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Exploring Lie Point Symmetries and Exact Solutions for (1+1) Dimensional Modified Thomas and (1+2) Dimensional Chaffee-Infante Equations

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Abstract. Lie symmetry analysis is a highly effective tool for finding exact solutions to differential equations, decreasing the number of independent variables, or at least reducing the equations order and nonlinearity. This article presents exact solutions for the (1+1)-dimensional modified

Thomas and (1+2)-dimensional Chaffee-Infante equations through the application of the symmetry reduction method. These equations yield exact solutions under specific parametric conditions. Multiple exact solutions, such as periodic, soliton, and solitary wave solutions, along with newly found solitary wave solutions, are derived to validate their physical relevance. The findings are graphically illustrated with appropriate parametric settings, shown in both 2D and 3D. The outcomes of this study are expected to have applications across a wide range of scientific fields.

AMS (MOS) Subject Classification Codes: 35B06; 58J70; 37J39
Key Words: Lie point symmetries, modified Thomas equation, Chaffee-Infante equation, exact solutions.

1. INTRODUCTION

Nonlinear partial differential equations (NPDEs) are used to model physical phenomena and abstract systems. The study of analytical solutions to NPDEs finds applications in fields like plasma physics, meteorology, quantum mechanics, fluid dynamics, biology, and more. The analysis of solutions, including traveling wave and soliton solutions, is a key area of research. Thus, the investigation of NPDEs plays a crucial role in both theoretical exploration and practical applications, Wang et al. [44], Li et al. [26], and Salamat et al. [38]. Widely used nonlinear equations include the Thomson, Burgers, Schamel, Chaffee-Infante, and Schamel–K-dV equations. Several methods have been proposed by researchers to find solutions, with notable approaches including a new ϕ^6 model expansion method, Expansion $(\frac{G'}{G})$ method [8], Bell polynomial method [41], Darboux transformation method [12], Variational iteration method [45], Bäcklund transformation [15], Extended direct algebraic technique [17], and Jacobi elliptic function expansion method [22]. Kumar and Singh [23] discover new precise solutions for a second-grade MHD flow through porous media using the traveling wave method. The work of Khater et al. delves into the propagation of new dynamic phenomena in the longitudinal bud equation within a magneto-electro-elastic cylindrical rod. The Lie symmetry method is employed to derive exact solutions for the (3+1)-dimensional Kadomtsev-Petviashvili equation. Considered one of the most effective methods for studying differential equations, the Lie group approach is often called Lie symmetry analysis. Its applications are vast, ranging from constructing analytical solutions to reducing the order and the number of independent variables. This approach, described in several textbooks, has been applied to investigate numerous physical and engineering models, focusing on their invariance properties and the formulation of exact solutions [30, 14, 43, 31, 32]. Additionally, several extensions of the Lie group analysis have been suggested, including the development of approximate symmetries [19], nonlocal residual symmetries [27] and nonclassical symmetries [28], for partial differential equations and systems. The soliton and complexiton solutions of the (1+2)-dimensional Date–Jimbo–Kashiwara–Miwa equation were found by Adem, Yildirim, and Yasar [2] who used the extended transformed rational function algorithm based on the Hirota bilinear form. Yıldırım [47] determined the optimal solutions for the Biswas-Arshed model through different techniques, which are important in various fields of engineering and mathematics. Amir et al. [6] apply the natural decomposition method (NDM) to solve the inviscid Burger equation and obtain an approximate solution, which they compare with the exact solution.

Since exact solutions to most PDEs are difficult to obtain, both analytical and numerical approaches [1, 34] must be utilized. Various authors have investigated the solutions of PDEs using powerful analytical methods. Several computational techniques, such as the Adomian decomposition method [33] and the homotopy perturbation method [13], have been recommended for solving PDEs. Despite the solutions provided by these methods being of a local nature, it is crucial to explore other approaches to find exact analytical solutions for PDEs. The Lie point symmetry method is fundamental in various scientific fields, particularly in integrable systems, where an infinite number of symmetries are involved. Lie symmetry analysis is widely recognized as a powerful technique for finding analytical solutions to NPDEs. Numerous studies focus on the theory of Lie point symmetry and its applications to differential equations. A concept in advanced mathematics, Lie point symmetry was introduced by Sophus Lie in the late nineteenth century through the notion of Lie groups, aimed at studying differential equations [24, 25]. Alexandrino and Bettiol [5] explored the connection between Lie groups and the geometric aspects of isometric actions. Schwichtenberg's work [40] focuses on the application of physics through symmetry. Agnus et al. [3] applied the Lie approach to derive the exact solutions of the Ramani equation. Using the Lie approach, Irshad et al. [16] determined the invariant solutions of a nonlinear fifth-order partial differential equation. By implementing the multiple exp-function approach, exact solutions are constructed for the shallow water wave equations, establishing crucial benchmarks for numerical simulations. Jarad et al. analyze the generalized Calogero–Bogoyavlenskii–Schiff equation using the Lie symmetry method, deriving wave solutions and identifying conservation laws. Examined nonlinear wave propagation phenomena in the (3+1)-dimensional generalized Boiti-Leon-Manna-Pempinelli equation using Lie symmetry analysis.

Ordinary differential equations (ODEs) and partial differential equations (PDEs) have gained significant popularity for describing various physical effects and nonlinear phenomena. Their effectiveness in capturing complex phenomena in fluid mechanics, viscoelasticity, electrochemistry, quantum biology, physics, and engineering explains their broad usage [29, 39, 7]. As a result, the study of ODEs and PDEs has become a prominent research area. Using derivative theory, a wide range of physical phenomena can be accurately modeled. Kumar and Kumar [21] applied Lie point symmetries to find solitary wave solutions of the pZK equation, while Faridi et al. [11] analyzed modulational instability and obtained power series solutions for a coupled system. Akhound-Sadegh et al. [4] explored the connection between Lie point symmetry and physics-informed networks. Yıldırım and Yaşar [46] utilized Lie group analysis to uncover the Lie point symmetry generators, symmetry reductions, and conservation laws for the (1+2) dimensional breaking soliton equation.

In 1944, Thomas introduced [42] the nonlinear equation. The (1+1)-dimensional Thomas equation can be represented conventionally as follows:

$$u_{xt} + \alpha u_t + \beta u_x + \gamma u_t u_x = 0. \tag{1. 1}$$

A modified version of this equation, which includes additional terms or adjustments based on specific applications, may take the following form:

$$u_{xt} + \alpha u_t + \beta u_x + \gamma u_t u_x + f(u) = 0. \qquad (1.\ 2)$$

The Thomas equation is a nonlinear partial differential equation used to model chemical reactions, wave propagation, and diffusion, focusing on soliton solutions and complex system dynamics [37]. To simplify the equation and obtain precise solutions that reveal the system's structure, we apply the symmetry method to identify its symmetries.

