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On the Elliptic Sombor and Euler Sombor indices of corona product of certain graphs

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Abstract. Graph theory has wide-ranging applications in chemistry, nano science and networking problems. The corona product operation effectively represents compounds like dendrimers, where a central molecule is encircled by multiple repeating units. By representing atoms as vertices and bonds as edges, graph theory provides a clear and structured way to analyze and understand the architecture of intricate chemical compounds through various graph operations. The corona product of two graphs G and H gives a new graph obtained by taking one copy of G and $|V(G)|$ copies of H by joining i^{th} vertex of G to every i^{th} copy of H. Elliptic Sombor and Euler Sombor indices are recently defined topological indices using Sombor index. Elliptic sombor index is defined as $ESO(G) = \sum_{uv \in E(G)} (d_u + d_v)\sqrt{d_u^2 + d_v^2}$ and Euler Sombor index is defined as $EU(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + d_u d_v}$, where d_u and d_v are degrees of vertices u and v in graph G. In this article, we compute the Elliptic Sombor and Euler Sombor indices of few resultant graphs using the operations join and corona product on standard graphs like path, cycle and complete graphs.

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Key Words: Elliptic Sombor index, Euler Sombor index, join, corona product.

1. INTRODUCTION

Degree-based topological indices are fundamental tools in the fields of mathematical chemistry and graph theory, serving as quantitative measures of a molecular graph structure. A molecular graph represents a chemical compound where vertices correspond to atoms and edges correspond to chemical bonds. The degree of a vertex in this context refers to the number of edges connected to it, representing the number of bonds of an atom. The primary utility of these indices lie in their ability to predict various physical, chemical, and biological properties of molecules [16, 18].

The Sombor index, introduced by Gutman in 2021, is a topological index derived from graph theory and Euclidean geometry. This index has gained significant traction in the fields such as chemistry and pharmacology for its utility in characterizing molecular structures and predicting biological activity. Let us delve into the formal definition, properties, and an example calculation of the Sombor index [12, 17].

The general form of a vertex degree-based index is a function which is chosen such that it satisfies symmetry property. The edge uv representation in 2-dimensional coordinate system is called the degree-point of the edge uv. The Euclidean distance between the degree-point (d_u, d_v) and the origin O is $\sqrt{d_u^2 + d_v^2}$ is the definition of Sombor index [13].

In 2023, Gutman et al., [11] introduced another version of Sombor index called elliptic Sombor index referring to the orbits of planets in the solar system which takes elliptic orbits with the sun as focus point. The concept of an elliptic Sombor index can be an extension of the traditional Sombor index by incorporating elliptic distance rather than the Euclidean distance between degree-points.

In 2024, Gutman et al., [14] showed that the lengths of semi major and the semi minor axes are equal in an ellipse. Leonard Euler found the approximate perimeter of the ellipse as $\pi\sqrt{2(d_u^2 + d_v^2)(d_u + d_v)^2}$. Using these relations, Euler Sombor index was proposed as $\sqrt{d_u^2 + d_v^2 + d_u.d_v}$. Algebraically, there is a geometric analogy of Sombor and Euler Sombor indices.

Bibhas et al., [1] studied structural properties of corona graphs including the statistics of signed links and types of signed triangles and degree distribution. They analyzed algebraic conflict of signed corona graphs generated by seed graphs. Sheeba et al., [2] established the explicit expressions of Y -index of different types of corona product of graphs. Khalid et al., [15] studied certain degree-based topological indices such as Randić index, Zagreb indices, multiplicative Zagreb indices, Narumi-Katayama index, atom-bond connectivity index, augmented Zagreb index, geometric-arithmetic index, harmonic index, and sum-connectivity index for the bistar graphs and the corona product. Sheeja et al., [8] established the explicit expressions for the SK index over different types of corona products on graphs are presented. Iswadi et al., [5] determined the metric dimension of corona product, and lower bound of metric dimension of join of graphs. Dhananjaya et al., [9] derived ABC index and the GA index of several corona products of graphs composed of path, cycle, and complete graphs. Arif et al., [4] computed the Sombor index of some graph operations namely, join and corona product of two graphs, for the standard graphs path, cycle, and complete graphs.

