

**Plithogenic Fuzzy Energy and Laplacian Energy Graphs with Application in
Healthcare Decision-Making Problems**

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Abstract. Plithogenic fuzzy graphs offer a flexible way to represent complex systems with uncertainty, multiple attributes, and conflicting relationships. This study develops their spectral theory through a plithogenic fuzzy adjacency matrix, its spectrum, and graph energy. We also define the Laplacian matrix, Laplacian spectrum, and Laplacian energy. Their mathematical properties are examined through several illustrative examples. These spectral measures help describe structural complexity, interaction strength, and conflicting relationships in uncertain networks. A healthcare decision-making model is developed to show how the proposed framework works in practice. It represents patient health concerns, treatment choices, clinical effectiveness, costs, side effects, and preferences through plithogenic fuzzy graph components. The related adjacency matrix and spectrum help measure the healthcare network's energy. This gives a quantitative view of treatment interactions, compatibility, and possible contradictions. The example shows how the framework can help identify coherent treatment strategies despite uncertain or conflicting information. Comparisons are also made with social networks, transportation, epidemiology, supply chains, telecommunications, and finance. These examples point to the wider usefulness of plithogenic fuzzy graph energy and Laplacian energy.

Key Words: Plithogenic fuzzy adjacency matrix, Spectrum and Energy of Plithogenic fuzzy graphs, Plithogenic Laplacian matrix, Laplacian Spectrum and Laplacian Energy of Plithogenic fuzzy graphs.

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1. INTRODUCTION

1.1. Fuzzy Set and Plithogenic Set. When it came to dealing with ambiguity and uncertainty, classical set theory was insufficient. The concept of a fuzzy set (FS) was introduced by Zadeh in 1965 [1]. Since then, fuzzy systems have been effectively applied for decades in a variety of fields to solve issues containing possible ambiguity [35, 36]. Zadeh expanded FS to interval-valued fuzzy sets in 1975 by permitting membership degrees to vary over intervals in $[0, 1]$. Atanassov introduced the concept of intuitionistic fuzzy sets (IFS) in 1986 [2]. As an extension of IFS, Smarandache subsequently proposed neutrosophic sets [3, 5], which are now a crucial tool in numerous domains. More recently, Smarandache (2017) [6, 7] introduced the idea of Plithogeny in philosophy, which led to the creation of Plithogenic set/statistics in engineering and mathematics. Plithogeny is currently widely employed to resolve decision-making problems because to its increasing popularity. In order to better manage ambiguous, inconsistent, and indeterminate data, a plithogenic fuzzy set generalises IFS, and neutrosophic fuzzy set theories. Martin and Priya [8, 9] employed plithogenic sets to define plithogenic numbers and later introduced plithogenic cubic sets; new plithogenic cognitive maps were also proposed by Martin et al. [10]. Furthermore, Smarandache extended plithogenic sets to plithogenic logic in order to explain multivariate concepts [11].

Table 1 provides an overview of set extensions.

TABLE 1. Mathematical Overview of Set Extensions

Set Type	Definition
Fuzzy Set	Membership function $\mu : X \rightarrow [0, 1]$.
Intuitionistic Fuzzy Set	Membership μ and non-membership ν with $\mu(x) + \nu(x) \leq 1$.
Neutrosophic Set	Truth T , indeterminacy I , and falsity F , each in $[0, 1]$.
Plithogenic Set	Tuple $(P, \chi, R_\chi, d_f, c_f)$; appurtenance $d_f : P \times R_\chi \rightarrow [0, 1]^s$ and contradiction $c_f : R_\chi \times R_\chi \rightarrow [0, 1]^t$.

By combining characteristics, appurtenance degrees, and contradiction degrees, plithogenic sets provide greater flexibility than fuzzy, intuitionistic, or neutrosophic frameworks, making it possible to effectively model ambiguous, inconsistent, and indeterminate data. According to this viewpoint, they must not only manage uncertainty in the real world but also accurately depict inconsistencies that result from shifts in uncertainty.

1.2. Plithogenic Fuzzy Graph. Graph theory, which began after the Konigsberg bridge issue in 1736, offers a mathematical foundation for paired relationships between objects and their objectives. The fuzzy graphs (FG) are proposed in 1975 and used fuzzy set theory to obtain analogues of several graph-theoretic ideas. In 1987, Bhattacharya [9] talked about comments on FG and pattern detection. In recent years, fuzzy graphs have been extensively investigated in various studies [42, 43, 44, 45]. Fuzzy graphs remain important in theoretical and applied research, including graph energy and other parameters examined later. Their study therefore continues to attract considerable attention. In 1995, Atanassov [12, 13] extended this idea to intuitionistic fuzzy graphs (IFGs)[41]. Kandasamy [14] introduced neutrosophic graphs in *Neutrosophic Graphs*. Various extensions

of neutrosophic graphs have also been investigated [24, 46]. For instance, neutrosophic cubic graphs with industrial applications were introduced by Gulistan et al. [15]; see also [16, 17] for similar works. Moreover, due to the importance of these concepts, various extensions based on HyperGraphs, SuperHyperGraphs, and related structures have also been investigated [37, 38, 39, 4]. The connections among plithogenic graphs and their corresponding graph models are shown in Table 2. Furthermore, a comparison of a Plithogenic Set and a Plithogenic Graph is demonstrated in Table 3. This comparison highlights the advantages of extending the plithogenic set framework to a graph-based structure.

TABLE 2. Plain-language definitions of graph extensions (very concise)

Graph	Concise description
Fuzzy Graph	Vertices and edges have degrees in $[0,1]$; an edges degree does not exceed the smaller degree of its endpoints.
Intuitionistic Fuzzy Graph	Each vertex/edge has membership and non-membership (sum at most 1); edge membership is bounded by the smaller endpoint membership.
Neutrosophic Graph (single-valued)	Each vertex/edge has truth, indeterminacy, and falsity in $[0,1]$; an edges truth is bounded by the smaller endpoint truth.
Plithogenic Graph	Vertices/edges have attributes with degrees and a symmetric contradiction measure; edge degrees are limited by endpoint degrees considering contradiction.

TABLE 3. Comparison of a Plithogenic Set and a Plithogenic Graph

Aspect	Plithogenic Set	Plithogenic Graph
Structure	$P^S = (P, \chi, R_\chi, d_f, c_f)$.	$P^G = (P_A, P_B)$ over $G^* = (V, E)$.
Objects	Elements $P \subseteq S$.	Vertices $A \subseteq V$ and edges $B \subseteq E$.
Attributes	Attribute χ with range R_χ .	Vertex and edge attributes α, β .
Appurtenance	$d_f : P \times R_\chi \rightarrow [0, 1]^s$.	Maps $d_{\alpha f}$ and $d_{\beta f}$ for vertices and edges.
Contradiction	$c_f : R_\chi \times R_\chi \rightarrow [0, 1]^t$.	Contradiction maps $c_{\alpha f}$ and $c_{\beta f}$.
Relation	No incidence structure is required.	Edge degrees are bounded by the corresponding endpoint degrees.
Modeling	Models uncertain and contradictory set information.	Models uncertain and contradictory network relationships.

Plithogenic fuzzy graphs were studied in 2022 [18], including applications to identifying reasons for the global spread of coronavirus disease (COVID-19). Plithogenic fuzzy graphs provide enhanced flexibility by integrating attributes, membership, and contradiction degrees, enabling effective modeling of uncertain, inconsistent, and indeterminate relationships in complex networked systems. Due to this scientific significance, several recent studies have been conducted on plithogenic graphs and plithogenic fuzzy graphs [28, 29].

1.3. The Concept of Graph Spectrum. The graph spectrum is a powerful tool in graph theory. Graph energy was first defined by Gutman [19, 22] as the total of the absolute values of the adjacency matrix's eigenvalues. Energy has been studied for many classes of graphs, including regular, irregular, signed, weighted, hyper-, and bipolar graphs (cf. [30, 31]). Akram [21] introduced some concepts of energy in single-valued neutrosophic graphs, and this idea was further expanded to intuitionistic fuzzy graphs in [20]. Moreover, research on Laplacian energy has also been conducted in graph theory and related areas [32, 33]. However, studies on graph energy and related concepts in plithogenic fuzzy graphs are still relatively limited [25].

1.4. Motivation and Main Contributions. Compared to fuzzy graphs, plithogenic fuzzy graphs offer the advantage of explicitly modeling contradictions that arise among attributes in uncertain environments. Although fuzzy graph energy has been investigated using membership information, intuitionistic fuzzy graph energy incorporates membership and non-membership information, while neutrosophic graph energy further considers truth, indeterminacy, and falsity. However, these existing graph-energy models do not explicitly incorporate the degree of contradiction among attribute values as an inherent component of the graph structure. In contrast, the plithogenic fuzzy framework simultaneously incorporates appurtenance degrees and contradiction degrees, allowing relationships with different levels of compatibility and conflict to be represented within the same graph. Therefore, the proposed plithogenic fuzzy energy provides a new spectral measure for quantifying the structural complexity and interaction intensity of networks characterized by uncertainty and contradiction. Similarly, the proposed plithogenic fuzzy Laplacian energy extends Laplacian-based spectral analysis to such contradictory fuzzy networks and provides additional information about their structural characteristics and irregularity. This distinction is important in decision-making environments, such as healthcare, where different treatments, patient conditions, risks, costs, and preferences may exhibit both degrees of suitability and conflicting relationships.

To address this research gap, in this study we establish the adjacency matrix and spectrum of a plithogenic fuzzy graph and introduce the corresponding notions of plithogenic fuzzy graph energy and Laplacian energy. We investigate their mathematical properties and derive related theorems, together with illustrative examples demonstrating their computation and interpretation. Furthermore, a healthcare application is presented to demonstrate how the proposed framework can represent treatment-related information, interactions, and contradictions and can support the analysis of complex healthcare decision-making problems. The main contributions can be summarized as follows:

- We establish the adjacency matrix and spectrum of a plithogenic fuzzy graph, providing a spectral representation for graphs involving uncertain and contradictory information.
- We introduce the concept of plithogenic fuzzy graph energy and define it through the eigenvalues of the proposed adjacency matrix, thereby extending the classical notion of graph energy to the plithogenic fuzzy environment.
- We define the Laplacian matrix, Laplacian spectrum, and plithogenic fuzzy Laplacian energy, providing additional spectral measures for analyzing the structural properties and irregularity of plithogenic fuzzy graphs.