The Chaffee-Infante equation is a fundamental model in nonlinear physics, particularly in reaction-diffusion processes. The standard form of the (2+1)-dimensional Chaffee-Infante equation is given by:

$$u_{xt} - u_{xxx} + 3\alpha u^2 u_x - \alpha u_x u_y + \sigma u_{yy} = 0. \qquad (1.\ 3)$$

The constant parameter α denotes the response rate. Equation (1. 3) serves as a basic model in heat conduction research. In this study, bright, dark, periodic, kink, anti-kink, and singular wave solutions to the (2+1)-dimensional Chaffee-Infante problem are derived using the extended sinh-Gordon equation expansion technique [36]. Additionally, we apply the symmetry method, which focuses on the equation's inherent symmetries to derive precise solutions and more thoroughly analyze its underlying dynamics and structure. The study by Yıldırım et al. focuses on dispersive optical solitons in birefringent fibers, utilizing a range of integration technologies. Nucci's reduction approaches were employed to investigate the Ivancevic option pricing model and extract soliton solutions. Through the application of the Laplace transform method, Jamil, Khan, and Shah [18] determine the exact analytical solutions of linear dissipative wave equations. The Painlevé analysis is used by Kudryashov et al. [20] to solve the Lakshmanan–Porsezian–Daniel model, nonlinear Schrödinger's equation, and Sasa–Satsuma equation, yielding soliton solutions. We apply the Lie symmetry method [9] in this study, known as one of the most commonly used and efficient methods for solving PDEs. In this case, a given PDEs are invariant under a transformation of Lie groups. The symmetries of the PDEs are derived by applying the invariance conditions. By employing various techniques, the invariant solutions of the symmetry reduced equations [10] are derived. The results highlight the methods efficiency, accuracy, and adaptability, underscoring its broad applicability in fields such as engineering, mathematical physics, and other scientific disciplines. The future direction of this research involves enhancing the methods for obtaining exact solutions to the given PDEs by leveraging Lie subalgebras to construct optimal systems. This approach has the potential to uncover more general and complex solutions, further advancing the application of Lie point symmetries in solving other nonlinear PDEs. It could also provide valuable insights for various fields, including fluid dynamics, quantum mechanics, and nonlinear optics. The Lie symmetry approach was employed to derive novel invariant solutions and examine their graphical characteristics for the generalized unstable nonlinear Schrödinger equation. Advances in mathematics, theoretical physics, computational techniques, interdisciplinary research, quantum symmetry, environmental studies, logistics, physiology, geometry, and topology are all likely to influence the future trajectory of Lie symmetry theory. The structure of this paper is as follows: Section 2 outlines the methodology for Lie symmetry analysis of PDEs. Sections 3 and 4 apply this approach to obtain symmetry reductions and exact solutions for Eq. (3. 17) and Eq. (4. 63), respectively. Section 5 discusses the evaluation and graphical representation of the results. The paper concludes in Section 6 with a summary of the main findings.

Objective: The primary goal of this study is to obtain new exact solutions for the (1+1) dimensional modified Thomas equation and the (1+2) dimensional Chaffee-Infante equation. These solutions are derived using the Lie point symmetry reduction method. By applying this approach, we aim to uncover previously unknown solutions to these nonlinear equations. This method offers a systematic way to reduce the complexity of the equations and find precise analytical solutions.

2. Basic Principles and the Application of Lie Symmetry Methods

This section outlines the fundamental notation and tools used throughout this work. Consider an nth-order partial differential equation (PDE) system with p independent variables, denoted by $x = (x^1, x^2, ..., x^p)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$. The system can be written as:

$$D_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0, \qquad \beta = 1, 2, ..., m\,, \tag{2. 4}$$

where $u_{(1)}, ..., u_{(l)}$ represent the collections of 1^{st}, 2^{nd},..., nth-order partial derivatives; specifically, u_i^β=$C_i(u^\beta)$, u_{ij}^β= C_iC_j,..., and so on. Correspondingly, the operator for the total derivative with respect to x^j is denoted as

$$C_i = \frac{\partial}{\partial x^j} + u_j^\beta \frac{\partial}{\partial u^\beta} + u_{ji}^\beta \frac{\partial}{\partial u_j^\beta} +, ... \qquad j = 1, 2, ..., p\,, \tag{2. 5}$$

and operator of Lie-Bäcklund is

$$P = \mu^i \frac{\partial}{\partial x^j} + \omega^\beta \frac{\partial}{\partial u^\beta}, \quad \mu^i, \omega^\beta \in \mathfrak{B}\,, \tag{2. 6}$$

where $\mathfrak{D}$ represents the functions of spatial differentials. The operator (2. 6) is a compact form of the infinite formal series:

$$P = \mu^i \frac{\partial}{\partial x^j} + \omega^\beta \frac{\partial}{\partial u^\beta} + \sum_{r \geq 1} \delta_{j_1, j_2, ..., j_r}^\beta \frac{\partial}{\partial u_{j_1, j_2, ..., j_r}^\beta}\,. \tag{2. 7}$$

Through the prolongation formulas, the additional coefficients are uniquely determined:

$$\begin{aligned} \delta_j^\beta &= D_j(E^\beta) + \mu^i u_{ji}^\beta\,, \\ \delta_{j_1, ..., j_r} &= D_{j1}, ..., D_{j_r}(E^\beta) + \mu^i u_{ij_1, ..., i_r}^\beta, \qquad r \geq 1\,, \end{aligned} \tag{2. 8}$$

in which (E^β) is a function of Lie characteristic :

$$E^\beta = \omega^\beta - \mu^i u_i^\beta\,. \tag{2. 9}$$

Next, we consider the successive transformations of the Lie group with independent variables t, x, y, and the dependent variable u:

$$\begin{aligned} u^* &= (x^*, x, y, t, u), \qquad t^* = (x^*, x, y, t, u), \\ x^* &= (x^*, x, y, t, u), \qquad y^* = (x^*, x, y, t, u)\,. \end{aligned} \tag{2. 10}$$

A one-parameter Lie transformation of the group is expressed in the following form (c):

$$\begin{aligned} \widetilde{x} &\approx x + \mu^1 H(x^*, x, y, t, u)\,, \\ \widetilde{y} &\approx y + \mu^2 H(x^*, x, y, t, u)\,, \\ \widetilde{t} &\approx t + \mu^3 H(x^*, x, y, t, u)\,, \\ \widetilde{u} &\approx u + \varpi^1 H(x^*, x, y, t, u)\,. \end{aligned} \tag{2. 11}$$

Here, O represents the group parameter, and the infinitesimal generator corresponding to the transformations above is given by:

$$P = \mu^1(u, x, y, t)\frac{\partial}{\partial t} + \mu^2(u, x, y, t)\frac{\partial}{\partial x} + \mu^3(u, x, y, t)\frac{\partial}{\partial y} + \varpi^1(u, x, y, t)\frac{\partial}{\partial u}. \tag{2. 12}$$

The infinitesimal generators of the Lie group transformations can be written in the following form.

$$P = \mu\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \varpi\frac{\partial}{\partial u}\,. \tag{2. 13}$$

Solving the Lie equations yields the group transformations of $\widetilde{x}, \widetilde{y}, \widetilde{t}$.

$$\begin{aligned} Os_t + Xs_x + Ys_y + Zs_x &= 0\,, \\ Or_t + Xr_x + Yr_y + Zr_x &= 0\,, \\ Ov_t + Xv_x + Yv_y + Zv_x &= 0\,, \\ Ou_t + Xu_x + Yu_y + Zu_x &= 0\,. \end{aligned} \tag{2. 14}$$

The addition of dependent variables u, v and independent variables r, s through equation (2. 14) facilitates the transformation of the nonlinear system into a linear one.

Lemma 1: The one-parameter (ε) Lie group of transformations $\mathbf{X}^*$=$\mathbf{X}$(x; ε) satisfies the relation

$$\mathbf{X}(x; \varepsilon + \Delta\varepsilon) = \mathbf{X}(\mathbf{X}(x; \varepsilon); \varphi(\varepsilon^{-1}, \varepsilon + \Delta\varepsilon)) \tag{2. 15}$$

Proof:

$$\begin{aligned} \mathbf{X}(\mathbf{X}(x; \varepsilon); \varphi(\varepsilon^{-1}, \varepsilon + \Delta\varepsilon)) &= \mathbf{X}(x; \varphi(\varepsilon, \varphi(\varepsilon^{-1}, \varepsilon + \Delta\varepsilon))) \\ &= \mathbf{X}(x; \varphi(, \varphi(\varepsilon, \varepsilon^{-1}), \varepsilon + \Delta\varepsilon)) \\ &= \mathbf{X}(x; \varphi(\varepsilon, 0), +\Delta\varepsilon) \\ &= \mathbf{X}(x; \varepsilon + \Delta\varepsilon) \end{aligned} \tag{2. 16}$$