Motivated by the above studies on graph operations, this study focuses on computing the elliptic Sombor and Euler Sombor index of few resultant graphs using the operations join and corona product on standard graphs like path, cycle and complete graphs.

Elliptic Sombor and Euler Sombor indices are defined as follows [14, 11]

$$ESO(G) = \sum_{uv \in E(G)} (d_u + d_v)(\sqrt{d_u^2 + d_v^2}) \tag{1. 1}$$

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + d_u d_v} \tag{1. 2}$$

In the above indices '$+$' represents mathematical operation sum.

2. Notations and Preliminaries

Graph operations, such as join, union, Cartesian product, and others, are fundamental tools in graph theory that significantly influence topological indices [10, 7, 6, 3]. Given two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the join $G_1 + G_2$ (See Figure 1) is defined as:

$$G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\})$$

Here $G_1 + G_2$ does not mean sum of two graphs, but it is the operation 'join' of two graphs G_1 and G_2.

Let G be a connected graph with n vertices and H be a graph with at least two vertices, the corona product of G and H, is defined as a graph which is formed by taking n copies of H and connecting i^{th} vertex of G to the vertices of H in each copies, and it is denoted by $G \odot H$ (See Figure 2).

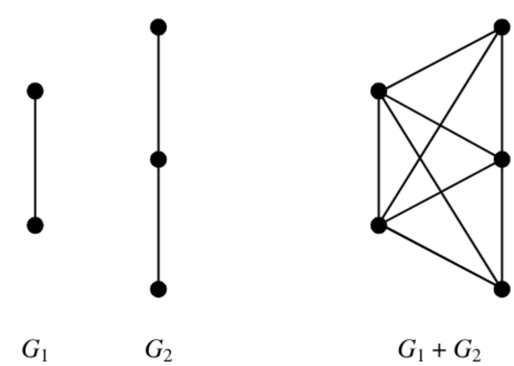

FIGURE 1. Join of two paths P_2 and P_3.

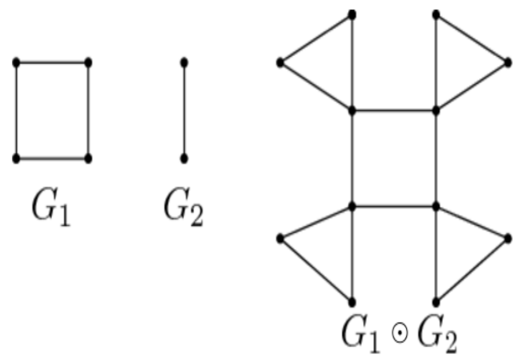

FIGURE 2. Corona product of two graphs C_4 and P_2.

3. RESULTS

In this section, Elliptic Sombor and Euler Sombor index of $P_r + P_s$, $C_r + C_s$, $K_r + K_s$, $P_r \odot P_s$, $C_r \odot C_s$ are obtained.

If P_r and P_s are two paths of order $|V(G)| = r + s$ and size $|E(G)| = (r-1) + (s-1) + rs = r + s + rs - 2$. In the following theorem, the Elliptic Sombor and Euler Sombor index of two path graphs P_r and P_s are computed.