- To show how to calculate and understand the suggested energy measures, we derive associated theoretical conclusions and provide illustrated examples.
- We develop a healthcare application in which patient conditions, treatment options, and treatment-related attributes are represented through a plithogenic fuzzy graph, demonstrating how the proposed framework can capture both suitability and contradiction in healthcare decision-making.
- We provide a comparative analysis with crisp, fuzzy, intuitionistic fuzzy, and neutrosophic graph models to highlight the additional capability of the proposed framework for representing uncertainty and explicit contradiction.

Overall, the proposed framework extends spectral graph theory to a more general setting in which uncertainty, multiple attributes, and contradictions can be modeled simultaneously. The healthcare application further demonstrates the practical potential of plithogenic fuzzy energy and Laplacian energy as quantitative tools for analyzing complex decision-making networks and identifying structural characteristics that may not be adequately represented by conventional graph-energy models.

1.5. Structure of this Paper. The structure of this paper is given below.

Section 2 presents the preliminary definitions, including Plithogenic Graphs, Graph Energy, Plithogenic Fuzzy Graphs, and other necessary notions. Section 3 discusses the results concerning the Energy of Plithogenic Fuzzy Graphs. Section 4 presents the Laplacian Energy of Plithogenic Fuzzy Graphs. Section 5 describes applications, while Section 6 provides a discussion. Section 7 concludes with recommendations for future research.

2. PRELIMINARIES

This section collects basic definitions concerning sets, graphs, and the energy of a graph. All graphs in this work are considered to be undirected, finite, and to have no more than one edge.

Definition 2.1 (Plithogenic Set[6]). *Let S be a universal set and let $P \subseteq S$. A plithogenic set on S , denoted P^S , is a tuple*

$$P^S = (P, \chi, R_\chi, d_f, c_f),$$

where χ is an appurtenance attribute, R_χ is the corresponding range of attribute values, $d_f : P \times R_\chi \rightarrow [0, 1]^s$ is the degree of appurtenance (daf), and $c_f : R_\chi \times R_\chi \rightarrow [0, 1]^t$ is the degree of contradiction (dcf), satisfying for all $a, b \in R_\chi$:

$$c_f(a, a) = 0, \quad c_f(a, b) = c_f(b, a).$$

Here $s, t \in \{1, 2, 3\}$. In particular, when $s = t = 1$ we obtain a plithogenic fuzzy set, denoted P^{FS} ; when $s = 2$ and $t = 1$ we obtain a plithogenic intuitionistic fuzzy set, denoted P^{IFS} ; and when $s = 3$ and $t = 1$ we obtain a plithogenic neutrosophic set, denoted P^{NS} .

Example 2.2 (Plithogenic Set). *Let $S = \{p_1, p_2, p_3\}$ and $P = \{p_1, p_2\} \subseteq S$. Let the attribute be $\chi = \text{"color"}$ with range $R_\chi = \{\text{red, blue}\}$. Take $s = t = 1$ (plithogenic fuzzy*

set). Then

$$d_f : P \times R_\chi \rightarrow [0, 1], \quad \begin{array}{c|cc} & \text{red} & \text{blue} \\ \hline p_1 & 0.8 & 0.2 \\ p_2 & 0.4 & 0.6 \end{array}$$

and a symmetric degree of contradiction $c_f : R_\chi \times R_\chi \rightarrow [0, 1]$ by

$$c_f(\text{red}, \text{red}) = c_f(\text{blue}, \text{blue}) = 0, \quad c_f(\text{red}, \text{blue}) = c_f(\text{blue}, \text{red}) = 0.7.$$

Then $P^S = (P, \chi, R_\chi, d_f, c_f)$ is a plithogenic set (with $s = t = 1$, a plithogenic fuzzy set).

In a Plithogenic Set, the equivalence relations hold as presented in Tables 4. From this, it is evident that the Plithogenic Set possesses a high degree of flexibility and has significant value in handling contradictions arising from uncertainty and philosophy (cf.[15]). Moreover, this perspective can likewise be applied to structures such as Plithogenic Graphs.

TABLE 4. Embedding of k partitioned sets into a plithogenic framework and their (s, t) counterparts.

Left label	k	Set Type	Attribute Values V
(1, 0)	1	FS	$\{v_1\}$ for $\{\mu\}$
(2, 0)	2	IFS	$\{v_1, v_2\}$ for $\{\mu, \nu\}$
(3, 0)	3	NFS	$\{v_1, v_2, v_3\}$ for $\{T, I, F\}$
(4, 0)	4	Quadripartitioned Neutrosophic Set [16]	$\{v_1, \dots, v_4\}$ for $\{T, C, G, F\}$
(5, 0)	5	Pentapartitioned Neutrosophic Set [14]	$\{v_1, \dots, v_5\}$ for $\{T, C, G, U, F\}$
(6, 0)	6	Hexapartitioned Neutrosophic Set [16]	$\{v_1, \dots, v_6\}$ for $\{T, C, G, U, H, F\}$
(7, 0)	7	Heptapartitioned Neutrosophic Set [16]	$\{v_1, \dots, v_7\}$ for $\{T, M, C, U, I, K, F\}$
(8, 0)	8	Octapartitioned Neutrosophic Set [16]	$\{v_1, \dots, v_8\}$ for $\{T, M, C, U, I, K, H, F\}$
(9, 0)	9	Nonapartitioned Neutrosophic Set [16]	$\{v_1, \dots, v_9\}$ for $\{T, ST, \dots, WF, F\}$
(10, 0)	10	Decapartitioned Neutrosophic Set [16]	$\{v_1, \dots, v_{10}\}$ for $\{T, SRT, \dots, WRF, F\}$
(1, 1)	–	Plithogenic Fuzzy Set	$\{v_1\}$ for $\{\mu\}$ with contradiction
(2, 1)	–	Plithogenic Intuitionistic Fuzzy Set	$\{v_1, v_2\}$ for $\{\mu, \nu\}$ with contradiction
(3, 1)	–	Plithogenic Neutrosophic Set	$\{v_1, v_2, v_3\}$ for $\{T, I, F\}$ with contradiction
(4, 1)	–	Plithogenic Quadripartitioned Neutrosophic Set	$\{v_1, \dots, v_4\}$ for $\{T, C, G, F\}$ with contradiction
(5, 1)	–	Plithogenic Pentapartitioned Neutrosophic Set	$\{v_1, \dots, v_5\}$ for $\{T, C, G, U, F\}$ with contradiction
(6, 1)	–	Plithogenic Hexapartitioned Neutrosophic Set	$\{v_1, \dots, v_6\}$ for $\{T, C, G, U, H, F\}$ with contradiction
(7, 1)	–	Plithogenic Heptapartitioned Neutrosophic Set	$\{v_1, \dots, v_7\}$ for $\{T, M, C, U, I, K, F\}$ with contradiction
(8, 1)	–	Plithogenic Octapartitioned Neutrosophic Set	$\{v_1, \dots, v_8\}$ for $\{T, M, C, U, I, K, H, F\}$ with contradiction
(9, 1)	–	Plithogenic Nonapartitioned Neutrosophic Set	$\{v_1, \dots, v_9\}$ for $\{T, ST, \dots, WF, F\}$ with contradiction
(10, 1)	–	Plithogenic Decapartitioned Neutrosophic Set	$\{v_1, \dots, v_{10}\}$ for $\{T, SRT, \dots, WRF, F\}$ with contradiction

2.3. Fuzzy Graph. [27] A *fuzzy graph* is a graph in which vertices and edges carry membership degrees in $[0, 1]$. The *spectrum of a fuzzy graph* is the multiset of eigenvalues of its fuzzy adjacency matrix. The *energy of a fuzzy graph* is the sum of the absolute values of the eigenvalues of its adjacency matrix. The *Laplacian matrix of a fuzzy graph* is

$L(G) = D(G) - A(G)$, where $D(G)$ is the degree matrix and $A(G)$ is the adjacency matrix.

Definition 2.4 (Fuzzy Graph[11]). A fuzzy graph on a (crisp) vertex set V with edge set $E \subseteq V \times V$ is a pair $G = (\mu, \nu)$, where $\mu : V \rightarrow [0, 1]$ and $\nu : E \rightarrow [0, 1]$ satisfy

$$\nu(xy) \leq \min\{\mu(x), \mu(y)\} \quad \text{for all } xy \in E.$$

Example 2.5 (Fuzzy Graph). Consider the fuzzy graph

$$G = (\mu, \nu)$$

on the vertex set

$$V = \{x_1, x_2, x_3\}$$

with edge set

$$E = \{x_1x_2, x_2x_3, x_1x_3\}.$$

Define the vertex-membership function

$$\mu : V \longrightarrow [0, 1]$$

by

$$\mu(x_1) = .9, \quad \mu(x_2) = .6, \quad \mu(x_3) = .4,$$

and define the edge-membership function

$$\nu : E \longrightarrow [0, 1]$$

by

$$\nu(x_1x_2) = .5, \quad \nu(x_2x_3) = .3, \quad \nu(x_1x_3) = .2.$$

Figure 1 displays the corresponding fuzzy graph.

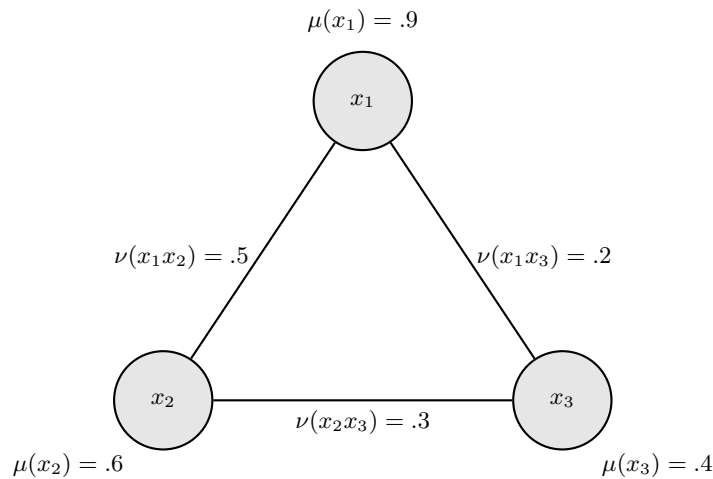


FIGURE 1. A fuzzy graph with vertex and edge memberships.

Now we verify the defining condition of a fuzzy graph:

$$\nu(xy) \leq \min\{\mu(x), \mu(y)\} \quad \text{for all } xy \in E.$$

Indeed,

$$\nu(x_1x_2) = .5 \leq \min\{.9, .6\} = .6,$$

$$\nu(x_2x_3) = .3 \leq \min\{.6, .4\} = .4,$$

and

$$\nu(x_1x_3) = .2 \leq \min\{.9, .4\} = .4.$$

Hence,

$$\nu(x_ix_j) \leq \min\{\mu(x_i), \mu(x_j)\}$$

for every edge $x_ix_j \in E$. Therefore, $G = (\mu, \nu)$ is a well-defined fuzzy graph.