3. Lie Point Symmetries and Exact Solutions of (1+1) Dimensional Modified Thomas Equation

The (1+1) dimensional modified Thomas equation is a nonlinear partial differential equation that is crucial in understanding various physical phenomena, especially in the study of soliton solutions, wave propagation, and integrable systems. This equation models the dynamics of fields or wave-like behaviors in one spatial dimension and one time dimension, classifying it as a (1+1) dimensional equation.

$$u_{xt} + \alpha u_t + \beta u_x + \gamma u_t u_x + f(u) = 0, \tag{3. 17}$$

where the coefficients α, β, and γ, are parameters, and $f(u)$ is arbitrary function while u varies with t and x. This equation is commonly studied in the context of physical sciences, particularly for modeling chemical processes (e.g., Henry Thomas model). Equation (3. 17) LPS generators are derived through the application of the invariance condition

$$W^{[2]}[u_{xt} + \alpha u_t + \beta u_x + \gamma u_t u_x + f(u) = 0] \,|_{(3.17)}= 0\,, \tag{3. 18}$$

here, $W^{[2]}$ is referred to as the second-order prolongation of the invariant transformation of W and $|_{(3.17)}$ applies the surface conditions that remain invariant. In equation After separation and expansion with respect to the different derivatives and powers of u, $W^{[2]}$ is found, and a linear PDE structure in terms of the new coefficients ξ^1, ξ^2, ξ^3 and ϕ is formed. The determining equations are:

$$\begin{aligned} \xi^1_u = 0, \xi^2_u = 0, \xi^1_t = 0, \xi^1_{uu} = 0, \xi^2_{uu} = 0, \xi^1_{tt} = 0, \\ \xi^1_{ux} = 0, \xi^2_{xt} = 0, \xi^2_x = 0, \xi^2_{ux} = 0, \phi_{uu} = 0\,. \end{aligned} \tag{3. 19}$$

The following determining equation, in which the function is involved, is given.

$$f_u\phi = 0. \tag{3. 20}$$

Taking the partial derivative of Eq. (3. 20) with respect to u:

$$f_u\phi_u + f_{uu}\phi = 0. \tag{3. 21}$$

Taking the again partial derivative of Eq. (3. 21) with respect to u:

$$f_u\phi_{uu} + f_{uu}\phi_u + \phi f_{uuu} + \phi_u f_{uu} = 0. \tag{3. 22}$$

Substitute ϕ_{uu}=0 in Eq. (3. 22)

$$f_{uu}\phi_u + \phi f_{uuu} + \phi_u f_{uu} = 0\,. \tag{3. 23}$$

From Eq. (3. 21)

$$f_u\phi_u = -f_{uu}\phi, \tag{3. 24}$$

$$\frac{f_u}{-f_{uu}}\phi_u = \phi. \tag{3. 25}$$

Substitute the value into Eq. (3. 23)

$$f_{uu}\phi_u + (\frac{-f_u}{f_{uu}}\phi_u)f_{uuu} + \phi_u f_{uu} = 0, \tag{3. 26}$$

$$2f_{uu}\phi_u + (\frac{-f_u}{f_{uu}}\phi_u)f_{uuu} = 0, \tag{3. 27}$$

$$2f_{uu}^2\phi_u + f_u f_{uuu}\phi_u = 0, \tag{3. 28}$$

$$(2f_{uu}^2 + f_u f_{uuu})\phi_u = 0, \tag{3. 29}$$

$$2f_{uu}^2 + f_u f_{uuu} = 0. \tag{3. 30}$$

From Eq. (3. 30), we have the following equations:

$$f(u) \;=\; au + b, \tag{3. 31}$$

$$f(u) \;=\; \frac{1}{2}au^2 + bu + c, \tag{3. 32}$$

$$f(u) \;=\; \frac{\ln(u+b)}{a} + c. \tag{3. 33}$$

When applying Lie theory to a differential equation, the process begins with introducing a symmetry generator. This leads to the formulation of determining equations, which help identify the Lie symmetries and invariants. These results are then used to derive the corresponding ordinary differential equation (ODE). Since the Thomas equation is a second-order partial differential equation (PDE), the symmetry generator must be extended to its second-order form for a thorough analysis.

3.1. Lie point symmetries and exact solutions of Eq. (3. 17) using $f(u) = au + b$.

$$u_{xt} + \alpha u_t + \beta u_x + \gamma u_t u_x + au + b = 0. \tag{3. 34}$$

Lie symmetries of Eq. (3. 34)

$$Z_1 = \frac{\partial}{\partial t} \quad \& \quad Z_2 = \frac{\partial}{\partial x}. \tag{3. 35}$$

Exact solutions of Eq. (3. 17) using Z_1 & Z_2 LPS:

$$Z = Z_1 + \alpha\, Z_2,$$

$$Z = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial x}, \tag{3. 36}$$

$$Z = \mu^1 \frac{\partial}{\partial t} + \mu^2 \frac{\partial}{\partial x} + \varpi^1 \frac{\partial}{\partial u}. \tag{3. 37}$$

The terms μ^1, μ^2, and ϖ^1 represent the infinitesimals associated with the variables x, t and u, respectively.
By comparing expressions, we find the relationships between these infinitesimals.

$$\mu^1 = 1, \mu^2 = \alpha, \varpi^1 = 0. \tag{3. 38}$$

The coordinates in their canonical form are defined as follows.

$$v(r) = u(t, x), \tag{3. 39}$$

$$r = -\alpha t + x, \quad s = t. \tag{3. 40}$$

Using these coordinates, find the derivatives and substitute them into Eq. (3. 34)

$$b + av(r) - \alpha^2 v_r + \beta v_r - \alpha\gamma v_r^2 - \alpha v_{rr} = 0. \tag{3. 41}$$

Solution of Eq. (3. 41) by substituting $a = b = 0$

$$v\left(r\right) = \ln\left(-\gamma \left(\mathit{C1}\, \alpha\, \mathrm{e}^{-\frac{\left(\alpha^2-\beta\right) r}{\alpha}} - \mathit{C2}\, \alpha^2 + \mathit{C2}\, \beta\right) \left(\alpha^2 - \beta\right)^{-1}\right) \gamma^{-1}.$$

Solution in the original variables u(t,x) form

$$u(t, x) = \ln\left(-\gamma \left(\mathit{C1}\, \alpha\, \mathrm{e}^{\frac{\left(\alpha^2-\beta\right)(\alpha\, t - x)}{\alpha}} - \mathit{C2}\, \alpha^2 + \mathit{C2}\, \beta\right) \left(\alpha^2 - \beta\right)^{-1}\right) \gamma^{-1}. \tag{3. 42}$$

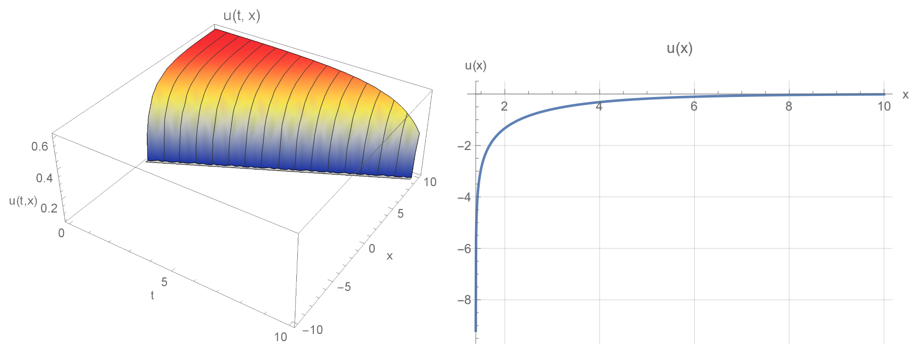

FIGURE 1. Analysis of the transverse displacement behavior of the beam wave solutions (3. 42) with all parameters equal to 1.