Theorem 3.1. *Elliptic Sombor index of $P_r + P_s$ is given by*

$$ESO(P_r + P_s) = \begin{cases} 108\sqrt{2};\ r = s = 2 \\ \\ 70 + 2\sqrt{2}(s+1)^2 + 32\sqrt{2}(s-3) \\ +4(s+4)\sqrt{9+(s+1)^2} \\ +2(s-2)(s+5)\sqrt{16+(s+1)^2};\ s > 2, r = 2 \\ \\ 70 + 2\sqrt{2}(r+1)^2 + 32\sqrt{2}(r-3) \\ +4(r+4)\sqrt{9+(r+1)^2} \\ +2(r-2)(s+5)\sqrt{16+(r+1)^2};\ r > 2,\ s = 2 \\ \\ 2(2s+3)\sqrt{(s+1)^2+(s+2)^2} \\ +(r-3)(2s+4(s+2)\sqrt{2} + 2(2r+3)\sqrt{(r+1)^2+(r+2)^2} \\ +(s-3)(2r+4)(r+2)\sqrt{2} + 4(r+s+2)\sqrt{(r+1)^2+(s+1)^2};\ r, s > 2. \end{cases}$$

Proof. Consider the join of two graphs P_r and P_s. If $r, s > 1$, based on the degrees of vertices of $P_r + P_s$, there are 4 types of vertices having degrees $s+1, s+2, r+1$and $r+2$. Then by considering the degrees of the vertices, there are 10 types of edge partitions of $P_r + P_s$ as shown in the Table 1.

TABLE 1. Number of edges for different cases in $P_r + P_s$.

Case	Number of edges	Condition
$(s+1, s+1)$	1	$r = 2$
	0	$r > 2$
$(s+1, s+2)$	0	$r = 2$
	2	$r > 2$
$(s+2, s+2)$	0	$r = 2$
	$r-3$	$r > 2$
$(r+1, r+1)$	1	$s = 2$
	0	$s > 2$
$(r+1, r+2)$	0	$s = 2$
	2	$s > 2$
$(r+2, r+2)$	0	$s = 2$
	$s-3$	$s > 2$
$(s+1, r+1)$	4	Always
$(s+1, r+2)$	0	$s = 2$
	$2(s-2)$	$s > 2$
$(s+2, r+1)$	0	$r = 2$
	$2(r-2)$	$r > 2$
$(s+2, r+2)$	0	$r = 2$ or $s = 2$
	$(r-2)(s-2)$	$r, s > 2$

Case 1: r=s=2

$$\begin{aligned} ESO(P_r + P_s) =& 2(s+1)\sqrt{(s+1)^2 + (s+1)^2} + 2(r+1)\sqrt{(r+1)^2 + (r+1)^2}, \\ &+ 4(s+1+r+1)\sqrt{(s+1)^2 + (r+1)^2} \\ ESO(P_r + P_s) =& 2(s+1)\sqrt{2(s+1)^2} + 2(r+1)\sqrt{2(r+1)^2} \\ &+ 4(r+s+2)\sqrt{(s+1)^2 + (r+1)^2}, \end{aligned}$$

substituting $r = 2, s = 2$, we get

$$ESO(P_r + P_s) = 108\sqrt{2}.$$

Case 2: r=2, s>2

$$\begin{aligned} ESO(P_r + P_s) =& 2(s+1)\sqrt{2(s+1)^2} + 2(r+1+r+2)\sqrt{(r+1)^2 + (r+2)^2} \\ &+ (s-3)2(r+2)\sqrt{2(r+2)^2} + 4(r+1+s+1)\sqrt{(r+1)^2 + (s+1)^2} \\ &+ 2(s-2)(r+2+s+1)\sqrt{(r+2)^2 + (s+1)^2}, \\ ESO(P_r + P_s) =& 2(s+1)^2\sqrt{2} + 2(2r+3)\sqrt{(r+1)^2 + (r+2)^2} \\ &+ 2(s-3)(r+2)(r+2)\sqrt{2} + 4(r+s+2)\sqrt{(r+1)^2 + (s+1)^2} \\ &+ 2(s-2)(r+s+3)\sqrt{(r+2)^2 + (s+1)^2}, \end{aligned}$$