Definition 2.6 (Adjacency Matrix[20]). Let $G = (\mu, \nu)$ be a fuzzy graph with $V = \{x_1, \dots, x_n\}$. The adjacency matrix of G is

$$A(G) := [\nu(x_ix_j)]_{1 \leq i, j \leq n}.$$

When all nonzero membership values $\nu(x_ix_j)$ equal 1, $A(G)$ reduces to the usual adjacency matrix of the underlying crisp graph.

Example 2.7 (Adjacency Matrix). For the fuzzy graph above with ordering (x_1, x_2, x_3) , the adjacency matrix is

$$A(G) = \begin{bmatrix} 0 & 0.5 & 0.2 \\ 0.5 & 0 & 0.3 \\ 0.2 & 0.3 & 0 \end{bmatrix}.$$

(Non-edges carry entry 0; here all three pairs are edges.)

Definition 2.8 (Spectrum of a Fuzzy Graph[20]). For a fuzzy graph $G = (\mu, \nu)$, the spectrum of G is the multiset of eigenvalues of its adjacency matrix $A(G)$; we denote it by $\text{Spec}(G) := \text{Spec}(A(G))$.

Example 2.9 (Spectrum of a Fuzzy Graph). Consider a fuzzy graph $G = (\mu, \nu)$ with vertex set $V = \{x_1, x_2, x_3\}$ and edges $E = \{x_1x_2, x_1x_3\}$. Let the edge memberships be $\nu(x_1x_2) = 0.5$, $\nu(x_1x_3) = 0.7$, and $\nu(x_2x_3) = 0$. The adjacency matrix is

$$A(G) = \begin{bmatrix} 0 & 0.5 & 0.7 \\ 0.5 & 0 & 0 \\ 0.7 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\lambda I - A(G)) = \det \begin{bmatrix} \lambda & -0.5 & -0.7 \\ -0.5 & \lambda & 0 \\ -0.7 & 0 & \lambda \end{bmatrix} = \lambda^3 - (0.5^2 + 0.7^2)\lambda = \lambda^3 - 0.74\lambda.$$

The eigenvalues (rounded) are

$$\lambda_1 \approx 0.860, \quad \lambda_2 = 0, \quad \lambda_3 \approx -0.860.$$

Hence, the spectrum is

$$\text{Spec}(G) = \{0.860, 0, -0.860\}.$$

Definition 2.10 (Energy of a Fuzzy Graph[20]). *For a fuzzy graph $G = (\mu, \nu)$ with adjacency matrix $A(G)$ having eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with algebraic multiplicity), the energy of G is*

$$E(G) := \sum_{i=1}^n |\lambda_i|.$$

Example 2.11 (Energy of a Fuzzy Graph). *Consider a fuzzy graph $G = (\mu, \nu)$ with vertex set $V = \{v_1, v_2\}$ and edge set $E = \{v_1 v_2\}$. Let the vertex memberships be $\mu(v_1) = \mu(v_2) = 1$, and the edge membership be $\nu(v_1 v_2) = 0.5$.*

Then the adjacency matrix is

$$A(G) = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}.$$

The eigenvalues of $A(G)$ are $\lambda_1 = 0.5$ and $\lambda_2 = -0.5$. Hence, the energy of G is

$$E(G) = |\lambda_1| + |\lambda_2| = 0.5 + 0.5 = 1.$$

Definition 2.12 ((Fuzzy) Degree[20]). *Let $G = (\mu, \nu)$ be a fuzzy graph. For a vertex $x \in V(G)$, its (fuzzy) degree is*

$$d_G(x) := \sum_{xy \in E(G)} \nu(xy).$$

Example 2.13 ((Fuzzy) Degrees). *The (fuzzy) degree of each vertex is the sum of incident edge memberships:*

$$d_G(x_1) = 0.5 + 0.2 = 0.7, \quad d_G(x_2) = 0.5 + 0.3 = 0.8, \quad d_G(x_3) = 0.3 + 0.2 = 0.5.$$

Definition 2.14 (Degree Matrix of a Fuzzy Graph[20]). *Let $G = (\mu, \nu)$ be a fuzzy graph on $V = \{x_1, \dots, x_n\}$. The degree matrix of G is the diagonal matrix*

$$D(G) := \text{diag}(d_G(x_1), \dots, d_G(x_n)).$$

Example 2.15 (Degree Matrix of a Fuzzy Graph[20]). *Using the degrees above and the ordering (x_1, x_2, x_3) ,*

$$D(G) = \text{diag}(d_G(x_1), d_G(x_2), d_G(x_3)) = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

Definition 2.16 (Laplacian Matrix of a Fuzzy Graph[20]). *The Laplacian matrix of a fuzzy graph G is*

$$L(G) := D(G) - A(G),$$

where $A(G)$ is the adjacency matrix and $D(G)$ is the degree matrix of G .

Example 2.17 (Laplacian Matrix of a Fuzzy Graph). *Consider a fuzzy graph $G = (\mu, \nu)$ on $V = \{v_1, v_2, v_3\}$ with*

$$\mu(v_1) = \mu(v_2) = \mu(v_3) = 1, \quad \nu(v_1 v_2) = 0.6, \quad \nu(v_2 v_3) = 0.3, \quad \nu(v_1 v_3) = 0.$$

(Thus $\nu(xy) \leq \min\{\mu(x), \mu(y)\}$ holds.)

The adjacency matrix is

$$A(G) = \begin{bmatrix} 0 & 0.6 & 0 \\ 0.6 & 0 & 0.3 \\ 0 & 0.3 & 0 \end{bmatrix}.$$

Using the (fuzzy) degree $d_G(v_i) = \sum_j \nu(v_i v_j)$, we obtain

$$d_G(v_1) = 0.6, \quad d_G(v_2) = 0.6 + 0.3 = 0.9, \quad d_G(v_3) = 0.3,$$

so the degree matrix is

$$D(G) = \text{diag}(0.6, 0.9, 0.3).$$

Therefore, the Laplacian matrix is

$$L(G) = D(G) - A(G) = \begin{bmatrix} 0.6 & -0.6 & 0 \\ -0.6 & 0.9 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix}.$$

2.18. Plithogenic Graph. A Plithogenic Graph extends fuzzy, intuitionistic, and neutrosophic graphs by integrating attributes, membership, and contradiction degrees to model complex uncertainties [18].

Definition 2.19 (Plithogenic Graph[18]). Let $G^* = (V, E)$ be a crisp graph. A plithogenic graph is a tuple

$$P^G := (P_A, P_B),$$

where $P_A = (A, \alpha, R_\alpha, d_{\alpha f}, c_{\alpha f})$ is a plithogenic vertex set with $A \subseteq V$, attribute α and range R_α , degree of appurtenance $d_{\alpha f} : A \times R_\alpha \rightarrow [0, 1]^s$, and degree of contradiction $c_{\alpha f} : R_\alpha \times R_\alpha \rightarrow [0, 1]^t$; and $P_B = (B, \beta, R_\beta, d_{\beta f}, c_{\beta f})$ is a plithogenic edge set with $B \subseteq E$, attribute β and range R_β , degree of appurtenance $d_{\beta f} : B \times R_\beta \rightarrow [0, 1]^s$, and degree of contradiction $c_{\beta f} : R_\beta \times R_\beta \rightarrow [0, 1]^t$. We require, for all $(x, a), (y, b) \in A \times R_\alpha$,

$$d_{\beta f}((x, a)(y, b)) \leq \min\{d_{\alpha f}(x, a), d_{\alpha f}(y, b)\},$$

and $c_{\beta f}$ is symmetric with

$$c_{\beta f}((a, b), (a, b)) = 0 \quad \text{for all } (a, b) \in R_\beta.$$

Here $s, t \in \{1, 2, 3\}$. In what follows we focus on the subclass of plithogenic fuzzy graphs.

Example 2.20 (Plithogenic Graph). Let the underlying crisp graph be $G^* = (V, E)$ with $V = \{x, y\}$ and $E = \{xy\}$. Define the plithogenic vertex set

$$P_A = (A, \alpha, R_\alpha, d_{\alpha f}, c_{\alpha f}),$$

$$A = \{x, y\}, \quad R_\alpha = \{u, v\},$$

with degrees of appurtenance

$d_{\alpha f}$	u	v
x	0.8	0.3
y	0.7	0.6

and a symmetric degree of contradiction on R_α given by

$$c_{\alpha f}(u, u) = c_{\alpha f}(v, v) = 0, \quad c_{\alpha f}(u, v) = c_{\alpha f}(v, u) = 0.4.$$

Define the plithogenic edge set

$$P_B = (B, \beta, R_\beta, d_{\beta f}, c_{\beta f}),$$

$$B = \{xy\}, \quad R_\beta = \{(u, u), (u, v), (v, u), (v, v)\}.$$

Set the edge appurtenance degrees (one value for each attribute pair) as

$$d_{\beta f}((x, u)(y, u)) = 0.6, \quad d_{\beta f}((x, u)(y, v)) = 0.5,$$

$$d_{\beta f}((x, v)(y, u)) = 0.3, \quad d_{\beta f}((x, v)(y, v)) = 0.3,$$

which satisfy, for example,

$$d_{\beta f}((x, u)(y, v)) = 0.5 \leq \min\{d_{\alpha f}(x, u), d_{\alpha f}(y, v)\} = \min\{0.8, 0.6\} = 0.6,$$

and similarly for the other cases. Let $c_{\beta f}$ be symmetric with zeros on the diagonal (e.g., $c_{\beta f}(r, r) = 0$ for all $r \in R_\beta$ and $c_{\beta f}(r_1, r_2) = 0.3$ for $r_1 \neq r_2$). Then $P^G = (P_A, P_B)$ is a plithogenic graph according to Definition 2.19.

2.21. Plithogenic Fuzzy Graph. A Plithogenic Fuzzy Graph models uncertain networks by combining fuzzy membership with plithogenic attributes and contradiction degrees for richer decision-making analysis [18].