3.2. **Lie point symmetries and exact solutions of Eq. (3. 17) using** $f(u) = \frac{1}{2}au^2 + bu + c$**.**

$$u_{xt} + \alpha u_t + \beta u_x + \gamma u_t u_x + \frac{1}{2}au^2 + bu + c = 0. \tag{3. 43}$$

Lie symmetries of Eq. (3. 43)

$$Z_1 = \frac{\partial}{\partial t} \quad \& \quad Z_2 = \frac{\partial}{\partial x}. \tag{3. 44}$$

Exact solutions of Eq. (3. 17) using Z_1 & Z_2 LPS:

$$Z{=}Z_1{+}\ \vartheta\ Z_2,$$

$$Z = \frac{\partial}{\partial t} + \vartheta\frac{\partial}{\partial x}. \tag{3. 45}$$

By comparing expressions, we find the relationships between these infinitesimals.

$$\mu^1 = 1, \ \ \mu^2 = \vartheta, \ \ \varpi^1 = 0. \tag{3. 46}$$

The coordinates in their canonical form are defined as follows.

$$v(r) = u(t, x), \tag{3. 47}$$

$$r = -\vartheta t + x, \quad s = t. \tag{3. 48}$$

Using these coordinates, find the derivatives and substitute them into Eq. (3. 43)

$$-\vartheta v_{rr} - \alpha\vartheta v_r + \beta v_r - \vartheta\gamma v_r^2 + \frac{1}{2}av(r)^2 + b(v(r)) + c = 0. \tag{3. 49}$$

Solution of Eq. (3. 49) by substituting $b = c = \vartheta$**=**$\alpha = 0$

$$v\left(r\right) = 2\,\frac{\beta}{ar + 2\,C1\,\beta}. \tag{3. 50}$$

Solution in the original variables $u(t, x)$ form

$$u\left(t, x\right) = 2\,\frac{\beta}{ax + 2\,C1\,\beta}. \tag{3. 51}$$

Solution of Eq. (3. 49) by substituting $b = \vartheta = 0$

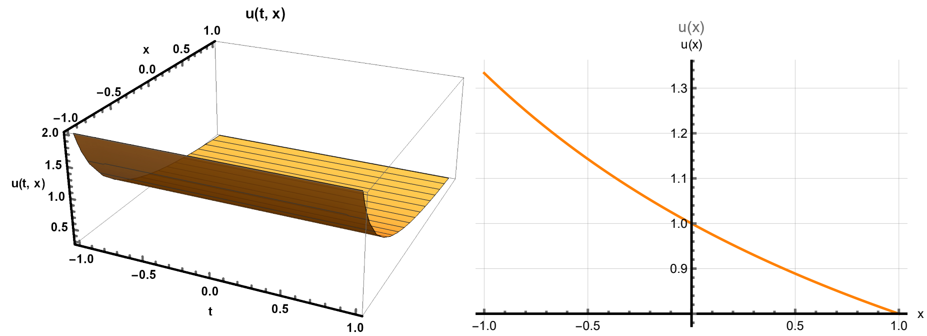

FIGURE 2. Analysis of the transverse displacement behavior of the beam wave solutions (3. 51) with all parameters equal to 1.

$$v\left(r\right)=-\tan\left(1/2\,\frac{\sqrt{ca}\left(r+C1\right)\sqrt{2}}{\beta}\right)\sqrt{ca}\sqrt{2}a^{-1}. \tag{3. 52}$$

Solution in the original variables $u(t,x)$ form

$$u\left(t,x\right)=-\tan\left(1/2\,\frac{\sqrt{ca}\left(x+C1\right)\sqrt{2}}{\beta}\right)\sqrt{ca}\sqrt{2}a^{-1}. \tag{3. 53}$$

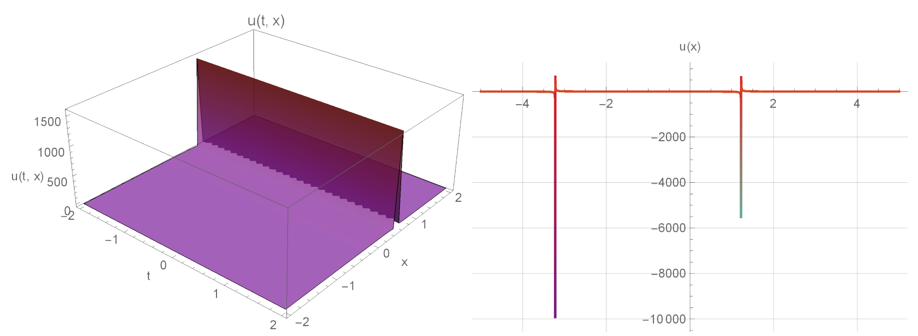

FIGURE 3. Analysis of the transverse displacement behavior of the beam wave solutions (3. 53)with all parameters equal to 1.

3.3. **Lie point symmetries and exact solutions of Eq. (3. 17) using** $f(u) = \frac{ln(u+b)}{a} + c$**.**

$$u_{xt} + \alpha u_t + \beta u_x + \gamma u_t u_x + -\frac{ln(u+b)}{a} + c = 0. \tag{3. 54}$$

Lie symmetries of Eq. (3. 54)

$$Z_1 = \frac{\partial}{\partial x} \quad \& \quad Z_2 = \alpha\frac{\partial}{\partial t} - \beta\frac{\partial}{\partial x}. \tag{3. 55}$$

Exact solutions of Eq. (3. 17) using Z_1 & Z_2 LPS:

$$Z = Z_1 + \theta\, Z_2,$$

$$Z = \theta\alpha\frac{\partial}{\partial t} + (1-\theta\beta)\frac{\partial}{\partial x}. \tag{3. 56}$$

By comparing expressions, we find the relationships between these infinitesimals.

$$\mu^1 = \alpha\theta, \quad \mu^2 = 1 - \theta\beta, \quad \varpi^1 = 0. \tag{3. 57}$$

The coordinates in their canonical form are defined as follows.

$$v(r) = u(t,x), \tag{3. 58}$$

$$r = \frac{-t + t\beta\theta + x\alpha\theta}{\theta\alpha}, \quad s = \frac{t}{\theta\alpha}. \tag{3. 59}$$

Using these coordinates, find the derivatives and substitute them into Eq. (3. 43)

$$\begin{gathered}\alpha\, a\theta\, c - \alpha\,\theta\,\ln\left(v\left(r\right)+b\right)+\\ a\left(\frac{d}{dr}v\left(r\right)\right)\left(\alpha\,\left(-1+2\,\beta\,\theta\right)+\gamma\,\left(-1+\beta\,\theta\right)\frac{d}{dr}v\left(r\right)\right)+\\ a\left(-1+\beta\,\theta\right)\frac{d^2}{dr^2}v\left(r\right)=0\end{gathered} \tag{3. 60}$$

Solution of Eq. (3. 60) by substituting $\theta = \alpha = \beta = b = 1, \gamma = 0$

$$v\left(r\right) = \mathrm{e}^{RootOf\left(r + a\mathrm{e}^{ac}Ei(1,-\_Z+ac)+C1\right)} - 1 \tag{3. 61}$$

Solution in the original variables $u(t,x)$ form

$$u(t,x) = \mathrm{e}^{RootOf\left(x + a\mathrm{e}^{ac}Ei(1,-\_Z+ac)+C1\right)} - 1 \tag{3. 62}$$

4. Lie Point Symmetries and Exact Solutions of (2+1) Dimensional Chaffee-Infante Equation

Riaz et al. [35] derive exact solutions and study conservation laws for the Chaffee-Infante (CI) equation. They determine optimal systems through Lie subalgebra techniques and derive conserved vector quantities using the multiplier method. By employing classical symmetry analysis and group classification, we derive exact solutions for the CI equation. This specific partial differential equation is analyzed for its symmetry properties to identify possible invariant forms and solutions.

$$u_{xt} - u_{xxx} + 3\alpha u^2 u_x - \alpha u_x + \sigma u_{yy} = 0\,. \tag{4. 63}$$

Various computer algebra systems provide packages that facilitate Lie symmetry calculations and related techniques. In this paper, we introduce a package designed for the symmetry analysis of differential equations (SADE), which is implemented in MAPLE, along with several other useful packages available for the MAPLE environment.

