]] & [[0.835, 0.847], [0.756, 0.760]] & [[0.849, 0.857], [0.750, 0.756]] & [[0.839, 0.859], [0.767, 0.795]] & [[0.835, 0.847], [0.758, 0.762]] & [[0.848, 0.856], [0.750, 0.756]] \\ [[0.846, 0.870], [0.780, 0.806]] & [[0.847, 0.861], [0.747, 0.753]] & [[0.842, 0.850], [0.754, 0.760]] & [[0.848, 0.872], [0.782, 0.808]] & [[0.847, 0.861], [0.748, 0.752]] & [[0.842, 0.848], [0.755, 0.761]] \\ [[0.850, 0.852], [0.739, 0.771]] & [[0.826, 0.840], [0.750, 0.756]] & [[0.853, 0.861], [0.746, 0.752]] & [[0.851, 0.853], [0.745, 0.779]] & [[0.824, 0.840], [0.749, 0.755]] & [[0.852, 0.860], [0.746, 0.754]] \\ [[0.845, 0.857], [0.729, 0.757]] & [[0.840, 0.852], [0.754, 0.760]] & [[0.845, 0.853], [0.758, 0.762]] & [[0.847, 0.859], [0.730, 0.758]] & [[0.840, 0.852], [0.755, 0.761]] & [[0.844, 0.852], [0.760, 0.762]] \\ [[0.846, 0.858], [0.729, 0.757]] & [[0.839, 0.851], [0.755, 0.759]] & [[0.845, 0.853], [0.758, 0.762]] & [[0.845, 0.857], [0.730, 0.758]] & [[0.840, 0.854], [0.756, 0.764]] & [[0.843, 0.853], [0.760, 0.764]] \end{bmatrix}$$

TABLE 19. Weighted decision matrix for lower approximation weights

$$D_l^* = \begin{bmatrix} [[0.130,0.148],[0.785,0.811],[0.041,0.085]] & [[0.146,0.158],[0.766,0.770],[0.072,0.088]] & [[0.139,0.147],[0.757,0.761],[0.092,0.104]] & [[0.129,0.147],[0.786,0.812],[0.041,0.085]] & [[0.146,0.158],[0.768,0.770],[0.072,0.086]] & [[0.141,0.149],[0.756,0.762],[0.089,0.103]] \\ [[0.118,0.140],[0.799,0.823],[0.037,0.083]] & [[0.133,0.147],[0.757,0.763],[0.090,0.110]] & [[0.146,0.154],[0.761,0.765],[0.081,0.093]] & [[0.117,0.139],[0.800,0.824],[0.037,0.083]] & [[0.133,0.147],[0.758,0.762],[0.091,0.109]] & [[0.148,0.154],[0.761,0.767],[0.079,0.091]] \\ [[0.135,0.137],[0.761,0.791],[0.072,0.104]] & [[0.153,0.167],[0.761,0.765],[0.068,0.086]] & [[0.135,0.143],[0.752,0.758],[0.099,0.113]] & [[0.135,0.137],[0.766,0.796],[0.067,0.099]] & [[0.154,0.168],[0.759,0.765],[0.067,0.087]] & [[0.137,0.143],[0.753,0.761],[0.096,0.110]] \\ [[0.131,0.141],[0.751,0.777],[0.082,0.118]] & [[0.142,0.154],[0.765,0.769],[0.077,0.093]] & [[0.143,0.151],[0.764,0.768],[0.081,0.093]] & [[0.130,0.140],[0.751,0.777],[0.083,0.119]] & [[0.141,0.153],[0.765,0.771],[0.076,0.094]] & [[0.144,0.152],[0.766,0.768],[0.080,0.090]] \\ [[0.130,0.140],[0.751,0.777],[0.083,0.119]] & [[0.142,0.154],[0.765,0.769],[0.077,0.093]] & [[0.143,0.151],[0.764,0.768],[0.081,0.093]] & [[0.130,0.142],[0.751,0.777],[0.081,0.119]] & [[0.139,0.153],[0.766,0.772],[0.075,0.095]] & [[0.142,0.152],[0.766,0.770],[0.078,0.092]] \end{bmatrix}$$