Definition 2.22 (Plithogenic Fuzzy Graph [18]). Taking $s = t = 1$ in Definition 2.19, a plithogenic fuzzy graph of a crisp graph $G^* = (V, E)$ is

$$P^{FG} := (P^{FA}, P^{FB}),$$

where $P^{FA} = (A, \alpha_F, R_{\alpha_F}, d_{\alpha_F}, c_{\alpha_F})$ is a plithogenic fuzzy vertex set with $A \subseteq V$, $d_{\alpha_F} : A \times R_{\alpha_F} \rightarrow [0, 1]$ and $c_{\alpha_F} : R_{\alpha_F} \times R_{\alpha_F} \rightarrow [0, 1]$; and $P^{FB} = (B, \beta_F, R_{\beta_F}, d_{\beta_F}, c_{\beta_F})$ is a plithogenic fuzzy edge set with $B \subseteq E$, $d_{\beta_F} : B \times R_{\beta_F} \rightarrow [0, 1]$ and $c_{\beta_F} : R_{\beta_F} \times R_{\beta_F} \rightarrow [0, 1]$. For all $(x, a), (y, b) \in A \times R_{\alpha_F}$,

$$d_{\beta_F}((x, a)(y, b)) \leq \min\{d_{\alpha_F}(x, a), d_{\alpha_F}(y, b)\},$$

and c_{β_F} is symmetric with

$$c_{\beta_F}((a, b), (a, b)) = 0 \quad \text{for all } (a, b) \in R_{\beta_F}.$$

In Table 5, we provide an overview and comparison, including the graph parameters that will be discussed later in this paper. The Plithogenic Fuzzy Graph offers several advantages for modeling real-world concepts with uncertainty, such as its ability to handle contradictions in uncertainty that cannot be addressed by a traditional Fuzzy Graph.

Example 2.23 (Plithogenic Fuzzy Graph). Consider the crisp graph

$$G^* = (V, E), \quad V = \{x, y, z\}, \quad E = \{xy, yz\}.$$

We define a plithogenic fuzzy graph

$$P^{FG} = (P^{FA}, P^{FB})$$

over G^* as follows.

First, define the plithogenic fuzzy vertex set

$$P^{FA} = (A, \alpha_F, R_{\alpha_F}, d_{\alpha_F}, c_{\alpha_F}),$$

where

$$A = \{x, y, z\}, \quad R_{\alpha_F} = \{\text{red}, \text{blue}\}.$$

TABLE 5. Concise comparison of a Fuzzy Graph and a Plithogenic Fuzzy Graph

Aspect	Fuzzy Graph	Plithogenic Fuzzy Graph
Underlying structure	$G = (\mu, \nu)$ on a crisp graph (V, E) .	$P^{FG} = (P^{FA}, P^{FB})$ over a crisp $G^* = (V, E)$, where P^{FA}, P^{FB} are plithogenic fuzzy vertex/edge sets.
Vertex information	$\mu : V \rightarrow [0, 1]$ (vertex membership).	$d_{\alpha F} : A \times R_{\alpha F} \rightarrow [0, 1]$ (attribute-dependent vertex membership).
Edge information	$\nu : E \rightarrow [0, 1]$ (edge membership).	$d_{\beta F} : B \times R_{\beta F} \rightarrow [0, 1]$ (attribute-dependent edge membership).
Attributes / contradiction	No attribute layer; no contradiction degree.	Attribute sets $R_{\alpha F}, R_{\beta F}$ and symmetric contradiction maps $c_{\alpha F}, c_{\beta F}$ (zero on the diagonal).
Consistency constraint	$\nu(xy) \leq \min\{\mu(x), \mu(y)\}$.	$d_{\beta F}((x, a)(y, b)) \leq \min\{d_{\alpha F}(x, a), d_{\alpha F}(y, b)\}$.
Adjacency matrix	$A(G) = [\nu(x_i x_j)]_{i,j}$.	$M^{PFG} = [d_{\beta F}((x_i, a_k)(y_j, b_\ell))]_{(i,k),(j,\ell)}$.
Vertex degree	$d_G(x) = \sum_{xy \in E} \nu(xy)$.	$d_{FG}(x, a) = \sum_{(y,b)} d_{\beta F}((x, a)(y, b))$.
Laplacian	$L = D - A$, with $D = \text{diag}(d_G(x_1), \dots, d_G(x_n))$.	$L(P^{FG}) = D - M^{PFG}$, with $D = \text{diag}(d_{FG}(x_1, a_{k_1}), \dots)$.
Spectrum / Energy	Eigenvalues of $A(G)$; energy $E(G) = \sum_i \lambda_i $.	Eigenvalues of M^{PFG} ; energy $E(P^{FG}) = \sum_i \lambda_i $ (similarly for Laplacian).
Modeling capability	Represents uncertainty via a single membership degree.	Handles multi-attribute uncertainty with explicit contradiction between attribute values.

The fuzzy degree of appurtenance

$$d_{\alpha F} : A \times R_{\alpha F} \rightarrow [0, 1]$$

is given by

	red	blue
x	0.8	0.2
y	0.6	0.5
z	0.4	0.7

that is,

$$\begin{aligned} d_{\alpha F}(x, \text{red}) &= 0.8, & d_{\alpha F}(x, \text{blue}) &= 0.2, \\ d_{\alpha F}(y, \text{red}) &= 0.6, & d_{\alpha F}(y, \text{blue}) &= 0.5, \\ d_{\alpha F}(z, \text{red}) &= 0.4, & d_{\alpha F}(z, \text{blue}) &= 0.7. \end{aligned}$$

The fuzzy degree of contradiction

$$c_{\alpha F} : R_{\alpha F} \times R_{\alpha F} \rightarrow [0, 1]$$

is defined by

$$c_{\alpha F}(\text{red}, \text{red}) = 0, \quad c_{\alpha F}(\text{blue}, \text{blue}) = 0,$$

$$c_{\alpha F}(\text{red}, \text{blue}) = c_{\alpha F}(\text{blue}, \text{red}) = 0.5.$$

Next, define the plithogenic fuzzy edge set

$$P^{FB} = (B, \beta_F, R_{\beta_F}, d_{\beta_F}, c_{\beta_F}),$$

where

$$B = \{xy, yz\}, \quad R_{\beta_F} = \{(\text{red}, \text{red}), (\text{red}, \text{blue}), (\text{blue}, \text{blue})\}.$$

The fuzzy degree of appurtenance

$$d_{\beta_F} : B \times R_{\beta_F} \rightarrow [0, 1]$$

is defined by

$$\begin{aligned} d_{\beta_F}(xy, (\text{red}, \text{red})) &= 0.6, & d_{\beta_F}(xy, (\text{red}, \text{blue})) &= 0.4, \\ d_{\beta_F}(yz, (\text{blue}, \text{blue})) &= 0.5, \end{aligned}$$

and we take all unspecified values of d_{β_F} to be 0.

For the contradiction map on edge-attribute values, let

$$c_{\beta_F} : R_{\beta_F} \times R_{\beta_F} \rightarrow [0, 1]$$

be symmetric with

$$c_{\beta_F}(r, r) = 0 \quad \text{for all } r \in R_{\beta_F},$$

and, for instance,

$$c_{\beta_F}(r_1, r_2) = 0.3 \quad \text{whenever } r_1 \neq r_2.$$

We now verify the defining condition. For example,

$$\begin{aligned} d_{\beta_F}(xy, (\text{red}, \text{red})) &= 0.6 \\ &\leq \min\{d_{\alpha F}(x, \text{red}), d_{\alpha F}(y, \text{red})\} \\ &= \min\{0.8, 0.6\} \\ &= 0.6, \\ d_{\beta_F}(xy, (\text{red}, \text{blue})) &= 0.4 \\ &\leq \min\{d_{\alpha F}(x, \text{red}), d_{\alpha F}(y, \text{blue})\} \\ &= \min\{0.8, 0.5\} \\ &= 0.5, \end{aligned}$$

and

$$\begin{aligned} d_{\beta_F}(yz, (\text{blue}, \text{blue})) &= 0.5 \\ &\leq \min\{d_{\alpha F}(y, \text{blue}), d_{\alpha F}(z, \text{blue})\} \\ &= \min\{0.5, 0.7\} \\ &= 0.5. \end{aligned}$$

Thus,

$$d_{\beta_F}((u, a)(v, b)) \leq \min\{d_{\alpha F}(u, a), d_{\alpha F}(v, b)\}$$

holds for all specified edge-attribute combinations, and both $c_{\alpha F}$ and c_{β_F} are symmetric with zero diagonal. Therefore,

$$P^{FG} = (P^{FA}, P^{FB})$$

is a well-defined plithogenic fuzzy graph of G^* .

The corresponding plithogenic fuzzy graph is illustrated in Figure 2. The vertices in the figure represent vertex–attribute pairs, while the edges represent the nonzero values of the edge appurtenance map $d_{\beta F}$.

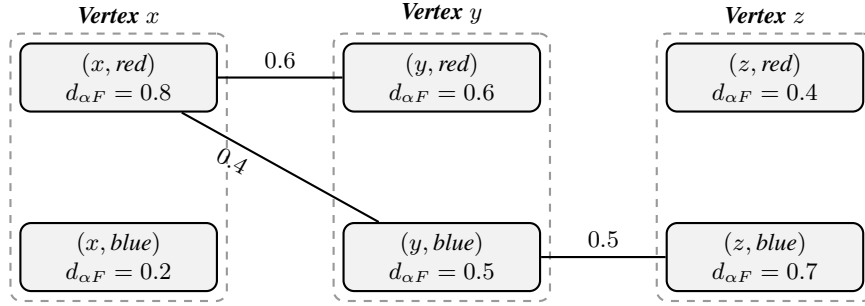


FIGURE 2. A Plithogenic Fuzzy Graph corresponding to Example 2.23. Each node represents a vertex–attribute pair together with its appurtenance degree. Only edge–attribute combinations having nonzero appurtenance degrees are displayed.

Example 2.24 (Another plithogenic fuzzy graph). Let the underlying crisp graph be

$$G^* = (V, E), \quad V = \{u, v\}, \quad E = \{uv\}.$$

We construct a plithogenic fuzzy graph

$$P^{FG} = (P^{FA}, P^{FB})$$

over G^* .