4.1. **Lie point symmetries and exact solutions of Eq. (4. 63).** Upon solving the corresponding equation (4. 63), we uncover three Lie point symmetries (LPS):

$$P_1 = \frac{\partial}{\partial t}, \quad P_2 = \frac{\partial}{\partial x}, \quad P_3 = \frac{\partial}{\partial y}. \tag{4. 64}$$

Exact solutions of Eq. (4. 63) using P_1 LPS:
Next, determine the exact solution of the LPS P_1.

$$P = P_1 = \frac{\partial}{\partial t}, \tag{4. 65}$$

$$P = \mu^1 \frac{\partial}{\partial t} + \mu^2 \frac{\partial}{\partial x} + \mu^3 \frac{\partial}{\partial y} + \varpi^1 \frac{\partial}{\partial u}. \tag{4. 66}$$

The terms μ^1, μ^2, μ^3 and ϖ^1 represent the infinitesimals associated with the variables x, y, t and u, respectively.
By comparing expressions, we find the relationships between these infinitesimals.

$$\mu^1 = 1, \mu^2 = 0, \mu^3 = 0, \varpi^1 = 0. \tag{4. 67}$$

The coordinates in their canonical form are defined as follows.

$$v(r, s) = u(t, x, y), \tag{4. 68}$$

$$r = y, \quad s = x, \quad w = t. \tag{4. 69}$$

Using these coordinates, find the derivatives and substitute them into Eq. (4. 63)

$$\alpha(-1 + 3v^2)v_s + \sigma 4 v_{rr} + v_{sss} = 0. \tag{4. 70}$$

Determine the LPS again and proceed to solve Eq. (4. 70)

$$P_1 = \frac{\partial}{\partial r}, \;\; P_2 = \frac{\partial}{\partial s}, \tag{4. 71}$$

Now, find the ES of given LPS P_1 & P_2.

$$P = P_1 + aP_2, \tag{4. 72}$$

$$P = \frac{\partial}{\partial r} + a\frac{\partial}{\partial s}, \tag{4. 73}$$

$$P = \mu^1 \frac{\partial}{\partial r} + \mu^2 \frac{\partial}{\partial s} + \varpi^1 \frac{\partial}{\partial v}, \tag{4. 74}$$

$$\mu^1 = 1, \;\mu^2 = a, \;\varpi^1 = 0. \tag{4. 75}$$

The coordinates in their canonical form are defined as follows.

$$v(r, s) = P(l), \tag{4. 76}$$

$$l = -ar + s, \quad m = r. \tag{4. 77}$$

Find the derivatives and put in Eq. (4. 70)

$$\alpha(-1 + 3p^2)P_l + a^2 \sigma P_{ll} - P_{lll} = 0. \tag{4. 78}$$

This form of the (2+1)-dimensional Chaffee-Infante equation is already simplified and cannot be further reduced. A general explanation for this reduction process isn't feasible, so we derive some conditional solutions by assigning specific values to the constants. We now proceed to examine several distinct cases of equation (4. 78).

Solution of Eq. (4. 78) by substituting $\alpha = 0$

$$P(l) = C_1 + C_2 l + C_3 e^{a^2 \sigma l}. \tag{4. 79}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1 + C_2(-ay + x) + C_3 e^{a^2 \sigma(-ay+x)}. \tag{4. 80}$$

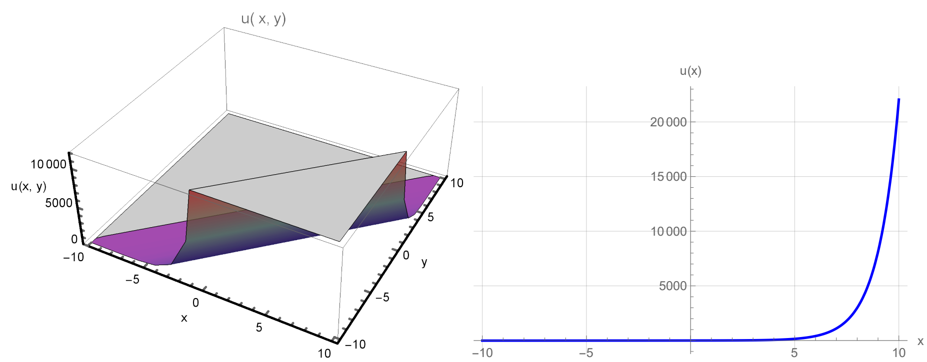

FIGURE 4. Analysis of the transverse displacement behavior of the beam wave solutions (4. 80)with all parameters equal to 1.

Solution of Eq. (4. 78) by substituting $a = 0$

$$P(l) = C_1 \int^{p(l)} (-\frac{2}{8C^2 - 4\alpha a^2 + 2\alpha a^4 - 8c_1\alpha})dl - l - C_3 = 0, \tag{4. 81}$$

$$P(l) = C_1 \int^{p(l)} (\frac{2}{8C^2 - 4\alpha a^2 + 2\alpha a^4 - 8c_1\alpha})dl - l - C_3 = 0. \tag{4. 82}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1 \int^{u(t,x,y)} (-\frac{2}{8C^2 + 2\alpha - 8c_1\alpha})dx - x - C_3 = 0, \tag{4. 83}$$

$$u(t, x, y) = C_1 \int^{u(t,x,y)} (\frac{2}{8C^2 + 2\alpha - 8c_1\alpha})dx - x - C_3 = 0. \tag{4. 84}$$

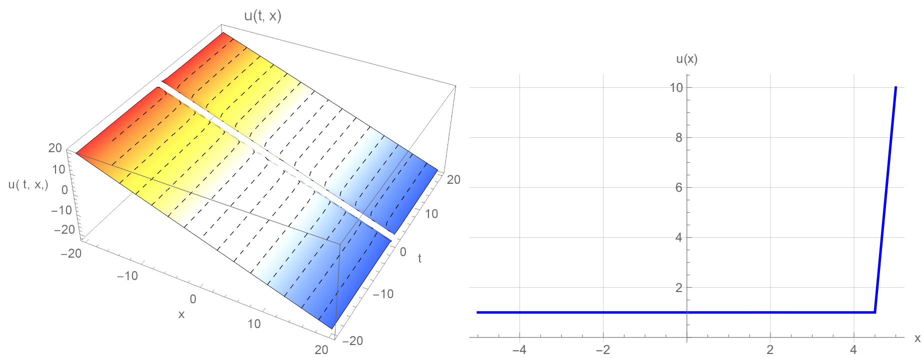

FIGURE 5. Analysis of the transverse displacement behavior of the beam wave solutions (4. 84) with all parameters equal to 1.

Solution of Eq. (4. 78) by substituting $a = \alpha = 0$

$$P(l) = \frac{1}{2}C_1 l^2 + C_2 l + C_3. \tag{4. 85}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = \frac{1}{2}C_1 x^2 + C_2 x + C_3. \tag{4. 86}$$

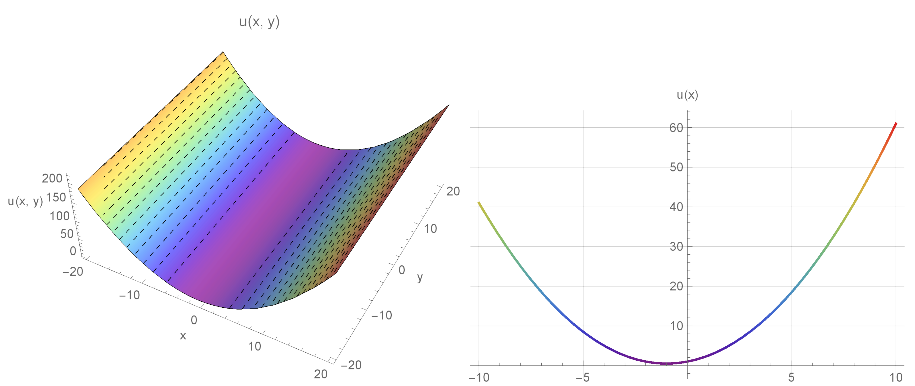

FIGURE 6. Analysis of the transverse displacement behavior of the beam wave solutions (4. 86) with all parameters equal to 1.