Substituting $r = 2$, we obtain

$$\begin{aligned}ESO(P_r + P_s) =& 2\sqrt{2}(s+1)^2 + 2\times 7\sqrt{3^2+4^2} + 32(s-3)\sqrt{2}\\ &+ 4(s+4)\sqrt{9+(s+1)^2} + 2(s-2)(s+5)\sqrt{16+(s+1)^2},\\ ESO(P_r + P_s) =& 70 + 2\sqrt{2}(s+1)^2 + 32(s-3)\sqrt{2} + 4(s+4)\sqrt{9+(s+1)^2}\\ &+ 2(s-2)(s+5)\sqrt{16+(s+1)^2}.\end{aligned}$$

Case 3: s=2, r>2

$$\begin{aligned}ESO(P_r + P_s) =& 2(r+1)\sqrt{2(r+1)^2} + 2(s+1+s+2)\sqrt{(s+1)^2+(s+2)^2}\\ &+ (r-3)2(s+2)\sqrt{2(s+2)^2} + 4(r+1+s+1)\sqrt{(r+1)^2+(s+1)^2}\\ &+ 2(r-2)(r+1+s+2)\sqrt{(r+1)^2+(s+2)^2},\\ ESO(P_r + P_s) =& 2(r+1)^2\sqrt{2} + 2(2s+3)\sqrt{(s+1)^2+(s+2)^2}\\ &+ 2(r-3)(s+2)^2\sqrt{2} + 4(r+s+2)\sqrt{(r+1)^2+(s+1)^2}\\ &+ 2(r-2)(r+s+3)\sqrt{(s+2)^2+(r+1)^2},\end{aligned}$$

substituting $s = 2$,

$$\begin{aligned}ESO(P_r + P_s) =& 2\sqrt{2}(r+1)^2 + 2\times 7\sqrt{3^2+4^2} + 32(r-3)\sqrt{2}\\ &+ 4(r+4)\sqrt{9+(r+1)^2} + 2(r-2)(s+5)\sqrt{16+(r+1)^2},\\ ESO(P_r + P_s) =& 70 + 2\sqrt{2}(r+1)^2 + 32\sqrt{2}(r-3) + 4(r+4)\sqrt{9+(r+1)^2}\\ &+ 2(r-2)(s+5)\sqrt{16+(r+1)^2}.\end{aligned}$$

Case 4: r, s>2

$$\begin{aligned}ESO(P_r + P_s) =& 2(2s+3)\sqrt{(s+1)^2+(s+2)^2} + 2(r-3)(s+2)\sqrt{2(s+2)^2}\\ &+ 2(2r+3)\sqrt{(r+1)^2+(r+2)^2} + 2(s-3)(r+2)\sqrt{2(r+2)^2}\\ &+ 4(r+s+2)\sqrt{(r+1)^2+(s+1)^2} + 2(s-2)(r+s+3)\\ &\sqrt{(r+2)^2+(s+1)^2} + (r-2)(s-2)(r+s+4)\sqrt{(r+2)^2+(s+2)^2}\\ &+ 2(r-2)(r+s+3)\sqrt{(r+1)^2+(s+2)^2}.\end{aligned}$$

□

Theorem 3.2. *The Euler Sombor index of $P_r + P_s$ is given by*

$$EU(P_r+P_s)=\begin{cases} 18\sqrt{3}, r=s=2 \\ \\ 2\sqrt{37}+(5s-11)\sqrt{3}+4\sqrt{(s+1)^2+3s+12} \\ +2(s-2)\sqrt{(s+1)^2+4s+20}; s>2, r=2 \\ \\ 2\sqrt{37}+(5s-11)\sqrt{3}+4\sqrt{(r+1)^2+3r+12} \\ +2(r-2)\sqrt{(r+1)^2+4r+20}; r>2, s=2 \\ \\ 2\sqrt{(s+1)^2+(s+2)^2+(s+1)(s+2)}+(r-3)\sqrt{3(s+2)^2} \\ +2\sqrt{(r+1)^2+(r+2)^2+(r+1)(r+2)}+(s-3)\sqrt{3(r+2)^2} \\ +4\sqrt{(r+1)^2+(s+1)^2+(r+1)(s+1)} \\ +2(s-2)\sqrt{(r+2)^2+(s+1)^2+(r+2)(s+1)} \\ +(r-2)(s-2)\sqrt{(r+2)^2+(s+2)^2+(r+2)(s+2)} \\ +2(r-2)\sqrt{(r+1)^2+(s+2)^2+(r+1)(s+2)}; r,s>2. \end{cases}$$