First, define the plithogenic fuzzy vertex set

$$P^{FA} = (A, \alpha_F, R_{\alpha_F}, d_{\alpha_F}, c_{\alpha_F}),$$

where

$$A = \{u, v\}, \quad R_{\alpha_F} = \{p, q\}.$$

The vertex appurtenance map

$$d_{\alpha_F} : A \times R_{\alpha_F} \rightarrow [0, 1]$$

is given by

d_{α_F}	u	v
p	0.8	0.5
q	0.3	0.6

that is,

$$\begin{aligned} d_{\alpha_F}(u, p) &= 0.8, & d_{\alpha_F}(u, q) &= 0.3, \\ d_{\alpha_F}(v, p) &= 0.5, & d_{\alpha_F}(v, q) &= 0.6. \end{aligned}$$

The vertex contradiction map

$$c_{\alpha_F} : R_{\alpha_F} \times R_{\alpha_F} \rightarrow [0, 1]$$

is defined by

$$c_{\alpha F}(p, p) = c_{\alpha F}(q, q) = 0, \quad c_{\alpha F}(p, q) = c_{\alpha F}(q, p) = 0.2.$$

Next, define the plithogenic fuzzy edge set

$$P^{FB} = (B, \beta_F, R_{\beta_F}, d_{\beta_F}, c_{\beta_F}),$$

where

$$B = \{uv\}, \quad R_{\beta_F} = \{pq, qp\}.$$

The edge appurtenance map

$$d_{\beta_F} : B \times R_{\beta_F} \rightarrow [0, 1]$$

is given by

$$d_{\beta_F}(uv, pq) = 0.4, \quad d_{\beta_F}(uv, qp) = 0.3,$$

and the edge contradiction map satisfies

$$c_{\beta_F}(r, r) = 0 \quad (r \in R_{\beta_F}),$$

$$c_{\beta_F}(pq, qp) = c_{\beta_F}(qp, pq) = 0.1.$$

Constraint check. For (uv, pq) , we have

$$d_{\beta_F}(uv, pq) = 0.4 \leq \min\{d_{\alpha F}(u, p), d_{\alpha F}(v, q)\} = \min\{0.8, 0.6\} = 0.6.$$

For (uv, qp) , we have

$$d_{\beta_F}(uv, qp) = 0.3 \leq \min\{d_{\alpha F}(u, q), d_{\alpha F}(v, p)\} = \min\{0.3, 0.5\} = 0.3.$$

Thus,

$$d_{\beta_F}((x, a)(y, b)) \leq \min\{d_{\alpha F}(x, a), d_{\alpha F}(y, b)\}$$

holds for all relevant edge-attribute pairs. Moreover, both $c_{\alpha F}$ and c_{β_F} are symmetric with zero diagonal.

Adjacency matrix. Index the vertex-attribute pairs as

$$(u, p), (u, q), (v, p), (v, q).$$

Then the adjacency matrix M^{PFG} is

$$M^{PFG} = \begin{bmatrix} 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding degrees are

$$d_{FG}(u, p) = 0.4, \quad d_{FG}(u, q) = 0.3, \quad d_{FG}(v, p) = 0.3, \quad d_{FG}(v, q) = 0.4,$$

so the degree matrix is

$$D(P^{FG}) = \text{diag}(0.4, 0.3, 0.3, 0.4).$$

Therefore,

$$P^{FG} = (P^{FA}, P^{FB})$$

is a concrete plithogenic fuzzy graph satisfying all defining axioms.

The corresponding plithogenic fuzzy graph is illustrated in Figure 3.

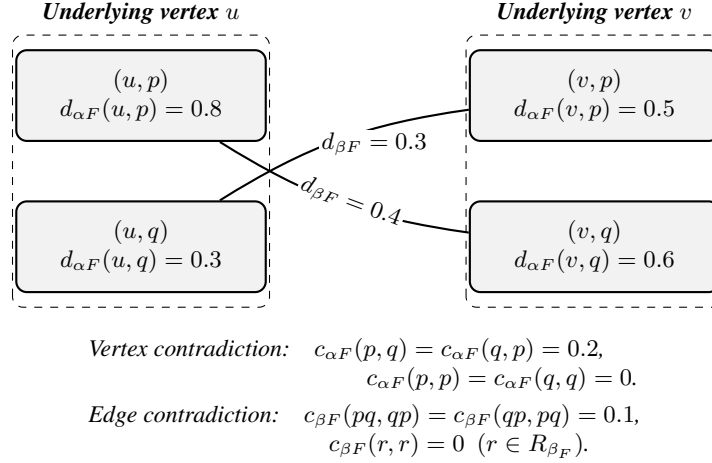


FIGURE 3. A plithogenic fuzzy graph corresponding to Example 2.24. The nodes represent vertex–attribute pairs, and the edges represent the nonzero appurtenance values in the adjacency matrix M^{PFG} .

3. ENERGY OF PLITHOGENIC FUZZY GRAPH

In this section we introduce the adjacency matrix, eigenvalues, and spectrum of a plithogenic fuzzy graph, together with an illustrative example. We also present several related theorems.

Definition 3.1 (Adjacency matrix of a plithogenic fuzzy graph). *Let $P^{FG} = (P^{FA}, P^{FB})$ be a plithogenic fuzzy graph associated with a crisp graph $G^* = (V, E)$, where*

$$P^{FA} = (A, \alpha_F, R_{\alpha_F}, d_{\alpha_F}, c_{\alpha_F}), \quad P^{FB} = (B, \beta_F, R_{\beta_F}, d_{\beta_F}, c_{\beta_F}).$$

Write $\mathcal{V} := A \times R_{\alpha_F}$ for the set of vertex–attribute pairs and set $n := |\mathcal{V}|$. The adjacency matrix of P^{FG} is the $n \times n$ real matrix $M^{PFG} = [m_{uv}]_{u,v \in \mathcal{V}}$ defined entrywise by

$$m_{(x,a),(y,b)} := d_{\beta_F}((x,a)(y,b)), \quad (x,a), (y,b) \in \mathcal{V}.$$

Equivalently, M^{PFG} records, for each pair of vertices together with their attributes, the (fuzzy) degree of appurtenance assigned to the corresponding edge.

Example 3.2. *Let $G^* = (V, E)$ with $A = \{x, y, z\} \subset V$, $B = \{xy, yz, xz\} \subset E$, and let $R_\alpha = \{a, b, c, d\}$. Define d_{α_F} and c_{α_F} on vertices and attributes by (see Table 6),*

These data satisfy, for all $(x, a), (y, b) \in A \times R_\alpha$,

$$d_{\beta_F}((x,a)(y,b)) \leq \min\{d_{\alpha_F}(x,a), d_{\alpha_F}(y,b)\},$$

and c_{β_F} is symmetric with $c_{\beta_F}((a,b), (a,b)) = 0$. Hence $P^{FG} = (P^{FA}, P^{FB})$ is a plithogenic fuzzy graph, as shown in Fig. 4.

TABLE 6. $d_{\alpha F}, c_{\alpha F}$ (vertex attributes) and $d_{\beta F}, c_{\beta F}$ (edge attributes).

$d_{\alpha F}$	x	y	z
a	.3	.5	.2
b	.4	.1	.3
c	.2	.2	.4
d	.1	.3	.1

$c_{\alpha F}$	a	b	c	d
a	0	.5	.6	.5
b	.5	0	.4	.4
c	.6	.4	0	.5
d	.5	.4	.5	0

$d_{\beta F}$	xy	yz	xz
ab	.1	.3	.2
bc	.2	.1	.3
cd	.2	.1	.1
ac	.1	.4	.3
ad	.3	.1	.1

$c_{\beta F}$	ab	bc	cd	ac	ad
ab	0	.4	.6	.5	.1
bc	.4	0	.4	.4	.3
cd	.6	.4	0	.5	.1
ac	.5	.4	.5	0	.2
ad	.1	.3	.1	.2	0

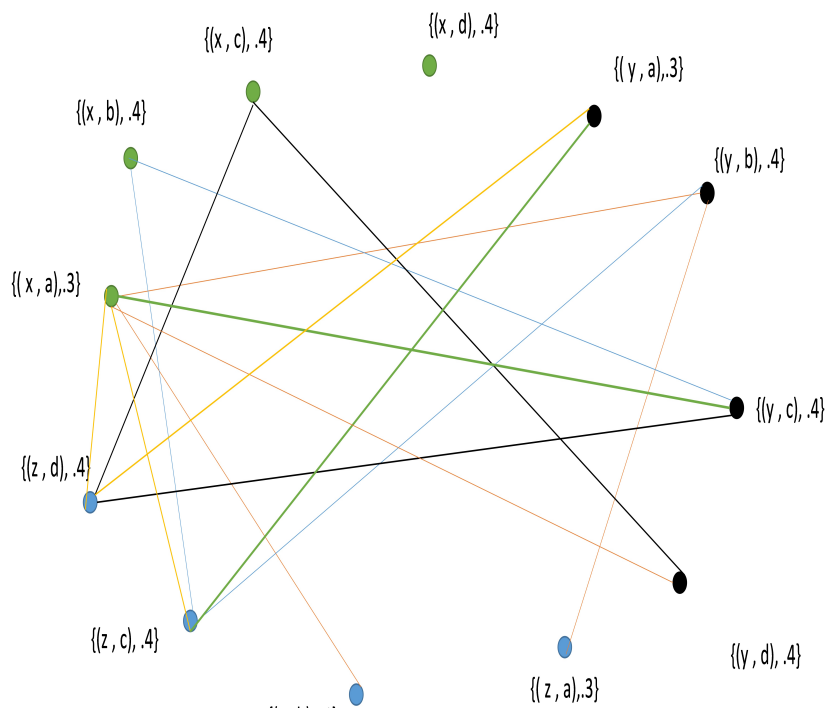


FIGURE 4. A Plithogenic fuzzy graph structure

Order the vertex–attribute pairs as

$(x, a), (x, b), (x, c), (x, d), (y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d).$

With this order the adjacency matrix M^{PFG} is the 12×12 matrix below:

$$M^{PFG} = \begin{bmatrix} 0 & 0 & 0 & 0 & .3 & .2 & 0 & .1 & 0 & 0 & .3 & .1 \\ 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & .4 & 0 & 0 & .1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .1 & 0 & 0 & 0 \\ .2 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .1 & 0 & .2 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .1 \\ .3 & .4 & 0 & .3 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .3 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 \\ .1 & .1 & 0 & 0 & 0 & 0 & .1 & .1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 3.3 (Spectrum and energy). Let P^{FG} be a plithogenic fuzzy graph with adjacency matrix $M^{PFG} \in \mathbb{R}^{n \times n}$. The spectrum of P^{FG} is the multiset of eigenvalues (counted with algebraic multiplicity)

$$\text{Spec}(P^{FG}) := \text{Spec}(M^{PFG}) = \{\lambda_1, \dots, \lambda_n\}.$$

The (graph) energy of P^{FG} is

$$E(P^{FG}) := \sum_{i=1}^n |\lambda_i|,$$

where for a complex eigenvalue $\lambda_i = a + bi$ we use $|\lambda_i| = \sqrt{a^2 + b^2}$.