The classical LPS transformations are derived as follows. way.
<u>**Exact solutions of Eq. (4. 63) using P_2 LPS:**</u>

$$P = P_2 = \frac{\partial}{\partial x}, \tag{4. 87}$$

$$P = \mu^1 \frac{\partial}{\partial t} + \mu^2 \frac{\partial}{\partial x} + \mu^3 \frac{\partial}{\partial y} + \varpi^1 \frac{\partial}{\partial u}, \tag{4. 88}$$

By comparing expressions, we find the relationships between these infinitesimals.

$$\mu^1 = 0, \mu^2 = 1, \mu^3 = 0, \varpi^1 = 0. \tag{4. 89}$$

The coordinates in their canonical form are defined as follows.

$$v(r, s) = u(t, x, y), \tag{4. 90}$$

$$r = y \quad s = t, \quad w = x. \tag{4. 91}$$

Using these coordinates, find the derivatives and substitute them into eq.(4. 63)

$$\sigma 4 v_{rr} = 0, \tag{4. 92}$$

Determine the LPS again and proceed to solve Eq. (4. 92)

$$P_1 = \frac{\partial}{\partial r}, \ \ P_2 = \frac{\partial}{\partial s}, \tag{4. 93}$$

Now, find the ES of given LPS P_1 & P_2.

$$P = P_1 + aP_2, \tag{4. 94}$$

$$P = \frac{\partial}{\partial r} + a\frac{\partial}{\partial s}. \tag{4. 95}$$

$$\mu^1 = 1, \ \mu^2 = a, \ \ \varpi^1 = 0. \tag{4. 96}$$

The coordinates in their canonical form are defined as follows.

$$v(r, s) = P(l), \tag{4. 97}$$

$$l = -ar + s, \quad m = r. \tag{4. 98}$$

find the derivatives and put in Eq. (4. 92)

$$p_{ll} = 0. \tag{4. 99}$$

This form of the (2+1)-dimensional Chaffee-Infante equation is already simplified and cannot be further reduced. A general explanation for this reduction process isn't feasible, so we derive some conditional solutions by assigning specific values to the constants. We now proceed to examine several distinct cases of equation (4. 99).

Solution of Eq. (4. 99)

$$P(l) = C_1 l + C_2. \tag{4. 100}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1(-ay + t) + C_2. \tag{4. 101}$$

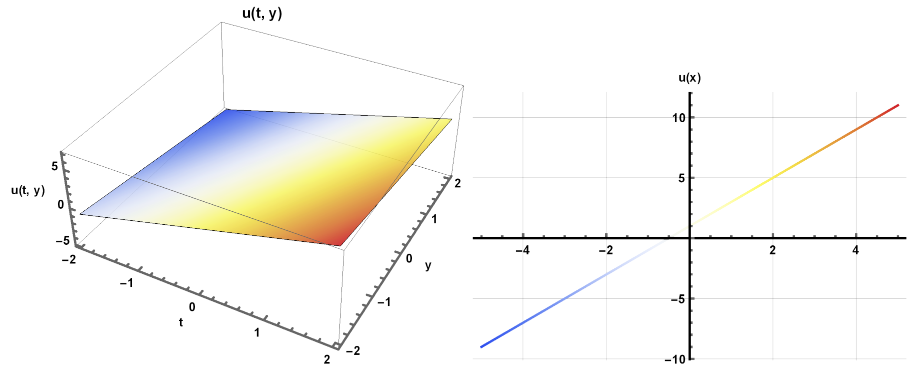

FIGURE 7. Analysis of the transverse displacement behavior of the beam wave solutions (4. 101) with all parameters equal to 1.

Exact solutions of Eq. (4. 63) using P_3 LPS:

$$P = P_3 = \frac{\partial}{\partial y}, \tag{4. 102}$$

$$P = \mu^1 \frac{\partial}{\partial t} + \mu^2 \frac{\partial}{\partial x} + \mu^3 \frac{\partial}{\partial y} + \varpi^1 \frac{\partial}{\partial u}, \tag{4. 103}$$

By comparing expressions, we find the relationships between these infinitesimals.

$$\mu^1 = 0, \mu^2 = 0, \mu^3 = 1, \varpi^1 = 0. \tag{4. 104}$$

The coordinates in their canonical form are defined as follows.

$$v(r, s) = u(t, x, y), \tag{4. 105}$$

$$r = x, \quad s = t, \quad w = y. \tag{4. 106}$$

Using these coordinates, find the derivatives and substitute them into eq.(4. 63)

$$\alpha(-1 + 3v^2)v_r + 4v_{rs} + v_{rrr} = 0, \tag{4. 107}$$

Determine the LPS again and proceed to solve Eq. (4. 107)

$$P_1 = \frac{\partial}{\partial r}, \quad P_2 = \frac{\partial}{\partial s}, \tag{4. 108}$$

Now, find the ES of given LPS Y_1 & Y_2.

$$P = P_1 + bP_2, \tag{4. 109}$$

$$P = \frac{\partial}{\partial r} + a\frac{\partial}{\partial s}. \tag{4. 110}$$

$$P = \mu^1 \frac{\partial}{\partial r} + \mu^2 \frac{\partial}{\partial s} + \varpi^1 \frac{\partial}{\partial v}, \tag{4. 111}$$

$$\mu^1 = 1, \ \mu^2 = a, \ \ \varpi^1 = 0. \tag{4. 112}$$

The coordinates in their canonical form are defined as follows.

$$v(r, s) = P(l), \tag{4. 113}$$

$$l = -br + s, \quad m = r. \tag{4. 114}$$

find the derivatives and put in Eq. (4. 107)

$$\alpha(1-3p^2)P_l + b^2 P_{lll} - P_{ll} = 0. \tag{4. 115}$$

This form of the (2+1)-dimensional Chaffee-Infante equation is already simplified and cannot be further reduced. A general explanation for this reduction process isn't feasible, so we derive some conditional solutions by assigning specific values to the constants. We now proceed to examine several distinct cases of equation (4. 115).

Solution of Eq. (4. 115) by substituting $\alpha = 0$

$$P(l) = C_1 + C_2 l + C_3 e^{\frac{l}{b^2}}. \tag{4. 116}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1 + C_2(-bx + t) + C_3 e^{\frac{(-bx+t)}{b^2}}. \tag{4. 117}$$

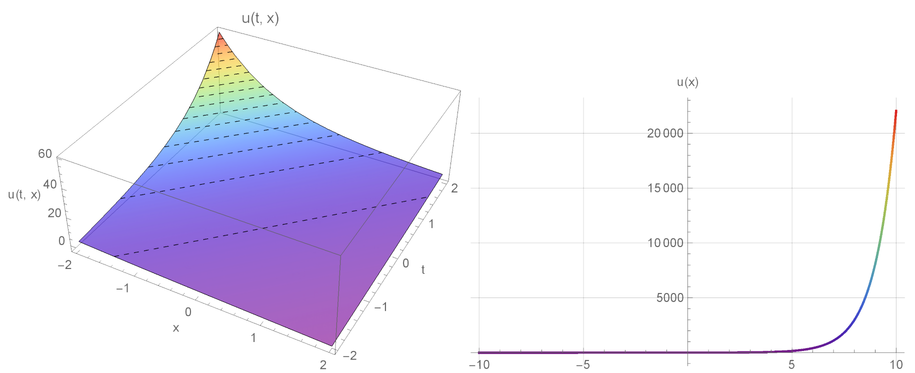

FIGURE 8. Analysis of the transverse displacement behavior of the beam wave solutions (4. 117)with all parameters equal to 1.