Proof. Using edge partition of join of two path graphs P_r and P_s from Table 1, Euler Sombor index is calculated using the equation 1.2. The proof is analogous to that of Theorem 3.1 and is omitted for brevity. □

Theorem 3.3. *Let r, s be two positive integers such that $r, s \geq 3$. Then Elliptic Sombor index and Euler Sombor index of $C_r + C_s$ are given by*

$$ESO(C_r+C_s)=2\sqrt{2}(s+2)^2+s(r+2)^2]+rs(r+s+4)\sqrt{(r+2)^2+(s+2)^2}.$$
$$EU(C_r+C_s)=2\sqrt{3}(r+s+rs)+rs\sqrt{(r+2)^2+(s+2)^2+(r+2)(s+2)}.$$

Proof. Consider join of two cycles C_r and C_s. The order of resultant graph $|V(C_r+C_s)| = r+s$ and size $|E(C_r+C_s)| = r+s+rs$. There are two vertices with degree $s+2$ and other type has $r+2$ in the resultant graph. Thus, there are three edge partitions.

TABLE 2. Number of edges for different cases in $C_r + C_s$.

Case	Number of edges
$(s+2, s+2)$	r
$(s+2, r+2)$	rs
$(r+2, r+2)$	s

Using above edge partitions (Table 2) and the equation (1.1) and (1.2),

$$\begin{aligned} ESO(C_r + C_s) =& r(s+2+s+2)\sqrt{2(s+2)^2} + rs(r+2+s+2)\sqrt{(r+2)^2+(s+2)^2} \\ &+ s(r+2+r+2)\sqrt{2(r+2)^2}, \\ ESO(C_r + C_s) =& 2\sqrt{2}r(s+2)^2 + rs(r+s+4\sqrt{(r+2)^2+(s+2)^2} + 2\sqrt{2}s(r+2)^2, \\ ESO(C_r + C_s) =& 2\sqrt{2}\left((r(s+2)^2 + s(r+2)^2\right) + rs(r+s+4)\sqrt{(r+2)^2+(s+2)^2}. \end{aligned}$$

$$\begin{aligned} EU(C_r + C_s) =& r\sqrt{3(s+2)^2} + rs\sqrt{(r+2)^2+(s+2)^2+(r+2)(s+2)} + s\sqrt{3(r+2)^2}, \\ EU(C_r + C_s) =& \sqrt{3}\left[r(s+2) + s(r+2)\right] + rs\sqrt{(r+2)^2+(s+2)^2+(r+2)(s+2)}, \\ EU(C_r + C_s) =& 2\sqrt{3}\left[r+s+rs\right] + rs\sqrt{(r+2)^2+(s+2)^2+(r+2)(s+2)}. \end{aligned}$$

□

Theorem 3.4. *If two complete graphs K_r and K_s are joined, the resultant is a complete graph K_{r+s} with order $|V(K_{r+s})| = r+s$ and size $|E(K_{r+s})| = \frac{(r+s)(r+s-1)}{2}$. Then*

$$ESO(K_r + K_s) = \sqrt{2}(r+s)(r+s-1)^3.$$
$$EU(K_r + K_s) = \frac{\sqrt{3}}{2}(r+s)(r+s-1)^2.$$

Proof. Each vertex of join of two graphs K_r and K_s has degree $r+s-1$. So there is only one type of edge that is $(r+s-1, r+s-1)$. Using equation (1.1), Elliptic Sombor index is

$$\begin{aligned} ESO(K_r + K_s) =& ESO(K_{r+s}) \\ =& \frac{(r+s)(r+s-1)}{2}(r+s-1+r+s-1)\sqrt{2(r+s-1)^2} \\ =& \sqrt{2}(r+s)(r+s-1)^3. \end{aligned}$$