Example 3.4 (Numeric illustration). For the matrix in Example 3.2, the spectrum (rounded) is

$$\text{Spec}(M^{PFG}) = \left\{ 0.660198, -0.600169, -0.402229, 0.3735, -0.233769, 0.190974, \right. \\ \left. 0.00870021 \pm 0.185283i, -0.00590602, 0, 0, 0 \right\}.$$

Hence the energy is the sum of the absolute values (zeros contribute 0, and

$$|0.00870021 \pm 0.185283i| = \sqrt{0.00870021^2 + 0.185283^2}$$

):

$$\begin{aligned} E(P^{FG}) &= 0.660198 + 0.600169 + 0.402229 + 0.3735 \\ &\quad + 0.233769 + 0.190974 + 0.00590602 \\ &\quad + 2\sqrt{0.00870021^2 + 0.185283^2} \\ &\approx 2.837719. \end{aligned}$$

We next record two basic properties of the spectrum and an energy bound.

Theorem 3.5. Let P^{FG} be a plithogenic fuzzy graph with adjacency matrix M^{PFG} . Suppose for each attribute a_k the eigenvalues are ordered $\lambda_{a_k}^1 \geq \lambda_{a_k}^2 \geq \dots \geq \lambda_{a_k}^n$. Then:

- (1) $\sum_{i=1}^n \lambda_{a_k}^i = 0$ for every k .
- (2) $\sum_{k=1}^{\ell} \sum_{i=1}^n (\lambda_{a_k}^i)^2 = 2 \sum_{1 \leq k, s \leq \ell} \sum_{(x, a_k), (y, a_s)} (d_{\beta F}((x, a_k)(y, a_s)))^2$.

- Proof.* (1) The matrix M^{FG} is real symmetric with zero diagonal, hence $\text{tr}(M^{FG}) = 0$, which equals the sum of its eigenvalues.
- (2) Using $\text{tr}((M^{FG})^2) = \sum_{i=1}^n \lambda_i^2$ and the definition of M^{FG} gives the stated identity. $\square$

Theorem 3.6 (Energy bounds). *Let P^{FG} be a plithogenic fuzzy graph on n vertex-attribute pairs with adjacency matrix M^{FG} . Then*

$$\sqrt{2 \sum_{(x,a),(y,b)} d_{\beta F}((x,a)(y,b))^2 + n(n-1) |M^{FG}|^{2/n}} \leq E(P^{FG})$$

$$\leq \sqrt{2n \sum_{(x,a),(y,b)} d_{\beta F}((x,a)(y,b))^2}, \quad (3.1)$$

where $|M^{FG}|$ denotes the determinant of M^{FG} .

Proof. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of M^{FG} . By Cauchy-Schwarz,

$$\sum_{i=1}^n |\lambda_i| \leq \sqrt{n} \left(\sum_{i=1}^n \lambda_i^2 \right)^{1/2}. \quad (3.2)$$

Since

$$\sum_{i=1}^n \lambda_i^2 = \text{tr}((M^{FG})^2) = 2 \sum_{(x,a),(y,b)} d_{\beta F}((x,a)(y,b))^2,$$

inequality (3.2) yields the upper bound in (3.1).

For the lower bound, write

$$E(P^{FG})^2 = \left(\sum_{i=1}^n |\lambda_i| \right)^2$$

$$= \sum_{i=1}^n \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j|.$$

By the AM-GM inequality applied to the multiset $\{|\lambda_i \lambda_j| : 1 \leq i < j \leq n\}$,

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \geq \left(\prod_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right)^{2/(n(n-1))}$$

$$= \left(\prod_{i=1}^n |\lambda_i| \right)^{2/n} = |M^{FG}|^{2/n}.$$

Combining this with $\sum_{i=1}^n \lambda_i^2 = 2 \sum d_{\beta F}((x,a)(y,b))^2$ gives the stated lower bound. $\square$

4. LAPLACIAN ENERGY OF A PLITHOGENIC FUZZY GRAPH

This section introduces the notions needed to define the Laplacian energy of a plithogenic fuzzy graph, gives illustrative examples, and records several basic results.

Definition 4.1 (Vertex degree in P^{FG}). *Let $P^{FG} = (P^{FA}, P^{FB})$ be a plithogenic fuzzy graph, and write $\mathcal{V} := A \times R_{\alpha F}$ for the set of vertex–attribute pairs. For $(x_i, a_k) \in \mathcal{V}$, the degree of (x_i, a_k) in P^{FG} is*

$$d_{FG}(x_i, a_k) := \sum_{(y_j, b_\ell) \in \mathcal{V}} d_{\beta F}((x_i, a_k)(y_j, b_\ell)).$$

Example 4.2 (Degrees in Example 3.2). *Using the data from Example 3.2, the degrees are*

$$\begin{aligned} d_{FG}(x, a) &= d_{\beta F}(xy, ab) + d_{\beta F}(xy, ac) + d_{\beta F}(xy, ad) \\ &\quad + d_{\beta F}(xz, ab) + d_{\beta F}(xz, ac) + d_{\beta F}(xz, ad) \\ &= 0.1 + 0.1 + 0.3 + 0.2 + 0.3 + 0.1 = 1, \\ d_{FG}(x, b) &= d_{\beta F}(xy, bc) + d_{\beta F}(xz, bc) = 0.2 + 0.3 = 0.5, \\ d_{FG}(x, c) &= d_{\beta F}(xy, cd) + d_{\beta F}(xz, cd) = 0.2 + 0.1 = 0.3, \\ d_{FG}(x, d) &= 0, \\ d_{FG}(y, a) &= d_{\beta F}(xy, ab) + d_{\beta F}(xy, ac) + d_{\beta F}(xy, ad) \\ &\quad + d_{\beta F}(yz, ab) + d_{\beta F}(yz, ac) + d_{\beta F}(yz, ad) \\ &= 0.1 + 0.1 + 0.3 + 0.3 + 0.4 + 0.1 = 1.3, \\ d_{FG}(y, b) &= d_{\beta F}(yz, bc) = 0.1, \\ d_{FG}(y, c) &= d_{\beta F}(xy, bc) + d_{\beta F}(yz, cd) + d_{\beta F}(xy, ac) + d_{\beta F}(yz, cd) \\ &= 0.2 + 0.1 + 0.1 + 0.4 = 0.8, \\ d_{FG}(y, d) &= d_{\beta F}(xy, cd) + d_{\beta F}(xy, ad) = 0.2 + 0.3 = 0.5, \\ d_{FG}(z, a) &= 0, \\ d_{FG}(z, b) &= d_{\beta F}(yz, ab) + d_{\beta F}(xz, ab) = 0.1 + 0.2 = 0.3, \\ d_{FG}(z, c) &= d_{\beta F}(yz, bc) + d_{\beta F}(yz, ac) = 0.2 + 0.4 = 0.6, \\ d_{FG}(z, d) &= d_{\beta F}(xy, cd) + d_{\beta F}(yz, cd) + d_{\beta F}(yz, ad) = 0.2 + 0.1 + 0.1 = 0.4. \end{aligned}$$

Definition 4.3 (Degree matrix of a Plithogenic Fuzzy Graph). *Let $n := |\mathcal{V}|$. The degree matrix of P^{FG} is the diagonal matrix $D(P^{FG}) = [d_{ij}]_{1 \leq i, j \leq n}$ defined by*

$$d_{ij} = \begin{cases} d_{FG}(x_i, a_k), & \text{if } (x_i, a_k) = (y_j, b_\ell), \\ 0, & \text{otherwise,} \end{cases}$$

for $(x_i, a_k), (y_j, b_\ell) \in \mathcal{V}$.

Example 4.4 (Degree matrix in Example 3.2). Ordering $\mathcal{V}$ as in Example 3.2, we have

$$D(P^{FG}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4 \end{bmatrix}$$

Definition 4.5 (Laplacian matrix of a Plithogenic Fuzzy Graph). Let M^{PFG} be the adjacency matrix of P^{FG} . The Laplacian matrix of P^{FG} is

$$L(P^{FG}) := D(P^{FG}) - M^{PFG}.$$

Example 4.6 (Laplacian in Example 3.2). With $D(P^{FG})$ as above and M^{PFG} from Example 3.2,

$$L(P^{FG}) = D(P^{FG}) - M^{PFG} = \begin{bmatrix} 1 & 0 & 0 & 0 & -.3 & -.2 & 0 & -.1 & 0 & 0 & -.3 & -.1 \\ 0 & .5 & 0 & 0 & 0 & -.3 & 0 & 0 & -.4 & 0 & 0 & -.1 \\ 0 & 0 & .3 & 0 & 0 & 0 & 0 & -.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.3 & 0 & 0 & 0 & 1.3 & 0 & 0 & 0 & -.1 & 0 & 0 & 0 \\ -.2 & -.3 & 0 & 0 & 0 & .1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .8 & 0 & 0 & 0 & 0 & 0 \\ -.1 & 0 & -.2 & -.2 & 0 & 0 & 0 & .5 & 0 & 0 & -.2 & -.1 \\ -.3 & -.4 & 0 & -.3 & -.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 \\ -.3 & 0 & 0 & 0 & 0 & 0 & -.2 & 0 & 0 & 0 & .6 & 0 \\ -.1 & -.1 & 0 & 0 & 0 & 0 & -.1 & -.1 & 0 & 0 & 0 & .4 \end{bmatrix}$$

Definition 4.7 (Laplacian spectrum and energy). Let P^{FG} be a plithogenic fuzzy graph with adjacency matrix $M^{PFG} = [m_{ij}]$, and let $D = \text{diag}(d_1, \dots, d_n)$ where $d_i := \sum_j m_{ij}$. The (row) Laplacian is

$$L(P^{FG}) := D - M^{PFG}.$$

If the eigenvalues of $L(P^{FG})$ (counted with algebraic multiplicity) are $\rho_1, \dots, \rho_n$, then the Laplacian spectrum is

$$\text{Spec}(L(P^{FG})) = \{\rho_1, \dots, \rho_n\},$$

and the Laplacian energy is

$$E(L(P^{FG})) := \sum_{i=1}^n |\rho_i|.$$

Example 4.8 (Laplacian spectrum and energy of a simple P^{FG}). Consider the plithogenic fuzzy graph whose adjacency matrix (on the ordered vertex-attribute

pairs $(u, p), (u, q), (v, p), (v, q)$ is

$$M^{PFG} = \begin{bmatrix} 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \end{bmatrix}.$$

Then the (row) degree vector is

$(d_1, d_2, d_3, d_4) = (0.4, 0.3, 0.3, 0.4)$, so $D = \text{diag}(0.4, 0.3, 0.3, 0.4)$ and

$$L(P^{FG}) = D - M^{PFG} = \begin{bmatrix} 0.4 & 0 & 0 & -0.4 \\ 0 & 0.3 & -0.3 & 0 \\ 0 & -0.3 & 0.3 & 0 \\ -0.4 & 0 & 0 & 0.4 \end{bmatrix}.$$

By permuting rows/columns to group the two edges

$\{(u, p), (v, q)\}$ (weight 0.4) and $\{(u, q), (v, p)\}$ (weight 0.3), the Laplacian becomes block diagonal with blocks $\begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix}$ and $\begin{bmatrix} 0.3 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}$. Hence the Laplacian eigenvalues are

$$\text{Spec}(L(P^{FG})) = \{0.8, 0.6, 0, 0\}.$$

Therefore the Laplacian energy is

$$E(L(P^{FG})) = |0.8| + |0.6| + |0| + |0| = 1.4.$$

Theorem 4.9 (Row-sum, zero eigenvalue, and trace). *For $L = D - M^{PFG}$ as above, the following hold:*

- (1) $L\mathbf{1} = \mathbf{0}$, where $\mathbf{1} = (1, \dots, 1)^T$. In particular, 0 is always an eigenvalue of L .
- (2) $\sum_{i=1}^n \rho_i = \text{tr}(L) = \sum_{i=1}^n d_i$.