Solution of Eq. (4. 115) by substituting $b = 0$

$$P(l) = C_1 \int^{p(l)} (-\frac{2}{C_1})\alpha - l - C_3 = 0. \tag{4. 118}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1 \int^{u(t,x,y)} (-\frac{2}{C_1})\alpha - t - C_3 = 0. \tag{4. 119}$$

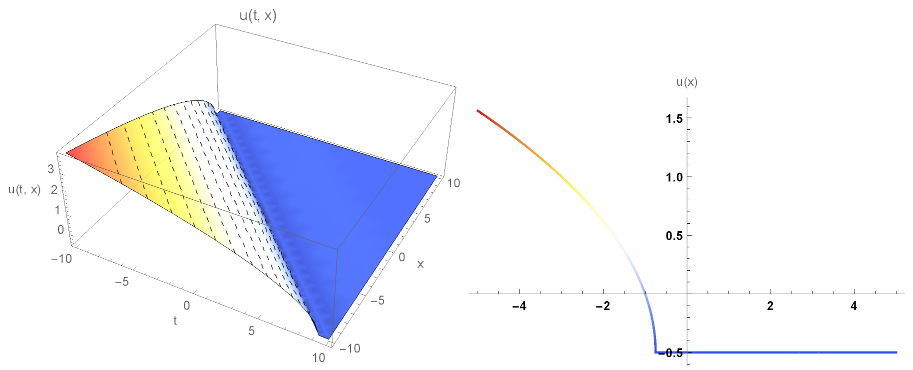

FIGURE 9. Analysis of the transverse displacement behavior of the beam wave solutions (4. 119)with all parameters equal to 1.

Solution of Eq. (4. 115) by substituting $b = \alpha = 0$

$$P(l) = C_1 l + C_2. \tag{4. 120}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1 t + C_2. \tag{4. 121}$$

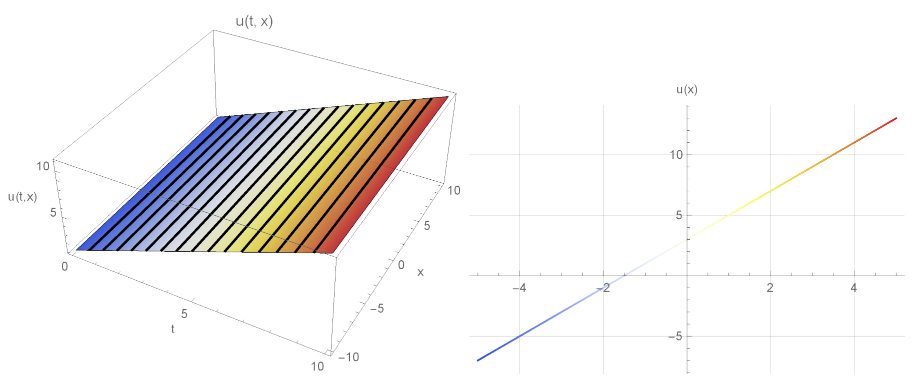

FIGURE 10. Analysis of the transverse displacement behavior of the beam wave solutions (4. 121)with all parameters equal to 1.

Exact solutions of Eq. (4. 63) using $P = P_1 + aP_2 + bP_3$ **LPS:**

$$P = P_1 + aP_2 + bP_3 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, \tag{4. 122}$$

$$P = \mu^1\frac{\partial}{\partial t} + \mu^2\frac{\partial}{\partial x} + \mu^3\frac{\partial}{\partial y} + \varpi^1\frac{\partial}{\partial u}, \tag{4. 123}$$

The terms μ^1, μ^2, μ^3 and ϖ^1 represent the infinitesimals associated with the variables x, y, t and u, respectively.
By comparing expressions, we find the relationships between these infinitesimals.

$$\mu^1 = 1, \mu^2 = a, \mu^3 = b, \varpi^1 = 0. \tag{4. 124}$$

The coordinates in their canonical form are defined as follows.

$$v(r,s) = u(t,x,y), \tag{4. 125}$$

$$r = -bt + y, \quad s = -at + x, \quad w = t. \tag{4. 126}$$

Using these coordinates, find the derivatives and substitute them into eq.(4. 63)

$$\alpha(-1+3v^2)v_s + \sigma 4 v_{rr} - a v_{ss} - b v_{rs} + v_{sss} = 0. \tag{4. 127}$$

Determine the LPS again and proceed to solve Eq.(4. 127)

$$P_1 = \frac{\partial}{\partial r}, \;\; P_2 = \frac{\partial}{\partial s}, \tag{4. 128}$$

Now, find the ES of given LPS Y_1 & Y_2.

$$P = P_1 + cP_2, \tag{4. 129}$$

$$P = \frac{\partial}{\partial r} + c\frac{\partial}{\partial s}, \tag{4. 130}$$

$$P = \mu^1\frac{\partial}{\partial r} + \mu^2\frac{\partial}{\partial s} + \varpi^1\frac{\partial}{\partial v}, \tag{4. 131}$$

$$\mu^1 = 1, \; \mu^2 = c, \;\; \varpi^1 = 0. \tag{4. 132}$$

The coordinates in their canonical form are defined as follows.

$$v(r,s) = P(l), \tag{4. 133}$$

$$l = -cr + s, \quad m = r. \tag{4. 134}$$

find the derivatives and put in eq.(4. 127)

$$\alpha(-1+3p^2)P_l + (-a + c(b + c\sigma))P_{ll} - P_{lll} = 0. \tag{4. 135}$$

This form of the (2+1)-dimensional Chaffee-Infante equation is already simplified and cannot be further reduced. A general explanation for this reduction process isn't feasible, so we derive some conditional solutions by assigning specific values to the constants. We now proceed to examine several distinct cases.

Solution of Eq. (4. 135) by substituting $a = b = c = \alpha = 0$

$$P(l) = \frac{1}{2}C_1 + C_2 l + C_3. \tag{4. 136}$$

Solution in the original variables $u(t,x,y)$ form

$$u(t,x,y) = \frac{1}{2}C_1 + C_2 x + C_3. \tag{4. 137}$$

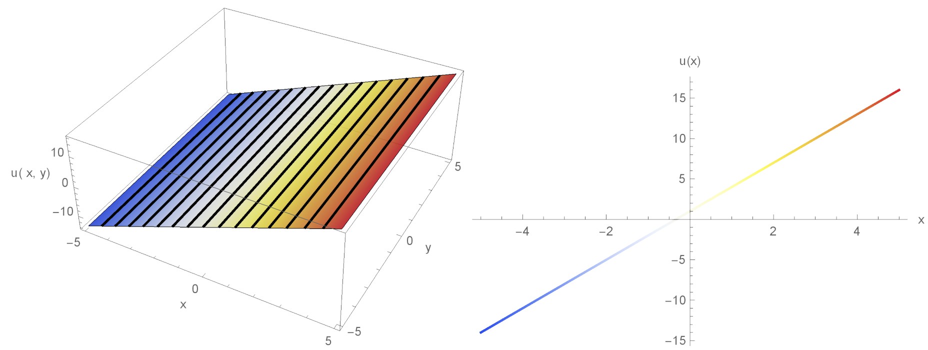

FIGURE 11. Analysis of the transverse displacement behavior of the beam wave solutions (4. 137) with all parameters equal to 1.