TABLE 20. Weighted decision matrix for upper approximation weights

$$D_u^* = \begin{bmatrix} [[0.142,0.162],[0.764,0.794],[0.044,0.094]] & [[0.153,0.165],[0.756,0.760],[0.075,0.091]] & [[0.143,0.151],[0.750,0.756],[0.093,0.107]] & [[0.141,0.161],[0.767,0.795],[0.044,0.092]] & [[0.153,0.165],[0.758,0.762],[0.073,0.089]] & [[0.144,0.152],[0.750,0.756],[0.092,0.106]] \\ [[0.130,0.154],[0.780,0.806],[0.040,0.090]] & [[0.139,0.153],[0.747,0.753],[0.094,0.114]] & [[0.150,0.158],[0.754,0.760],[0.082,0.096]] & [[0.128,0.152],[0.782,0.808],[0.040,0.090]] & [[0.139,0.153],[0.748,0.752],[0.095,0.113]] & [[0.152,0.158],[0.755,0.761],[0.081,0.093]] \\ [[0.148,0.150],[0.739,0.771],[0.079,0.113]] & [[0.160,0.174],[0.750,0.756],[0.070,0.090]] & [[0.139,0.147],[0.746,0.752],[0.101,0.115]] & [[0.147,0.149],[0.745,0.779],[0.072,0.108]] & [[0.160,0.176],[0.749,0.755],[0.069,0.091]] & [[0.140,0.148],[0.746,0.754],[0.098,0.114]] \\ [[0.143,0.155],[0.729,0.757],[0.088,0.128]] & [[0.148,0.160],[0.754,0.760],[0.080,0.098]] & [[0.147,0.155],[0.758,0.762],[0.083,0.095]] & [[0.141,0.153],[0.730,0.758],[0.089,0.129]] & [[0.148,0.160],[0.755,0.761],[0.079,0.097]] & [[0.148,0.156],[0.760,0.762],[0.082,0.092]] \\ [[0.142,0.154],[0.729,0.757],[0.089,0.129]] & [[0.149,0.161],[0.755,0.759],[0.080,0.096]] & [[0.147,0.155],[0.758,0.762],[0.083,0.095]] & [[0.143,0.155],[0.730,0.758],[0.087,0.127]] & [[0.146,0.160],[0.756,0.764],[0.076,0.098]] & [[0.147,0.157],[0.760,0.764],[0.079,0.093]] \end{bmatrix}$$

9. Results

Rank	Alternative	Energy Source	Score	Interpretation
1	$A2$	**Solar energy**	0.5240	Highest-ranked and most preferred alternative
2	$A3$	**Wind energy**	0.5165	Second-best alternative with strong performance
3	$A1$	**Biomass-energy**	0.5100	Moderately preferred alternative
4	$A4$	**Ocean energy**	0.4870	Lower-ranked alternative with weaker performance
5	$A5$	**Petroleum & natural gas**	0.4775	Least preferred alternative

The results clearly demonstrated that solar energy $(A2)$ as the most sustainable choices, excelling in energy savings and emission reductions. Although initial costs were higher, long-term benefits justified their selection. These findings underscore the utility of our methodology in addressing energy optimization challenges in all industries. The ability to evaluate alternatives based on multiple criteria ensures adaptability to diverse industrial contexts, supporting sustainable practices and efficient resource management.

10. Sensitivity Analysis

Alternatives	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$
Biomass energy	0.5128	0.5123	0.5118	0.5113	0.5100
Wind energy	0.5362	0.5318	0.5286	0.5261	0.5240
Solar energy	0.5210	0.5188	0.5172	0.5161	0.5165
Ocean energy	0.4804	0.4824	0.4845	0.4865	0.4870
Petroleum and natural gas	0.4568	0.4642	0.4696	0.4737	0.4775

TABLE 21. Performance Values for Varying recognition coefficient ρ by Energy Source

The sensitivity analysis was carried out by varying the parameter ρ from 0.1 to 0.5. The numerical results reveal that the scores of alternatives change only slightly with different values of ρ. However, the overall ranking order remains unchanged throughout the analysis.