Proof. (1) The i th row sum of L equals $\sum_j (d_i \delta_{ij} - m_{ij}) = d_i - \sum_j m_{ij} = 0$, hence $L\mathbf{1} = \mathbf{0}$.

(2) The sum of the eigenvalues equals the trace, and $\text{tr}(L) = \text{tr}(D) - \text{tr}(M^{PFG}) = \sum_i d_i$ since the diagonal of M^{PFG} is zero. $\square$

Theorem 4.10 (Gershgorin localization and basic bounds). *Let $d_{\max} := \max_i d_i$. Every eigenvalue ρ of L lies in at least one Gershgorin disc*

$$\mathbb{D}(d_i, R_i) \quad \text{with} \quad R_i := \sum_{j \neq i} |-m_{ij}| = \sum_{j \neq i} m_{ij} = d_i.$$

Consequently, $\Re(\rho) \geq 0$ and $|\rho| \leq 2d_{\max}$.

Proof. By Gershgorin's theorem (row version), the disc centered at $l_{ii} = d_i$ has radius $\sum_{j \neq i} |l_{ij}| = \sum_{j \neq i} m_{ij} = d_i$. Each such disc is contained in the closed right half-plane (it touches the origin and extends to $2d_i$ on the positive real axis). Hence $\Re(\rho) \geq 0$ and $|\rho| \leq 2d_{\max}$ for all eigenvalues. $\square$

Theorem 4.11 (Two universal bounds for Laplacian energy). *For the Laplacian energy $E(L) := \sum_{i=1}^n |\rho_i|$ we have*

$$\text{tr}(L) \leq E(L) \leq \sqrt{n} \|L\|_F,$$

where $\|L\|_F = (\sum_{i,j} |l_{ij}|^2)^{1/2}$ is the Frobenius norm.

Proof. The lower bound follows from the triangle inequality:

$$E(L) = \sum_i |\rho_i| \geq \left| \sum_i \rho_i \right| = |\text{tr}(L)| = \text{tr}(L) \quad (\text{since } \text{tr}(L) \geq 0).$$

For the upper bound, Cauchy–Schwarz yields

$$E(L)^2 = \left(\sum_i |\rho_i| \right)^2 \leq n \sum_i |\rho_i|^2.$$

Let $L = QUQ^*$ be a Schur decomposition (unitary Q , upper triangular U with diagonal ρ_i). Then $\sum_i |\rho_i|^2 = \sum_i |u_{ii}|^2 \leq \sum_{i,j} |u_{ij}|^2 = \|U\|_F^2 = \|L\|_F^2$, using unitary invariance of the Frobenius norm. Hence $E(L) \leq \sqrt{n} \|L\|_F$. $\square$

Corollary 4.12 (Undirected (symmetric) case). *If M^{PFG} is symmetric (i.e., the underlying plithogenic fuzzy graph is undirected), then:*

- (1) L is real symmetric and positive semidefinite; hence all ρ_i are real and $\rho_i \geq 0$.
- (2) For all $x \in \mathbb{R}^n$,

$$x^T L x = \frac{1}{2} \sum_{i,j} m_{ij} (x_i - x_j)^2 \geq 0.$$

- (3) The multiplicity of the eigenvalue 0 equals the number of connected components of the underlying (weighted) graph.

In particular, $E(L) = \sum_i \rho_i = \text{tr}(L) = \sum_i d_i$ in the symmetric case.

Proof. Symmetry of M^{PFG} implies symmetry of $L = D - M^{PFG}$. The quadratic-form identity is standard:

$$x^T L x = \sum_i d_i x_i^2 - \sum_{i,j} m_{ij} x_i x_j = \frac{1}{2} \sum_{i,j} m_{ij} (x_i - x_j)^2 \geq 0.$$

Thus L is positive semidefinite and has real, nonnegative spectrum. The statement on the multiplicity of 0 follows from the usual Laplacian theory for connected components of an undirected weighted graph. Finally, when $\rho_i \geq 0$, $E(L) = \sum_i |\rho_i| = \sum_i \rho_i = \text{tr}(L)$. $\square$

Example 4.13 (A 2-vertex plithogenic fuzzy graph). *Let*

$$M^{PFG} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}, \quad a \in [0, 1], \quad \text{so} \quad D = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad L = D - M^{PFG} = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}.$$

Then $\text{Spec}(L) = \{0, 2a\}$, and the Laplacian energy is

$$E(L) = |0| + |2a| = 2a.$$

This illustrates Theorem 4.9 (the zero eigenvalue) and Corollary 4.12 (nonnegativity and $E(L) = \text{tr}(L) = 2a$).

5. APPLICATION IN HEALTHCARE DATA

In this section, we apply plithogenic fuzzy energy graphs to healthcare decision-making problems. We propose a mathematical model that captures complexity, uncertainty, and contradictions in healthcare data, treatments, and decision processes, together with a corresponding graphical representation. Since applications in healthcare have already been explored using fuzzy graphs [34] and neutrosophic graphs[35], it is a natural consideration to investigate similar applications within the framework of plithogenic fuzzy energy graphs.

Let $P^{FG} = (P^{FV}, P^{FE})$ be a plithogenic fuzzy graph for a crisp graph $G^* = (V, E)$, where V is the set of healthcare providers (e.g., hospitals, clinics, or general practitioners near the patient's home) and E encodes popularity/effectiveness relationships among elements of V .

Vertex (patient–issue) component. Define

$$P^{FV} = (M, \mu, M_\mu, \alpha_{df}, \alpha_{cf}),$$

where M denotes a (fixed) patient, μ is the set of multiple health issues of M , say

$$\begin{aligned} x &= \text{diabetes}, & y &= \text{hypertension}, \\ z &= \text{kidney disease}, & r &= \text{patient condition}, \\ s &= \text{age factor}. \end{aligned}$$

so that $\mu = \{x, y, z, r, s\}$. Let M_μ be the set of treatments/medical advice given to the patient (as appropriate for the disease):

$a = \text{medication}, \quad b = \text{lifestyle changes}, \quad c = \text{food management}, \quad d = \text{exercises/walk},$

so $M_\mu = \{a, b, c, d\}$. The degree of appurtenance $\alpha_{df} : M \times M_\mu \rightarrow [0, 1]$ represents the membership (suitability) of each treatment for M 's issues, and the degree of contradiction $\alpha_{cf} : M_\mu \times M_\mu \rightarrow [0, 1]$ represents pairwise contradictions among treatments. For illustration (see Table 7):

TABLE 7. α_{df} and α_{cf} tables.

α_{df}	x	y	z	r	s
a	.3	.5	.2	.1	.6
b	.4	.4	.3	.2	.7
c	.2	.2	.4	.5	.3
d	.1	.3	.5	.4	.2

α_{cf}	a	b	c	d
a	0	.5	.6	.3
b	.5	0	.4	.2
c	.6	.4	0	.1
d	.3	.2	.1	0

Edge (treatment–interaction) component. Define

$$P^{FE} = (N, \nu, N_\nu, \beta_{df}, \beta_{cf}),$$

where N models choosing the best treatment option(s) for the patient. Here, ν collects edge attributes such as: *clinical effectiveness*, *cost*, *side effects/risks*, and *patient preference/tolerance*. Let N_ν be the range of such attributes across multiple diseases. The appurtenance $\beta_{df} : N \times N_\nu \rightarrow [0, 1]$ measures edge membership, and $\beta_{cf} : N_\nu \times N_\nu \rightarrow [0, 1]$ measures pairwise contradictions (compatibility) among treatment attributes. For example, with edge labels xy, yz, zr, ys and attribute pairs ab, bc, cd, ad (as shown in Table 8):

TABLE 8. β_{df} and β_{cf} tables.

β_{df}	xy	yz	zr	ys
ab	.3	.2	.2	.5
bc	.2	.2	.3	.3
cd	.1	.2	.4	.2
ad	.3	.2	.1	.2

β_{cf}	ab	bc	cd	ad
ab	0	.5	.6	.5
bc	.5	0	.4	.4
cd	.6	.4	0	.5
ad	.5	.4	.5	0

The graph helps in modeling the best combination of treatments, so the given criteria is shown graphically in Figure 5.

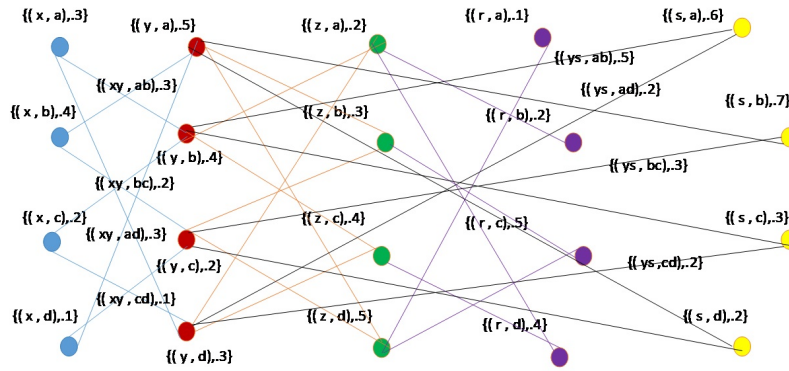


FIGURE 5. A Plithogenic fuzzy graph structure used in healthcare decision making problems.