Using equation (1.2), Euler Sombor index is

$$\begin{aligned} EU(K_r + K_s) =& EU(K_{r+s}) \\ =& \frac{(r+s)(r+s-1)}{2}\sqrt{3(r+s-1)^2} \\ =& \sqrt{3}\frac{(r+s)(r+s-1)^2}{2}. \end{aligned}$$

□

Theorem 3.5. *Let C_r and K_S be the cycle and complete graph respectively. Then Elliptic Sombor index and Euler sombor index of $C_r + K_s$ are*

$$\begin{aligned} ESO(C_r + K_s) =& 2\sqrt{2}r(s+2)^2 + rs(r+2s+1)\sqrt{(s+2)^2 + (r+s-1)^2} \\ &+ C(s,2)(r+s-1)\sqrt{2}. \\ EU(C_r + K_s) =& r(s+2)\sqrt{3} + rs\sqrt{(s+2)^2 + (r+s-1)^2 + (s+2)(r+s-1)} \\ &+ C(s,2)(r+s-1)\sqrt{3}. \end{aligned}$$

Proof. The graph $C_r + K_s$ contains $r + s$ number of vertices and $r + rs + C(s, 2)$ number of edges. Here C_r has r number of edges and K_s has $C(s, 2)$ number of edges. There are three types of edge partitions that are tabulated in Table 3.

TABLE 3. Number of edges for different cases in $C_r + K_s$.

Case	Number of edges
$(s+2, s+2)$	r
$(s+2, s+r-1)$	rs
$(r+s-1, r+s-1)$	$C(s,2)$

$$\begin{aligned} ESO(C_r + K_s) =& r(s+2+s+2)\sqrt{2(s+2)^2} + rs(s+2+r+s-1)\sqrt{(s+2)^2 + (r+s-1)^2} \\ &+ C(s,2)(r+s-1+r+s-1)\sqrt{2(r+s-1)^2}, \\ ESO(C_r + K_s) =& 2\sqrt{2}r(s+2)^2 + rs(r+2s+1)\sqrt{(s+2)^2 + (r+s-1)^2} \\ &+ 2\sqrt{2}C(s,2)(r+s-1)^2. \\ EU(C_r + K_s) =& r\sqrt{3(s+2)^2} + rs\sqrt{(s+2)^2 + (r+s-1)^2 + (s+2)(r+s-1)} \\ &+ C(s,2)\sqrt{3(r+s-1)^2}, \\ EU(C_r + K_s) =& \sqrt{3}r(s+2) + rs\sqrt{(s+2)^2 + (r+s-1)^2 + (s+2)(r+s-1)} \\ &+ \sqrt{3}C(s,2)(r+s-1). \end{aligned}$$

□

Theorem 3.6. *Elliptic Sombor index of corona product of two paths are given by*

$$ESO(P_r \odot P_s) = \begin{cases} 34\sqrt{2} + 20\sqrt{13}, r = s = 2 \\ \\ 20\sqrt{13} + 2\sqrt{2}(s+1)^2 + 4(s+3)\sqrt{4+(s+1)^2} \\ +2(s-2)(s+4)\sqrt{9+(s+1)^2}; s > 2, r = 2 \\ \\ 70 + 20\sqrt{13} + 8\sqrt{2}r + 32\sqrt{2}(r-3) + 24(r-2)\sqrt{5}; r > 2, s = 2 \\ \\ 18\sqrt{2}r(s-3) + 2\sqrt{2}(r-3)(s+2)^2 + 10\sqrt{13}r + 4(s+3)\sqrt{4+(s+1)^2} \\ +2(r-1)(s+4)\sqrt{4+(s+2)^2} + 2(s-2)(s+4)\sqrt{9+(s+1)^2} \\ +(r-2)(s-2)(s+5)\sqrt{9+(s+2)^2} + 2(2s+3)\sqrt{(s+1)^2+(s+2)^2}; r, s > 2. \end{cases}$$

Proof. Consider the graph $P_r \odot P_s$ having $|V(P_r \odot P_s)| = rs + r$ and $|E(P_r \odot P_s)| = 2rs - 1$. There are four types of degrees of vertices of graph $P_r \odot P_s$ namely $2, 3, s+1, s+2$. Using edge partition listed in Table 4 and equation (1.1),

Case 1: r=s=2

TABLE 4. Number of edges for different cases in $P_r \odot P_s$.