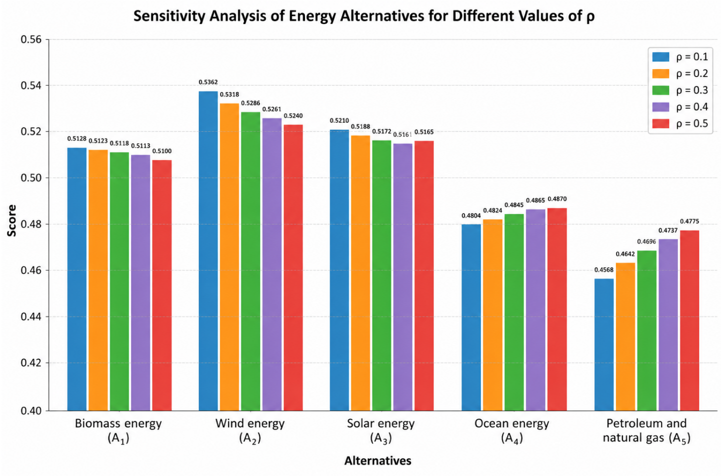


FIGURE 2. Performance analysis

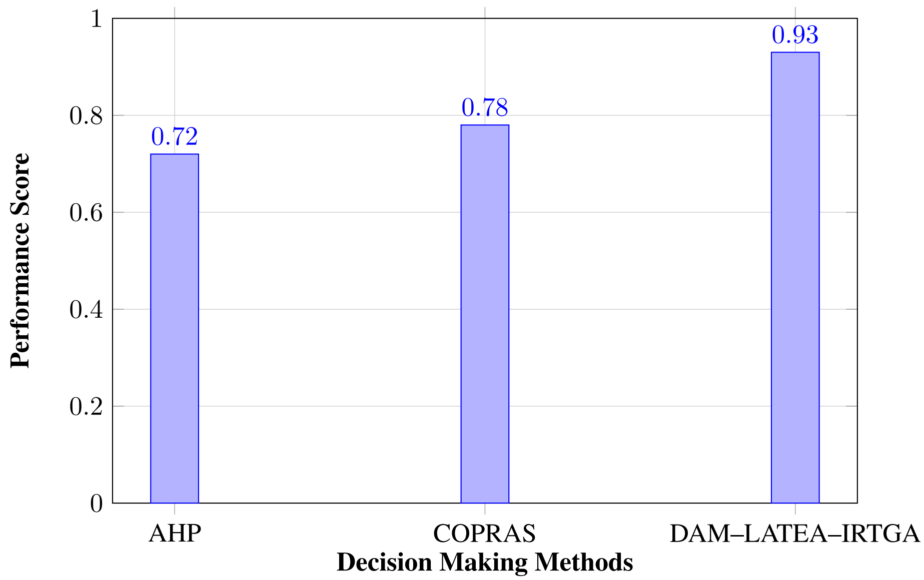
The final ranking of alternatives is obtained as:

$$A_2 > A_3 > A_1 > A_4 > A_5,$$

i.e.,

Solar energy $>$ Wind energy $>$ Biomass energy $>$ Ocean energy $>$ Petroleum and natural gas.

Therefore, the proposed framework demonstrates strong robustness and stability against parameter variation, confirming the reliability of the decision-making process.



(1) DAM–LATEA–IRTGA = highest dominance (showing superiority of the proposed approach due to its dynamic and objective weighting mechanism).
(2) AHP = subjective weighting limitation.
(3) COPRAS = moderate improvement but still static.