Solution of Eq. (4. 135) by substituting $a = b = \alpha = 0$

$$P(l) = C_1 + C_2 l + C_3 e^{c^2\sigma l}. \quad (4.\ 138)$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1 + C_2(-cy + x) + C_3 e^{c^2\sigma(-cy+x)}. \quad (4.\ 139)$$

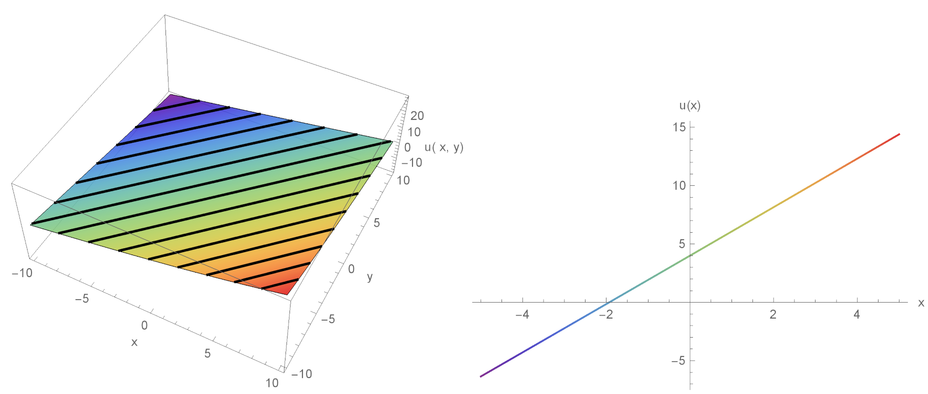

FIGURE 12. Analysis of the transverse displacement behavior of the beam wave solutions (4. 139) with all parameters equal to 1.

Solution of Eq. (4. 135) by substituting $a = \alpha = 0$

$$P(l) = C_1 l^2 + C_2 l + C_3 e^{(cb+c^2\sigma)l}. \quad (4.\ 140)$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1(cbt - cy + x)^2 + C_2(cbt - cy + x) + C_3 e^{(cb+c^2\sigma)(cbt-cy+x)}. \quad (4.\ 141)$$

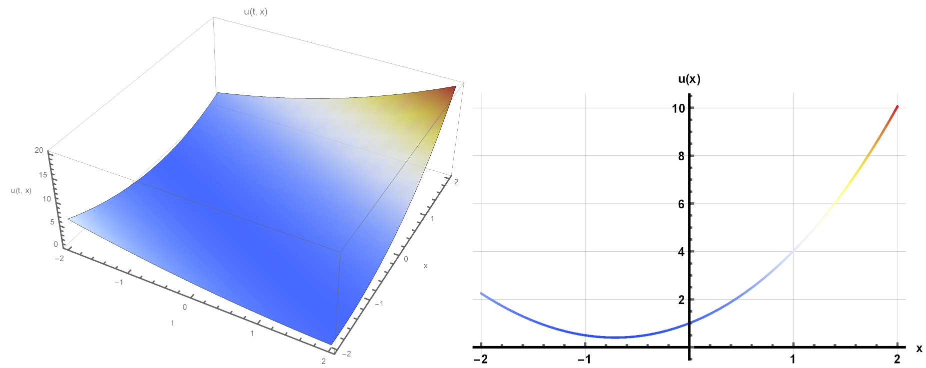

FIGURE 13. Analysis of the transverse displacement behavior of the beam wave solutions (4. 141) with all parameters equal to 1.

Solution of Eq. (4. 135) by substituting $a = b = \alpha = 0$

$$P(l) = C_1 + C_2 l + C_3 e^{(-a+cb+c^2\sigma)l}. \tag{4. 142}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = C_1 + C_2(cbt - cy - at + x) + C_3 e^{(-a+cb+c^2\sigma)(cbt-cy-at+x)}. \tag{4. 143}$$

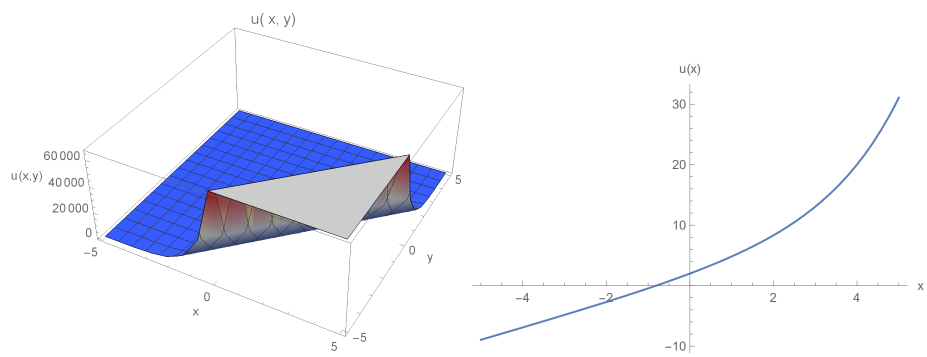

FIGURE 14. Analysis of the transverse displacement behavior of the beam wave solutions (4. 143) with all parameters equal to 1.

Solution of Eq. (4. 135) by substituting $a = b = \alpha = \sigma = 0$

$$P(l) = \frac{1}{c} C_1 l^2 + C_2 l + C_3. \tag{4. 144}$$

Solution in the original variables $u(t, x, y)$ form

$$u(t, x, y) = \frac{1}{c} C_1 (y + x)^2 + C_2(y + x) + C_3. \tag{4. 145}$$

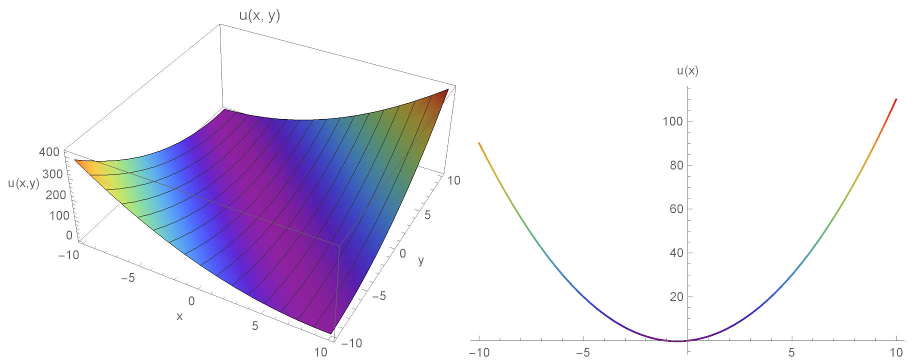

FIGURE 15. Analysis of the transverse displacement behavior of the beam wave solutions (4. 145) with all parameters equal to 1.

5. EVALUATION AND DISCUSSION

In this article, we investigate the geometric representation of group solutions for the (1+1)-dimensional Modified Thomas equation and the (1+2)-dimensional Chaffee-Infante equation. We apply the Lie method to solve the given PDEs and construct their symmetries. The application of these symmetries is crucial in various scientific fields, such as biological modeling, nonlinear optics, and fluid dynamics. By applying symmetry reductions, we obtain exact solutions. The derived solutions, which include wave solitons, double solitons, and parabolic solitons, reveal various dynamical behaviors, such as soliton interactions and annihilations. These behaviors are visualized through 3D and 2D plots created using Mathematica, as shown in Figures (1-15). Graphically representing invariant solutions in 2D and 3D demonstrates how transformations preserve the same solution patterns or structures across various dimensions. In a 2D graph, curves that are unchanged by symmetry operations can illustrate this concept, while in a 3D graph, surfaces that exhibit invariant behavior despite transformations convey the same idea. Through these graphical representations, one gains insight into the fundamental symmetries and stability of the solution systems.

6. CONCLUSION

In this study, we explored the (1+1)-dimensional modified Thomas equation and the (1+2)-dimensional Chaffee-Infante equation using Lie group analysis. We identified Lie point symmetries and applied symmetry reduction to transform the PDEs into ODEs, which were further simplified using canonical coordinates. Through group classification, the exact solutions for the equations under Lie analysis were successfully derived. These solutions are efficient, accurate, and versatile, with significant applications in engineering and mathematical physics. The Lie symmetry method proved to be highly effective, robust, and powerful for simplifying and solving nonlinear PDEs. This approach is also applicable to other nonlinear evolution equations and Schrödinger-type systems. Beyond their practical applications, closed-form solutions of nonlinear PDEs assist numerical solvers in stability analysis and provide a benchmark for validating the accuracy of their results. In the next phase, we will utilize the Lie symmetry analysis method to study this newly extended system.

Conflict of Interest

All authors confirmed that there is no conflict of interest regarding the publication of this research.

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