Edge partitions	Number of edges	Condition
$(2,2)$	r	$s=2$
	0	$s>2$
$(3,3)$	0	$s=2$
	$r(s-3)$	$s>2$
$(s+1,s+1)$	1	$r=2$
	0	$s>2$
$(s+2,s+2)$	0	$r=2$
	$n-3$	$r>2$
$(2,3)$	0	$s=2$
	$2r$	$s>2$
$(2,s+1)$	4	all
$(2,s+2)$	0	$r=2$
	$2(r-2)$	$r>2$
$(3,s+1)$	0	$s=2$
	$2(s-2)$	$s>2$
$(3,s+2)$	0	$r=2$
	$(r-2)(s-2)$	$r>2$
$(s+1,s+2)$	0	$r=2$
	2	$s>2$

$$\begin{aligned}ESO(P_r \odot P_s) =& r(2+2)\sqrt{2^2+2^2}+(s+1+s+1)\sqrt{2(s+1)^2}\\ &+4(s+1+2)\sqrt{(s+1)^2+2^2},\end{aligned}$$

Subsitute $r = s = 2$

$$ESO(P_r \odot P_s) = 34\sqrt{2}+20\sqrt{13}.$$

Case 2: $r = 2, s > 2$

$$\begin{aligned}ESO(P_r \odot P_s) =& (s+1+s+1)\sqrt{2(s+1)^2}+2r(2+3)\sqrt{2^2+3^2}\\ &+4(2+s+1)\sqrt{2^2+(s+1)^2}+2(s-2)(3+s+1)\sqrt{3^2+(s+1)^2}\end{aligned}$$

substitute r=2

$$\begin{aligned}ESO(P_r \odot P_s) =& 20\sqrt{13}+2\sqrt{2}(s+1)^2+4(s+3)\sqrt{4+(s+1)^2}\\ &+2(s-2)(s+4)\sqrt{9+(s+1)^2}.\end{aligned}$$

Case 3: $s = 2$,$r > 2$

$$\begin{aligned}ESO(P_r \odot P_s) =& r(2+2)\sqrt{2^2+2^2}+(r-3)(s+2+s+2)\sqrt{2(s+2)^2}\\ &+4(2+s+1)\sqrt{2^2+(s+1)^2}+2(r-2)(s+2+2)\sqrt{2^2+(s+2)^2}\\ &+(r-2)(s-2)(3+s+2)\sqrt{3^2+(s+2)^2}\\ &+2(s+1+s+2)\sqrt{(s+1)^2+(s+2)^2}\end{aligned}$$

Substituting $s = 2$, we get

$$ESO(P_r \odot P_s) = 70+20\sqrt{13}8\sqrt{2}r+32\sqrt{2}(r-3)+24\sqrt{5}(r-2).$$

Case 4: $r, s > 2$

$$\begin{aligned}ESO(P_r \odot P_s) =& r(s-3)(3+3)\sqrt{3^2+3^2}+(r-3)(s+2+s+2)\sqrt{2(s+2)^2}\\ &+2r(2+3)\sqrt{2^2+3^2}+4(s+1+2)\sqrt{(s+1)^2+2^2}\\ &+2(r-1)(s+2+2)\sqrt{(s+2)^2+2^2}+2(s-2)(s+1+3)\sqrt{(s+1)^2+3^2}\\ &+(r-2)(s-2)(s+2+3)\sqrt{(s+2)^2+3^2}\\ &+2(s+1+s+2)\sqrt{(s+1)^2+(s+2)^2}.\end{aligned}$$