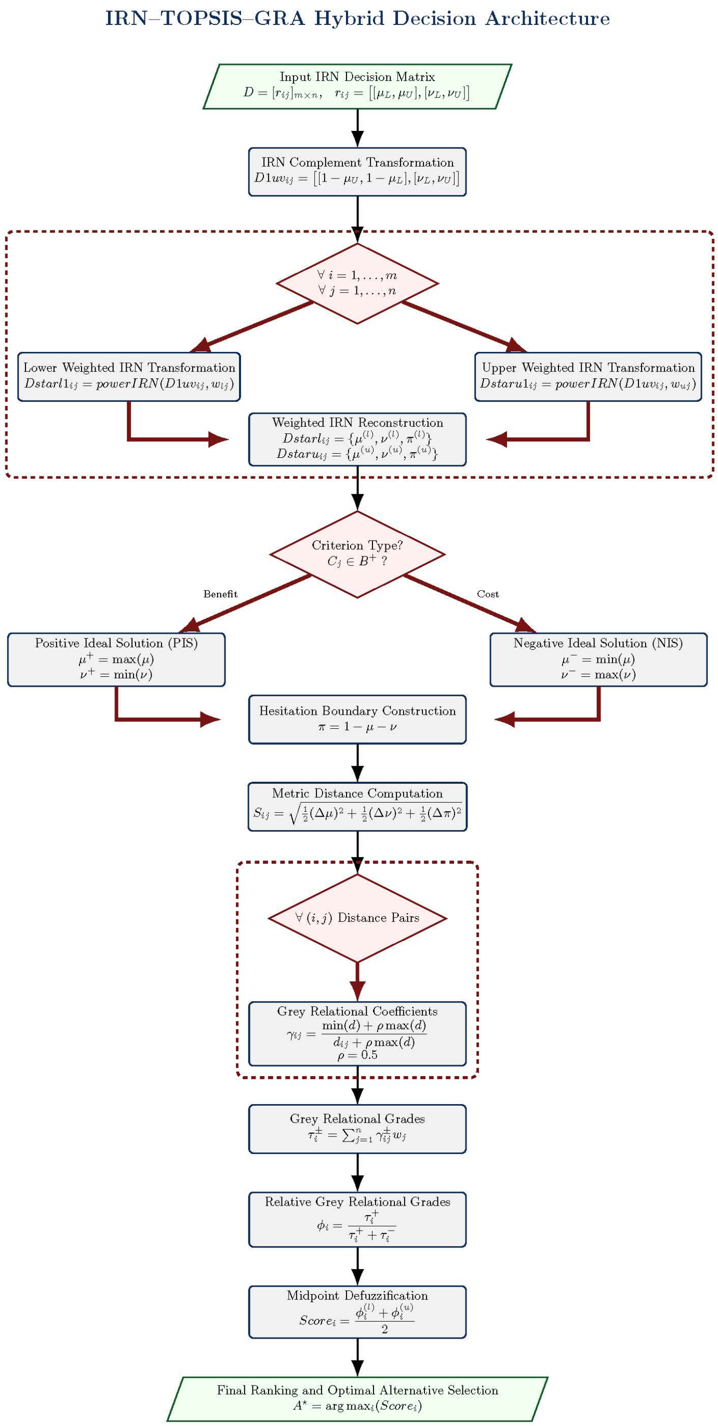


FIGURE 3. Optimal alternative algorithm

11. Conclusion

This study introduces a robust hybrid methodology—DAM-LATEA-IRTGA—for evaluating industrial energy alternatives. By integrating IRNs, entropy, and temporal expert assessments, the model enhances transparency and adaptability. Application to a real-world textile factory confirms solar energy as the most viable alternative based on a comprehensive set of criteria. The framework supports informed, strategic decision-making in uncertain industrial environments.

12. Future Work

Future research may integrate this methodology with real-time data acquisition systems such as IoT sensors and AI-based predictive models. The coupling of IRNs with machine learning and big data platforms will enable real-time adaptation and forecasting, expanding the scope of sustainable energy decision-making frameworks across industrial domains.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Ruby Jabbar: Conceptualization, methodology development, formal analysis, validation, investigation, visualization, manuscript drafting, and revision.
Dr. Athar Kharal: Supervision, conceptual guidance, methodology refinement, validation, critical review and project administration, and final approval of the submitted version.
Both authors, read and approved the final manuscript and agreed to be responsible for all aspects of the work.

DECLARATIONS

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