Adjacency matrix and spectrum. The (weighted) adjacency matrix M^{PFG} of this plithogenic fuzzy graph is

$$M^{PFG} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & .3 & 0 & .3 & 0 & 0 & .3 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .3 & 0 & .2 & 0 & 0 & 0 & 0 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .3 & .3 & 0 & .1 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 & .5 & 0 & .2 \\ 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & .1 & 0 & 0 & .2 & 0 & 0 & 0 & .5 & 0 & .3 & 0 \\ .3 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 \\ 0 & .2 & 0 & .1 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & .2 & 0 & 0 & 0 & .2 & 0 & 0 & 0 \\ .3 & 0 & .1 & .2 & 0 & 0 & 0 & 0 & .2 & 0 & .2 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .3 & .4 & 0 & .3 & .1 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & .4 & 0 & 0 & 0 \\ .3 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 & 0 & 0 & .1 & 0 & .4 & 0 & 0 & 0 & 0 & 0 \\ .1 & .1 & 0 & 0 & .2 & 0 & .1 & .1 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & .4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .1 & 0 & .4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Its eigenvalues (numerically, including several values ≈ 0 due to rounding) are

$$\text{Spec}(M^{PFG}) = \left\{ \begin{array}{l} 1.00231, \quad -0.828908, \quad 0.407123, \\ -0.36744 \pm 0.0928797i, \quad -0.104829 \pm 0.35374i, \\ 0.240343 \pm 0.180577i, \quad -0.184689 \pm 0.131167i, \\ 0.125676 \pm 0.160889i, \quad -0.00880596 \pm 0.152336i, \\ 0.0189598, \quad 1.60892 \times 10^{-16}, \\ -9.59094 \times 10^{-17}, \quad 6.44097 \times 10^{-18}, \quad 0, \quad 0 \end{array} \right\}.$$

Hence the (adjacency) energy is

$$E(M^{PFG}) = \sum_i |\lambda_i|.$$

Remark (Interpretation and use). Higher energy indicates more complex interactions, stronger influences, and possibly more contradictions in the decision-making process; lower energy suggests more straightforward decisions with fewer conflicts.

(1) *Treatment-plan selection.* Compute and compare the energies of candidate treatment strategies (graphs); prefer the lower-energy strategy to reduce contradictions or risks.

(2) *Resource allocation.* In hospital management, graph energy can guide allocation of beds, staff, and equipment by minimizing conflicts and improving efficiency.

(3) *Personalized healthcare.* Model treatments as vertices and compatibilities as edges; for a given patient's history and preferences, the lowest-energy option (or subgraph) indicates the most coherent overall plan.

Finally, we note that *Laplacian energy* is a versatile diagnostic across domains (e.g., social networks, power grids, biological and communication systems, transportation, finance, ecology, and smart cities). It is particularly helpful for measuring irregularity, efficiency, and robustness of networks, thereby supporting optimization and the management of complex systems.

6. DISCUSSION: COMPARISON ANALYSIS

The findings show that plithogenic fuzzy graphs can model complex networks quite flexibly. They handle uncertainty, several attributes and contradictory relationships at the same time. The proposed adjacency matrix represents connections between vertices numerically. Its spectrum also reveals important structural features of the network. Graph energy, based on absolute eigenvalue values, reflects interaction intensity and complexity. Laplacian matrices and their spectra offer further insight into connectivity, organization and network irregularity. These developments extend conventional graph-theoretic analysis to circumstance in which relationships cannot be adequately represented by binary or single-valued memberships. The proposed framework becomes clearer when compared with common graph models. Table 9 outlines the main representational features of crisp, fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic fuzzy graphs. Crisp graphs use binary values to represent relationships. They cannot capture gradual membership or uncertainty directly. Fuzzy graphs address this by assigning membership values in $[0, 1]$. This makes them useful when relationships vary in strength. Intuitionistic fuzzy graphs add membership and non-membership degrees. Neutrosophic graphs go a step further by representing indeterminacy. However, in many real-world decision-making environments, uncertainty and indeterminacy may coexist with explicit contradictions among attributes or alternatives. The plithogenic fuzzy framework is particularly suitable for such situations because contradiction can be incorporated explicitly into the representation rather than being treated only indirectly.

TABLE 9. Comparative features across graph models

Feature	Crisp	Fuzzy	Intuitionistic Fuzzy	Neutrosophic	Plithogenic Fuzzy
Edge membership	Binary (0 or 1)	Single value in $[0, 1]$	T & N	T, I & N	T, I, N & C
Uncertainty handling	None	Moderate	High	Very high	Extremely high
Contradiction handling	None	None	None	Implicit	Explicit
Multi-criteria modeling	No	No	Limited	Limited	Yes
Complexity handling	Low	Moderate	Moderate	High	Very high
Best application	Simple networks	Uncertainty in networks	Risk assessments	Uncertain DM	Conflicting and uncertain DM

Table 9 also makes one point fairly clear. The proposed framework isn't meant to replace every existing graph model. Its value is more specific. It fits problems where several information types, sometimes conflicting, must be handled together. If available information is binary and unambiguous, a crisp graph may be enough. A conventional fuzzy graph could work better when only graded uncertainty matters. The plithogenic fuzzy model becomes useful in more complicated decision settings. Here, attributes interact and their contradictions need to be represented explicitly. The numerical example makes the usefulness of proposed approach clearer. Matrix M^{PFG} represents weighted relationships, not only binary connections. Its eigenvalues therefore reveal aspects of the network's overall weighted

structure. Both real and complex eigenvalues appear, reflecting the network's nontrivial organization. The eigenvalue with greatest magnitude has a strong effect on graph energy. The other eigenvalues also capture different interaction patterns across network. So, graph energy is more than just a numerical result. It summarizes the network's spectral structure and offers a useful way to compare different configurations.

For the numerical example considered in this study, the adjacency energy is calculated as

$$E(M^{PFG}) = \sum_i |\lambda_i|,$$

where λ_i denotes the eigenvalues of the plithogenic fuzzy adjacency matrix. The obtained value gives a useful baseline for network complexity. Changes in edge weights or relationships can alter the spectrum. So can shifts in the underlying contradiction structure. These changes affect the graph energy too. This makes energy helpful when comparing different network configurations, not just describing one graph.

Furthermore, the Laplacian-based measures provide complementary structural information. Whereas adjacency energy is primarily associated with the weighted interaction structure represented by the adjacency matrix, Laplacian energy is related to the degree distribution and connectivity characteristics of the network. Hence, considering both adjacency energy and Laplacian energy can provide a more comprehensive assessment. Two networks may have comparable adjacency energies but differ substantially in their degree distributions or structural regularity; Laplacian-based analysis can help distinguish such cases. This complementary perspective is particularly useful in decision-making problems where both interaction strength and network organization are important. Table 10 outlines energy and Laplacian aspects of the graph models considered. The comparison shows that proposed framework supports spectral analysis while handling fuzzy information and contradictions. Its contribution isn't just another graph energy measure. Rather, it offers a spectral setting where uncertainty and conflicting relationships can be examined through adjacency and Laplacian structures.

TABLE 10. Energy- and Laplacian-related aspects across graph models

Aspect	Crisp	Fuzzy	Intuitionistic Fuzzy	Neutrosophic	Plithogenic Fuzzy
Edge representation	Binary (0 or 1)	Single value in $[0, 1]$	T & N	T, I & N	T, I, N & C
Uncertainty handling	None	Partial	High	Advanced	Extremely high
Energy calculation	Basic	Moderate	None	Advanced	Rich
Laplacian energy	Connectivity	Connectivity + uncertainty	Connectivity + uncertainty	Limited	Yes
Insights from energy	Simple	Moderate	Moderate	High	Very high
Best application	Simple networks	Moderate uncertainty	Risk assessments in DM	Uncertain DM	Conflicting and uncertain decision-making

7. CONCLUSION

In this study, we developed a spectral framework for plithogenic fuzzy graphs. It introduces the plithogenic fuzzy adjacency matrix, spectrum, and graph energy. We also defined the Laplacian matrix, its spectrum, and Laplacian energy, using numerical examples for illustration. The results suggest these measures can offer useful information about network strength, complexity, connectivity, and structure, especially when relationships are uncertain or contradictory. The comparison also shows that plithogenic fuzzy graphs can represent multiple criteria, uncertainty, and clear contradictions more flexibly. The proposed energy measures help compare graph structures and identify different levels of interaction and complexity. The healthcare example shows why the proposed framework matters in practice. Patient conditions, treatment options, clinical outcomes, costs, side effects, and preferences can fit into one plithogenic fuzzy graph. Its spectrum and energy may support decisions about treatment relationships and conflicting healthcare factors. The model offers a mathematical basis for using spectral graph ideas in complex healthcare decisions. Overall, this study links plithogenic fuzzy theory with spectral graph analysis. It also creates opportunities for future work in uncertain and conflicting real-world systems.

7.1. Future Works. The proposed framework opens several directions for future research. The energy and Laplacian energy of plithogenic fuzzy graphs can be further investigated in different application domains, including:

- Social Networks: Measuring connectivity, interaction intensity, and conflicting relationships [40].
- Chemistry: Investigating molecular structures and stability [40].
- Collaboration Networks: Analyzing teamwork and communication efficiency.
- Transportation: Assessing traffic flow, connectivity, and network robustness [40].
- Recommendation Systems: Modeling uncertain and conflicting preferences.
- Epidemiology: Studying disease transmission and interactions among risk factors.
- Supply Chains: Analyzing network efficiency, uncertainty, and potential risks [40].
- Telecommunication: Optimizing traffic, connectivity, and network performance.
- Financial Markets: Analyzing uncertain relationships, volatility, and asset correlations.

Future studies could extend these ideas to other advanced graph structures. HyperGraphs [11, 23], SuperHyperGraphs [17, 18], Bidirected Graphs [10, 42], and HyperUncertain Sets [26] seem especially promising. They may help broaden plithogenic spectral analysis. Larger real-world datasets could also be examined. Sensitivity analysis and different weighting methods may offer useful insights. Optimization-based decision-support approaches might further test the practical value of plithogenic fuzzy graph energy and Laplacian energy. In this paper, we mainly discussed the proposed concepts from a theoretical perspective. As future work, we plan to investigate the computational complexity of constructing the adjacency matrix, computing eigenvalues, and evaluating graph energy for large-scale plithogenic fuzzy graphs.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Fazeelat Sultana: Conceptualization, methodology, mathematical analysis, software, and writing original draft. Tahir Abbas: Methodology, validation, formal analysis, and investigation. Muhammad Gulistan: Supervision, conceptualization, methodology, critical review, and editing of the manuscript. Takaaki Fujita: Formal analysis, visualization, validation, and software. Dr. Muhammad Usman Jamil: Conceptualization, formal analysis, algorithm verification, technical validation, resources, and critical review and editing of the manuscript. All authors contributed to the discussion of the results, reviewed the manuscript critically, and approved the final version for publication.

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