□

Theorem 3.7. *Euler Sombor index of corona product of two paths are given by*

$$EU(P_r \odot P_s) = \begin{cases} 7\sqrt{3}+4\sqrt{19}, r=s=2 \\ \\ 4\sqrt{19}+(s+1)\sqrt{3}+4\sqrt{4+(s+1)^2+2(s+1)} \\ +2(s-2)\sqrt{9+(s+1)^2+3(s+1)}; s>2, r=2 \\ \\ 4\sqrt{19}+2\sqrt{3}r+8\sqrt{2}r+32\sqrt{2}(r-3)+24(r-2)\sqrt{5}; r>2, s=2 \\ \\ 3\sqrt{3}r(s-3)+\sqrt{3}(r-3)(s+2)+2\sqrt{19}r+4\sqrt{4+(s+1)^2+2(s+1)} \\ +2(r-1)\sqrt{4+(s+2)^2+2(s+2)}+2(s-2)\sqrt{9+(s+1)^2+3(s+1)} \\ +(r-2)(s-2)\sqrt{9+(s+2)^2+3(s+2)}+2\sqrt{(s+1)^2+(s+2)^2+(s+1)(s+2)}. \end{cases}$$

Proof. Using the edge partitions in Table 4 and equation (1.2) Euler Sombor index is obtained. Proof is same as Theorem 3.6. □

Theorem 3.8. *The Elliptic Sombor index and Euler Sombor index of corona product of two cycles ($C_r \odot C_s$) are given by*

$$ESO(C_r \odot C_s) = 18rs\sqrt{2}+rs(s+5)\sqrt{3^2+(s+2)^2}+2\sqrt{2}r(s+2)^2.$$

$$EU(C_r \odot C_s) = 3\sqrt{3}+rs\sqrt{3^2+(s+2)^2+3(s+2)}+\sqrt{3}r(s+2).$$

Proof. The graph $C_r \odot C_s$ with order $rs+r$ and size $2rs+r$ has two types of vertices, such as 3-degree vertex and $s+2$ degree vertex. The edges of $C_r \odot C_s$ can be partitioned into three types.

$$ESO(C_r \odot C_s) = rs(3+3)\sqrt{3^2+3^2}+rs(s+2+3)\sqrt{(s+2)^2+3^2}$$
$$+\,r(s+2+s+2)\sqrt{2(s+2)^2},$$
$$ESO(C_r \odot C_s) = 18\sqrt{2}rs+rs(s+5)\sqrt{3^2+(s+2)^2}+2\sqrt{2}r(s+2)^2.$$
$$EU(C_r \odot C_s) = rs\sqrt{3^2+3^2+3^2}+rs\sqrt{(s+2)^2+3^2+3(s+2)}+r\sqrt{3(s+2)^2},$$
$$EU(C_r \odot C_s) = 3\sqrt{3}rs+rs\sqrt{3^2+(s+2)^2+3(s+2)}+\sqrt{3}r(s+2).$$

□

4. Conclusion

In this article, Elliptic Sombor and Euler Sombor indices of join and corona product of certain graphs are obtained for standard graphs such as path, cycle and complete graphs. Graph operations such as the corona product and join play a significant role in the study of chemical graph theory. These operations are instrumental in modeling complex molecular architectures by simulating the structural formation of larger compounds from simpler molecular units. The corona product is particularly effective in representing molecular structures with a central scaffold bonded to multiple identical functional groups, a common motif in organic and pharmaceutical compounds. Similarly, the join operation models complete interactions between distinct molecular entities, such as those found in polymer blends or multi-sub unit biological systems. These results can be further extended for corona product and join of other standard graphs.

AUTHOR CONTRIBUTIONS

Dr. Kirana B - Conceptualization, Methodology, Article writing.

Dr. Shanmukha M C - Conceptualization, Methodology, Article writing, Formal analysis.

Dr. Usha A : Review of the manuscript, Formal analysis, Suggestions given for correction of manuscript.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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